"Mémoire sur la propagation du movement dans un fluide indéfini (première Partie)," Journ. de Math. (4th series), tome III (1887), 477-492.

Memoir on the propagation of motion in an indefinite fluid (Part one)

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1. The theory of propagation of motion in an indefinite fluid remains incomplete up to now. One can scarcely treat the case of the perfect gas, at least if one seeks to study the phenomenon with some degree of rigor. Moreover, one has introduced some hypotheses into the equations that hydrodynamics has provided for the representation of the motion of these bodies, that are, it is true, disguised behind the name of *approximations*, but which singularly alter the value of the results that one may deduce.

The lack of success of the tentative efforts made up to this day seems to suggest that geometers may only obtain the expression for the velocity of propagation of motion crudely by means of the integrals of the equations, integrals that remain unknown up to now, at least in a general form, and that we do not seem close to discovering.

In this work, I would like to show that the analytical expression for the velocity of propagation of motion in an indefinite fluid is easily obtained in the most general manner by the simple consideration of the equations of hydrodynamics, without actually needing to be preoccupied with the form of the integrals.

For this, it will suffice for me to generalize the principles that I made use of in a previous work ([2]), which was, moreover, dedicated in a large part to a particular case of fluid motion where the motion takes place in parallel sections, in such a manner that in each section the velocity of all points is the same at each instant and normal to the plane of the section.

2. In this first part, I will take for my point of departure the well-known equations of hydrodynamics, such as were established by Euler. In the second part, I will re-address the question by taking the equations of Lagrange, which permit us to extend the theory to some cases where the Euler equations become inapplicable.

Referring the fluid to three rectangular coordinate axes Ox, Oy, and Oz, let x, y, z be the coordinates of a point of the fluid at the instant t, and let u, v, w be the components of

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([1]) This posthumous memoir, which was recovered from the papers left by Hugoniot, was communicated to us by Léauté.

([2]) *Mémoire sur la propagation du movement dans le corps, et spécialement dans les gaz parfaits,* presented at l'Académie des Sciences on 26 October 1885.

its velocity. Finally, for the same point let ρ be the density, p be the pressure, and X, Y, Z be the components of the exterior force acting on a unit mass.

The components X, Y, Z may be given functions of x, y, z, and t, but they may also depend upon the state of the fluid at the instant considered. This is what happens, for example, when one takes into account the mutual attractions that are exerted by the various molecules. In the sequel, one will always suppose that the components X, Y, Z are independent of the acceleration; the results will still remain the same if these components are functions of the velocities.

Having said this, the Euler equations are:

$$(1) \qquad \begin{cases} \dfrac{1}{\rho}\dfrac{\partial\rho}{\partial x} = \mathrm{X} - \dfrac{\partial u}{\partial t} - u\dfrac{\partial u}{\partial x} - v\dfrac{\partial u}{\partial y} - w\dfrac{\partial u}{\partial z}, \\ \dfrac{1}{\rho}\dfrac{\partial\rho}{\partial y} = \mathrm{Y} - \dfrac{\partial v}{\partial t} - u\dfrac{\partial v}{\partial x} - v\dfrac{\partial v}{\partial y} - w\dfrac{\partial v}{\partial z}, \\ \dfrac{1}{\rho}\dfrac{\partial\rho}{\partial z} = \mathrm{Z} - \dfrac{\partial w}{\partial t} - u\dfrac{\partial w}{\partial x} - v\dfrac{\partial w}{\partial y} - w\dfrac{\partial w}{\partial z}, \\ \dfrac{\partial\rho}{\partial t} + \dfrac{\partial(\rho u)}{\partial x} + \dfrac{\partial(\rho v)}{\partial y} + \dfrac{\partial(\rho w)}{\partial z} = 0. \end{cases}$$

3. The preceding four equations are the same for all fluids, i.e., for all bodies in which the pressure on an element is normal to the surface. They refer to five unknown functions of x, y, z, and t and may not, as a result, determine them. One must therefore add a fifth equation, but it will depend upon the nature of the fluid.

Here, I will occupy myself with only the case where the heat conductivity of the body may be neglected. The relation between pressure and density of a given element of mass is then always the same and depends only upon the initial state, although this is provided that it does not produce discontinuities, i.e., that the velocity is never subjected to a finite variation in an infinitely small time interval. The relation in question is nothing but the one that corresponds to what one calls *adiabatic relaxation* in thermodynamics. One will suppose in what follows that this relation is the same at all points of the fluid, which is true, for example, when it is homogeneous and has the same temperature at all of its points in the initial state, and one represents it by the equation:

$$(2) \qquad \rho = \mathrm{F}(p),$$

which becomes, when the fluid is a perfect gas:

$$(3) \qquad \rho = \mathrm{K}p^{1/m},$$

K denoting a constant and m, the ratio of the two specific heats.

The number of equations is then equal to that of the unknown functions.

4. When one desires to apply the preceding equations to the study of the propagation of motion in perfect gases, one assumes that the exterior forces are null, and that the expression:

$$u\,dx + v\,dy + w\,dz$$

is the exact differential of a function φ of the coordinates. From a well-known theorem of Lagrange, if this condition is satisfied at an arbitrary instant then it is true for all later instants.

Therefore, denoting the initial density by ρ_0 and the initial pressure by p_0, one has:

$$p = p_0(1 + \gamma)^{-m}, \qquad \rho = \rho_0(1 + \gamma)^{-1},$$

γ representing the cubic dilatation. One then deduces the following two equations from the Euler equations:

$$\frac{m}{m-1}\frac{p_0}{\rho_0}[(1+\gamma)^{-m+1}-1]+\frac{\partial\varphi}{\partial t}+\frac{1}{2}\left[\left(\frac{\partial\varphi}{\partial x}\right)^2+\left(\frac{\partial\varphi}{\partial y}\right)^2+\left(\frac{\partial\varphi}{\partial z}\right)^2\right]=0,$$

$$(1+\gamma)^{-1}\left(\frac{\partial\gamma}{\partial t}+\frac{\partial\varphi}{\partial x}\frac{\partial\gamma}{\partial x}+\frac{\partial\varphi}{\partial y}\frac{\partial\gamma}{\partial y}+\frac{\partial\varphi}{\partial z}\frac{\partial\gamma}{\partial z}\right)=\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}+\frac{\partial^2\varphi}{\partial z^2}.$$

Considering only small motions that alter the density of the fluid very weakly, one neglects the squares of the velocities and the powers of γ greater than unity in the first equation; it then becomes:

$$\gamma=\frac{\rho_0}{mp_0}\frac{\partial\varphi}{\partial t}=\frac{1}{a^2}\frac{\partial\varphi}{\partial t},$$

upon setting:

$$a^2=\frac{mp_0}{\rho_0}.$$

In the second equation, one neglects γ when compared to unity and the products such as $\frac{\partial\varphi}{\partial t}\frac{\partial\gamma}{\partial x}$ when compared to $\frac{\partial\gamma}{\partial x}$; it then reduces to:

$$\frac{\partial\gamma}{\partial t}=\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}+\frac{\partial^2\varphi}{\partial z^2},$$

or, by reason of the previous value that was obtained for γ:

$$\frac{\partial^2\varphi}{\partial t^2}=a^2\left(\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}+\frac{\partial^2\varphi}{\partial z^2}\right). \tag{4}$$

This is the equation that Poisson made use of in his research ([1]). One sees from the preceding that it is established only by means of approximations that must singularly distort real phenomena, and which will be very difficult to justify. Moreover, they amount to changing the hypotheses that were originally made about the properties of the gas.

It was by integrating the preceding equation by means of definite integrals that Poisson studied the propagation of motion in gases. I will show later on that if one limits oneself to the study of the velocity of propagation of motion then it is quite useless to determine the integrals of equation (4); however, before one does this it is important to define the *velocity of propagation* with precision.

5. When a motion is governed by a system of partial differential equations, any system of integrals of these equations represents a possible motion. One must intend this to mean that if an indefinite mass of fluid is found, at a given moment, to be animated with a motion that the systems of integrals represents then this will be true for all later moments. Moreover, if one considers a finite portion of fluid bounded by a surface and animated, at a given moment, with a motion that is defined by a system of integrals, then the motion always continues to be represented by the same integrals if the conditions imposed on the surface are compatible with the latter ones. This is what happens, for example, when the exterior pressure applied to the surface has precisely the value that furnishes these integrals at each instant.

However, if one wishes to modify the conditions imposed on a certain part of the bounded surface then it is necessary that it give rise to a motion that is different from the first one in a neighborhood of that surface portion, which is developed into the fluid and extended gradually by substituting for the original fluid in that part. There thus exists at each instant a certain surface in the fluid mass that separates the parts of this fluid, each of which are animated with a different motion, and which displace by deforming in time. One will then give that surface the name of *wave surface*, and one will say that the second motion *propagates* into the first one.

Let S be the wave surface at the instant t and S′, the position that it occupies at the instant $t + dt$. Construct the normal to a point x, y, z of the first surface and let dn denote the portion of that normal that lies between S and S′. The ration dn/dt represents the *velocity of propagation* of the second motion into the first one.

The preceding definitions generalize the ones that I gave in my previously-cited memoir *Sur la propagation du mouvement dans les corps.* In that work, I showed that for a gas that is confined to a pipe, for example, it is not sufficient to juxtapose two arbitrary possible motions in order for the phenomenon of propagation to result. It is further necessary that the representative surfaces agree along a common *characteristic.*

This is obviously true, moreover, when, instead of motion in parallel sections, one considers the most general motion of a body. If, at a given moment, two contiguous portions of the same body are each animated by a different motion then the phenomenon that takes place at the separation surface is generally complex and has, as a consequence, the birth of new motions that are represented by systems of integrals that are different from the first ones.

([1]) *Journal de l'École Polytechnique,* t. VII.

There is propagation when, at the instant $t + dt$, the motion of the body is again represented by the two original systems of integrals, the separation surface being nevertheless displaced and deformed only infinitesimally.

When one motion may propagate in another then one will say that the two motions are *compatible* with each other.

6. In the study that follows, one will essentially suppose that continuity is not disturbed. Now, when a motion A is propagated in a motion B then a point that was first animated by the motion A will be instantaneously animated with the motion B at the moment when the wave surface passes through this point. To the two motions A and B, there must correspond the same values of the velocities for the points that, at the instant t, are found on the wave surface.

7. Before starting with the general equations of hydrodynamics, one first considers, for more simplicity, just the equation that Poisson made use of:

$$\frac{\partial^2 \varphi}{\partial t^2} = a^2 \left(\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} \right). \tag{4}$$

The cubic dilatation is proportional to $\frac{\partial \varphi}{\partial t}$, and the components of the velocity are equal to $\frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y}, \frac{\partial \varphi}{\partial z}$, respectively.

Let φ_1 and φ_2 be two integrals that represent compatible motions. If one lets:

$$\varphi_1 - \varphi_2 = \Phi$$

then the function Φ obviously satisfies equation (4), in such a way that one has:

$$\frac{\partial^2 \Phi}{\partial t^2} = a^2 \left(\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} \right). \tag{5}$$

Continuity of the velocities demands that one have, for all points of the wave surface:

$$\frac{\partial \Phi}{\partial x} = 0, \qquad \frac{\partial \Phi}{\partial y} = 0, \qquad \frac{\partial \Phi}{\partial z} = 0.$$

Moreover, it is easy to see that if the velocities do not experience any brief variation then the dilatation cannot vary briefly, either. Therefore, along the wave surface the two values of the dilatation must be the same, in such a way that:

$$\frac{\partial \varphi_1}{\partial t} - \frac{\partial \varphi_2}{\partial t} = 0, \qquad \frac{\partial \Phi}{\partial t} = 0.$$

The four partial derivatives of the function Φ are thus null along the wave surface. Therefore, if λ, μ, and ν denote the direction cosines of the normal to this surface at a point then one has three systems of equations:

$$(6)\qquad \begin{cases} \dfrac{\lambda}{\dfrac{\partial^2\Phi}{\partial x^2}} = \dfrac{\mu}{\dfrac{\partial^2\Phi}{\partial x\partial y}} = \dfrac{\nu}{\dfrac{\partial^2\Phi}{\partial x\partial z}}, \\ \dfrac{\lambda}{\dfrac{\partial^2\Phi}{\partial x\partial y}} = \dfrac{\mu}{\dfrac{\partial^2\Phi}{\partial y^2}} = \dfrac{\nu}{\dfrac{\partial^2\Phi}{\partial y\partial z}}, \\ \dfrac{\lambda}{\dfrac{\partial^2\Phi}{\partial x\partial z}} = \dfrac{\mu}{\dfrac{\partial^2\Phi}{\partial y\partial z}} = \dfrac{\nu}{\dfrac{\partial^2\Phi}{\partial z^2}}. \end{cases}$$

Suppose one of the partial derivatives of Φ is null at the point x, y, z of the wave surface that corresponds to the point t. The same must be true for the instant $t + dt$ at the points whose coordinates are:

$$x + \lambda\, dn, \qquad y + \mu\, dn, \qquad z + \nu\, dn,$$

since this point belongs to the wave surface that corresponds to the time $t + dt$. One has the four equations:

$$\frac{\partial^2\Phi}{\partial t^2} + \frac{dn}{dt}\left(\lambda\frac{\partial^2\Phi}{\partial x\partial t} + \mu\frac{\partial^2\Phi}{\partial y\partial t} + \nu\frac{\partial^2\Phi}{\partial z\partial t}\right) = 0,$$
$$\frac{\partial^2\Phi}{\partial x\partial t} + \frac{dn}{dt}\left(\lambda\frac{\partial^2\Phi}{\partial x^2} + \mu\frac{\partial^2\Phi}{\partial x\partial y} + \nu\frac{\partial^2\Phi}{\partial x\partial z}\right) = 0,$$
$$\frac{\partial^2\Phi}{\partial y\partial t} + \frac{dn}{dt}\left(\lambda\frac{\partial^2\Phi}{\partial x\partial y} + \mu\frac{\partial^2\Phi}{\partial y^2} + \nu\frac{\partial^2\Phi}{\partial y\partial z}\right) = 0,$$
$$\frac{\partial^2\Phi}{\partial z\partial t} + \frac{dn}{dt}\left(\lambda\frac{\partial^2\Phi}{\partial x\partial z} + \mu\frac{\partial^2\Phi}{\partial y\partial z} + \nu\frac{\partial^2\Phi}{\partial z^2}\right) = 0.$$

The elimination of the quantities $\dfrac{\partial^2\Phi}{\partial x\,\partial t}, \dfrac{\partial^2\Phi}{\partial y\,\partial t}, \dfrac{\partial^2\Phi}{\partial z\,\partial t}$ gives:

$$\frac{\partial^2\Phi}{\partial t^2} = \left(\frac{dn}{dt}\right)^2\left[\lambda^2\left(\frac{\partial^2\Phi}{\partial x^2} + \frac{\mu}{\lambda}\frac{\partial^2\Phi}{\partial x\,\partial y} + \frac{\nu}{\lambda}\frac{\partial^2\Phi}{\partial x\,\partial z}\right) + \mu^2\left(\frac{\lambda}{\mu}\frac{\partial^2\Phi}{\partial x\,\partial y} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\nu}{\mu}\frac{\partial^2\Phi}{\partial y\,\partial z}\right)\right.$$

$$+\nu^2\left(\frac{\lambda}{\nu}\frac{\partial^2\Phi}{\partial x\,\partial y}+\frac{\mu}{\nu}\frac{\partial^2\Phi}{\partial y^2}+\frac{\partial^2\Phi}{\partial z^2}\right)\Bigg],$$

or, taking equations (6) into account:

$$\begin{aligned}\frac{\partial^2\Phi}{\partial t^2}&=\left(\frac{dn}{dt}\right)^2(\lambda^2+\mu^2+\nu^2)\left(\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}+\frac{\partial^2\Phi}{\partial z^2}\right)\\&=\left(\frac{dn}{dt}\right)^2\left(\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}+\frac{\partial^2\Phi}{\partial z^2}\right).\end{aligned}$$

The comparison of this equation with equation (5) demands that one have:

$$\left(\frac{dn}{dt}\right)^2=a^2.$$

The formula that gives the velocity of sound or, more generally, the velocity of propagation of one motion in another, is thus found to be established without having to be preoccupied with the form of the integrals.

8. I would now like to apply an analogous method to the general equations of hydrodynamics:

$$(1')\qquad\begin{cases}\dfrac{1}{\rho}\dfrac{\partial\rho}{\partial x}=\mathrm{X}-\dfrac{\partial u}{\partial t}-u\dfrac{\partial u}{\partial x}-v\dfrac{\partial u}{\partial y}-w\dfrac{\partial u}{\partial z},\\[2mm]\dfrac{1}{\rho}\dfrac{\partial\rho}{\partial y}=\mathrm{Y}-\dfrac{\partial v}{\partial t}-u\dfrac{\partial v}{\partial x}-v\dfrac{\partial v}{\partial y}-w\dfrac{\partial v}{\partial z},\\[2mm]\dfrac{1}{\rho}\dfrac{\partial\rho}{\partial z}=\mathrm{Z}-\dfrac{\partial w}{\partial t}-u\dfrac{\partial w}{\partial x}-v\dfrac{\partial w}{\partial y}-w\dfrac{\partial w}{\partial z},\\[2mm]\dfrac{\mathrm{F}'(p)}{\mathrm{F}(p)}\left(\dfrac{\partial p}{\partial t}+u\dfrac{\partial p}{\partial x}+v\dfrac{\partial p}{\partial y}+w\dfrac{\partial p}{\partial z}\right)+\dfrac{\partial u}{\partial x}+\dfrac{\partial v}{\partial y}+\dfrac{\partial w}{\partial z}=0.\end{cases}$$

The last of equations (1′) is the equation of continuity where we have replaced ρ with its value F(p).

Let u_1, v_1, w_1, ρ_1, p_1; u_2, v_2, w_2, ρ_2, p_2 be two systems of integrals that represent compatible motions. Setting:

$$u_1-u_2=\mathrm{U},\qquad v_1-v_2=\mathrm{V},\qquad w_1-w_2=\mathrm{W},\qquad p_1-p_2=\mathrm{P},$$

continuity demands that one have along the wave surface:

$$(7)\qquad \mathrm{U}=0,\qquad \mathrm{V}=0,\qquad \mathrm{W}=0.$$

Moreover, since the velocity varies in a continuous manner, it is necessary that the same be true for the density. Thus, along the wave surface one has:

$$\rho_1 = \rho_2 ,$$

and, since $\rho = \mathrm{F}(p)$ it then results that:

(8) $$p_1 - p_2 = 0, \qquad \mathrm{P} = 0$$

along that surface.

If one then successively substitutes the two systems of integrals in any of equations (1′), and if one attributes values to the variables x, y, z, and t that correspond to a point of the surface then when one takes the difference the quantities X, Y, and Z will disappear. The functions u, v, w, ρ, and p take unchanged values, in such a way that one may dispense with the indices; one thus obtains four equations:

(9) $$\begin{cases} \dfrac{1}{\rho}\dfrac{\partial \mathrm{P}}{\partial x} = -\dfrac{\partial \mathrm{U}}{\partial t} - u\dfrac{\partial \mathrm{U}}{\partial x} - v\dfrac{\partial \mathrm{U}}{\partial y} - w\dfrac{\partial \mathrm{U}}{\partial z}, \\ \dfrac{1}{\rho}\dfrac{\partial \mathrm{P}}{\partial y} = -\dfrac{\partial \mathrm{V}}{\partial t} - u\dfrac{\partial \mathrm{V}}{\partial x} - v\dfrac{\partial \mathrm{V}}{\partial y} - w\dfrac{\partial \mathrm{V}}{\partial z}, \\ \dfrac{1}{\rho}\dfrac{\partial \mathrm{P}}{\partial z} = -\dfrac{\partial \mathrm{W}}{\partial t} - u\dfrac{\partial \mathrm{W}}{\partial x} - v\dfrac{\partial \mathrm{W}}{\partial y} - w\dfrac{\partial \mathrm{W}}{\partial z}, \\ \dfrac{\mathrm{F}'(p)}{\mathrm{F}(p)}\left(\dfrac{\partial \mathrm{P}}{\partial t} + u\dfrac{\partial \mathrm{P}}{\partial x} + v\dfrac{\partial \mathrm{P}}{\partial y} + w\dfrac{\partial \mathrm{P}}{\partial z}\right) + \dfrac{\partial \mathrm{U}}{\partial x} + \dfrac{\partial \mathrm{V}}{\partial y} + \dfrac{\partial \mathrm{W}}{\partial z} = 0, \end{cases}$$

which are valid for each point of the wave surface.

9. Denoting, as in the preceding, the direction cosines of the normal at a point of the wave surface by λ, μ, ν, equations (7) and (8) demand that one have:

(10) $$\begin{cases} \dfrac{\lambda}{\frac{\partial \mathrm{U}}{\partial x}} = \dfrac{\mu}{\frac{\partial \mathrm{U}}{\partial y}} = \dfrac{\nu}{\frac{\partial \mathrm{U}}{\partial z}}, \\ \dfrac{\lambda}{\frac{\partial \mathrm{V}}{\partial x}} = \dfrac{\mu}{\frac{\partial \mathrm{V}}{\partial y}} = \dfrac{\nu}{\frac{\partial \mathrm{V}}{\partial z}}, \\ \dfrac{\lambda}{\frac{\partial \mathrm{W}}{\partial x}} = \dfrac{\mu}{\frac{\partial \mathrm{W}}{\partial y}} = \dfrac{\nu}{\frac{\partial \mathrm{W}}{\partial z}}, \\ \dfrac{\lambda}{\frac{\partial \mathrm{P}}{\partial x}} = \dfrac{\mu}{\frac{\partial \mathrm{P}}{\partial y}} = \dfrac{\nu}{\frac{\partial \mathrm{P}}{\partial z}}. \end{cases}$$

On the other hand, by differentiating each of equations (7) and (8) along the normal to the wave surface one obtains:

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(11)\qquad \begin{cases}
\dfrac{\partial \mathrm{U}}{\partial t}+\dfrac{\partial n}{\partial t}\left(\lambda\dfrac{\partial \mathrm{U}}{\partial x}+\mu\dfrac{\partial \mathrm{U}}{\partial y}+\nu\dfrac{\partial \mathrm{U}}{\partial z}\right)=0,\\[2ex]
\dfrac{\partial \mathrm{V}}{\partial t}+\dfrac{\partial n}{\partial t}\left(\lambda\dfrac{\partial \mathrm{V}}{\partial x}+\mu\dfrac{\partial \mathrm{V}}{\partial y}+\nu\dfrac{\partial \mathrm{V}}{\partial z}\right)=0,\\[2ex]
\dfrac{\partial \mathrm{W}}{\partial t}+\dfrac{\partial n}{\partial t}\left(\lambda\dfrac{\partial \mathrm{W}}{\partial x}+\mu\dfrac{\partial \mathrm{W}}{\partial y}+\nu\dfrac{\partial \mathrm{W}}{\partial z}\right)=0,\\[2ex]
\dfrac{\partial \mathrm{P}}{\partial t}+\dfrac{\partial n}{\partial t}\left(\lambda\dfrac{\partial \mathrm{P}}{\partial x}+\mu\dfrac{\partial \mathrm{P}}{\partial y}+\nu\dfrac{\partial \mathrm{P}}{\partial z}\right)=0.
\end{cases}
$$

Equations (9), (10), and (11) are sixteen in number. They are, moreover, homogeneous with respect to the sixteen derivatives $\dfrac{\partial \mathrm{U}}{\partial x},\dfrac{\partial \mathrm{U}}{\partial y}, \ldots$ Therefore, if all of these derivatives are simultaneously null then the determinant must be annulled.

In other words, the elimination of the sixteen derivatives will furnish an equation that gives a condition between the values of the functions u, v, w, ρ, and p and the velocity of propagation $\dfrac{dn}{dt}$, from which one can deduce the value of that velocity of propagation.

10. The elimination is carried out very simply by first expressing all of the derivatives of a given function by means of only one of them and transporting the values thus obtained into equations (9).

Equations (10) first give:

$$\frac{\partial \mathrm{U}}{\partial y}=\frac{\mu}{\lambda}\frac{\partial \mathrm{U}}{\partial x},\quad \frac{\partial \mathrm{U}}{\partial z}=\frac{\nu}{\lambda}\frac{\partial \mathrm{U}}{\partial x}.$$

The first of equations (11) then gives:

$$\frac{\partial \mathrm{U}}{\partial t}+\frac{1}{\lambda}\frac{dn}{dt}\frac{\partial \mathrm{U}}{\partial x}=0.$$

Likewise, for V, W, and P the partial derivatives are expressed linearly as functions of $\dfrac{\partial \mathrm{V}}{\partial x},\dfrac{\partial \mathrm{W}}{\partial x}$, and $\dfrac{\partial \mathrm{P}}{\partial x}$. Substituting in equations (9), one obtains:

$$\frac{\lambda}{\rho}\frac{\partial \mathrm{P}}{\partial x}=\left[\frac{dn}{dt}-(\lambda u+\mu v+\nu w)\right]\frac{\partial \mathrm{U}}{\partial x},$$
$$\frac{\mu}{\rho}\frac{\partial \mathrm{P}}{\partial x}=\left[\frac{dn}{dt}-(\lambda u+\mu v+\nu w)\right]\frac{\partial \mathrm{V}}{\partial x},$$

$$\frac{\nu}{\rho}\frac{\partial \mathrm{P}}{\partial x} = \left[\frac{dn}{dt} - (\lambda u + \mu v + \nu w)\right]\frac{\partial \mathrm{W}}{\partial x},$$

$$\frac{\mathrm{F}'(p)}{\mathrm{F}(p)}\left[\frac{dn}{dt} - (\lambda u + \mu v + \nu w)\right]\frac{\partial \mathrm{P}}{\partial x} = \lambda\frac{\partial \mathrm{U}}{\partial x} + \mu\frac{\partial \mathrm{V}}{\partial x} + \nu\frac{\partial \mathrm{W}}{\partial x}.$$

Multiplying the first three equations by λ, μ, ν, respectively, adding then, and then dividing the sum by the fourth equation, one finds, upon remarking that $\rho = \mathrm{F}(p)$:

$$\mathrm{F}'(p)\left[\frac{dn}{dt} - (\lambda u + \mu v + \nu x)\right]^2 = 1.$$

From this, one deduces the following two values for the velocity of propagation:

$$\frac{dn}{dt} = \lambda u + \mu v + \nu w \pm \sqrt{\frac{1}{\mathrm{F}'(p)}}.$$

The quantity $\lambda u + \mu v + \nu w$ is nothing but the projection of the velocity at the point considered onto the normal to the wave surface. Representing this projection by N and remarking:

$$\frac{1}{\mathrm{F}'(p)} = \frac{dp}{d\rho},$$

the preceding formula may be written:

$$\frac{dn}{dt} = \mathrm{N} \pm \sqrt{\frac{dp}{d\rho}}. \tag{12}$$

We remark that the right-hand side of that equation is determined as soon as one of the motions is given, which gives us this theorem of remarkable generality:

The velocity of propagation of a motion in a fluid depends upon the state of the fluid, but it is independent of the nature of the motion that is propagating, provided that it produces no discontinuities.

11. The preceding formula gives the *absolute* value of the velocity of propagation; it is found to be referred to fixed coordinate axes. The two values are different according to whether the propagation takes place in one sense of the unit normal or the other.

However, it is more natural to refer this velocity to the fluid itself, which must be regarded as displacing in the direction of the normal with a velocity N. The velocity of propagation is then to be taken equal to:

$$\pm\sqrt{\frac{1}{\mathrm{F}'(p)}} = \pm\sqrt{\frac{dp}{d\rho}}\,;$$

these two values are then equal and of opposite sign. The analytic expression is, moreover, the same as the one that was obtained by considering the motion of fluids in pipes.

The general formula (12) gives an account of a fact that is well-known to physicists: When the atmosphere is agitated the normal velocity of sound is augmented or diminished by a quantity equal to the projection of the wind velocity onto the direction of propagation.

In the particular case where the exterior forces are null, the fluid may obviously remain at rest, in such a way that $u = 0$, $v = 0$, $w = 0$, $\rho = \rho_0$, $p = p_0$ constitute a system of integrals. If, in formula (12), one attributes the value p_0 to p, and if one sets, in addition, $\mathrm{N} = 0$ then one obtains:

$$\frac{dn}{dt} = \pm\frac{1}{\sqrt{\mathrm{F}'(p_0)}}\,,$$

which is the velocity of propagation of motion in a fluid at rest. This velocity is independent of the nature of the motion that propagates, provided that it produces no discontinuities.

12. The right-hand side of the first three equations (9) represents the components of the acceleration relative to the point considered of the wave surface. Now one has, from (10):

$$\frac{\lambda}{\frac{1}{\rho}\frac{\partial \mathrm{P}}{\partial x}} = \frac{\mu}{\frac{1}{\rho}\frac{\partial \mathrm{P}}{\partial y}} = \frac{\nu}{\frac{1}{\rho}\frac{\partial \mathrm{P}}{\partial z}}\,,$$

which shows that *the relative acceleration is directed along the normal to the wave surface.*

It is well-known that if one considers an element of volume in the fluid then the quantities:

$$\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\,, \qquad \frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\,, \qquad \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}$$

are annulled for any point of the wave surface.

Now, upon multiplying the second of equations (9) by V and the third one by $-\mu$ and adding, the left-hand side is annulled by virtue of equations (10). As for the right-hand side, it becomes, taking into account equations (10) and (11):

$$\left[\frac{dn}{dt} - (\lambda u + \mu v + \nu w)\right]\left(\frac{\partial \mathrm{V}}{\partial z} - \frac{\partial \mathrm{W}}{\partial y}\right).$$

Since the velocity of propagation $\frac{dn}{dt}$ is different from $\lambda u + \mu v + \nu w$, it then results that one must have:

$$\frac{\partial \mathrm{V}}{\partial z} - \frac{\partial \mathrm{W}}{\partial y} = 0,$$

which proves the theorem.

13. When the fluid considered is a perfect gas one has:

$$\rho = \mathrm{F}(p) = \mathrm{K}p^{1/m},$$
$$\mathrm{F}'(p) = \frac{\mathrm{K}}{m} p^{1/m-1} = \frac{\rho}{mp}.$$

The velocity of propagation referred to the fluid is thus represented by:

$$\sqrt{\frac{mp}{\rho}}$$

and by $\sqrt{\frac{mp_0}{\rho_0}}$ when the propagation is in a medium at rest. The ordinary formula for the velocity of propagation is thus found to be established in an entirely rigorous manner; its analytic expression is, moreover, the same when one considers the propagation of one motion in another. However, the numerical value depends upon the pressure and the density that corresponds to the original motion.

14. When the fluid considered is a liquid the relation between the pressure and the density reduces to essentially:

$$\rho = \mathrm{A}p + \mathrm{B},$$

A and B denoting constants. The velocity of propagation referred to this fluid is then equal to:

$$\sqrt{\frac{1}{\mathrm{A}}}.$$

It is an absolute constant, independent of not only the motion that propagates, but also the original motion.

15. Despite its general character, the preceding theory is still incomplete, because one must suppose that the relation $\rho = \mathrm{F}(p)$ is the same for all of the points of the fluid. To go further, it is necessary to regard the function $\mathrm{F}(p)$ as depending upon the initial coordinates of the molecules. In the second Part of this memoir, I will show how one arrives at this result by taking the equations of hydrodynamics in the form that they were given by Lagrange.