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On the moment of impulse of an electromagnetic wave

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Summary

1. The decomposition of the angular momentum of the electromagnetic radiation into three terms is studied: Two of those terms present a formal resemblance with an orbital momentum and a spin momentum, respectively. The third one, which is a surface term, has often been wrongfully neglected. A new form for the flux of angular momentum is given that corresponds to that decomposition. It allows one to make the calculation of that flux without it being necessary to know the terms in the potentials and fields that have a magnitude of $1 / R^2$ (in particular, without knowing the longitudinal fields).

2. The general considerations of the first paragraph are applied to the electric dipole radiation in order to elucidate a paradox in **de Broglie**'s theory of the photon that was pointed out by **J. Géhéniau**.

3. The case of the plane wave is also studied within the context of the first paragraph. The **Abraham-Sommerfeld** formula is established for a quasi-plane wave. Finally, it is pointed out that there exists an ambiguity in the definition of the angular momentum of a wave that is rigorously planar.

1. Moment of impulse and flux of moment of impulse. –

a) The classical expression for the moment of impulse of an electromagnetic field that is contained in a volume *v* is well known. It is defined by ([1]):

$$\mathcal{M} = \frac{1}{c}\int_v \mathbf{R} \wedge (\mathbf{E} \wedge \mathbf{H})\, dv \tag{1}$$

if the moment is defined with respect to the origin of the coordinate axes, and **R**, **E**, and **H** denote the radius vector, electric field, and magnetic field, respectively. Furthermore, let **A** be the vector potential. Up to an interaction term between the radiation and the possible sources that are present in *v*, the expression (1) can be decomposed into the sum of three terms ([*]):

([1]) [Translator's note: Apparently, the author is using ^ for the cross product and × for the dot product, while the dot means scalar multiplication.]

([*]) It is in that decomposed form that one obtains the partial contribution to the moment of total impulse that is provided by the Lagrange function of radiation $-\frac{1}{2}(E^2 - H^2)$. That method of obtaining the moment of impulse was presented independently by **F. J. Belinfante** [**1**] and **L. Rosenfeld** [**2**]; the former

$$\mathbf{M}_0 = \frac{1}{c}\sum_{i=x}^{y,z} \int_v E_i\, \mathbf{R} \wedge \operatorname{grad} A_i\, dv\,, \tag{2a}$$

$$\mathbf{M}_s = \frac{1}{c}\int_v \mathbf{E} \wedge \mathbf{A}\, dv\,, \tag{2b}$$

$$\mathbf{m} = -\frac{1}{c}\int_s \mathbf{R} \wedge \mathbf{A} \cdot \mathbf{E} \times \mathbf{n}\, ds\,, \tag{2c}$$

in which s is the surface that bounds the volume, and **n** is the unit vector that is carried by the exterior normal to the surface element ds. If one sets:

$$\mathbf{M} = \mathbf{M}_0 + \mathbf{M}_s + \mathbf{m} \tag{3}$$

then one will have, rigorously:

$$\mathcal{M} = \mathbf{M} + \frac{1}{c}\int_v \mathbf{R} \wedge \mathbf{A} \operatorname{div} \mathbf{E}\, dv\,. \tag{4}$$

The first term $\mathbf{M}_0$ of **M**, in which the operator **R** ^ grad appears, has the form of an orbital moment, while the second one $\mathbf{M}_s$ has the form of an intrinsic moment or spin, since the quantity that appears under the integration sign is independent of the point with respect to which the moment is calculated.

In general, it is not sufficient to take a volume that extends to infinity in order to make **m** become negligible, since at a great distance R from the sources of radiation, **A** will have order of magnitude $1 / R$, while the product $\mathbf{R} \wedge \mathbf{A} \cdot \mathbf{E}\ \ \mathbf{n}$ will have order $1 / R^2$, and the surface integral **m** cannot possible be canceled by angular integration. If that is, in fact, the case then we cannot, however, further say that the decomposition that is then realized by $\mathcal{M} = \mathbf{M}_0 + \mathbf{M}_s$ is unique and that it is physically meaningful, since the terms $\mathbf{M}_0$ and $\mathbf{M}_s$ (and **m**) do not separately possess gauge invariance (Ger: *Eichinvarianz*). That is why one cannot change **m**, **M**, and $\mathcal{M}$ by adding the gradient of $k \log R$, which must be $k\,\mathbf{R} / R^2$ (k is a constant with suitable dimensions), to **A**, although $\mathbf{M}_0$ and $\mathbf{M}_s$ can become individually very different from what they would be with another choice of potential.

The case of an electromagnetic field is therefore essentially different from the case of the meson field, for which, due to the uniqueness in the definition of the potentials, the decomposition into an orbital moment and a spin moment will be unique. In the domain of frequencies that are presently observable, meson fields decrease exponentially, and the surface term **m** will disappear when one integrates over all space. On the contrary, if the frequency of the meson field is sufficiently large for there to be a "radiation field" then there will no longer be an exponential decrease in the meson field [**3**], and the surface integral **m** will not be zero, as was shown by **J. Serpe** [**4**]. This case of the meson field

systematically neglected all surface integrals. Formulas (2), (3), and (4) are the application of formulas (74), (75), and (96) in the paper by **L. Rosenfeld** to the particular case of electromagnetism.

will then become similar to that of the electromagnetic field, and since there is a transport of moment of impulse, it will be the flux of that quantity that crosses a given surface that will be interesting to know, rather than the moment of impulse that is contained in the volume that is bounded by the surface.

In electromagnetism, it is indispensable to *define* the moment of impulse with the surface term **m**, since the sum $\mathbf{M}_0 + \mathbf{M}_s$ does not possess gauge invariance. On the contrary, it is permissible to drop that term from the definition of the moment of impulse of a field – such as the meson field, whose potentials are defined uniquely, since $\mathbf{M}_0 + \mathbf{M}_s$, like **M**, satisfies a conservation law. In regard to that, we point that **L**. **Rosenfeld** [**5**] has studied the conditions under which the moment of impulse flux of a radiation field will be independent of whether one has defined the moment of impulse of such a field with or without the surface term **m**.

b) Returning to the case of electromagnetism more especially, it is then natural to now look into whether it is possible to put the moment of impulse flux into a form that is analogous to the one that was just given (4) for the total moment of impulse that is contained in a volume.

If the charges do not cross the surface *s* for which it is calculated then we will know that the moment of impulse flux that leaves the volume per unit time will be:

$$\Phi = -\int_s \mathbf{R} \wedge T\,\mathbf{n}\,ds\,, \tag{5}$$

in which T is the **Maxwell** tensor, and **n** is the unit vector that is carried by the exterior normal to the surface element *ds*. We will then have:

$$T\,\mathbf{n} = \mathbf{E}\cdot\mathbf{E}\times\mathbf{n} + \mathbf{H}\cdot\mathbf{H}\times\mathbf{n} - \tfrac{1}{2}(E^2 + H^2)\,\mathbf{n}\,. \tag{6}$$

From now on, we shall suppose that the sources are situated in a neighborhood of the origin of the reference axes, and that the surface *s* is a sphere of radius R that is centered at that origin. It will result from this that $\mathbf{n} = \mathbf{R}^0$, if $\mathbf{R}^0 = \mathbf{R} / R$. Moreover, we suppose that the radius R is very large, and from now on, we shall neglect the terms in $\mathbf{R} \wedge T\,\mathbf{n}$ that have order $1 / R^3$, as opposed to the ones that have order $1 / R^2$. Since *ds* has order R^2, that amounts to neglecting the terms in the flux that decrease when the surface *s* is stretched to infinity, in order to keep only the constant terms: viz., "the flux at infinity." If one then decomposes **E** and **H** into their transversal components (i.e., normal to **R**) in $1 / R$ and $1 / R^2$ and their longitudinal ones (i.e., parallel to **R**) in $1 / R^2$, without it being necessary to calculate them as functions of the sources, then it will be easy to verify that one will have:

$$\mathbf{E} \wedge \mathbf{H} = -T\,\mathbf{n}, \tag{7}$$

up to terms in $1 / R^4$.

From this, we deduce that, with the adopted approximation, the flux that leaves *s* can be written:

$$\Phi = \int \mathbf{R} \wedge (\mathbf{E} \wedge \mathbf{H})\,ds \tag{8}$$

or

$$\Phi = \sum_i \int E_i \, \mathbf{R} \wedge \text{grad}\, A_i \, ds + \int \mathbf{E} \wedge \mathbf{A} \, ds - \int (\mathbf{E} \times \text{grad}) (\mathbf{R} \wedge \mathbf{A}), \tag{9}$$

since we have:

$$\mathbf{R} \wedge (\mathbf{E} \wedge \text{rot}\, \mathbf{A}) = \sum_i E_i \, \mathbf{R} \wedge \text{grad}\, A_i + \mathbf{E} \wedge \mathbf{A} - (\mathbf{E} \times \text{grad}) (\mathbf{R} \wedge \mathbf{A}),$$

identically.

We let $\mathcal{M}(s + \Delta s)$ denote the moment of impulse that is contained in a sphere of radius $R + c\, \Delta t$, which will be represented by $s + \Delta s$, and let $\mathcal{M}(s)$ denote the moment of impulse that is contained in the sphere s. Up to higher-order infinitesimals in Δt, the flux that leaves s during the time interval t, $t + \Delta t$ will be:

$$\Phi \, \Delta t = \mathcal{M}(s + \Delta s) - \mathcal{M}(s), \tag{10}$$

since a comparison of (1) and (8) will show that it is the moment of impulse that is contained between the spheres $s + \Delta s$ and s. Since the charge density $\rho = \text{div}\, \mathbf{E}$ is zero on s, we will have, by virtue of (10) and (4), and upon utilizing the decomposition (3) of **M**:

$$\Phi \, \Delta t = \sum_i \int E_i \, \mathbf{R} \wedge \text{grad}\, A_i \, ds \, \Delta t + \int \mathbf{E} \wedge \mathbf{A} \, ds \, \Delta t$$
$$- \frac{1}{c}\left[\int_{s+\Delta s} \mathbf{R} \wedge \mathbf{A} \cdot \mathbf{E} \times \mathbf{n} \, ds - \int_s \mathbf{R} \wedge \mathbf{A} \cdot \mathbf{E} \times \mathbf{n} \, ds\right],$$

which is a formula that can be deduced immediately from (9), moreover.

We have supposed that we can arrange, by gauge invariance, for the vector potential **A** to have order of magnitude $1 / R$. Since it is a "retarded quantity," that means that (*):

$$\mathbf{A}(\mathbf{R}, t) = \frac{1}{R} \mathbf{a}(\mathbf{R}^0, t - R / c) + O(1 / R^2).$$

Under those conditions, the longitudinal component of the electric field can be written in the form:

$$\mathbf{E}(\mathbf{R}, t) \times \mathbf{n} = \frac{1}{R^2} b(\mathbf{R}^0, t - R / c) + O(1 / R^3).$$

It will result from this that:

$$\int_{s+\Delta s} \mathbf{R} \wedge \mathbf{A} \cdot \mathbf{E} \times \mathbf{n} \, ds = \int_s \mathbf{R} \wedge \mathbf{A} \cdot \mathbf{E} \times \mathbf{n} \, ds - \Delta t \left[\frac{d}{dt} \int_s \mathbf{R} \wedge \frac{\mathbf{a}(\mathbf{R}^0, t)}{R} \cdot \frac{b(\mathbf{R}^0, t)}{R^2} ds\right]_{t \to t - R/c} + \ldots$$

in which the unwritten terms have order:

(*) $O(1 / R^i)$ means: + terms of order $1 / R^i$, $1 / R^{i+1}$, …

$$\left(\Delta t \frac{d}{dt}\right)^2, -\left(\Delta t \frac{d}{dt}\right)^3, \ldots$$

It is, moreover, obvious that if we calculate the temporal mean of the flux at infinity when the state of the sources is a periodic function with respect to time (or *almost-periodic*, in the sense of **H. Bohr** [**6**]) then we can write:

$$\int_{s+\Delta s} \overline{\mathbf{R} \wedge \mathbf{A} \cdot \mathbf{E} \times \mathbf{n}} \cdot ds = \int_s \overline{\mathbf{R} \wedge \mathbf{A} \cdot \mathbf{E} \times \mathbf{n}} \cdot ds = \text{vector that is constant at infinity.}$$

The symbol ____ over a quantity indicates its temporal mean. The mean flux will then reduce to the remarkable expression:

$$\bar{\Phi} = \sum_i \int \overline{E_i \, \mathbf{R} \wedge \text{grad } A_i} \, ds + \int \overline{\mathbf{E} \wedge \mathbf{A}} \, ds. \tag{11}$$

We then have to show indirectly that the temporal mean of the third term in (9) is zero.

The expression (11) shows us that it is not necessary to calculate the terms of the fields that are in $1 / R^2$ in order to obtain the temporal mean of the flux of moment of impulse at infinity. Indeed, since ds is in R^2, it is obvious that one can appeal solely to the terms of **E** and **A** that are in $1 / R$ in order to calculate the second term of (11). The same thing will be true for the first term, since the components of the operator $\mathbf{k} = \mathbf{R} \wedge$ grad at a point of s (with polar coordinates R, θ, ϕ) are such that:

$$k_x \pm i\, k_y = \pm i\, e^{\pm i\phi} \frac{\partial}{\partial \theta} - e^{\pm i\phi} \cot \theta \frac{\partial}{\partial \phi},$$

$$k_z = \frac{\partial}{\partial \phi},$$

and contain neither the factor R nor derivatives with respect to R. $\mathbf{R} \wedge \text{grad } A_i$ is then in $1 / R$, like A_i itself. That result can seem surprising, because, from (5) and (6), it would seem that it would be absolutely necessary to calculate the longitudinal components of the fields $\mathbf{E} \times \mathbf{n}$ and $\mathbf{H} \times \mathbf{n}$, which are in $1 / R^2$. However, that paradox is easily explained, because it is in the calculation of the third term of (9) (whose temporal mean we have shown to be zero *a priori*) that the longitudinal fields enter explicitly. We shall verify that in the following paragraph for the case of an electric dipole field. However, in a general manner, we easily remark that this results from the fact that the gradient of $(\mathbf{R} \wedge \mathbf{A})_i$ will contain, along with the terms in $1 / R$, some other ones in R^0 that are introduced because the gradient of A_i is not in R^2, but only in $1 / R$, like A_i itself. Indeed, the gradient of the term A_i, which is in $1 / R$, is written:

$$\text{grad}\left[\frac{1}{R} a_i(\mathbf{R}^0, t - R/c)\right] = \left\{ grad\left[\frac{1}{R} a_i(\mathbf{R}^0, t)\right] - \frac{1}{R} \dot{a}_i(\mathbf{R}^0, t) \cdot \mathbf{R}^0 \right\}_{t \to t - R/c} \qquad \left(\dot{a} = \frac{1}{c}\frac{\partial a_i}{\partial t}\right),$$

in which the first term is in $1 / R^2$ and the second one is in $1 / R$.

The expression (11) for the flux remains valid for the case of a meson radiation field, and contrary to the case of the electromagnetic field, will then yield a unique decomposition of the moment of impulse flux into an "orbital moment flux" and a "spin moment flux."

2. The electric dipole wave. – Consider the electric dipole wave that is emitted by a source whose state is a periodic function of time, such that we can write the charge density as:

$$\rho(\mathbf{r}, t) = \mathrm{Re}\,[\rho\, e^{i\omega t}],$$

in which $\rho = \rho(\mathbf{r})$. The symbol "Re [ ]" refers to "the real part of," and will generally be omitted in what follows. If we set:

$$\mathbf{R} = \int \mathbf{r}\ \rho\, e^{i\,\omega\, t}\, dv \tag{12}$$

then we will know that with the **Lorentz** choice, the potentials for the electric dipolar wave will be [**7**]:

$$\begin{aligned} \varphi &= \frac{1}{4\pi R} ik\, e^{-ikR}\left(1-\frac{i}{kR}\right)\mathbf{R}^0 \times \mathbf{P}, \\ \mathbf{A} &= \frac{1}{4\pi R} ik\, e^{-ikR} \cdot \mathbf{P}, \end{aligned} \tag{13}$$

if $k = \omega / c$ is the wave number. If we use the property of gauge invariance of **E** and **H** then we will see that these two quantities, which are written:

$$\begin{aligned} \mathbf{E} &= \frac{1}{4\pi R} e^{-ikR}\left[\left(k^2 - \frac{ik}{R}\right)\mathbf{P} + \left(\frac{3ik}{R} - k^2\right)\mathbf{R}^0 \cdot \mathbf{R}^0 \times \mathbf{P}\right], \\ \mathbf{H} &= \frac{1}{4\pi R} e^{-ikR}\left(k^2 - \frac{ik}{R}\right)\mathbf{R}^0 \wedge \mathbf{P}, \end{aligned} \tag{14}$$

can likewise be deduced from the potentials:

$$\begin{aligned} \varphi &= 0, \\ \mathbf{A} &= \frac{i}{4\pi kR} e^{-ikR}\left[\left(k^2 - \frac{ik}{R}\right)\mathbf{P} + \left(\frac{3ik}{R} - k^2\right)\mathbf{R}^0 \cdot \mathbf{R}^0 \times \mathbf{P}\right]. \end{aligned} \tag{15}$$

The expressions (13), (14), and (15) are exact, up to terms in $1 / R^3$. Let F^* denote the complex conjugate of F. From (5), the temporal mean of the flux that leaves the sphere s will be:

$$\begin{aligned}\bar{\Phi} &= \tfrac{1}{2}\int \mathbf{R}\wedge(\mathbf{E}\wedge\mathbf{H}^*)\,ds \\ &= -\frac{ik^3}{16\pi^2}\int\frac{1}{R^2}\mathbf{R}^0\wedge P^*\cdot\mathbf{R}^0\times\mathbf{P}\,ds\,, \\ &= ik^3\,\mathbf{P}^*\wedge\mathbf{P}\cdot\frac{1}{12\pi}. \end{aligned} \tag{16}$$

That result can also be obtained by starting with (9). We then have the following expression for the temporal mean of the flux:

$$\bar{\Phi} = \tfrac{1}{2}\sum_i \int E_i\mathbf{R}\wedge\operatorname{grad} A_i^*\,ds + \tfrac{1}{2}\int \mathbf{E}\wedge\mathbf{A}^*\,ds - \tfrac{1}{2}\int(\mathbf{E}\times\operatorname{grad})(\mathbf{R}\wedge\mathbf{A}^*)\,ds\,,$$

in which the three terms on the right-hand side will be referred to as the first, second, and third flux terms, respectively.

One easily verifies that the first two terms of the flux are calculated under the conditions that were predicted in the preceding paragraph and that the decomposition that appears in the expression (11) is not unique. Indeed, one finds that these two terms are equal to:

$$0 \quad\text{and}\quad ik^3\cdot\mathbf{P}^*\wedge\mathbf{P}\cdot\frac{1}{12\pi}\,,$$

respectively, with the potentials (13), and:

$$\tfrac{1}{2}ik^3\cdot\mathbf{P}^*\wedge\mathbf{P}\cdot\frac{1}{12\pi} \quad\text{and}\quad \tfrac{1}{2}ik^3\cdot\mathbf{P}^*\wedge\mathbf{P}\cdot\frac{1}{12\pi}\,,$$

respectively, with the potentials (15). In order to illustrate our results in the preceding paragraph, we further carry out the calculation of the third flux term. With the potentials (13), for example, we will have successively:

$$\mathbf{R}\wedge\dot{\mathbf{A}} = \frac{1}{4\pi R}ik\,e^{-ikR}\cdot\mathbf{R}\wedge\mathbf{P},$$

$$\operatorname{grad}(\mathbf{R}\wedge\mathbf{A})_x = \frac{1}{4\pi R}\,ik\,e^{-ikR}\,[(yP_z - zP_y)(-ik\,\dot{\mathbf{R}} - \mathbf{R}^0/R) + (0, P_z, -P_y)],$$

and if one preserves only the terms in $1 / R^2$:

$$(\mathbf{E}\times\operatorname{grad})(\mathbf{R}\wedge\mathbf{A}^*) = \frac{1}{16\pi^2 R}\,ik\,e^{-ikR}\,(-ik\,\mathbf{R}\wedge\mathbf{P}^*\cdot\mathbf{E}\times\mathbf{R}^0 - \mathbf{E}\wedge\mathbf{P}^*).$$

The integration of that expression over s demands that one must know $\mathbf{E} \times \mathbf{R}^0$, which is in $1 / R^2$, and since we can write:

$$(\mathbf{E} \times \mathrm{grad})(\mathbf{R} \wedge \mathbf{A}^*) = \frac{1}{16\pi^2 R^2} \; (3ik\, \mathbf{R}^0 \wedge \mathbf{P}^* \cdot \mathbf{R}^0 \times \mathbf{P} - ik^3 \cdot \mathbf{P} \wedge \mathbf{P}^*),$$

the result of that integration will be:

$$\frac{1}{16\pi^2} \; ik^3 \cdot \left(3 \cdot \frac{4\pi}{3} - 4\pi\right) \mathbf{P} \wedge \mathbf{P}^* = 0,$$

as we could have predicted.

These considerations allow us to explain certain results that relate to the electric dipole wave that were published by **J**. **Géhéniau** [**8**] (*). The surface term **m** does not enter into the expression that he adopted for the definition of the moment of impulse. It results from this that there is no third term in the expression for the flux that he obtained. He then likewise obtained the expression (11) without, however, remarking that its use *for the electromagnetic field* is permissible only when the vector potential is in $1 / R$, since otherwise one would have a flux at infinity that did not possess gauge invariance! The fact that the longitudinal components of **E** intervene only in the calculation of the third flux term explains how **J**. **Géhéniau** was led to say that it was characteristic of his theory that it permitted the calculation of the total flux while employing only the terms in the fields and potentials that are in $1 / R$. It is appropriate to remark that if one remarks that if **Géhéniau** did not obtain the same moment of impulse flux with the two choices of potentials (13) and (15) then that would be due to the fact that **de Broglie** did not adopt the usual definition [**15**] of the moment of spin (**). He added the term $- 1 / c \int \varphi \mathbf{H} \, dv$ to them, which is obviously zero for the potentials (13), but non-zero with the potentials (15). The origin of the present work was to explain these results of **Géhéniau**.

3. The plane wave. –

a) The electric and magnetic fields of a rigorously-elliptic, monochromatic plane wave that propagates in the Oz direction can be deduced from the complex potentials:

$$\varphi = 0, \tag{17}$$

(*) It was an application of **L. de Broglie**'s theory of the photon that **Géhéniau** treated, but he limited his calculations to an approximation that was equivalent, in practice, to annulling the rest mass of the photon. It results from this that it seems natural to expect from this that his results would coincide exactly with those of **Maxwell**'s classical theory. He confirmed that this prediction is justified for the energy flux, but not for the moment of impulse flux. That anomaly is due to the fact that the classical approximation ($\mu_0 = 0$) that **L. de Broglie** used in his definition of the moment of impulse is not gauge-invariant.

(**) Cf. [**8**], pages 17 and 37.

$$\mathbf{A} = \frac{i}{k} e^{-ikz} \mathbf{P},$$

in which:

$$\mathbf{P} = (ia, b, 0)\, e^{i\omega t} \tag{18a}$$

if the elliptic polarization is right, and:

$$\mathbf{P} = (a, ib, 0)\, e^{i\omega t} \tag{18b}$$

if it is left; a and b are real, and $\omega = ck$. Indeed, those fields are the real parts of:

$$\mathbf{E} = e^{-ikt}\, \mathbf{P}, \qquad \mathbf{H} = e^{-ikt} \cdot \mathbf{k}^0 \wedge \mathbf{P}, \tag{19}$$

respectively, in which $\mathbf{k}^0 = (0, 0, 1)$ is the unit vector in the direction of propagation of the wave.

For quite some time, **P. Ehrenfest** [**9**] has said that a plane wave – thus-defined – cannot give rise to any moment of impulse flux in the direction of the propagation of the wave; indeed, we have $(\mathbf{R} \wedge T\,\mathbf{n})_z = 0$, $\mathbf{n} \equiv \mathbf{k}^0$. Moreover, the component of the moment of impulse itself along that direction is already zero. We therefore do not find, as we would for the spherical electric dipole wave, a ratio of the mean flux Φ_z^M of moment of impulse in the direction Oz to the mean energy flux Φ^E that is given by the **Abraham-Sommerfeld** formula [**10**]:

$$\frac{\Phi_z^M}{\Phi^E} = \pm \frac{2ab}{\omega(a^2 + b^2)}, \tag{20}$$

according to whether the electric dipole moment (12) has the form (18*a*) or (18*b*), resp.

We can apply formula (9) to the calculation of the moment of impulse flux of the plane wave, since the formula (7) is rigorously verified by the fields (19). A simple calculation shows that the temporal means of the component along Oz of the three terms in (9), in which the symbol "Re [ ]" is implied, are equal to:

$$0, \ \pm\frac{abs}{k}, \ \mp\frac{abs}{k}, \tag{21}$$

respectively, for a finite portion s of the plane wave, according to whether $\mathbf{P}$ is defined by (18*a*) or (18*b*), respectively.

The fact that the third term in the flux is not zero shows us the importance of the condition that is required for the utilization of formula (11), namely, that $\mathbf{A}$ must be in $1 / R$. The results (21) likewise show us that the last two terms of the flux cancel on each unit of surface, but a rigorously plane and monochromatic wave such as (19) must be considered to extend over all space, in such a way that the decomposition (9) is, in reality, indeterminate. It is nevertheless interesting, because it will allow us to understand why, in calculating the flux per unit surface for the plane wave (19), **J. Géhéniau** [**8**] obtained a moment of impulse flux whose component along Oz is coupled to the energy flux by formula (20) precisely, if one takes into account the difference in notations. We then see that there is no reason to include that result, as well as what **J. Géhéniau** did, from some considerations of **E. Henriot** [11] on "radiation couples." One must simply attribute it to the fact that the surface integral **m** here plays an essential role in the definition of the moment of impulse, since the temporal mean of the third flux term that corresponds to it is not zero.

b) Meanwhile, one must recognize that if the **Abraham-Sommerfeld** formula is valid for the wave that is defined by the fields (19) then it will exhibit the interesting fact that a plane wave can give up its moment of impulse to a perfectly absorbing body, and thus make it begin rotating (*). However, it is important to remark that the plane wave that has been in question up to now extends over all space and does not possess any physical reality (it constitutes a simple solution to the **Maxwell** equations for a pure field). We must then rather expect from this that the **Abraham-Sommerfeld** formula will be verified by a physical plane wave; i.e., by a wave *of finite spatial extent*, which cannot be the case for either a rigorously-plane wave or a rigorously-monochromatic one, but only for a packet of waves with different frequencies and directions of propagation (†).

The considerations that follow have precisely the objective of showing that the moment of impulse and energy that are contained in such a wave will verify the ratio (20), and that will become all the more rigorous when the wave considered differs even less from the ideal wave (19). We can hope that (as would be true for a monochromatic dipole wave) the temporal means of the flux per unit time would verify that ratio (20), because those temporal means are zero *a priori* here (**).

We assume that we can write the electric and magnetic field that constitute the wave packet that we propose to study in the form of **Fourier** integrals. Those fields will then be the real part of:

$$\mathbf{E}\,(\mathbf{R},t) = \frac{1}{(2\pi)^{3/2}}\int_{-\infty}^{+\infty}\mathbf{F}(\mathbf{k})\,e^{i\omega t - i\mathbf{k}\times\mathbf{R}}\,dk^{(3)}\,, \tag{22a}$$

and the real part of:

$$\mathbf{H}\,(\mathbf{R},t) = \frac{1}{(2\pi)^{3/2}}\int_{-\infty}^{+\infty}\mathbf{k}^0\wedge\mathbf{F}(\mathbf{k})\,e^{i\omega t - i\mathbf{k}\times\mathbf{R}}\,dk^{(3)}\,, \tag{22b}$$

(*) In quantum theory, we are led to interpret the **Abraham-Sommerfeld** formula by saying: For a radiated photon of energy $\hbar\omega$, the component along *Oz* of the moment of impulse that is transported by a circularly-polarized plane wave is $\pm\hbar$ ($\hbar = h/2\pi$).

(†) Here, we point out that **W. Westphal** [**12**] gave a discussion of the set of difficulties that were formerly raised by the question of the moment of impulse of a rigorously-plane wave. The reader will find bibliographic references in the papers of **Sadowsky**, **Poynting**, **Epstein**, … Since then, **R. A. Beth** [**13**] has experimentally verified the fact that a physical plane wave can put a birefringent crystal into rotation.

(**) Indeed, since the existence of a packet that is not limited in time and monochromaticity is not realized rigorously, one must take the temporal mean per unit time for the entire interval $t = -\infty$, $t = +\infty$. Now, the energy *flux* that traverses the plane *xOy* during the *entire* temporal interval $t = -T$, $t = +T$ (*T* very large) is equal to at most the *total* energy in the entire packet. Since the latter is spatially limited, that total will be essentially finite. It will then result that the mean flux *per unit time* will be very small, and zero in the limit $T = \infty$. That argument applies just the same to the moment of impulse flux.

respectively, in which $dk^{(3)} = dk_x\, dk_y\, dk_z$, $\omega = |\,\mathbf{k}\,|\, c$ and $\mathbf{k}^0 = \mathbf{k} / k$. The relations div **E** = div **H** = 0 demand that:

$$\mathbf{k} \times \mathbf{F}\,(\mathbf{k}) \equiv 0. \tag{23}$$

From **C. G. Darwin** [**14**], the energy and moment of impulse of such a packet, which are defined by:

$$\mathcal{E} = \tfrac{1}{2}\int_{-\infty}^{+\infty} [(\mathrm{Re}\,\mathbf{E})^2 + (\mathrm{Re}\,\mathbf{H})^2]\, dk^{(3)}, \tag{24}$$

$$\mathcal{M} = \frac{1}{c}\int_{-\infty}^{+\infty} \mathbf{R} \wedge (\mathrm{Re}\,\mathbf{E} \wedge \mathrm{Re}\,\mathbf{H})\, dk^{(3)}, \tag{25}$$

respectively, can be likewise written (*):

$$\mathcal{E} = \tfrac{1}{2}\int \mathbf{F}^* \times \mathbf{F}\, dk^{(3)}, \tag{24′}$$

$$\mathcal{M} = \mathrm{Re}\,\frac{i}{2c}\sum_{i=x}^{y,z}\int F_i^*\, \mathbf{k}^0 \wedge \mathrm{grad}_{(\mathbf{k})}\, F_i\, dk^{(3)} + \mathrm{Re}\,\frac{i}{2c}\int \frac{1}{k}\, \mathbf{F}^* \wedge \mathbf{F}\, dk^{(3)}. \tag{25′}$$

With the goal of calculating these expressions when the packet is reasonably planar and elliptically polarized, we shall first perform the calculations for the case in which **F** has the form:

$$\mathbf{F} = (i\alpha,\, n\alpha,\, -\beta - i\gamma),$$

and α, β, γ, and the constant n are real. The relation (24) demands that:

$$\alpha\, k_x - \gamma k_z = n\ \alpha\, k_y - \beta\, k_z = 0,$$

so

$$k_z = \frac{\alpha\, k_x}{\gamma} = \frac{n\alpha\, k_y}{\beta},$$

and finally:

$$\mathbf{F} = \left(i\alpha, n\alpha, -\frac{\alpha}{k_z}(ik_x + nk_y)\right). \tag{26}$$

A simple calculation shows that the energy $\mathcal{E}$ has the expression:

$$\mathcal{E} = \tfrac{1}{2}\int \alpha^2 \left[1 + \frac{k_x^2}{k_z^2} + n^2\left(1 + \frac{k_y^2}{k_z^2}\right)\right] dk^{(3)}.$$

(*) There is good reason to compare the expressions (25′) and (3); the term in (25′) that corresponds to the term **m** in (3) will be zero, due to (23).

Similarly, the components along *Oz* of the two terms of $\mathcal{M}$ are:

$$\mathcal{M}_z^0 = \frac{1}{2c}\int \frac{n\alpha^2}{k}\left(\frac{k^2}{k_z^2}-1\right)\, dk^{(3)},$$

$$\mathcal{M}_z^s = \frac{1}{2c}\int \frac{2n\alpha^2}{k}\, dk^{(3)}.$$

Now suppose that the **Fourier** amplitude $\alpha(\mathbf{k})$ is non-zero only in a domain *D* that is as small as one desires and situated in a neighborhood of the point $k_x = 0$, $k_y = 0$, $k_z = k_0$. One easily sees that this implies that the wave must be "**reasonably planar,"** because the amplitude χ will remain bounded in *D*, and one will see immediately that E_z **and** H_z **can become less than any arbitrarily small quantity in modulus, provided that the extend of** *D* **is itself sufficiently small.**

On the other hand, consider the ratio:

$$\frac{\mathcal{M}_z^s}{\mathcal{E}} = \frac{1}{c}\frac{\int \frac{2n\alpha^2}{k}\, dk^{(3)}}{\int \alpha^2\left[1+\frac{k_x^2}{k_z^2}+n^2\left(1+\frac{k_y^2}{k_z^2}\right)\right]dk^{(3)}}. \tag{27}$$

It is finite and non-zero, and with an approximation that will get better as the domain *D* gets smaller, one can write:

$$\frac{\mathcal{M}_z^s}{\mathcal{E}} = \frac{2n}{\omega_0(1+n^2)} = \frac{2ab}{\omega_0(a^2+b^2)},$$

if $n = a / b$ and $\omega_0 = k_0\, c$. One can likewise show that the ratios:

$$\frac{\mathcal{M}_z^s}{\mathcal{E}}, \quad \frac{\mathcal{M}_x}{\mathcal{E}}, \quad \text{and } \frac{\mathcal{M}_y}{\mathcal{E}}$$

will be zero in the same approximation. From this, we deduce that the moment of impulse and energy *content* of the wave packet verify the relation:

$$\frac{|\mathcal{M}|}{\mathcal{E}} = \frac{\mathcal{M}_z}{\mathcal{E}} = \frac{2ab}{\omega_0(a^2+b^2)}, \tag{28a}$$

and that this will be true *with an approximation that gets better as the extent of the domain D gets smaller.*

If, instead of performing the calculations for **F**, as defined by (26), we perform them for:

$$\mathbf{F} = \left(\alpha, in\alpha, -\frac{\alpha}{k_z}(k_x + ink_y)\right),$$

then we will get a wave whose polarization is inverse, and is such that:

$$-\frac{|\mathcal{M}|}{\mathcal{E}} = \frac{\mathcal{M}_z}{\mathcal{E}} = \frac{2ab}{\omega_0(a^2+b^2)}. \tag{28b}$$

There is good reason to recall that formulas (28) and (20) differ by the fact that the latter relates to ratios of flux per unit time. However, since formulas (28) were established for spatially-bounded waves, they will nonetheless give us the right to say that when a physical plane wave that is elliptically polarized is absorbed by a material body, it will not only give up its energy, but also a moment of impulse, and the ratio of those two quantities will verify the **Abraham-Sommerfeld** formula (*). It results from this that the absorbing bodies will enter into rotation around an axis that is parallel to the direction of propagation – namely, *Oz*.

Here, we point out that upon denoting the energy flux at time t by $\Phi^E(t)$:

$$\Phi^E(t) = c \iint (\mathrm{Re}\,\mathbf{E} \wedge \mathrm{Re}\,\mathbf{H}) \times \mathbf{n}\, dx\, dy,$$

the *total energy flux* that is defined by:

$$\Phi^E_{tot} = \lim_{T=\infty} \int_{-T}^{+T} \Phi^E(t)\, dt,$$

can be put into a form that is analogous to that of the content, namely:

$$\Phi^E_{tot} = \tfrac{1}{2}\int \mathbf{k}^0 \times \mathbf{n} \cdot \mathbf{F}^* \times \mathbf{F}\, dk^{(3)}.$$

That expression must reduce to that of the content for a packet whose wave numbers **k** are all reasonable parallel to the normal **n** to the plane for which the flux is calculated; that is what we confirm, since $\mathbf{k}^0 \sim \mathbf{n}$ implies that $\mathbf{k}^0 \times \mathbf{n} \sim 1$.

By contrast, we have confirmed that the integration by parts that allowed us to make **R** disappear from the expression (25) for the moment of impulse content – in order to put it into the form (25′) – will not, at the same time, allow us to do that with the expression:

(*) In order to avoid the fact that the **Abraham-Sommerfeld** formula will break down for the plane wave, **E. Henriot** [**11**] believed that there was good reason to introduce some new quantities into electromagnetism – in particular, an "impulse moment of the second kind" or "momentor." In addition to the fact that *those quantities are not gauge-invariant*, we think that the considerations above, which use only classical definitions and have more real physical character, render them superfluous. The fields (19) do not change when one adds $id / k\, e^{i\omega t - ikx}$ to the component A_x of the vector potential and to the scalar potential (17); however, the "momentor flux" will not be the same in both cases. Cf. [**11**], page 15.

$$\Phi_{tot}^{E} = -\int_{-\infty}^{+\infty} dt \iint \mathbf{R} \wedge T\,\mathbf{n}\, dx\, dy,$$

which is that of the *total flux* of the moment of impulse.

On the other hand, the latter cannot be calculated by starting with the expression (9), which is valid only for the "wave zone" of a spherical wave.

c) The result (28) that we just established cannot be the opposite of the one at the beginning of that paragraph (*a*), because we found it for a problem that is clearly different; namely, that of the moment of impulse of a spatially-unlimited wave. Nevertheless, for the sake of completeness, it still remains for us to see what the result (28) will become when there exists a passage to the limit that rigorously permits one to pass from the fields (22) to the fields (19) when the domain D reduces to the point (0, 0, k_0). We shall formally realize such a passage to the limit by introducing a delta function into (22) for the **Fourier** amplitude α, namely:

$$\frac{1}{(2\pi)^{3/2}}\int \alpha\, dk^{(3)} = a\int \delta(k_x)\,\delta(k_x)\,\delta(k_z - k_0)\, dk^{(3)} = a,$$

in such a way that the fields **E** and **H** will become:

$$\mathbf{E} = e^{-ik_0 z}\cdot\mathbf{P},$$
$$\mathbf{H} = e^{-ik_0 z}\cdot\mathbf{k}_0^0 \wedge \mathbf{P}, \tag{19′}$$

in which we have set $\mathbf{P} = (ia, b, 0)\,e^{i\omega_0 t}$ or $\mathbf{P} = (a, ib, 0)\,e^{i\omega_0 t}$, $b = na$, $\mathbf{k}_0 = (0, 0, k_0)$ and $\mathbf{k}_0^0 = \mathbf{k}_0 / k_0$. They are then indeed identical with the fields (19), up to notations. The limit *of the ratio* (28) when the packet (22) becomes the plane wave (19′) is then *rigorously:*

$$\pm\frac{|\mathcal{M}|}{\mathcal{E}} = \frac{\mathcal{M}_z}{\mathcal{E}} = \pm\frac{2ab}{\omega_0(a^2+b^2)}. \tag{30}$$

On the other hand, we know that:

$$\frac{\dfrac{1}{c}\displaystyle\int\left[\mathbf{R}\wedge(\mathrm{Re}\,\mathbf{E}\wedge\mathrm{Re}\,\mathbf{H})\right]_z dv}{\dfrac{1}{c}\displaystyle\int\left[(\mathrm{Re}\,\mathbf{E})^2+(\mathrm{Re}\,\mathbf{H})^2\right]dv} = 0, \tag{31}$$

when **E** and **H** are given by (19) or (19′).

The last two results (30) and (31) then correspond to a moment of impulse that is infinite and zero, respectively, for the rigorously-planar wave (19′)! However, that is easily explained, and does not in any way weaken the results of paragraphs *a*) and *b*). Indeed, the points of a physical plane wave that contribute the most towards a non-zero moment of impulse for the entire packet are situated on the boundaries of the latter, because, due to diffraction, it is there that it will differ the most from a wave that is rigorously planar and monochromatic. If we imagine a series of wave packets whose fields increasingly approach the ones that constitute the plane, monochromatic solution to the **Maxwell** equations then their spatial extent will automatically become increasingly greater, respectively. *In the limit, for the rigorously-monochromatic case, the spatial extent will be necessarily infinite, and the boundaries of the packet, as well as its moment of impulse content, will be pushed out to infinity.* That explains why *defining* the moment of impulse of a *rigorously-planar* wave by $1 / c \int \mathbf{R} \wedge (\mathrm{Re}\, \mathbf{E} \wedge \mathrm{Re}\, \mathbf{H})\, dv$ is not equivalent to defining it by the limiting expression (25′), which is a fact that will introduce a singular behavior to the electromagnetic field at infinity when it is defined by the limit in (22) (*). How does one choose one definition, rather than the other one? It seems to us that the first one is the most natural, but that is a question of personal taste that has no great significance, since *it does not refer to a real physical situation; it is essential to remark that they are not equivalent.*

We can then summarize this third paragraph as follows:

1. A circularly-polarized physical plane wave possesses a moment of impulse such that its ratio to the energy content will verify the **Abraham-Sommerfeld** formula; that will become increasingly exact as the monochromaticity becomes better realized. It will result from this that its absorption by a material body will make it enter into rotation around an axis that is parallel to the mean direction of propagation (§).

2. By contrast, if one defines the moment of impulse of a rigorously-planar wave as one usually does then there will not exist a formula that is analogous to that of **Abraham-Sommerfeld**. Meanwhile, one can reestablish that analogy with the dipolar spherical wave by defining the moment of impulse of the rigorously-planar wave to be the limit of that of a wave packet that becomes rigorously-planar and monochromatic. There is reason to insist upon the fact that the ambiguity that is present in the definition of the moment of impulse of a rigorously-planar wave has no practical significance (†).

In concluding this paper, I would like to warmly acknowledge Professor **L. Rosenfeld** and Doctor **J. Serpe**, who have allowed me to bring this work to a good conclusion, thanks to numerous conversations that we have had and the correspondences that we have exchanged.

(*) This result can be compared to the fact that the limit of Rr when R tends to infinity and r tends to zero is not defined uniquely. One has $\lim_{R=\infty} (\lim_{r=0} Rr) = 0$ and $\lim_{r=0} (\lim_{R=\infty} Rr) = \infty$. In the problem that we are concerned with, we likewise have to make two passages to the limit that do not commute: one of them, over the domain D of the frequencies, and the other one, over a spatial domain v that must be taken to be infinite if one wants to have the moment of impulse content of the entire wave.

(§) This interpretation is essentially classical. *From the quantum viewpoint,* we can interpret formulas (28) by saying that they establish the existence of a moment of impulse for a photon that is associated with the *continuous* spectrum of an electromagnetic wave. That result is essential in order to be able to account for the conservation of the moment of impulse during the emission or absorption of a photon by an optical electron whose axial quantum number m varies from ∓ 1.

(†) It is, however, possible that this ambiguity can play a role from the mathematical viewpoint – for example, when one specifies the "boundary conditions" in the development of an arbitrary wave into a system of rigorously-planar waves.

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