"Sur la transformation d'une équation différentielle de l'ordre pair à la forme d'une équation isopérimétrique," Bull. Acad. imp. Sci. St.-Petersbourg (3) **31** (1887), 283-291.

On the transformation of a differential equation of even order to the form of an isoperimetric equation.

By B. Imchenetsky

(13 May 1886)

Translated by D. H. Delphenich

---

1. – **Jacobi** proved that an isoperimetric differential equation:

$$\frac{\partial V}{\partial y}-\frac{d}{dx}\frac{\partial V}{\partial y'}+\frac{d^2}{dx^2}\frac{\partial V}{\partial y''}-\cdots+(-1)^n\frac{d^n}{dx^n}\frac{\partial V}{\partial y^{(n)}}=0\,, \tag{1}$$

in which V is a function of $x, y, y', \ldots, y^{(n)}$ that is nonlinear with respect to $y^{(n)} = d^n y / dx^n$ can always be transformed into a system of equations with the *canonical* form:

$$\frac{dp_i}{dx}=-\frac{\partial H}{\partial q_i}\,, \qquad \frac{dq_i}{dx}=\frac{\partial H}{\partial p_i}\,, \tag{2}$$

in which H is a known function of $x, p_1, q_1, \ldots, p_n, q_n$, and $i = 1, 2, \ldots, n$ ([1]).

In this article, we propose to show the conditions that are necessary and sufficient for the transformation of a differential equations of even order:

$$y^{(2n)} = f(x, y, y', \ldots, y^{(2n-1)}) \tag{3}$$

into an equation of isoperimetric form (1), and thereupon into a system of equations in canonical form (2).

One finds oneself to be in possession of the general method of that transformation, and upon applying it (when possible) to the problem of integrating the given equation (3) of even order, one

---

([1]) **Jacobi** (*Vorles. über Dynamik*), Nachgelassene Abhandlung: *De aequationum differentialium isoperimetricarum transformationibus earumque reductione ad aequationem differentialem partialem primi ordinis non linearem*.

can profit from some considerable advantages that one knows are intrinsic to the canonical form of differential equations.

2. – Our transformation problem that we mentioned above is expressed by the equation:

$$\frac{\partial V}{\partial y} - \frac{d}{dx}\frac{\partial V}{\partial y'} + \frac{d^2}{dx^2}\frac{\partial V}{\partial y''} - \cdots + (-1)^n \frac{d^n}{dx^n}\frac{\partial V}{\partial y^{(n)}} = \mu\,(y^{(2n)} - f)\,, \tag{4}$$

in which V and μ denote two unknown functions of x, y, y', …, $y^{(n)}$, and f is the given function in the right-hand side of equation (3).

When convenient values of V and μ have been obtained and substituted in equation (4), it will become an identity in x, y, y', …, $y^{(2n)}$, and it will give some other identities when it is partially-differentiated with respect to those variables.

Upon performing those differentiations, in order to avoid complications, one must follow some special rules that are not ordinarily given in the textbooks on differential calculus, but which can be expressed by the following formula:

$$\frac{\partial}{\partial y^{(k)}}\frac{d^m}{dx^m}\varphi(x, y, y', \ldots, y^{(n)})$$

$$= \frac{d^m}{dx^m}\frac{\partial \varphi}{\partial y^{(k)}} + m\frac{d^{m-1}}{dx^{m-1}}\frac{\partial \varphi}{\partial y^{(k-1)}} + \frac{m(m-1)}{1\cdot 2}\frac{d^{m-2}}{dx^{m-2}}\frac{\partial \varphi}{\partial y^{(k-2)}} + \cdots + \frac{m(m-1)\cdots(m-i+1)}{1\cdot 2\cdots i}\frac{d^{m-i}}{dx^{m-i}}\frac{\partial \varphi}{\partial y^{(k-i)}}$$

$$+ \cdots m\frac{d}{dx}\frac{\partial \varphi}{\partial y^{(k-m+1)}} + \frac{\partial \varphi}{\partial y^{(k-m)}}\,. \tag{A}$$

The number of terms in that formula is not always equal to $m + 1$ because the index $k - i$ in the variable $y^{(k-i)}$ in the derivative $\frac{\partial \varphi}{\partial y^{(k-i)}}$ is never negative or greater than n, so one must suppose that all of the terms in the formula (A) in which the conditions $0 < k - i \le n$ are not fulfilled for $i = 0, 1, 2, \ldots, m$ are equal to zero.

The proof of formula (A) is quite simple: First of all, it is easy to verify it for $m = 1$, and then if one supposes that it is true for any well-defined number m, one can easily assure oneself that it will then exist for that value of m when it is increased by one unit.

3. – Upon differentiating equation (4) with respect to $y^{(2n)}$ and $y^{(2n-1)}$ in succession with the aid of formula (A), one will have:

$$(-1)^n \frac{\partial^2 V}{\partial y^{(n)}\,\partial y^{(n)}} = \mu \tag{5}$$

and

$$(-1)^n n \frac{d}{dx} \frac{\partial^2 V}{\partial y^{(n)}\,\partial y^{(n)}} = -\ \mu \frac{\partial f}{\partial y^{(2n-1)}} . \tag{6}$$

It follows from equation (5) that the value of μ cannot contain the derivatives of y of order higher than n. That is why one supposed that $\frac{\partial \mu}{\partial y^{(2n-1)}} = 0$ in deducing equation (6).

If one substitutes the value (5) of μ in equation (6) then one will get:

$$n \frac{d\mu}{dx} + \mu \frac{\partial f}{\partial y^{(2n-1)}} = 0$$

or

$$\frac{d \log \mu^n}{dx} + \mu \frac{\partial f}{\partial y^{(2n-1)}} = 0 , \tag{7}$$

which is an equation that defines the factor μ of our isoperimetric transformation.

Upon comparing equation (7) with the known equation:

$$\frac{d \log M}{dx} + \frac{\partial f}{\partial y^{(2n-1)}} = 0 \tag{8}$$

that defines the last factor M in the system of equations:

$$dx = \frac{dy}{y'} = \ldots = \frac{dy^{(2n-2)}}{y^{(2n-1)}} = \frac{dy^{(2n-1)}}{f} \tag{9}$$

according to Jacobi's theory, which is equivalent to the given equation (3), one will find only the difference between μ^n and M : The value of M can generally be a function of x, y, y', …, $y^{(2n-1)}$, while from the remark that was already made above, the value of μ can contain the derivatives of y only up to an order that is not greater than n. Therefore:

If one can obtain a value of the last factor M for the system of equations (9) *that does not contain derivatives of y of order higher than n then when one sets:*

$$\mu = \sqrt[n]{M} , \tag{10}$$

one will have the μ *in the isoperimetric transformation.*

The required value of the last factor M is easily obtained from equation (8) is the partial derivative $\frac{\partial f}{\partial y^{(2n-1)}}$ is, at the same time, the complete derivative with respect to x of a certain expression in x, y, y', … whose differential order is no greater than n, i.e., the equation for the **Euler** condition:

$$\frac{\partial^2 f}{\partial y^{(2n-1)}\,\partial y} - \frac{d}{dx}\frac{\partial^2 f}{\partial y^{(2n-1)}\,\partial y'} + \cdots + (-1)^m \frac{d^m}{dx^m}\frac{\partial^2 f}{\partial y^{(2n-1)}\,\partial y^{(m)}} = 0 \tag{11}$$

Hence, the necessary condition (11) can be considered to be equivalent to (7).

4. – We now pass on to the search for the value of V by supposing that the condition (11) is satisfied and that the value of μ that we infer from equation (7) has been substituted in equations (4) and (5).

One must first remark that it suffices to obtain the particular value (V) of V that satisfies equation (4) in order to have its general value immediately.

Indeed, that general value will be:

$$V = (V) + \frac{d \cdot \Pi}{dx}, \tag{12}$$

if Π denotes an arbitrary function of x, y, y', …, $y^{(n-1)}$. That is because one knows that:

$$V = \frac{d \cdot \Pi}{dx}$$

is the most-general value that satisfies equation (1), i.e., the right-hand side of equation (4).

One can compute the particular value (V) of the integral of equation (4) as a sum:

$$(V) = V_n + V_{n-1} + \ldots + V_1 + V_0 \tag{13}$$

that is composed of differential expressions V_n, V_{n-1}, …, V_1, V_0 of orders n, $n-1$, …, 1, 0, respectively, in the derivatives of y, in which V_0 denotes simply a function of x, y. One easily notes that the term V_n will necessarily exist in that sum, while the other ones can be missing, either partially or totally.

5. – In order to first obtain the term V_n in the sum (13), one deduces from equation (5) that:

$$V = (-1)^n \int \partial y^{(n)} \int \mu\, \partial y^{(n)} + U . \tag{14}$$

Of course, it is understood that during the partial integration with respect to $y^{(n)}$, one considers the other variables $x, y, y', \ldots, y^{(n-1)}$ to be constants. One will then have:

$$U = \alpha + \beta\, y^{(n)}, \tag{15}$$

in which α and β are unknown functions of the $x, y, y', \ldots, y^{(n-1)}$.

Let W denote the double integral $(-1)^n \int \partial y^{(n)} \int \mu\, \partial y^{(n)}$, to abbreviate the notation, and agree to let the symbol $I^{(n)}$ represent the operation:

$$\frac{\partial}{\partial y} - \frac{d}{dx}\frac{\partial}{\partial y'} + \frac{d^2}{dx^2}\frac{\partial}{\partial y''} - \cdots + (-1)^n \frac{d^n}{dx^n}\frac{\partial}{\partial y^{(n)}},$$

which is applied to the function V in the left-hand side of the isoperimetric equation (1).

With the aid of that symbol, equation (4) will be written:

$$I^{(n)}[V] = \mu\,(y^{(2n)} - f), \tag{4$'$}$$

and upon substituting:

$$V = (-1)^n \int \partial y^{(n)} \int \mu\, \partial y^{(n)} + U = W + U$$

in it, one will have:

$$I^{(n)}[W] + I^{(n)}[U] = \mu\,(y^{(2n)} - f),$$

from which, one can infer that:

$$I^{(n)}[U] = F, \tag{16}$$

upon setting:

$$F = \mu\,(y^{(2n)} - f) - I^{(n)}[W]\,. \tag{17}$$

It is easy to see that the left-hand side of equation (16) do not contain the derivatives of y or order higher than $2n - 2$. Indeed, since U generally has the form (15), one will have:

$$I^{(n)}[U]$$

$$= \frac{\partial\alpha}{\partial y} + \frac{\partial\beta}{\partial y} y^{(n)} - \frac{d}{dx}\left(\frac{\partial\alpha}{\partial y} + \frac{\partial\beta}{\partial y} y^{(n)}\right) + \cdots + (-1)^{n-1}\frac{d^{n-1}}{dx^{n-1}}\left(\frac{\partial\alpha}{\partial y^{(n-1)}} + \frac{\partial\beta}{\partial y^{(n-1)}} y^{(n)}\right) + (-1)^n \frac{d^n}{dx^n}(\beta)\,.$$

However, one can write:

$$\frac{d^n}{dx^n}(\beta) = \frac{d^{n-1}}{dx^{n-1}}\left(\frac{\partial\beta}{\partial x} + \frac{\partial\beta}{\partial y} y' + \cdots + \frac{\partial\beta}{\partial y^{(n-2)}} y^{(n-1)} + \frac{\partial\beta}{\partial y^{(n-1)}} y^{(n)}\right).$$

Hence, the preceding equality will become:

$$I^{(n)}[U] = \frac{\partial\alpha}{\partial y} + \frac{\partial\beta}{\partial y} y^{(n)} - \frac{d}{dx}\left(\frac{\partial\alpha}{\partial y} + \frac{\partial\beta}{\partial y} y^{(n)}\right) + \cdots + (-1)^{n-1} \frac{d^{n-1}}{dx^{n-1}}\left(\frac{\partial\alpha}{\partial y^{(n-1)}}\right)$$
$$+ (-1)^n \frac{d^{n-1}}{dx^{n-1}}\left(\frac{\partial\beta}{\partial x} + \frac{\partial\beta}{\partial y} y' + \cdots + \frac{\partial\beta}{\partial y^{(n-2)}} y^{(n-1)} + \frac{\partial\beta}{\partial y^{(n-1)}} y^{(n)}\right),$$

in which one sees that $y^{(n-2)}$ is the highest derivative of y that can enter into $I^{(n)}[U]$, and the same thing would be true if one were to suppose that $\beta = 0$.

We assume that hypothesis, viz., $\beta = 0$, in order to simplify the search for the particular value of V, and by virtue of that, we must write $I^{(n-1)}[U]$, instead of $I^{(n)}[V]$, if $U = \alpha$ is a differential expression of order $n - 1$, or rather, when we change $I^{(n)}[U]$ into $I^{(n-m)}[U]$, if $U = \alpha$ is a differential expression of order $n - m$.

However, in order to be able to make a well-defined hypothesis about the differential order of the unknown function U, one must first know the differential order of the expression F in the right-hand side of equation (16).

6. – One effortlessly finds that the terms in $y^{(2n)}$ and $y^{(2n-1)}$ in the expression (17) for F mutually cancel. Indeed, with the aid of formula (A), one has:

$$\frac{\partial F}{\partial y^{(2n)}} = \mu - \frac{\partial^2 W}{\partial y^{(n)}\,\partial y^{(n)}}$$

and

$$\frac{\partial F}{\partial y^{(2n-1)}} = -\,\mu \frac{\partial f}{\partial y^{(2n-1)}} - (-1)^n n \frac{d}{dx}\frac{\partial^2 W}{\partial y^{(n)}\,\partial y^{(n)}} = -\left[\mu \frac{\partial f}{\partial y^{(2n-1)}} + n\frac{d\mu}{dx}\right],$$

which are two expressions that reduce to zero identically by virtue of equations (5) and (7).

On the other hand, no matter what hypothesis that one might make on the differential order $n - m$ of U, it is clear that $I^{(n-m)}[U]$ will always have even order $2(n - m)$ and be linear with respect to the highest derivative $y^{2(n-m)}$.

Therefore, our problem of isoperimetric transformation will be impossible to solve when the expression (17) *for F does not have the following form:*

$$F = \mu_m [y^{2(n-m)} - f_m(x, y, y', \ldots, y^{2(n-m)-1})], \tag{18}$$

in which m can have one of the values 1, 2, …, *n*.

Upon supposing the necessary condition (18) is satisfied, one must set:

$$U = V_{n-m}\,,$$

in which V_{n-m} denotes an unknown function of x, y, y', …, $y^{(n-m)}$.

In that way, one will have a new problem that is expressed by the equation:

$$I^{(n-m)}[V_{n-m}] = \mu_m\,(y^{(n-m)} - f_m)\,, \tag{19}$$

which is analogous to the original problem that was expressed by equation (4′). However, there is a difference between them, namely, that the order of equation (19) is lower by $2m$ units, and the factor μ_m in its right-hand side is known *a priori*. The latter situation gives rise to two new necessary conditions:

$$\left.\begin{aligned} &(-1)^{n-m}\frac{\partial^2 V_{n-m}}{\partial y^{(n-m)}\,\partial y^{(n-m)}} = \mu_m, \\ \text{and}\qquad &(n-m)\frac{d\mu_m}{dx} + \mu_m\,\frac{\partial f_m}{\partial y^{2(n-m)-1}} = 0\,, \end{aligned}\right\} \tag{20}$$

which are analogous to (5) and (7), and which one obtains by differentiating equation (19) with respect to $y^{2(n-m)}$ and $y^{2(n-m)-1}$.

The first of equations (20) shows that μ_m must have differential order at most $n - m$, and if that condition is fulfilled then the value of μ_m must once more satisfy the second of equations (20).

7. – In the foregoing, we obtained the general form for the necessary conditions for the solution to the problem considered, and it is easy to see that they are, at the same time, sufficient.

Upon supposing that the condition (11) is satisfied, one will infer the value of the factor μ from equation (7), and one will then have:

$$V_n = (-1)^n \int \partial y^{(n)} \int \mu\, \partial y^{(n)}$$

for the term of highest differential order in the sum (13).

If one supposes that the conditions (18) and (20) are fulfilled then one will have:

$$V_{n-m} = (-1)^{n-m} \int \partial y^{(n-m)} \int \mu_m\, \partial y^{(n-m)}$$

for the term of order $n - m$ that follows V_n in the sum (13).

One can transform equation (19) with the aid of the value of V_{n-m} in the same way that one transformed equation (4′) into (16) with the aid of the value $W = V_n$ in no. **5**.

If one finds that conditions analogous to (18) and (20) are satisfied for the transformed equation then one will have a new term in the sum (13) whose order is less than $n - m$ and which follows the term V_{n-m} .

Upon continuing similarly and supposing that conditions analogous to (18) and (20) are always satisfied, which will necessarily be the case when it is possible to solve the problem, one will get the general solution:

$$V = V_n + \sum_m V_{n-m} + \frac{d \cdot \Pi}{dx},$$

in which, as was said above, Π is an arbitrary function of $x, y, y', \ldots, y^{(n-1)}$, and the number of terms in the sum $\sum_m V_{n-m}$ can vary from zero to n.

The simplest case will then present itself when one finds from formula (17) that the expression F is a function f_0 of x and y, which might reduce to zero, in particular.

It is obvious that the sum $\sum_m V_{n-m}$ will then reduce to just one term $V_0 = \int f_0 \, \partial y$ that will vanish when $f_0 = 0$.

That situation is always encountered in the special case of $n = 1$.

Consequently, if one knows a value M of the last fact factor of the system:

$$dx = \frac{dy}{y'} = \frac{dy'}{f(x, y, y')}$$

then the second-order equation:

$$y'' - f(x, y, y') = 0 ,$$

multiplied by M, will always reduce to the isoperimetric form:

$$\frac{\partial V}{\partial y} - \frac{d}{dx}\frac{\partial V}{\partial y'} = 0 .$$

I hasten to cite a recent article by Professor **N. Sonine** ([2]) in which that same remark was made in the particular case of the isoperimetric transformation. However, to my knowledge, the general problem of that transformation has not been discussed any further.

8. – In order to complete the explanation of the preceding theory, we shall apply it to a particular example. Let:

([2]) **N. Ya. Sonina**, "Ob opredlenii maksimalnuykh i minimalnuykh svoistv ploskikh krivuykh," Univ. Isv. (1886), r. no. 1, Varshava.

$$y^{\mathrm{IV}} - \left(2\frac{y'}{y} + 2\frac{y''}{y'} - \frac{y'''}{y''}\right) y''' - \frac{2}{3}\frac{y''^3}{y'^2} + \frac{1}{6}\frac{y''^2}{y} + \frac{y'^2 y''}{y^2} - y^2 y' - \frac{1}{2}\frac{y\,y'^2}{y''} + \frac{y^2 y'}{y''} = 0$$

be the fourth-order differential equation that one demands to reduce (if possible) to the isoperimetric form.

One must pose the equation:

$$I^{(2)}[V] = \mu(y^{\mathrm{IV}} - f), \tag{a}$$

in which V and μ denote unknown functions of x, y, y', y'', and:

$$f = \left(2\frac{y'}{y} + 2\frac{y''}{y'} - \frac{y'''}{y''}\right) y''' + \frac{2}{3}\frac{y''^3}{y'^2} - \frac{1}{6}\frac{y''^2}{y} - \frac{y'^2 y''}{y^2} + y^2 y' + \frac{1}{2}\frac{y\,y'^2}{y''} - \frac{y^2 y'}{y''}.$$

One has the equation:

$$\frac{d\cdot\log\mu^2}{dx} + \frac{\partial f}{\partial y'''} = 0,$$

from which one will obtain μ. Indeed:

$$\frac{\partial f}{\partial y'''} = \frac{dy''}{y''} - \frac{dy'}{y'} - \frac{dy}{y},$$

and consequently:

$$\mu = C\frac{y''}{y\,y'},$$

in which C denotes an arbitrary constant.

One can then set:

$$V = V_2 + V_1 + V_0 + \frac{d\cdot\Pi}{dx},$$

in which V_2, V_1, V_0 denote differential expressions that are expressed in terms of derivatives of y with orders 2, 1, 0, respectively, and Π is an arbitrary function of x, y, y'.

One has:

$$V_2 = \int \partial y'' \int \mu\, \partial y'' = \frac{1}{6}C\frac{y''^3}{y\,y'},$$

and if one sets:

$$V = V_2 + U$$

then equation (a) will become:

$$I^{(2)}[U] = F, \tag{b}$$

in which:

$$F = \mu(y^{\mathrm{IV}} - f) - I^{(2)}[V_2].$$

It is easy to express the quantities $\mu(y^{\mathrm{IV}} - f)$ and $I^{(2)}[V_2]$ in terms of y and its derivatives, and their difference will be:

$$F = C(-y\,y'' - \tfrac{1}{2}y'^2 + y)\ .$$

One sees that F does not contain the derivatives y^{IV} and y''', and it is linear with respect to y'', so the condition that the theory requires is fulfilled.

One can then suppose that U denotes an unknown function of x, y, y' and change $I^{(2)}[U]$ into $I^{(1)}[U]$.

As a result, equation (b) will become:

$$I^{(1)}[U] = \mu_1(y'' - f_1), \tag{c}$$

in which $\mu_1 = -C\,y$ and $f_1 = -\frac{1}{2}\frac{y'^2}{y} + 1$. Those two quantities likewise satisfy the conditions that are required by the theory, namely, μ_1 does not contain the derivatives of y higher than y', while μ_1 and f_1 satisfy the equation:

$$\frac{d\mu_1}{dx} + \mu_1\frac{\partial f}{\partial y'} = 0\ .$$

One then has:

$$V_1 = -\int \partial y' \int \mu_1\,\partial y' = \tfrac{1}{2}C\,y\,y'^2\,,$$

and one can set:

$$U = V_1 + U_1\ .$$

Consequently, in place of equation (c), we will have:

$$I^{(1)}[U] = F_1\,, \tag{d}$$

in which:

$$F_1 = \mu_1(y'' - f_1) - I^{(1)}[V_1]\ .$$

One easily finds from the last formula that:

$$F = C\,y\,,$$

so one can suppose that U_1 does not contain the derivatives y', in such a way that equation (d) will become:

$$I^{(0)}[U_1] = C\,y \qquad \text{or} \qquad \frac{\partial U_1}{\partial y} = C\,y\ .$$

Upon integrating that, one will have:

$$U_1 = V_0 = \tfrac{1}{2} C\, y^2 \,.$$

Therefore, upon combining the preceding results, one can conclude that when one sets:

$$V = \tfrac{1}{2} C \left(\frac{y''^3}{3y\,y'} + y\,y'^2 + y^2 \right) + \frac{d \cdot \Pi(x, y, y')}{dx},$$

the equation:

$$\frac{\partial V}{\partial y} - \frac{d}{dx}\frac{\partial V}{\partial y'} + \frac{d^2}{dx^2}\frac{\partial V}{\partial y''} = 0$$

will be equivalent to the proposed differential equation.

Received 18 October 1886
