"Die Wellengleichung des Elektrons in der allgemeinen Relativitätstheorie," Sitz. preuss. Akad. Wiss., phys.-math. Klasse (1933), 380-401; errata on pp. 474.

# The wave equation of the electron in the general theory of relativity.

By Priv.-Doz. Dr. L. Infeld In Lemberg

and Prof. Dr. B. L. van der Waerden in Leipzig

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Translated by D. H. Delphenich

### § 1. – Introduction.

It was previously believed that the adaptation of Dirac's theory of the electron to the general theory of relativity would be possible only on the basis of teleparallelism (<sup>1</sup>). **V. Fock** (<sup>2</sup>) had contradicted that opinion by introducing an orthogonal *n*-bein at every point *arbitrarily* relative to an arbitrary metric field and showed the independence of the equations that he exhibited of the choice of *n*-bein. **Schrödinger** (<sup>3</sup>) sought to avoid the somewhat-unfamiliar *n*-bein construction by following the example of **Tetrode** and generalized the Dirac four-rowed matrices by making them vary from point to point. However, the **Schrödinger** formalism seems unnecessarily complicated to us, which is also manifest in the fact that for **Schrödinger** the transformation law for the quantities in question was not once given explicitly, but one would have to be content with being given the infinitesimal transformations.

In the present article, a formalism will be developed that achieves the same thing as the theories of **Fock** and **Schrödinger**, while avoiding the aforementioned limitation. In place of four-rowed matrices, only two-rowed ones will be introduced, for whose representation only the ordinary rules of tensor calculus will suffice. Along with the world-vectors, two-component "spin vectors" will appear that [in contrast to the earlier "specialized" spinor analysis (<sup>4</sup>)] are transformed independently of the world-vectors, and in that way the formal rules of constructing invariant equations are known to become very simple. Certain mixed quantities  $\sigma^{k\lambda\mu}$  (generalized Pauli matrices) will serve as a connecting link between world-vectors and spinors that will assume the role of Fock's *n*-bein components and will, just like the latter, determine the metric field  $g_{kl}$ . After introducing a covariant differentiation for spinors, the Dirac equations can be written down

<sup>(&</sup>lt;sup>1</sup>) Cf., say, **E. Wigner**, Zeit. Phys. **53** (1929), 592.

<sup>(2)</sup> V. Fock, Zeit. Phys. 57 (1929), pp. 261. Cf., also H. Weyl, Zeit. Phys. 56 (1929), 330.

<sup>(&</sup>lt;sup>3</sup>) **E. Schrödinger**, Sitz. preuss. Akad. Wiss. (1932), 105.

<sup>(&</sup>lt;sup>4</sup>) See **B. L. van der Waerden**, Göttinger Nachrichten (1929), 100.

immediately. An arbitrary world-vector  $\Phi_k$  appears in the expression for the covariant derivatives that will be identified with the electromagnetic potential. In that way, the electromagnetic field will be geometrized in a manner that is similar to what is done in the known unified field theories of gravitation and electricity.

Our paper exhibits many points of contact with a paper by **J. A. Schouten** (<sup>1</sup>). **Schouten** ultimately arrived at almost the same formalism that will be developed in this article, except that he unnecessarily employed *n*-bein components and theorems about "sedenions" in order to introduce his formalism, while the formalism was later burdened with auxiliary variables and pseudo-quantities. We have adopted **Schouten**'s introduction of "spinor densities."

In the present work, two formalisms will be presented side-by-side: The simpler " $\gamma$ -formalism," which will be based upon an alternating spinor  $\gamma_{\lambda\mu}$ , and the somewhat-morecomplicated " $\varepsilon$ -formalism," which borrows from **Schouten** in employing spinor densities. The principle of gauge invariance of the potential is valid in both formalisms. The fact that introducing "pseudo-quantities" will be unnecessary for the purpose of gauge invariance here is based in the fact that we (in contrast to **Schouten**) will also admit transformations with complex determinants in spin space: They will replace the Weyl-Schouten changes of gauge. A knowledge of the articles that were cited in the introduction will not be required of the reader in what follows.

### § 2. – Metric space and spin space.

We shall establish the usual concepts of the general theory of relativity: viz., four-dimensional Riemannian space, metric tensor with the components  $g_{kl}$  ( $g_{11}$  to  $g_{33} < 0$ ,  $g_{44} > 0$ ), covariant and contravariant world-vectors and world-tensors.

Every point in the metric space with the *real* components  $x^1$ ,  $x^2$ ,  $x^3$ ,  $x^4$  shall now be further assigned a *complex two-dimensional spin space* whose vectors and tensors will be denoted by spinors with the same names. To abbreviate, we denote the indices of world-tensors by *Latin* symbols and those of spinors with *Greek* ones. Indices that appear twice will always be summed over, and indeed the Latin ones will be summed from 1 to 4, while Greek ones will be summed from 1 to 2.

Let  $a^{\lambda}$  ( $\lambda = 1, 2$ ) be the components of a contravariant spin vector. It will transform in spin space as follows:

$$a^{\prime\lambda} = \Lambda^{\lambda}_{\mu} a^{\mu} \tag{1.a}$$

under coordinate transformations.  $a^{\lambda}$ , as well as  $\Lambda^{\lambda}_{\rho}$ , are complex functions of the world-point that appear as parameters, in general. The single condition (in addition to differentiability) that the transformation coefficients  $\Lambda^{\lambda}_{\rho}$  must satisfy is the following one: The determinant  $|\Lambda^{\lambda}_{\rho}|$  shall be everywhere non-zero:

$$\Delta = \text{Det} |\Lambda_{\alpha}^{\lambda}| \neq 0.$$
<sup>(2)</sup>

<sup>(&</sup>lt;sup>1</sup>) J. A. Schouten, Journal of Math. and Phys. 10 (1931), 239.

The transformation of the four-dimensional metric space and the spin space are considered to be completely independent of each other.

The spin vector that is complex conjugate to  $a^{\lambda}$  shall be denoted by  $a^{\lambda}$ . Its transformation equations will be:

$$a^{\prime \dot{\lambda}} = \bar{\Lambda}^{\lambda}_{\rho} a^{\dot{\rho}} \,. \tag{1.b}$$

The spinors:

 $\beta^{\dot{\lambda}\mu}, \beta^{\dot{\lambda}\mu}, \beta^{\dot{\lambda}\mu\nu}, \text{ etc.}$ 

transform like:

$$a^{\lambda}a^{\mu}$$
,  $a^{\lambda}a^{\dot{\mu}}$ ,  $a^{\dot{\lambda}}a^{\mu}a^{\nu}$ , etc.

The spinors  $\beta^{\lambda\mu\nu}$  and  $\beta^{\lambda\mu\nu}$  are complex-conjugate to each other, where all undotted indices are replaced with dotted ones and conversely. One then has:

$$\beta^{\dot{\lambda}\mu\dot{\nu}} = \overline{\beta^{\dot{\lambda}\mu\nu}} \,. \tag{3}$$

Up to now, it has been customary in spinor analysis to restrict oneself to transformations in spin space with constant coefficients and unity determinant. In that way, two spin vectors  $\alpha^{\lambda}$ ,  $\beta^{\lambda}$  will have the absolute invariant:

$$\varepsilon_{\lambda\mu} \alpha^{\lambda} \beta^{\mu} = \alpha^{1} \beta^{2} - \alpha^{2} \beta^{1}$$

$$(\varepsilon_{12} = -\varepsilon_{21} = 1; \quad \varepsilon_{11} = \varepsilon_{22} = 0).$$
(4)

When one allows arbitrary linear transformations, that expression will no longer be absolutely invariant. However, since one cannot very well demand the construction  $\varepsilon_{\lambda\mu} \alpha^{\mu}$ , which involves the raising and lowering of indices, in spinor calculus, a skew-symmetric quantity  $\gamma_{\lambda\mu} = -\gamma_{\mu\lambda}$  will be introduced that takes on the function of the  $\varepsilon_{\lambda\mu}$  and plays a similar role in spin space to what the metric fundamental tensor  $g_{ik}$  plays in x-space (<sup>1</sup>). Since  $\gamma_{\lambda\mu} = -\gamma_{\mu\lambda}$ ,  $\gamma_{12} = -\gamma_{21}$  will be the single non-zero component of  $\gamma_{\lambda\mu}$ . However, it can be an entirely-arbitrary non-zero function of the world-point. Let the complex-conjugate quantity be  $\gamma_{\lambda\mu}$ . Only the product  $\gamma_{\lambda\mu} \gamma_{\rho\sigma}$  will appear in all physically-important formulas. That product defines an invariant volume measure in spin space (<sup>2</sup>).

The inverse matrix to the matrix  $\gamma_{\lambda\mu}$  is  $\gamma^{\lambda\mu}$ , where:

(<sup>1</sup>) Cf., L. Infeld, Phys. Zeit. 33 (1932), 475.

(2) Namely, the volume element is given by  $\gamma_{12} \gamma_{12} \frac{\partial(\mu^1, \mu^2, \mu^1, \mu^2)}{\partial(q, r, s, t)} dq dr ds dt$ , where q, r, s, t are real parameters on which the complex spin variables  $\mu^1, \mu^2$  depend in some way.

$$\gamma^{12} = -\gamma^{21} = \frac{1}{\gamma_{12}}, \qquad \gamma^{11} = \gamma^{22} = 0.$$
 (5)

The spinor  $\gamma_{\lambda\mu}$  characterizes a complex number magnitude. Therefore, let:

$$\gamma_{12} \gamma_{\dot{1}\dot{2}} = \gamma, \quad \gamma_{12} = \sqrt{\gamma} e^{i\vartheta}, \quad \gamma_{\dot{1}\dot{2}} = \sqrt{\gamma} e^{-i\vartheta}.$$

One can now define the transition from the covariant to contravariant components of a vector or tensor with help of the quantities  $\gamma^{\lambda\mu}$  and  $\gamma_{\lambda\mu}$ :

$$\begin{array}{l} \alpha_{\mu} = \alpha^{\rho} \gamma_{\rho \mu}, \qquad \alpha^{\mu} = \gamma^{\mu \rho} \alpha_{\rho}, \\ \alpha_{\mu} = \alpha^{\rho} \gamma_{\rho \mu}, \qquad \alpha^{\mu} = \gamma^{\mu \rho} \alpha_{\rho}, \\ \alpha_{\lambda \mu} = \alpha^{\rho \sigma} \gamma_{\mu \lambda} \gamma_{\sigma \mu}, \qquad \text{etc.} \end{array}$$

$$(6)$$

The scalar product:

$$-\alpha_{\mu}\beta^{\mu} = \alpha^{\mu}\beta_{\mu} = \gamma^{\rho\sigma}\alpha_{\sigma}\beta_{\rho} = \gamma_{\rho\sigma}\alpha^{\sigma}\beta'$$

is invariant; in particular:

$$\alpha_{\lambda} \alpha^{\lambda} = 0$$

The use of the quantities  $\gamma_{\lambda\mu}$  is indeed formally the simplest way of enabling one to lower indices, but it is not the only one possible. The concept of a spin density offers a second possibility, which we would now like to explain.

A (scalar) *spin density* of *weight*  $\mathfrak{k}$  is a numerical quantity that transforms like a scalar under space transformations but will be multiplied by  $\Delta^{-1}$  under a spin transformation of determinant  $\Delta$ , where  $\mathfrak{k}$  is a whole number. A *spinor density* (e.g., spin vector density, spin tensor density) of weight  $\mathfrak{k}$  is a quantity that transforms line a product of a spinor (e.g., spin vector, spin tensor) and a spin density. For example, the quantity  $\varepsilon_{\lambda\mu}$  that was introduced in (4) is a spinor density of weight – 1.

If one now employs the concept of spinor density then one can also employ  $\varepsilon_{\lambda\mu}$  in place of  $\gamma_{\lambda\mu}$  to lower indices, except that when  $\alpha^{\rho}$  is a spin vector:

$$\alpha_{\mu} = \alpha^{\rho} \varepsilon_{\rho\mu}$$

will not be a vector then, but a vector density of weight – 1. Naturally, the inverse quantity  $\varepsilon^{\lambda\mu}$  will likewise be a spinor density of weight + 1.

We will refer to those quantities that take on the factor  $\overline{\Delta}^{-1}$  under spin transformations as *spin densities of weight*  $\dot{\mathfrak{k}}$ . Thus, the complex-conjugate quantity  $\varepsilon_{\dot{\lambda}\dot{\mu}}$  to  $\varepsilon_{\lambda\mu}$  (which has the same components  $\pm 1$  and 0 as  $\varepsilon_{\lambda\mu}$  and  $\varepsilon^{\lambda\mu}$  in any coordinate system, moreover) is a spinor density of weight -1. Ultimately, we will need only those quantities that take on the factor  $|\Delta|^{-1}$  under spin transformations ( $\mathfrak{k}$  can also be fractional in that case). We shall call them *spin densities* (*spinor densities*, resp.) of *absolute weight*  $\mathfrak{k}$ . Thus,  $\varepsilon_{\lambda\mu}\varepsilon_{\rho\sigma}$  will be a spinor density of absolute weight -2 and  $\varepsilon^{\lambda\mu}\varepsilon^{\rho\sigma}$  will be one of absolute weight +2.

As one sees, the introduction of densities is formally burdensome. However, it has the advantage that one will avoid the introduction of spinors  $\gamma^{\lambda\mu}$  whose physical reality is not established (cf., § 4 below). In what follows, we

will distinguish the two possibilities that we discussed as the  $\gamma$ -formalism and the  $\varepsilon$ -formalism. We will leave the question of which of those formalisms offers the advantage to the further development of the theory.

It should be remarked that any algebraic relationship between spinors that is formulated in the  $\gamma$ -formalism can be adapted to the  $\varepsilon$ -formalism. If one simultaneously divides and multiplies, e.g.,  $\gamma_{\lambda\mu}$ , by  $\gamma_{12} = e^{i\theta} \sqrt{\gamma}$  then one can replace  $\frac{\gamma_{\mu\lambda}}{\sqrt{\gamma}} e^{-i\theta}$  with  $\varepsilon_{\lambda\mu}$  and spinors with corresponding spinor densities.

Only world-tensors transform under space transformations, and only the spinors transform under transformations of spin space. However, in addition to world-vectors and spin tensors, one can also introduce *mixed quantities*. A mixed quantity of the type:

$$\sigma^{\scriptscriptstyle k}_{\scriptscriptstyle \lambda\mu}$$

behaves like a contravariant world-vector under space transformations and like a spin tensor under transformations of spin space. In complete generality, the mixed quantities will transform like world-tensors in the Latin indices and like spinors in the Greek indices.

### § 3. – World-vectors and spin tensors.

The possibility of relating spinors to spatial quantities at all is known  $(^1)$  to be based upon the fact that one can define a one-to-one correspondence between any Hermitian symmetric spin tensor:

$$a_{\lambda\mu} = a_{\mu\lambda} \tag{7}$$

(with two real components  $a_{i_1}$ ,  $a_{i_2}$  and two complex-conjugate ones  $a_{i_2}$  and  $a_{i_1}$ ) with a real world-vector  $a^k$  whose four components are linear functions of the  $a_{i_{2i}}$ :

$$a^{k} = \sigma^{k\lambda\mu} a_{\lambda\mu}. \tag{8}$$

Moreover, that correspondence can be arranged so that the linear transformations of the  $a_{\lambda\mu}$  that they suffer under a coordinate transformation with constant coefficients and unit determinant in spin space will correspond to Lorentz transformations of the associated vectors  $a^k$ .

In the case of the special theory of relativity, in which the numerical constants (that determine the Lorentz group) are  $g_{kl}$ , one can also choose numerical constants for the  $\sigma^{k\lambda\mu}$ . Naturally, that will no longer be true in the case of the general theory of relativity, when the  $\sigma^{k\lambda\mu}$  will be arbitrary

<sup>(&</sup>lt;sup>1</sup>) Cf., **B. L. van der Waerden**, "Spinoranalyse," Nachr. Ges. Wiss. Göttingen (1929), 100 or *Die Gruppentheoretische Methode in der Quantenmechanik*, Berlin, 1932, pp. 82, § **20**.

functions of position that must satisfy only the condition that they should mediate a one-to-one correspondence between the real vectors  $a^k$  and the symmetric tensors  $a_{in}$ .

In order for the  $a^k$  in (8) to always prove to be real, one must also choose the mixed quantities  $\sigma^{k\lambda\mu}$  to be Hermitian symmetric:

$$\sigma^{k\lambda\mu} = \sigma^{k\mu\lambda}.\tag{9}$$

The fact that the linear transformations of spin space always induce Lorentz transformations of the  $a^k$ , i.e., linear transformations with an invariant quadratic form of signature --+, is based upon the fact that the Hermitian matrices  $a_{\lambda\mu}$  possess an invariant of the same signature, namely, their determinant (<sup>1</sup>):

$$a_{i_1}a_{i_2}-a_{i_2}a_{i_1}$$

in place of which we can also consider the expression:

$$a_{\dot{\lambda}\mu}a^{\dot{\lambda}\mu} = \gamma^{\dot{\lambda}\dot{\rho}}\gamma^{\mu\sigma}a_{\dot{\lambda}\mu}a_{\dot{\rho}\sigma}, \qquad (10)$$

which agrees with it, up to a real numerical factor. The association  $a_{\lambda\mu} \rightarrow a^k$  converts that quadratic form into a quadratic form in the  $a^k$ , which we will require to coincide with the metric fundamental form  $g_{kl} a^k a^l$ . We thus demand that:

$$g_{kl} a^k a^l = g_{kl} \sigma^{k \dot{\lambda} \mu} \sigma^{l \dot{\rho} \sigma} a_{\dot{\lambda} \mu} a_{\dot{\rho} \sigma} = \gamma^{\dot{\lambda} \dot{\rho}} \gamma^{\mu \sigma} a_{\dot{\lambda} \mu} a_{\dot{\rho} \sigma}$$

identically in  $a_{\dot{\lambda}\mu}$ . However, that means:

$$g_{kl}\,\sigma^{k\dot{\lambda}\mu}\,\sigma^{l\dot{\rho}\sigma} = \gamma^{\dot{\lambda}\dot{\rho}}\,\gamma^{\mu\sigma}.$$
(11)

If one observes the rules for the raising and lowering of indices then one can also write (11) as:

$$\sigma^{k\dot{\lambda}\mu}\sigma_{k\dot{\sigma}\rho} = \delta^{\dot{\lambda}}_{\dot{\sigma}}\delta^{\mu}_{\rho} \qquad \left\{ \begin{array}{ccc} \delta^{\dot{\lambda}}_{\dot{\sigma}} = \delta^{\mu}_{\rho} = \begin{cases} +1 & \text{for } \dot{\lambda} = \dot{\sigma} & \text{and } \mu = \sigma \\ 0 & " & \dot{\lambda} \neq \dot{\sigma} & " & \mu \neq \sigma \end{cases} \right\}$$
(12)

Based upon (12), one can also solve (8) for  $a^{\lambda\mu}$ :

$$a_{\lambda\mu} = \sigma_{k\lambda\mu} a^k.$$

<sup>(1)</sup> In order to determine signature of that form, one sets  $a_{11} = a + b$ ,  $a_{22} = a - b$ ,  $a_{12} = c + i d$ ,  $a_{21} = c - i d$ . One will then have  $\Delta = a^2 - b^2 - c^2 - d^2$ .

If one substitutes that solution in (8) and compares the coefficients on both sides then it will follow that:

$$\sigma^{k\lambda\mu}\sigma_{l\lambda\mu} = \delta_l^k. \tag{13}$$

When one raises the index l on both sides of (13), it will follow that:

$$g^{kl} = \sigma^{k\lambda\mu} \sigma^l_{\lambda\mu}. \tag{14}$$

On the basis of formulas (12), (13), one can not only associate world-vectors, but any world-tensors at all, with a unique spin quantity in an invertible way, e.g., for a tensor  $p^{kl}$ :

$$\pi_{\dot{\lambda}\mu,\dot{
ho}\sigma} = \sigma_{k\dot{\lambda}\mu}\sigma_{l\dot{
ho}\sigma}p^{kl}; \qquad p^{kl} = \sigma^{k\dot{\lambda}\mu}\sigma^{l\dot{
ho}\sigma}\pi_{\dot{\lambda}\mu,\dot{
ho}\sigma}.$$

Formula (14) once more shows explicitly that the quantities  $\sigma^{k\lambda\mu}$  and  $\gamma_{\lambda\mu}$  collectively determine the metric field. Naturally, the converse is not true.

Under the aforementioned association  $\sigma_{\dot{\lambda}\mu} \rightarrow a^k$ , the time-like vectors  $(g_{kl} a^k a^l > 0)$  correspond to matrices  $a_{\dot{\lambda}\mu}$  that belong to *definite* Hermitian forms. We would always like to choose the  $\sigma^{k\dot{\lambda}\mu}$  such that the future-pointing vectors  $(a^4 > 0)$  correspond to positive-definite forms. Furthermore, we can always normalize the  $\sigma^{k\dot{\lambda}\mu}$  such that the pure-imaginary four-rowed determinants of the vectors:

$$\sigma^{ki1}, \sigma^{ki2}, \sigma^{k21}, \sigma^{k22}$$

prove to be positive-imaginary. If those conditions are fulfilled then we multiply the by -1 or go to the complex-conjugate values. With that normalization, we will succeed in determining the  $\sigma^{k\lambda\mu}$  uniquely, up to an arbitrary spin transformation, in terms of the metric (cf., § 5).

The expression  $\gamma_{\dot{\lambda}\dot{\rho}}(\sigma^{k\dot{\lambda}\mu}\sigma^{l\dot{\rho}\nu} + \sigma^{l\dot{\lambda}\mu}\sigma^{k\dot{\rho}\nu})$  is obviously anti-symmetric in the indices  $\mu$  and  $\nu$ , so it will be equal to a multiple of  $\gamma^{\mu\nu}$ , say,  $h^{kl}\gamma^{\mu\nu}$ . If one then multiplies by  $\gamma_{\nu\rho}$  and observes the rules for the lowering of indices then it will follow that:

$$\sigma^{k\dot{\lambda}\mu}\sigma^l_{\dot{\lambda}\sigma} + \sigma^{l\dot{\lambda}\mu}\sigma^k_{\dot{\lambda}\sigma} = h^{kl}\,\delta^{\mu}_{\sigma}\,.$$

In order to determine  $h^{kl}$ , we set  $\mu = \sigma$ , sum over  $\mu$ , and compare the result with (14). We will find that  $h^{kl} = g^{kl}$ . The first of the following two formulas is proved with that:

The second one following in an entirely analogous way.

In the  $\varepsilon$ -formalism, one must replace  $\gamma^{\dot{k}\dot{\rho}}\gamma^{\mu\sigma}$  with  $\varepsilon^{\dot{k}\dot{\rho}}\varepsilon^{\mu\sigma}$  on the right-hand sides of (10) and (11). In order for the product  $\varepsilon^{\dot{k}\dot{\rho}}\varepsilon^{\mu\sigma}$  in (11) to have the correct weight, one must choose  $\sigma^{k\dot{k}\rho}$  to be a mixed density of absolute weight + 1. Naturally,  $\sigma_{k\dot{k}\rho}$  will then be a mixed density of absolute weight – 1. In the  $\varepsilon$ -formalism, only the spinor density  $a_{\dot{k}\mu}$  of absolute weight – 1 the spinor density  $a^{\dot{k}\mu}$  of absolute weight + 1 will be associated with an actual world-vector in a one-to-one manner, since only then will  $\sigma^{k\dot{k}\mu}a_{\dot{k}\mu}$  and  $\sigma^{k}_{\dot{k}\mu}a^{\dot{k}\mu}$  suffer no alteration under a transformation spin space.

The derivation of formulas (12) to (15) will remain valid with no changes in the  $\varepsilon$ -formalism.

## § 4. – The affine connection.

In order to be able to exhibit invariant differential equations for spinors, it is necessary to introduce a covariant differentiation (or a parallel displacement) for spinors. As usual, it will be defined for spin vectors by linear formulas of the type  $(^1)$ :

$$\begin{array}{c}
\psi_{\alpha|k} = \partial_{k} \psi_{\alpha} - \Gamma^{\rho}_{\alpha k} \psi_{\rho}, \\
\psi^{\alpha}_{|k} = \partial_{k} \psi^{\alpha} + \Gamma^{\alpha}_{\rho k} \psi^{\rho}.
\end{array}$$
(16)

If one transforms the formulas (16) to another coordinate system and demands that the  $\psi_{\alpha|k}$  or  $\psi_{|k}^{\alpha}$  should transform covariantly, i.e., like mixed quantities, then that will give the transformation rules for the  $\Gamma$ :

$$\Lambda^{\rho}_{\sigma} \Gamma^{\alpha}_{\rho k} = \Lambda^{\rho}_{\sigma} \Gamma'^{\alpha}_{\rho k} - \partial_k \Lambda^{\alpha}_{\sigma} \ .$$

The parallel displacement of the complex-conjugate vectors  $\psi_{\dot{\alpha}}$ ,  $\psi^{\dot{\alpha}}$  is given along with that of the vectors  $\psi_{\alpha}$ ,  $\psi^{\alpha}$  as soon as one demands that the relation between a vector  $\psi^{\alpha}$  and its complex-conjugate vector  $\psi^{\dot{\alpha}}$  should remain preserved under parallel translation. One will then obtain a covariant differentiation:

$$(\psi_{\alpha} \chi^{\alpha})_{|s} = \psi_{\alpha|s} \chi^{\alpha} + \psi_{\alpha} \chi^{\alpha}_{|s}.$$

<sup>(&</sup>lt;sup>1</sup>) We have set  $\partial_k = \partial / \partial x^k$ , to abbreviate. The differentiation  $\partial_k$  will not have an invariant character due to the arbitrariness of the choice of coordinates in spin space. That implies the necessity of introducing the  $\Gamma^{\mu}_{\alpha k}$ . The fact that one chooses the same  $\Gamma^{\rho}_{\alpha k}$  with opposite signs in both cases of formulas (16) is justified by the fact that the invariant relation  $\psi_{\alpha} \chi^{\alpha}$  = const. should remain conserved under parallel translation of the vectors  $\psi_{\alpha}$ ,  $\chi^{\alpha}$  or (what amounts to the same thing) that the differentiation rule for a product  $\psi_{\alpha} \chi^{\alpha}$  should be true:

$$\begin{split} \psi_{\dot{\alpha}|k} &= \partial_k \, \psi_{\dot{\alpha}} - \Gamma_{\dot{\alpha}k}^{\,\dot{\rho}} \, \psi_{\dot{\rho}}, \\ \psi_{|k}^{\dot{\alpha}} &= \partial_k \, \psi^{\dot{\alpha}} + \Gamma_{\dot{\rho}k}^{\,\dot{\alpha}} \, \psi^{\dot{\rho}}, \end{split} \\ \Gamma_{\dot{\alpha}k}^{\,\dot{\beta}} &= \overline{\Gamma_{\alpha k}^{\,\beta}} \; . \end{split}$$

with

We will further demand that an arbitrary tensor should be differentiated like a product of vectors, so e.g.:

$$a_{\dot{\lambda}\mu|k} = \partial_k a_{\dot{\lambda}\mu} - \Gamma^{\dot{\rho}}_{\dot{\lambda}k} a_{\dot{\rho}\mu} - \Gamma^{\sigma}_{\mu k} a_{\dot{\lambda}\sigma} .$$

We demand of the covariant differentiation that it should be *volume-preserving*, i.e., that the tensor  $\gamma_{\alpha\beta} \gamma_{\dot{\alpha}\dot{\beta}}$  should have zero covariant derivative:

$$\gamma_{\alpha\beta|k}\gamma_{\dot{\alpha}\dot{\beta}} + \gamma_{\alpha\beta}\gamma_{\dot{\alpha}\dot{\beta}|k} = 0.$$
<sup>(17)</sup>

The condition (17) is actually meaningful for only the one non-zero component  $\gamma_{12} \gamma_{12}$ ; it then says that (<sup>1</sup>):

$$\partial_k (\gamma_{12} \gamma_{1\dot{2}}) - \Gamma_{1k}^1 \gamma_{12} \gamma_{1\dot{2}} - \Gamma_{2k}^2 \gamma_{12} \gamma_{1\dot{2}} - \Gamma_{1k}^i \gamma_{12} \gamma_{1\dot{2}} - \Gamma_{\dot{2}k}^2 \gamma_{12} \gamma_{\dot{1}\dot{2}} = 0,$$

or when one sets  $\gamma_{12} \gamma_{1\dot{2}} = \gamma$ :

$$\left. \begin{array}{c} \partial_{k}\gamma - (\Gamma_{\alpha k}^{\ \alpha} + \Gamma_{\dot{\alpha} k}^{\ \dot{\alpha}})\gamma = 0, \\ \Gamma_{\alpha k}^{\ \alpha} + \Gamma_{\dot{\alpha} k}^{\ \dot{\alpha}} = \partial_{k}\log\gamma. \end{array} \right\}$$
(18)

Since (8) implies that any tensor  $a_{\dot{\lambda}\mu}$  is coupled with a world-vector  $a^k$ , the parallel translation of the tensor  $a_{\dot{\lambda}\mu}$  likewise implies a parallel translation of the vector  $a^k$ . The associated covariant derivative is defined by:

$$a_{|l}^{k} = \sigma^{k\lambda\mu} a_{\lambda\mu|l} . \tag{19}$$

One will obtain the same result when one established the covariant differentiation of  $a^k$  in the usual way by means of a  $\Gamma_{kl}^s$  and determines that  $\Gamma_{kl}^s$  in such a way that the covariant derivative of  $\sigma^{k\lambda\mu}$  is equal to zero:

$$\sigma^{k\dot{\lambda}\mu}{}_{|s} = \partial_s \sigma^{k\dot{\lambda}\mu} + \Gamma^k_{rs} \sigma^{r\dot{\lambda}s} + \Gamma^{\dot{\lambda}}_{\dot{\rho}s} \sigma^{k\dot{\rho}\mu} + \Gamma^\mu_{\sigma s} \sigma^{k\dot{\lambda}\sigma} = 0.$$
(20)

<sup>(&</sup>lt;sup>1</sup>) Equation (17) can also be derived (as will be shown later) from the demand that  $g_{kl|s} = 0$ , i.e., from the condition that the length of a rigid yardstick will not change under parallel displacement.

Formula (20) obviously determines the  $\Gamma_{rs}^{k}$  uniquely and has (19) as a consequence. A further consequence of (17), (20), and the differentiation rule for products in that the covariant derivative of:

$$g^{kl} = \sigma^{k\dot{\lambda}\mu} \sigma^{l}_{\dot{\lambda}\mu} = \sigma^{k\dot{\lambda}\mu} \sigma^{l\dot{\rho}\sigma} \gamma_{\dot{\lambda}\dot{\rho}} \gamma_{\mu\sigma}$$

$$g^{kl} = 0$$
(21)

will be equal to zero  $(^1)$ :

$$g^{kl}_{\ |s} = 0$$
. (21)

If one now requires that the displacement  $\Gamma_{rs}^{k}$  should be symmetric:

$$\Gamma_{rs}^{k} = \Gamma_{sr}^{k} \tag{22}$$

then it is known that it will follow from (21) that the  $\Gamma_{rs}^{k}$  must be the Christoffel symbols  $\begin{cases} rs \\ k \end{cases}$  that are determined by  $g_{kl}$ . The symmetry assumption is the simplest one that one can make: If one did not make it then one will get a tensor  $S_{rs}^{k}$  from the difference  $\Gamma_{rs}^{k} - \Gamma_{sr}^{k}$ , whose physical meaning is unknown.

When one calculates the  $\Gamma_{rs}^{k}$  using (20), the symmetry condition (22) will be equivalent to 6×4 linear conditions for 8×4 real and imaginary parts of the 4×4 components  $\Gamma_{\rho s}^{\mu}$ . Together with the four conditions (18), that still does not suffice to determine the  $\Gamma_{\rho s}^{\mu}$ , since four real parameters will still remain arbitrary. In fact: If the equations (18), (20) are fulfilled then that will remain the same when one leaves the  $\Gamma_{rs}^{k}$  unchanged, but replaces the  $\Gamma_{\rho s}^{\mu}$  and  $\Gamma_{\rho s}^{\mu}$  with:

$$\Gamma^{\mu}_{\rho s} + \frac{i}{2} \varphi_s \, \delta^{\mu}_{\rho} \text{ and } \Gamma^{\mu}_{\rho s} - \frac{i}{2} \varphi_s \, \delta^{\rho}_{\mu}$$

resp. Thus, an arbitrary real quantity  $\Phi_s$  appears here entirely by itself that is defined by:

$$\Gamma^{\alpha}_{\alpha s} - \Gamma^{\dot{\alpha}}_{\dot{\alpha} s} = 2i \, \Phi_s \,. \tag{23}$$

We would now like to investigate the transformation properties of  $\Phi_s$ .  $\Phi_s$  behaves like a worldvector under space transformations. By contrast, under transformations in spin space, one will get:

$$\Phi'_{s} = \Phi_{s} - \partial_{s} \varphi$$

when  $\Delta = |\Delta| e^{i\varphi}$ . That follows from the equations of transformation:

<sup>(1)</sup> One can also start, conversely, from the demand (21) derive (17) and (18) from it.

$$\Gamma_{\alpha s}^{\prime \alpha} = \Gamma_{\alpha s}^{\alpha} - \partial_s \log \Delta$$
 and  $\Gamma_{\dot{\alpha} s}^{\prime \dot{\alpha}} = \Gamma_{\dot{\alpha} s}^{\dot{\alpha}} - \partial_s \log \bar{\Delta}$ .

The  $\Phi_s$  thus transform under a transformation of the form  $\alpha'^{\lambda} = e^{\frac{1}{2}i\varphi} \alpha^{\lambda}$  in precisely the same way that a potential should according to Weyl's principle of gauge invariance.  $\Phi_s$  will then be identified with the electromagnetic potential.

It follows from the equation  $\gamma_{12} = \sqrt{\gamma} e^{i\vartheta}$  that  $\partial_s \vartheta$  transforms in precisely the same way that  $\Phi_s$  transforms, i.e.:  $\partial_s \vartheta' = \partial_s \vartheta - \partial_s \varphi$ .

Therefore:

$$\Phi_s^* = \Phi_s - \partial_s \vartheta$$

in an *actual* vector. It differs from the electromagnetic potential by only the gradient of a function. The vector  $\Phi_s^*$  does not seem to have any physical meaning.

The contracted quantities  $\Gamma^{\alpha}_{\alpha s}$  and  $\Gamma^{\dot{\alpha}}_{\dot{\alpha} s}$  can be calculated explicitly from (18) and (23), resp.:

$$\Gamma^{\alpha}_{\alpha s} = i \Phi_{s} + \partial_{s} \log \sqrt{\gamma} = i \Phi^{*}_{s} + \partial_{s} \log \sqrt{\gamma} e^{i \vartheta},$$

$$\Gamma^{\dot{\alpha}}_{\dot{\alpha} s} = -i \Phi_{s} + \partial_{s} \log \sqrt{\gamma} = -i \Phi^{*}_{s} + \partial_{s} \log \sqrt{\gamma} e^{-i \vartheta}.$$

One then finds the covariant derivatives of the spinor  $\gamma_{\lambda\mu}$  from that:

$$\gamma^{\lambda\mu}_{\ |s} = i \gamma^{\lambda\mu} \Phi^*_s, \quad \gamma^{\dot{\lambda}\mu}_{\ |s} = -i \gamma^{\dot{\lambda}\mu} \Phi^*_s,$$

$$\gamma_{\lambda\mu|s} = -i \gamma_{\lambda\mu} \Phi^*_s, \quad \gamma_{\dot{\lambda}\mu|s} = i \gamma_{\dot{\lambda}\mu} \Phi^*_s.$$

$$(24)$$

In the  $\varepsilon$ -formalism, one cannot adapt formulas (17) to (24) above with no further analysis, because one must first define a covariant derivative for spinor densities. That will happen when one first makes the Ansatz:

$$\alpha_k = \partial_k \; \alpha - \mathfrak{k} \; \Gamma_k \; \alpha \tag{25}$$

for the covariant differentiation of a spin density of weight  $\mathfrak{k}$ . The factor  $\mathfrak{k}$  is added because a density of weight  $\mathfrak{k}$  is the  $\mathfrak{k}^{th}$  power of a density of weight one. In order for the  $\alpha_k$  thus-defined to be a vector density, the  $\Gamma_k$  must transform like vector components under a space transformation, but according to the formula:

$$\Gamma'_{k} = \Gamma_{k} - \partial_{k} \log \Delta \tag{26}$$

under a spin transformation. The covariant differentiation of arbitrary spinor densities will now be defined in the way that would be given for products of spinors with scalar densities, so from the usual formulas for covariant spinor differentiation with an extra term of  $- \notin \Gamma_k$  times the spinor in question.

The  $\Gamma_k$ , which were initially introduced arbitrarily, can be fixed uniquely by the invariant demand that the covariant derivative of the spinor density  $\varepsilon_{\lambda\mu}$  (of weight – 1) should prove to be equal to zero:

$$\varepsilon_{12|k} = -\Gamma_{1k}^{1} \varepsilon_{12} - \Gamma_{2k}^{2} \varepsilon_{12} + \Gamma_{k} \varepsilon_{12} = 0,$$

$$\Gamma_{k} = \Gamma_{\alpha k}^{\alpha}.$$

$$(27)$$

For the complex-conjugate densities of weight  $\dot{\mathfrak{k}}$ ,  $\Gamma_k$  must be replaced with its complex-conjugate  $\overline{\Gamma}_k$  everywhere in the formulas above.

The product of two densities of weights  $\mathfrak{k}$ ,  $\dot{\mathfrak{k}}$  is a density of absolute weight  $2\mathfrak{k}$  whose covariant derivative would read:

$$\alpha_k = \partial_k \alpha - \mathfrak{k}(\Gamma_{\mu} + \overline{\Gamma}_{\mu})\alpha.$$

We now set:

$$\Gamma_{k} + \overline{\Gamma}_{k} = \Gamma_{\alpha k}^{\alpha} + \Gamma_{\dot{\alpha} k}^{\dot{\alpha}} = 2 \Pi_{k}$$

and define the covariant derivative for arbitrary densities of absolute weight *k* by:

$$\alpha_k = \partial_k \alpha - \mathfrak{k} \prod_k \alpha.$$

How one must differentiate spinor densities of absolute weight *t* and mixed densities is now clear.

On the same basis as above, we will seek to coordinate the covariant differentiation of spinors and world-vectors with each other in such a way that the covariant derivative of  $\sigma^{k\lambda\mu}$  proves to be equal to zero. Since  $\sigma^{k\lambda\mu}$  is a quantity of absolute weight 1 in the  $\varepsilon$ -formalism, that will give the condition:

$$\sigma^{k\dot{\lambda}\mu}_{\ |s} = \partial_s \sigma^{k\dot{\lambda}\mu} + \Gamma^k_{rs} \sigma^{k\dot{\lambda}\mu} + \Gamma^{\dot{\lambda}}_{\dot{\rho}s} \sigma^{k\dot{\lambda}\mu} + \Gamma^\mu_{\sigma s} \sigma^{k\dot{\lambda}\mu} - \Pi_s \sigma^{k\dot{\lambda}\mu} = 0, \qquad (28)$$

which also allows one to express the  $\Gamma_{rs}^{k}$  in terms of the  $\Gamma_{\sigma s}^{\mu}$  and  $\Gamma_{\rho s}^{\dot{\lambda}}$ . The demand that  $\Gamma_{rs}^{k}$  should be symmetric yields 6×4 conditions for the 8×4 real and imaginary parts of the  $\Gamma_{\sigma s}^{\mu}$ . Eight real parameters will then remain arbitrary now. In fact: (28) will remain fulfilled when one replaces  $\Gamma_{\sigma s}^{\mu}$  and  $\Gamma_{\sigma s}^{\mu}$  with:

$$\Gamma^{\mu}_{\rho s} + \frac{1}{2}(\pi_s + i\varphi_s)\delta^{\mu}_{\rho} \quad \text{and} \quad \Gamma^{\mu}_{\rho s} + \frac{1}{2}(\pi_s - i\varphi_s)\delta^{\mu}_{\rho},$$

resp. If we set:

$$\Gamma_s = \Gamma^{\alpha}_{\alpha s} = \Pi_s + i \Phi_s , \qquad \overline{\Gamma}_s = \Gamma^{\dot{\alpha}}_{\dot{\alpha} s} = \Pi_s - i \Phi_s$$

then we will find that not only the  $\Phi_s$ , but also the  $\Pi_s$ , are arbitrary in this formalism.

However, we will see that this latter arbitrariness drops out of the physically-important formulas completely, since the  $\Gamma^{\mu}_{\sigma\tau}$  always enter into those formulas only in the combinations:

$$\Gamma^{\mu}_{\rho s} - \frac{1}{2} \delta^{\mu}_{\rho} \Pi_{s}$$
 and  $\Gamma^{\mu}_{\dot{\rho} s} - \frac{1}{2} \delta^{\mu}_{\dot{\rho}} \Pi_{s}$ .

Those combinations are, in fact, definitive of the covariant differentiation of the contravariant spin vector densities of weight 1/2 and the covariant spin vector densities of weight -1/2, and that will show (cf., § 7) that only those vector densities will play an actual role. However, the quantities  $\Phi_s$  appear in the formulas in an essential way that also transforms like a vector under space transformations in this formalism, but transforms according to the following formula:

$$\Phi'_{s} = \Phi_{s} - \partial_{s} \varphi \tag{30}$$

under spin transformations of determinant  $\Delta = |\Delta| e^{i\varphi}$ .

3.

Those quantities  $\Phi_s$  shall again be identified with the electromagnetic potential.

### § 5. – Completely-geodetic systems.

We would now like to calculate the components of the quantities  $\sigma^{k\lambda\mu}$  and the parallel translation in the neighborhood of a world-point  $P_0$  in a suitably-chosen coordinate system.

If space is *Euclidian* then one can satisfy the conditions (11), (18), (20), and (22) in a suitablychosen coordinate system in the world and in spin space by way of the following values of the quantities that appear in them:

1. 
$$g_{kl} = \overset{\circ}{g}_{kl}$$
;  $\overset{\circ}{g}_{11} = \overset{\circ}{g}_{22} = \overset{\circ}{g}_{33} = -\overset{\circ}{g}_{44} = -1$ , otherwise  $\overset{\circ}{g}_{kl} = 0$ , (31.a)

2. 
$$\gamma^{\lambda\mu} = \varepsilon^{\lambda\mu}; \quad \varepsilon^{12} = -\varepsilon^{21} = 1, \quad \varepsilon^{11} = \varepsilon^{22} = 0,$$
 (31.b)

$$\sigma^{k\dot{\lambda}\mu} = \overset{\circ}{\sigma}{}^{k\dot{\lambda}\mu}; \quad \overset{\circ}{\sigma}{}^{1\dot{\lambda}\mu} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1\\ 1 & 0 \end{pmatrix}; \quad \overset{\circ}{\sigma}{}^{2\dot{\lambda}\mu} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i\\ i & 0 \end{pmatrix}; \\ \overset{\circ}{\sigma}{}^{3\dot{\lambda}\mu} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0\\ 0 & -1 \end{pmatrix}; \quad \overset{\circ}{\sigma}{}^{4\dot{\lambda}\mu} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0\\ 0 & 1 \end{pmatrix}$$

 $(0, \cdot)$ 

4. 
$$\Gamma_{sk}^{l} = \begin{cases} s \ k \\ l \end{cases} = 0$$
, (31.d)

5. 
$$\Gamma^{\beta}_{\alpha k} = \frac{1}{2} i \Phi_s \delta^{\beta}_{\alpha}$$
. (31.e)

Those values will remain the same when one performs an arbitrary Lorentz transformation on the world and simultaneously performs a suitable constant linear transformation of unit determinant in spin space  $(^1)$ .

For given  $g_{kl} = \overset{\circ}{g}_{kl}$  and  $\gamma^{\lambda\mu} = \varepsilon^{\lambda\mu}$ , the  $\sigma^{k\lambda\mu}$  that were given in (31.c) are not the only ones that fulfill the conditions (11), but one still has the freedom to perform an arbitrary Lorentz

(31.c)

<sup>(&</sup>lt;sup>1</sup>) For the proof of this, see perhaps, **B. L. van der Waerden**, Die Gruppentheoretische Methode in der Quantenmechanik, § 20.

transformation on the index k or also an arbitrary spin transformation of unit determinant on the indices  $\dot{\lambda}$ ,  $\mu$ . However, that is also *the only* freedom that one has, because, from its derivation, the condition (11) says that under the association  $a_{\lambda\mu} \rightarrow a^k$ , the quadratic form  $\gamma^{\dot{\lambda}\dot{\rho}}\gamma^{\mu\sigma}a_{\dot{\lambda}\mu}a_{\dot{\rho}\sigma}$  will go

to  $\overset{\circ}{g}_{kl}a^ka^l$ , and such a linear association is completely determined, up to a linear transformation that leaves the latter quadratic form invariant, i.e., up to a Lorentz transformation (<sup>1</sup>).

We now move on to the general case of Riemannian space. We imagine that we use the  $\gamma$ -symbolism as our basis and assume that the coordinate system in spin space is always chosen such that  $\gamma^{12} = 1$ . There will then be no difference between  $\gamma^{\lambda\mu}$  and  $\varepsilon^{\lambda\mu}$ . In addition, for an *isolated* world-point  $P_0$ , one can take the  $g_{kl}$  to  $\overset{\circ}{g}_{kl}$  by a suitable linear transformation:

$$x^k = c^k_{\ p} \overset{\mathrm{o}}{x}^p.$$

The vectors  $c_1^k$ ,  $c_2^k$ ,  $c_3^k$ ,  $c_4^k$  must define an orthogonal *n*-bein in order for that to be true. One can then determine a system of components  $\sigma^{k\lambda\mu}$  that satisfy equation (11) by the Ansatz:

$$\sigma^{k\dot{\lambda}\mu}=c^k_{\ p}\overset{\mathrm{o}}{\sigma}^{p\dot{\lambda}\mu},$$

so it would be suitable to a metric field. From the remarks above, the system  $\sigma^{k\dot{\lambda}\mu}$  is also essentially the only one (i.e., up to a linear spin transformation). If one lets the  $c_p^k$  depend differentiably upon the world-point in question then the  $\sigma^{k\dot{\lambda}\mu}$ -field that arises will also be differentiable.

However, one can achieve even more in the immediate neighborhood of a point  $P_0$ . If one chooses the coordinate system in  $P_0$  such that one has  $g_{kl} = \overset{\circ}{g}_{kl}^{k}$  then one can also choose  $c_p^k = \delta_p^k$ ,  $\sigma^{k\dot{\lambda}\mu} = \overset{\circ}{\sigma}^{k\dot{\lambda}\mu}$  at that point. Now one can determine the orthogonal *n*-bein  $c_p^k$  for every point *P* in the immediate neighborhood of  $P_0$  by parallel-translating the *n*-bein  $\delta_p^k$  from  $P_0$  to *P* and correspondingly establishing the  $\sigma^{k\dot{\lambda}\mu}$ . Analytically, it is easy to perform that parallel translation in such a way one defines a geodetic coordinate system at  $P_0$ , so one for which:

$$(\partial_s g_{kl})_{P_0} = 0, \qquad (\Gamma_{sk}^l)_{P_0} = 0.$$

The parallel-displaced vectors  $c_{p}^{k}$  will then satisfy the simple condition that:

<sup>(&</sup>lt;sup>1</sup>) The "improper" Lorentz transformations, which invert the direction of the flow of time or the orientation of space, cannot be treated here on the basis of the normalization that was used in § 3.

$$(\partial_s c_p^k)_{P_0} = 0.$$

It then follows immediately from this that:

$$(\partial_s \sigma^{k\lambda\mu})_{P_0} = 0,$$

i.e.: the  $\sigma^{k\lambda\mu}$  are constant in the first approximation relative to the chosen coordinate system, as are the  $g_{kl}$  and  $\gamma^{\lambda\mu}$ . The most general  $\sigma^{k\lambda\mu}$ -field results from the one that is found in that way by a linear spin transformation for the space coordinates, once they are chosen.

We would like to call a coordinate system (whose existence was just proved) relative to which the conditions:

$$(g_{kl})_{P_0} = g_{kl}$$
,  $(\partial_s g_{kl})_{P_0} = 0$ , (32.a)

$$(\gamma_{\lambda\mu})_{P_0} = \mathcal{E}_{\lambda\mu} , \qquad (\partial_s \gamma_{\lambda\mu})_{P_0} = 0 , \qquad (32.b)$$

$$(\sigma^{k\dot{\lambda}\mu})_{P_0} = \overset{\circ}{\sigma}^{k\dot{\lambda}\mu} , \qquad (\partial_s \, \sigma^{k\dot{\lambda}\mu})_{P_0} = 0 \qquad (32.c)$$

are fulfilled a completely-geodetic coordinate system at  $P_0$ .

We can now calculate the components  $\Gamma^{\rho}_{\alpha k}$  of the parallel translation of spinors relative to a completely-geodesic coordinate system at  $P_0$ . Equations (20) and (23) assume the following simple form at  $P_0$ :

$$\Gamma^{\dot{\lambda}}_{\rho s} \overset{o}{\sigma}{}^{k \dot{\rho} \mu} + \Gamma^{\mu}_{\rho s} \overset{o}{\sigma}{}^{k \dot{\lambda} \rho} = 0, \qquad (33)$$

$$\Gamma^{\dot{\alpha}}_{\dot{\alpha}s} - \Gamma^{\alpha}_{\alpha s} = -2i\,\Phi_s\,. \tag{34}$$

That elementary system of linear equations has only one solution, namely:

$$(\Gamma^{\beta}_{\alpha s})_{P_0} = \frac{1}{2} i \delta^{\beta}_{\alpha} \Phi_s.$$
(35)

With that, the proof that  $\Gamma_{\alpha s}^{\beta}$  is actually determined uniquely (up to the arbitrariness in the  $\Phi_s$ ) by equations (20), which remained pending up to now, is achieved. Namely, it is true in a completely-geodetic coordinate system, so it will also be true in any other one.

The condition (32.b) drops out in the  $\varepsilon$ -formalism, and that has the consequence that a completely-geodetic reference system will admit the arbitrary spin transformation  $a'^{\lambda} = \beta a^{\lambda}$ ,  $\beta = \rho e^{i\theta}$ .

In place of (20), one will now have equation (28), which arises from (20) when one replaces  $\Gamma^{\mu}_{\rho s}$  and  $\Gamma^{\dot{\mu}}_{\dot{\rho}s}$  with  $\Gamma^{\mu}_{\rho s} - \frac{1}{2}\Pi_s \delta^{\mu}_{\sigma}$  and  $\Gamma^{\dot{\lambda}}_{\dot{\rho}s} - \frac{1}{2}\Pi_s \delta^{\dot{\lambda}}_{\dot{\rho}}$ , resp. If one also makes that substitution in the solution (35) then one will get the general solution in a completely-geodetic coordinate system:

$$\begin{split} & \left(\Gamma^{\mu}_{\sigma s}\right)_{P_0} = \frac{1}{2}(\Pi_s + i\Phi_s)\delta^{\mu}_{\sigma}, \\ & \left(\Gamma^{\dot{\lambda}}_{\dot{\rho}s}\right)_{P_0} = \frac{1}{2}(\Pi_s - i\Phi_s)\delta^{\dot{\lambda}}_{\dot{\rho}}. \end{split}$$

Here, as well, there is no arbitrariness in the determination of the  $\Gamma^{\mu}_{\rho s}$  besides the arbitrariness in the  $\Pi_s$  and  $\Phi_s$ . As remarked before, the arbitrariness in the  $\Pi_s$  will drop out when one considers only the quantities:

$$^{*}_{\Gamma \rho s}^{\mu} = \Gamma^{\mu}_{\rho s} - \frac{1}{2} \delta^{\mu}_{\rho} \Pi_{s} \quad \text{and} \quad \Gamma^{\mu}_{\dot{\rho} s} = \Gamma^{\dot{\mu}}_{\dot{\rho} s} - \frac{1}{2} \delta^{\dot{\mu}}_{\dot{\rho}} \Pi_{s} , \quad (36)$$

which are definitive of the covariant differentiation of contra-(co-) variant spin vector densities of weight  $\frac{1}{2}(-\frac{1}{2}, \text{resp.})$ , instead of the  $\Gamma^{\mu}_{\rho s}$  and  $\Gamma^{\mu}_{\dot{\rho}s}$ , resp.

## § 6. – The curvature tensors.

In Riemannian space, there exists the well-known curvature tensor:

$$R^{r}_{kps} = -\partial_{s}\Gamma^{r}_{kp} + \partial_{p}\Gamma^{r}_{ks} - \Gamma^{h}_{kp}\Gamma^{r}_{hs} + \Gamma^{h}_{ks}\Gamma^{r}_{hp} .$$

The mixed curvature tensor for spin space can also be constructed similarly. Its components are:

$$P^{\mu}_{\lambda ps} = -\partial_{s} \Gamma^{\mu}_{\lambda p} + \partial_{p} \Gamma^{\mu}_{\lambda s} - \Gamma^{\rho}_{\lambda p} \Gamma^{\mu}_{\rho s} + \Gamma^{\mu}_{\rho s} \Gamma^{\rho}_{\lambda p},$$

$$P^{\mu}_{\lambda ps} = -\partial_{s} \Gamma^{\mu}_{\lambda p} + \partial_{p} \Gamma^{\mu}_{\lambda s} - \Gamma^{\rho}_{\lambda p} \Gamma^{\mu}_{\rho s} + \Gamma^{\mu}_{\rho s} \Gamma^{\rho}_{\lambda p}.$$

$$(37)$$

By contraction, while recalling (23.a), one will get the skew-symmetric world-tensor:

$$P^{\rho}_{\rho ps} = i(\partial_{p}\Phi_{s} - \partial_{s}\Phi_{p}) = iF_{ps},$$

$$P^{\dot{\rho}}_{\dot{\rho}ps} = -i(\partial_{p}\Phi_{s} - \partial_{s}\Phi_{p}) = -iF_{ps}.$$
(38)

Here, the world-tensor  $F_{ps} = -F_{sp}$  of the *electromagnetic field strengths* will appear automatically.

The relationship that exists between the mixed and Riemannian curvature tensor will now be sought. One first finds the customary formulas by calculation:

$$\begin{array}{c} \psi^{\rho}_{|kl} - \psi^{\rho}_{|lk} = \psi^{\sigma} P^{\rho}_{\sigma lk}, \\ \psi_{\dot{\rho}|kl} - \psi_{\dot{\rho}|lk} = \psi_{\dot{\sigma}} P^{\dot{\sigma}}_{\dot{\rho}lk}, \end{array} \right\}$$
(39.a)

$$\sigma^{k\dot{\lambda}\mu}{}_{|ps} - \sigma^{k\dot{\lambda}\mu}{}_{|sp} = \sigma^{k\dot{\rho}\mu} P^{\dot{\lambda}}{}_{\dot{\rho}sp} + \sigma^{k\dot{\lambda}\mu} P^{\mu}{}_{\rho sp} + \sigma^{r\dot{\lambda}\mu} R^{k}{}_{rsp} = 0.$$
(39.b)

The left-hand side of equation (39.b) vanishes identically as a result of (20). One will then have:

$$\sigma^{k\dot{\rho}\mu} P^{\dot{\lambda}}_{\ \dot{\rho}sp} + \sigma^{k\dot{\lambda}\mu} P^{\mu}_{\ \rho sp} + \sigma^{r\dot{\lambda}\mu} R^{k}_{\ rsp} = 0.$$

$$\tag{40}$$

One can express  $R^{k}_{rsp}$  in terms of  $P^{\dot{\lambda}}_{\dot{\rho}sp}$  and  $P^{\lambda}_{\rho sp}$  with no further discussion. Conversely, one can assume that  $R^{k}_{rsp}$  and seek to determine the  $P^{\lambda}_{\rho sp}$  from (40) and (38). Equations (38) and (40) will be satisfied when one sets:

$$P^{\lambda}_{\rho sp} = \frac{1}{2} R_{krsp} \sigma^{k\lambda\nu} \sigma^{r}_{\nu\rho} + \frac{1}{2} i F_{sp} \delta^{\lambda}_{\rho},$$

$$P^{\lambda}_{\rho sp} = \frac{1}{2} R_{krsp} \sigma^{k\lambda\nu} \sigma^{r}_{\nu\rho} - \frac{1}{2} i F_{sp} \delta^{\lambda}_{\rho}.$$

$$(41)$$

Another solution cannot be given when  $R_{rsp}^{k}$  and  $F_{sp}$  are given, since the difference of two solutions must satisfy the system of equations:

which will coincide with the system of equations (33), (34) (with  $\Phi_s = 0$ ) for fixed *p* (and  $\sigma^{k\dot{\lambda}\mu} = \sigma^{o}_{k\dot{\lambda}\mu}$ ), from which we can already establish that it possesses only the zero solution.

One can define a curvature tensor in the  $\varepsilon$ -formalism in terms of the  $\Gamma^{\mu}_{\lambda s}$ , as well as the  $\Gamma^{\mu}_{\lambda s}$  that are defined by (36). If one does the latter, so one replaces  $\Gamma^{\mu}_{\lambda s}$  with  $\Gamma^{\mu}_{\lambda s}$  everywhere in (37), then formulas (38) will also be valid. We would like to decide upon the latter possibility, since the parallel translation  $\Gamma^{\mu}_{\lambda s}$ , into whose curvature the undetermined quantities  $\Pi_s$  enter, seems to have no physical meaning. The formulas (39) will also be valid as long as one understands  $\psi^{\rho}$  and  $\psi_{\mu}$  to mean spinor densities of absolute weights 1/2 and – 1/2. (40) and (41) will be likewise true.

#### § 7. – The Dirac equations.

The starting point of our considerations is the law of conservation. Let  $\mathfrak{J}^k$  be the current vector that is supposed to obey the conservation law. Therefore, in a Riemannian continuum, one will have:

$$\frac{1}{\sqrt{-g}}\partial_k\sqrt{-g}\,\widetilde{\mathfrak{J}}^k\,=\,\widetilde{\mathfrak{J}}^k_{\ |k}=0\;.$$

The world-vector  $\hat{\mathfrak{J}}^k$  induces a spin tensor  $\kappa^{\lambda\mu}$ , whereby:

$$\begin{aligned} \widehat{\mathbf{J}}^{k} &= \sigma^{k\dot{\lambda}\mu} \kappa_{\dot{\lambda}\mu} = \sigma^{k}_{\dot{\lambda}\mu} \kappa^{\dot{\lambda}\mu} \,. \\ \sigma^{k\dot{\lambda}\mu} \kappa_{\dot{\lambda}\mu|k} &= \sigma^{k}_{\dot{\lambda}\mu} \kappa^{\dot{\lambda}\mu}_{|k} = 0 \,. \end{aligned}$$

$$\tag{42}$$

Since  $\mathfrak{J}^k$  is a real world-vector,  $\kappa^{\lambda\mu}$  must commute in the indices  $\dot{\lambda}$ ,  $\mu$ . One satisfies that condition, along with the other one that  $\mathfrak{J}^k$  should be a time-like vector with  $\mathfrak{J}^4 \ge 0$ , by the following simple Ansatz:

$$\kappa^{\dot{\lambda}\mu} = \psi^{\dot{\lambda}}\psi^{\mu} + \chi^{\dot{\lambda}}\chi^{\mu}, \qquad (43)$$

in which  $\psi^{\lambda}$ ,  $\chi^{\mu}$  denote the components of two spin vectors. Equation (42) can be written in the following form:

$$(\sigma^{k}_{\dot{\lambda}\mu}\psi^{\dot{\lambda}}\psi^{\mu} + \sigma^{k\dot{\lambda}\mu}\chi_{\dot{\lambda}}\chi_{\mu})_{|k} = 0.$$
(44)

Upon developing that equation, one finds that:

We then demand that:

$$\left. \begin{array}{c} \psi^{\mu} \sigma^{k}_{\ \dot{\lambda}\mu} \psi^{\dot{\lambda}}_{\ |k} + \chi_{\mu} \sigma^{k\dot{\lambda}\mu} \chi_{\dot{\lambda}|k} \\ + \psi^{\dot{\lambda}} \sigma^{k}_{\ \dot{\lambda}\mu} \psi^{\mu}_{\ |k} + \chi_{\dot{\lambda}} \sigma^{k\dot{\lambda}\mu} \chi_{\mu|k} = 0. \end{array} \right\}$$
(45)

The conservation law will then be fulfilled when:

$$\left. \begin{array}{c} \sigma^{k}_{\lambda\mu}\psi^{\lambda}_{|k} = \alpha \chi_{\mu}, \\ \sigma^{k\lambda\mu}\chi_{\lambda|k} = -\alpha \psi^{\mu}. \end{array} \right\}$$
(46.a)

The equations that are complex-conjugate to these are:

$$\left. \begin{array}{c} \sigma^{k}_{\lambda\dot{\mu}}\psi^{\lambda}_{\ |k} = \bar{\alpha}\,\chi_{\dot{\mu}}, \\ \sigma^{k\lambda\dot{\mu}}\,\chi_{\lambda|k} = -\bar{\alpha}\,\psi^{\dot{\mu}}. \end{array} \right\}$$
(46.b)

Equations (46) define the general-relativistic generalization of the Dirac equations. One next sees that they possess a generally-covariant form, and indeed under arbitrary transformations in Riemannian space, as well as under arbitrary spin transformations. From the derivation up to now, the factor  $\alpha$  can be any function of position. However, since the phases of  $\psi$  and  $\chi$  are completely arbitrary in (43), one can always arrange them such that  $\alpha$  will be pure imaginary. Since we have

no basis for ascribing a mass to the electron that varies from point to point, we will assume that  $\alpha$  is a constant.

If the coordinate system is chosen such that it is completely-geodetic at the point  $P_0$  (which is always possible, from § 5) then one will get the Dirac equations in the usual form in the immediate neighborhood of  $P_0$  when one sets  $\alpha$  and  $\Phi_s$  equal to:

$$\alpha = \frac{2\pi i mc}{h\sqrt{2}}, \quad \Phi_s = \frac{2\pi}{h}\varphi_s,$$

resp., in which  $\varphi_s$  is the electromagnetic potential of the external field. The world-tensors  $\Phi_s$ ,  $F_{pq}$ , which determine the parallel displacement and the curvature in spin space, shall then be identified with the electromagnetic potential and the field strength (up to constant factors).

Equations (46) define a self-adjoint linear eigenvalue problem for the four components  $\psi^{\lambda}$ ,  $\chi_{\dot{\lambda}}$ . In order to see the linearity, one must take, say, the second equation in (46.a) with the first equation in (46.b) together:

$$\left. \begin{array}{c} \sigma^{k\mu\lambda} \chi_{\lambda|k} = -\alpha \psi^{\mu}, \\ \sigma^{k}_{\ \mu\lambda} \psi^{\lambda}_{\ |k} = \overline{\alpha} \chi_{\mu} = -\alpha \chi_{\mu}. \end{array} \right\}$$
(47)

If one introduces the matrices:

$$\sigma^{\prime k} = (\sigma^{k\alpha\beta}), \quad \sigma^{k} = (\sigma^{k}_{\dot{\alpha}\beta}), \\ \psi = \begin{pmatrix} \psi^{1} \\ \psi^{2} \end{pmatrix}, \quad \chi = \begin{pmatrix} \chi_{1} \\ \chi_{2} \end{pmatrix}$$

$$(48)$$

then one can also write (47) as:

$$\begin{array}{c} \sigma^{\prime k} \chi_{|k} = -\alpha \psi, \\ \sigma^{k} \psi_{|k} = -\alpha \chi, \end{array} \right)$$

or after multiplying by  $\frac{h\sqrt{2}}{2\pi i}$ :

$$\begin{cases} \sqrt{2} \,\sigma^{\prime k} \, p_k \, \chi = -mc \, \psi, \\ \sqrt{2} \,\sigma^k \, p_k \, \psi = -mc \, \chi, \end{cases}$$

$$(49)$$

in which  $p_k$  is  $\frac{h}{2\pi i}$  times the covariant differentiation operator.

In order to see the self-adjointness of the problem, one rewrites (49) in terms of four-rowed matrices:

$$\sum_{k} p_{k} \begin{pmatrix} \sqrt{2} \sigma^{k} & 0 \\ 0 & \sqrt{2} \sigma^{\prime k} \end{pmatrix} \begin{pmatrix} \psi \\ \chi \end{pmatrix} + \begin{pmatrix} 0 & mc \\ mc & 0 \end{pmatrix} \begin{pmatrix} \psi \\ \chi \end{pmatrix} = 0.$$
 (50.a)

The matrices that appear here all have the property of Hermitian symmetry. If one now chooses the coordinate system in spin space such that  $\sigma^4$  is equal to the identity matrix (which is always possible, since  $\sigma^4$  defines a positive-definite Hermitian form that can always be transformed into the unit form) then equation (50) will have the form:

$$\left(\frac{h}{2\pi i}\frac{\partial}{\partial t} + H\right)\Psi = 0$$

with self-adjoint *H* and  $\Psi = \begin{pmatrix} \psi \\ \chi \end{pmatrix}$ .

A different matrix form of equations (49) is better suited to a comparison with the four-rowed matrix theories, namely:

$$\sum_{k} p_{k} \begin{pmatrix} 0 & \sqrt{2} \sigma'^{k} \\ \sqrt{2} \sigma^{k} & 0 \end{pmatrix} \begin{pmatrix} \psi \\ \chi \end{pmatrix} = -mc \begin{pmatrix} \psi \\ \chi \end{pmatrix}.$$
 (50.b)

The matrices that appear on the left-hand side are Tetrode's  $\gamma^k$ , which fulfill then known relations:

$$\gamma^k \gamma^l + \gamma^l \gamma^k = 2 g^{kl}.$$

Namely, they emerge from:

$$\sigma^{\prime k} \sigma^{l} + \sigma^{\prime l} \sigma^{k} = g^{kl}, \qquad \Big\}$$
$$\sigma^{k'} \sigma^{l} + \sigma^{l'} \sigma^{k} = g^{kl}, \qquad \Big\}$$

so from (15). Those matrices are not Hermitian. However, they can be made Hermitian when one either (with **Schrödinger**) multiplies by  $\gamma_0$  and restricts the permissible coordinate system by a suitable prescription or [with **V. Bargmann** (<sup>1</sup>)] forces the Hermiticity independently of the coordinate system by multiplying by another matrix  $\alpha$ . The latter essentially derives from the fact that one goes over to the other matrix form (50.a), in which (as we see) all difficulties concerned with Hermiticity will vanish. Our theory then differs from the four-rowed matrix theories by a special choice of coordinates in four-dimensional spin space that makes the matrices  $\gamma^k$  assume the form (50.b), and the four components of  $\Psi$  split into two pairs  $\psi^{\lambda}$ ,  $\chi_{\lambda}$ . However, that special choice is distinguished in an invariant way by the fact that both pairs are transformed individually under Lorentz transformations.

In the *ɛ*-formalism:

$$\kappa^{\dot{\lambda}\mu} = \psi^{\dot{\lambda}}\psi^{\mu} + \chi^{\dot{\lambda}}\chi^{\mu}$$

<sup>(&</sup>lt;sup>1</sup>) **V. Bargmann**, Sitzber. presuss. Akad. Wiss. (1932), pp. 346.

is not a spinor, but a spin density of weight + 1, since it is only in that case that a one-to-one relationship exists between  $\kappa^{\dot{\lambda}\mu}$  and the world-vector  $\mathfrak{J}^k$ . The  $\psi^{\lambda}$ ,  $\chi^{\lambda}$  will then be spinor densities of absolute weight + 1/2, while  $\psi_{\lambda}$ ,  $\chi_{\lambda}$  have absolute weight – 1/2. Only the  $\Gamma^{\rho}_{\lambda s}$  will appear while performing the covariant differentiations and in the curvature tensor. Otherwise, all results of this and the following two paragraphs are also true in the  $\varepsilon$ -formalism.

# § 8. – The transition to second-order equations.

If one substitutes  $\chi_{\lambda}$  from (46.b) in (46.a) then one will get:

$$\sigma^{k\lambda\mu}\sigma^{l}_{\lambda\rho}\psi^{\rho}_{|lk} = -\alpha\,\overline{\alpha}\,\psi^{\mu}\,. \tag{51}$$

The left-hand side of this equation shall now be calculated out. By a simple rewriting and application of (39.a), one will get:

$$-\alpha \,\overline{\alpha} \,\psi^{\mu} = \frac{1}{2} \sigma^{k\dot{\lambda}\mu} \sigma^{l}_{\dot{\lambda}\rho} \psi^{\rho}_{|lk} + \frac{1}{2} \sigma^{l\dot{\lambda}\mu} \sigma^{k}_{\dot{\lambda}\rho} \psi^{\rho}_{|kl} = \frac{1}{2} (\sigma^{k\dot{\lambda}\mu} \sigma^{l}_{\dot{\lambda}\rho} + \sigma^{l\dot{\lambda}\mu} \sigma^{k}_{\dot{\lambda}\rho}) \psi^{\rho}_{|lk} + \frac{1}{2} \sigma^{l\dot{\lambda}\mu} \sigma^{k}_{\dot{\lambda}\rho} \psi^{\sigma} P^{\rho}_{\sigma lk}.$$

$$(52)$$

From (15), the first term on the right gives:

$$\frac{1}{2}g^{kl}\psi^{\mu}_{\ |lk}$$
.

We now move on to calculate the second expression. It follows from (41) that:

$$\frac{1}{2}\sigma^{l\dot{\lambda}\mu}\sigma^{k}_{\dot{\lambda}\rho}\psi^{\sigma}P^{\rho}_{\sigma lk} = \frac{1}{4}\sigma^{l\dot{\lambda}\mu}\sigma^{k}_{\dot{\lambda}\rho}\psi^{\sigma}R_{pslk}\sigma^{\rho\rho\dot{\nu}}\sigma^{s}_{\dot{\nu}\sigma} + \frac{i}{4}\sigma^{l\dot{\lambda}\mu}\sigma^{k}_{\dot{\lambda}\rho}\psi^{\rho}F_{lk}.$$
(53)

In conclusion, we would like to prove that:

$$\frac{1}{4}R_{pslk}\,\sigma^{\rho\rho\dot{\nu}}\sigma^{s}_{\dot{\nu}\sigma}\,\sigma^{l\dot{\lambda}\mu}\,\sigma^{k}_{\dot{\lambda}\rho}\,\psi^{\sigma} = \frac{1}{8}R\psi^{\mu} \qquad (R = \text{curvature scalar}).$$
(54)

Rewriting the left-hand side of this equation will yield:

$$\frac{1}{4}R_{pslk}\sigma^{l\dot{\lambda}\mu}\sigma^{k}_{\dot{\lambda}\rho}\sigma^{p\rho\dot{\nu}}\sigma^{s}_{\dot{\nu}\sigma} = \frac{1}{4}R_{pslk}\sigma^{l\dot{\lambda}\mu}\sigma^{k}_{\dot{\lambda}\rho}\sigma^{p\rho\dot{\nu}}\sigma^{s}_{\dot{\nu}\sigma} + \frac{1}{4}R_{pslk}\sigma^{l\dot{\lambda}\mu}\sigma^{p}_{\dot{\lambda}\rho}\sigma^{k\rho\dot{\nu}}\sigma^{s}_{\dot{\nu}\sigma} - \frac{1}{4}R_{kslp}\sigma^{l\dot{\lambda}\mu}\sigma^{k}_{\dot{\lambda}\rho}\sigma^{p\rho\dot{\nu}}\sigma^{s}_{\dot{\nu}\sigma}.$$
(55)

We write the last term in the form:

$$- \frac{1}{12} \sigma^{l\dot{\lambda}\mu} \sigma^{k}_{\dot{\lambda}\rho} \sigma^{\rho\rho\dot{\nu}} \sigma^{s}_{\dot{\nu}\sigma} (R_{kslp} + R_{kslp} + R_{kslp}) .$$
<sup>(56)</sup>

If we now repeatedly apply a conversion that is entirely analogous to (55) then we will get from (56) that:

$$-rac{1}{12}\sigma^{l\lambda\mu}\sigma^k_{\lambda
ho}\sigma^{p
hov}_{\sigma^s}\sigma^s_{\sigma^s}[R_{kslp}-R_{pslk}-R_{lskp}]$$

As a result of the symmetry properties of the curvature tensor, the expression in brackets can also be written in the form:

$$R_{kslp} + R_{klps} + R_{kpsl}$$
.

However, that vanishes identically, and therefore (54) is also proved.

Equation (51), which defines the starting point of those calculations, can then be written in the form:

$$g^{lk}\psi^{\mu}_{\ |lk} + \frac{1}{4}R\psi^{\mu} + \frac{i}{2}F_{lk}\sigma^{l\lambda\mu}\sigma^{k}_{\lambda\sigma}\psi^{\sigma} = -2\alpha\,\overline{\alpha}\,\psi^{\mu}.$$
(57.a)

One finds, analogously, for  $\chi_{\mu}$  that:

$$g^{lk} \chi_{\mu llk} + \frac{1}{4} R \chi_{\mu} + \frac{i}{2} F_{lk} \sigma^{l}_{\lambda\mu} \sigma^{k\lambda\sigma} \chi_{\sigma} = -2\alpha \overline{\alpha} \chi_{\mu}.$$
(57.b)

The first term on the left-hand side and the right-hand side of (57) correspond to the expressions that appear in the **Klein-Gordon** equation. The curvature scalar appears in those equations, along with the latter expressions and the spin term (which include the electromagnetic field strength).

### § 9. – The energy-impulse tensor.

Can the energy-impulse tensor be constructed from the quantities that characterize the *wave field*? It must be real and satisfy the following equations:

$$\left. \begin{array}{c} T^{kl} = T^{lk}, \\ T^{lk}_{\ \ |l} = F^{sk} J_s. \end{array} \right\}$$
(58)

One can show that the world-tensor:

$$T'_{k}^{l} = i(\psi^{\lambda} \sigma_{\lambda\mu}^{l} \psi_{|k}^{\mu} - \psi^{\lambda} \sigma_{\lambda\mu}^{l} \psi_{|k}^{\mu}) -i(\chi_{\lambda} \sigma^{l\lambda\mu} \chi_{\mu|k} - \chi_{\lambda} \sigma^{l\lambda\mu} \chi_{\mu|k}).$$
(59)

satisfies the second of those conditions. We immediately see that  $T'_{k}^{l}$  is real. For the sake of calculating  $T'_{k|l}^{l}$ , we would next like to establish the contribution that the first term in (59) makes to  $T'_{k|l}^{l}$ . It is:

$$i(\psi^{\lambda}\sigma^{l}_{\lambda\mu}\psi^{\mu}_{|k})_{|l} = i\,\overline{\alpha}\,\chi_{\mu}\psi^{\mu}_{|k} + i\,\alpha\psi^{\mu}\chi_{\mu|k} + \frac{i}{2}\sigma^{l}_{\lambda\mu}R_{prlk}\sigma^{p\mu\nu}\sigma^{r}_{\mu\nu}\psi^{\lambda}\psi^{\mu} + \frac{1}{2}F_{lk}\sigma^{l}_{\lambda\mu}\psi^{\lambda}\psi^{\mu}.$$

$$(60)$$

If the term that includes the curvature tensor is converted in a completely-analogous way to what was done in the previous § then we will get:

$$i(\psi^{\lambda}\sigma^{l}_{\lambda\mu}\psi^{\mu}_{|k})_{|l} = i\,\overline{\alpha}\,\chi_{\mu}\psi^{\mu}_{|k} + i\,\alpha\psi^{\mu}\chi_{\mu|k} \\ -\frac{i}{2}R_{kl}\sigma^{l}_{\lambda\mu}\psi^{\lambda}\psi^{\mu} + \frac{1}{2}F_{lk}\sigma^{l}_{\lambda\mu}\psi^{\lambda}\psi^{\mu}.$$

$$(61)$$

Calculating the other terms and summing will then yield:

$$T'_{kl}^{l} = F^{sk} \,\mathfrak{J}_{s} \,. \tag{62}$$

The tensor  $T'^{lk}$  is real and satisfies the conservation law, but it is not symmetric. Can the tensor  $T'^{lk}$  be symmetrized? I.e.: Does the tensor:

$$T^{lk} = \frac{1}{2} (T'^{lk} + T'^{kl}) \tag{63}$$

also satisfy the conservation law? It will be shown that this is, in fact, the case. We would now like to calculate  $T'_{ll}^{kl}$ , not  $T'_{ll}^{lk}$ , as before.

We have the following expression for  $T'^{kl}$ :

$$T'^{kl} = i g^{sl} (\psi^{\lambda} \sigma^{k}_{\lambda \dot{\mu}} \psi^{\dot{\mu}}_{|s} - \psi^{\lambda} \sigma^{k}_{\lambda \dot{\mu}} \psi^{\mu}_{|s}) - i g^{sl} (\chi_{\lambda} \sigma^{k \lambda \dot{\mu}} \chi_{\dot{\mu}|s} - \chi_{\dot{\lambda}} \sigma^{k \lambda \mu} \chi_{\mu|s}).$$

$$(64)$$

$$i g^{sl} (\psi^{\lambda} \sigma^{k}_{\lambda \dot{\mu}} \psi^{\dot{\mu}}_{|s})_{|l} = i g^{sl} \psi^{\lambda}_{|l} \sigma^{k}_{\lambda \dot{\mu}} \psi^{\dot{\mu}}_{|s} + i g^{sl} \psi^{\lambda} \sigma^{k}_{\lambda \dot{\mu}} \psi^{\dot{\mu}}_{|sl}$$

$$= i g^{sl} \psi^{\lambda}_{|l} \sigma^{k}_{\lambda \dot{\mu}} \psi^{\mu}_{|s} - 2\alpha \overline{\alpha} i \psi^{\dot{\mu}} \psi^{\lambda} \sigma^{k}_{\lambda \dot{\mu}}$$

$$- \frac{i}{4} R \psi^{\mu} \psi^{\lambda} \sigma^{k}_{\lambda \dot{\mu}} - \frac{1}{2} F_{pr} \sigma^{p\rho \dot{\mu}} \sigma^{r}_{\rho \phi} \psi^{\dot{\sigma}} \psi^{\lambda} \sigma^{k}_{\lambda \dot{\mu}}.$$

$$(65)$$

By carrying out analogous calculations with the other terms and summing, we will get:

$$T'^{kl}_{\ |l} = F^{sk} \, \tilde{\mathfrak{J}}_s \,. \tag{66}$$

However, that follows from the fact that the tensor that is defined by (63) is symmetric and satisfies the conservation laws, i.e., it fulfills the conditions (58). It gives the contribution to the energyimpulse tensor of the world of the electron considered.

We would now like to assume the simplest case, which is, in fact, the case of the free electron. The tensor  $F_{sk}$  will vanish then, and we will have simply:

knowing the energy-impulse tensor makes it possible to exhibit the gravitational equations. For the case of the free electron and in the absence of external masses in the region considered, one has:

$$R_k^l - \frac{1}{2} \delta_k^l R = \kappa T_k^l \tag{68}$$

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( $\kappa$  = gravitational constant). Contracting the indices k and l implies that:

$$-R = \kappa T = 2i\kappa(\alpha \chi_{\lambda}\psi^{\lambda} - \overline{\alpha}\chi_{\dot{\lambda}}\psi^{\dot{\lambda}}).$$
(69)

The scalar  $T = T_k^k$  can then be identified with the mass density (with a corresponding normalization of  $\psi^{\mu}$  and  $\chi_{\mu}$ ).

If it is permissible to calculate the gravitational action of the free electron by itself in such a way that one substitutes the value of R that (57) implies in (69) then that will give the following nonlinear equations:

$$g^{kl}\psi^{\mu}{}_{|lk} - \frac{i}{2}\kappa\psi^{\mu}(\alpha\,\chi_{\lambda}\psi^{\lambda} - \overline{\alpha}\,\chi_{\dot{\lambda}}\psi^{\dot{\lambda}}) = -2\,\alpha\,\overline{\alpha}\,\psi^{\mu}, \\g^{kl}\psi_{\mu|lk} - \frac{i}{2}\kappa\,\chi_{\mu}(\alpha\,\chi_{\lambda}\psi^{\lambda} - \overline{\alpha}\,\chi_{\dot{\lambda}}\psi^{\dot{\lambda}}) = -2\,\alpha\,\overline{\alpha}\,\chi_{\mu}.$$

$$(70)$$

The term that is endowed with  $\kappa$  is obviously very small in comparison to  $2\alpha \overline{\alpha}$ , such that one will obtain a completely-geodetic system for the Klein-Gordon equation at  $P_0$  in the first approximation.

Received on 30 May

#### Correction

# to the paper by L. Infeld and B. L. van der Waerden, "Die Wellengleichung des Elektrons in der allgemeinen Relativitätstheorie."

Hr. Prof. J. A. Schouten was kind enough to point out the following error:

In § 4, last paragraph (lowercase type), instead of "weight" and "weight -1/2," it should read "absolute weight" and "absolute weight -1/2," similarly, in the last paragraph of § 5.

The last paragraph of § 7 (lowercase type) should read:

"In the *ɛ*-formalism:

$$\kappa^{\dot{\lambda}\mu} = \psi^{\dot{\lambda}}\psi^{\mu} + \chi^{\dot{\lambda}}\chi^{\mu} = \psi^{\dot{\lambda}}\psi^{\mu} + \varepsilon^{\dot{\lambda}\dot{\rho}}\varepsilon^{\mu\sigma}\chi_{\dot{\rho}}\chi_{\sigma}$$
(50.c)

is not a spinor, but a spinor density of absolute weight + 1, since it is only in that case that a oneto-one relationship between  $\kappa^{\lambda\mu}$  and the world-vector  $\mathfrak{J}^s$  will exist. That suggests that we should understand  $\psi^{\lambda}$  (and  $\psi^{\lambda}$ ) to mean spinors of absolute weight 1/2. The Dirac equations (47) then demand that  $\chi_{\mu}$  and  $\chi_{\mu}$  should be spinors of absolute weight -1/2. However, the numerical factors  $|\Delta|^{1/2} \Delta^{-1} (|\Delta|^{-1/2} \Delta, \text{ resp.})$  will then enter into the transformation formulas for  $\psi^{\lambda}$  and  $\chi_{\mu}$ , resp. If we then agree to consider the spinor densities  $\psi^{\lambda}$  and  $\chi_{\mu}$  of absolute weight  $\pm 1/2$ , which also enter into (47) alone, as physically-meaningful quantity, while the  $\psi_{\lambda}$  and  $\chi^{\mu}$  are purely computational quantities that can always be eliminated from the final formulas [as in (50.c)], then the statement in § **4** will be true for the physically-meaningful quantities that they are all vector densities of weight  $\pm 1/2$  (+ for contravariant, – for covariant) or can be composed of such densities. Only the  $\Gamma_{\lambda}^{\rho}$  will appear in the formulas for their covariant derivatives."

#### Received on 3 July