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Remark on quantum-mechanical velocity

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A commentary on the previous paper by Fock [*] is given.

We now believe that we may assume that the matrix coefficients of the Dirac equation:

$$\sum_{i=1}^{5} \gamma_i p_i \psi = 0 \tag{1}$$

mean something more that a mere technical tool for giving a concise representation to the system of four equations. In the cited work, Fock has, in fact, proved that the γ_i , 1, 2, 3 are to be regarded as the quantum-mechanical operators that correspond to the classical three-dimensional velocity. One thus proves that the eigenvalues of the quantum velocity yield only two roots $\pm c$, which obviously ascribes an electrodynamical nature to the operator. Along with this, however, there exists yet another analogue of the velocity with a more mechanical character that has a continuous spectrum from $-c$ to $+c$.

The question arises: If one is given a classical formula that involves velocity then which operations are to be used for the transition to quanta? The answer is already included in the aforementioned assumption on the electrodynamical nature of the operator and reads: One must replace any current velocity (which is thought of as the velocity of electricity, and not a massive particle) with the quantum quantities γ_i . As simple and lucid as the principle of such a recipe might seem, it has still not been used up to now for obtaining quantum-theoretic relations. The reason for this was obviously the usual explanation of the γ_i as ordinary technical “symbols.” In the following, we would like to remark that the interpretation of the γ_i that is proposed here (admittedly, not in a rigorous way) allows one to obtain various relations completely by itself.

Breit has already proved the fact that the corpuscular current immediately confronts a quantum-theoretical one:

$$j_i = \rho\, V_i \cong \bar{\psi}\, \gamma_i\, \psi. \tag{2}$$

On the same grounds, we can write down the tensor T_{ik} as follows:

$$T_{ik} = \rho\, V_i\, V_k \cong \tfrac{1}{2}\{\bar{\psi}\, \gamma_i\, p_k \psi + \psi\, p_k\, \gamma_i\, \bar{\psi}\}, \tag{3}$$

[*] V. Fock, ZS. f. Phys. **55**, 127, 1929.

if we apply the usual symmetrization [*]. If we would like to regard the four terms in the Dirac equations – following Eddington's proposal [**] – as $T\psi$ then the symmetrization also seems imperative here [***]. Fock [****] has shown, moreover, that the expression for the Lorentz force also requires that the matrices γ_i replace the V_i .

We then translate the classical formula for the interaction energy as follows:

$$\frac{e^2}{r} V_i' V_i'' \simeq \frac{e^2}{r} \gamma_i' \gamma_i'' \tag{4}$$ [†]

(where the primes mean the first and second particle, respectively). By a corresponding normalization and with the addition of the fourth scalar term e^2/r, (4) includes the summation over all four component, and represents the total binding energy. Since the classical formula is only approximated, we cannot consider (4) as rigorous. The action-at-a-distance character of the expression is interesting. From what was said, it does not seem to be necessary interpret this term in the Gaunt-Eddington equation [††] for two electrons as an identity term, and nothing stands in the way of its use for the proton-electron system (although one could scarcely speak of the identity here).

If we restrict ourselves to the consideration of a world with only two electrical particles (the restriction to an electron seems impossible due to the appearance of e^2) then all potentials φ_i are proportional to the charge e. Instead of using the operator:

$$p_i = \frac{h}{2\pi i} \frac{\partial}{\partial x_i} + e\,\varphi_i ,$$

we demand that the coupling given by the terms φ_i yields the same value for the interaction energy as (4). That is, we must set:

$$e\gamma_i\,\varphi_i \simeq \frac{e^2}{r} \gamma_i \gamma_i' , \tag{5.1}$$

$$\varphi_i = \frac{e^2 \gamma_i'}{r} , \tag{5.2}$$

from which, we obtain the operator for the potential. We can deduce this from the formula:

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[*] H. Tetrode, ZS. f. Phys. **50**, 336, 1928.

[**] A. S. Eddington, Proc. Roy. Soc. (A) **122**, 358, 1929.

[***] Rem. by the editor: In the meantime, V. Fock and the author have proposed the Dirac equation as an invariant of linear geometry. – E. Wigner, ZS. f. Phys. **53**, 592, 1929 also symmetrized the equation, but on different grounds. In a very interesting paper, J. A. Gaunt, Proc. Roy. Soc. (A) **122**, 513, 1929 found the same expression for the potential as the author (5.2).

[****] V. Fock, *loc. cit.*

[†] D. Iwanenko, C. R. **188**, 616, 1929.

[††] A. S. Eddington, *loc. cit.*

$$A_i = \int \frac{\rho \mathfrak{v}_i}{r} d\tau \simeq \int \frac{\bar{\psi}\gamma_i\psi}{r} d\tau ,$$

because the latter may indeed be interpreted as the mathematical expectation value for the quantity γ_i / r. With this bridge between γ_i and the potentials, the restriction to the eigenvalues $\pm c$ seem understandable *.

We would like to further remark that the coefficients of the Dirac equation can be given a geometric meaning, and it is not entirely without interest to also catch sight of an electrodynamical sense to the same quantities. The recently-used fundamental form of linear geometry thus acquires one, as well, when it is also extended by similarity to the Weyl fundamental form. The path to the construction of the field operators also seems to be open.

Addendum by the editor: The approximate value of the quantum potential γ_μ / r must correspond to the Einstein potential $\Lambda^{\alpha}_{\mu\alpha}$. $\Lambda^{\alpha}_{\mu\alpha}$ is constructed from the Ricci γ_i k_i (see V. Fock and D. Iwanenko, *loc. cit.*), which is equal to the projection of the vector of geodetic curvature onto the congruence. In three-dimensional Euclidian space, this vector is simply equivalent to the first curvature W / R. One can draw a certain analogy between this value and the magnitude of φ_μ . The connection between the acceleration (i.e., curvature) and the potential seems somewhat unnatural (V. Fock has voiced this remark), since otherwise the Einstein Hamilton function is quadratic in the field strengths. Naturally, the potential should be associated with a constant with the dimensions of an electric charge. Let it be mentioned that Einstein's infinitely small quantities $\varepsilon_1\varepsilon_2$ and $\sigma = \varepsilon_2 / \varepsilon_1$ can be regarded as constructed from the universal constants. In essence, we have only two of them: $\dfrac{2\pi e^2}{hc}$ and $\dfrac{2\pi x m^2}{hc}$, or their combinations. These quantities are equal to zero in a vacuum.

The term $\dfrac{2\mathfrak{H}^{x\mu\alpha}}{\sigma}$ can be interpreted as a current with the e proportional to σ. Interpreting the introduction of such a constant h as the quantization of geometry seems all too naïve; the gap between quanta and gravitation can presumably be filled only by a rigorous operator geometry.

I would like to warmly thank H. Mandel for making it possible for the editor to be able to read through the recent work of Albert Einstein.

Leningrad, March 1929.

* On this, cf., V. Fock, *loc. cit*., where the velocity $\pm c$ is interpreted as the phase velocity of the wave.