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Remarks on a unified, nonlinear theory of matter

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Summary

The contemporary search for a unified description of all elementary particles in the context of a nonlinear spinor theory begins with three different Ansätzen: First, the nonlinear generalization of electrodynamics by MIE, BORN, and INFELD, and an analogous extension of mesodynamics, second, the fusion theory of DE BROGLIE, and third, the theory of composite particles of FERMI-YANG, GOLDHABER, SAKATA, et al. The treatment of various possibilities for the nonlinear extension of the DIRAC equation gives a brief glimpse of the further development of nonlinear theory in the work of Heisenberg and his colleagues. Moreover, we will go into the induced vacuum nonlinearity of the meson and gravitational theories and some simple solutions of the nonlinear field equations.

I

DIFFICULTIES IN FIELD THEORY AND THE PROBLEM OF THE UNIFIED DESCRIPTION OF MATTER

It is well-known that regardless of the tremendous progress in the description of atomic and free electrons and the electromagnetic field, the prediction of positrons and other particles, the discovery of isotopy and strangeness properties, as well as the satisfactory description of atomic nuclei, quantum electrodynamics and mesodynamics, and above all, contemporary relativistic quantum field theory is by no means complete. In fact, up to recent times, the deep-rooted difficulties have not been overcome that exist relative to the divergence of field masses of charged particles point particles, and furthermore, a unified description is lacking of all known particles, and finally, the provisional theory of elementary particles stands apart from the cosmological problems, and is, e.g., not capable of explaining the asymmetry between matter and anti-matter in our part of the universe. The general opinion is that the tools of the contemporary theory are not sufficient to resolve the aforementioned difficulties. Thus, the necessity arises of going to a newer, more general theory that includes a new constant in the form of a characteristic length in order to arrive at a satisfactory description of elementary particles. Here, let me be permitted to sketch out a brief overview of the various aspects of the problem of a unified theory of matter that might perhaps serve to show a broader audience how deeply the ideas of unified, nonlinear spinorial theory are rooted in modern physics. The discussion of such questions seems especially appropriate to a collection volume that is dedicated to MAX PLANCK, the great discoverer of a new fundamental constant – the quantum of action.

Without pausing to go into a detailed analysis of the aforementioned difficulties in quantum field theory, we remark that the hope that emerged in due course that one might remove the divergences by the process of renormalization has proved to be unfounded. LEHMANN and KÄLLEN have shown that the divergence in the GREEN functions of interacting particles is not weaker than the GREEN functions of free fields, which possess a δ-character on the light cone. Renormalization does not remove the divergences, but only isolates them; the problem of an ultimate lifting of the divergences is obviously essentially more deeply-rooted than the class of problems that is ordinarily circumscribed by the concept of regularization. Divergence problems relating to the field theory of a point charge are already known from classical electrodynamics, where one introduces the electron radius to remove them, which is nonetheless not possible to do in a relativistically invariant way. Formally free of objections, but still arbitrary relative to the choice of LAGRANGE function, the electrodynamics of BORN and INFELD, which represents a nonlinear generalization of MAXWELL's theory, is in a position to arrive at the finitude of the field energy. The theory of BORN and INFELD (1934), which builds upon an idea of G. MIE, gave a strong push in the direction of introducing nonlinear generalizations into meson theory and the theory of spinor fields, as well as the study of various nonlinear phenomena.

In connection with that, and independently of the removal of difficulties from classical theory, the presence of a null-point energy and vacuum fluctuations in quantum field theory, in turn, proves to be just as problematic in regard to self-energy; in addition, the charge also proves to be divergent.

There is an entire series of regularization procedures for the removal of these divergences, such as the introduction of various arbitrary cutoff factors or the smearing of the δ-function on the right-hand side of the commutation relations. However, the property of the right-hand side of the commutation relations that they must be solutions of the initial equations of field theory is then lost. The cutoff factors ultimately mean the introduction of non-locality, which was first proposed by G. WATAGHIN, and which find their expression in the non-local interaction and the ultimate dimensions of particles. Apparently, the idea of non-locality, which many papers have been devoted to – despite the fact that its introduction has led to no concrete results up to now – still contains something valid in regard to the elementary length and the ultimate dimensions of particles.

A second possibility for removing the divergences exists in connection with the hypothesis of a discontinuous, in some way quantized, spacetime manifold. This idea of a nuclear spacetime structure was developed simultaneously by HEISENBERG and myself, together with V. A. AMBARZUMYAN, in a series of papers. It rests upon the notion that the four component axes are associated with operators. This picture, which also carries the presence of a smallest length in nature (the requirement of a "softening" of the limit in the light-cone calculation, resp.), might possibly contain some truth. An analogous idea was already expressed much earlier by Arabian philosophers in the Middle Ages in a crude, qualitative way.

We now turn briefly from the sketching of the first problem – viz., the divergence difficulties – to the second problem of the unified description of all matter. The fact that all particles can be converted into each other implies that they are based upon a certain unified sub-stratum that appears in some modifications. The isotopy properties, the possibility of combining the elementary particles into a semi-empirical schema (e.g., GELL-MANN, NISHIJIMA) and being capable of classifying photons (γ), leptons (e_-, e_+, ν, $\bar{\nu}$, m_+, m_-), mesons p and κ; nucleons (p, $\bar{p}$, n, $\bar{n}$), and hyperons Y (Λ, Σ, Ξ), likewise suggests that all particles must have something in common.

In connection with that, we mention the very interesting classifications of SACHS, E. RAYSKI, and H. DALLAPORTE. The situation recalls the time before the development of the periodic system by MENDELEYEV and the explanation for the chemical elements on the basis of the quantum theory of atomic shells, and then the development of a periodic system for the atomic nucleus, and its explanation on the basis of the shell model. The proposal of the existence of a substratum that is common to all elementary particles led to two attempts to present a unified theory of elementary particles. On the one hand, DE BROGLIE showed that DIRAC spinors, which indeed describe particles of spin $\frac{1}{2}\frac{h}{2\pi}$, must therefore play a fundamental role, from which one can construct all other particles with spin 0, 1, etc., by "fusion." The wave functions of composite particles ψ can be expressed by products of spinor functions $\psi_{1/2}$, $\psi^*_{1/2}$:

$$\psi = \psi_{1/2} \cdot \psi^*_{1/2} .$$

Speaking intuitively, they can be regarded as "rotating" objects by addition or subtraction of the angular momenta, as well as objects with a large angular impulse, as well as ones for which it vanishes, while one can obtain no rotation by composition from "non-rotating" particles of spin 0, which can be described by the simpler, one-component, KLEIN-GORDON equation. In this, we recognize, in modern form, a further development of the ideas of KELVIN and HELMHOLTZ, or previously by DESCARTES, on the composition of matter from vorticial structures. A somewhat more general treatment of fusion on the basis of group theory was considered by H. A. SOKOLIK [**1**]. An example of the theory of fusion is the DE BROGLIE hypothesis of the possibility of obtaining a photon by the union of two neutrinos. The neutrino theory of light was developed further by JORDAN, KRONIG, and A. SOKOLOV. The weakest aspect of the original theory of fusion was that it ignored the interaction energy between two fields.

In the construction of a unified theory that requires two-component or four-component spinors, one must consider that, in addition to ordinary spinors, as one employs in, e.g., the theory of electrons ("spinors of the second kind," according to E. CARTAN), there also exist spinors of the first kind (according to CARTAN), whose transformations differ by only inversions, and indeed one has:

$$\psi \to \psi' = \pm a_\mu \gamma_5 \gamma_\mu \psi,$$

while:

$$\psi \rightarrow \psi' = \pm\, a_\mu\, \gamma_\mu\, \psi,$$

resp., under a reflection in a hyperplane that is normal to the unit vector a_μ .

If one does not permanently leave aside those extraordinary spinors whose corresponding DIRAC equation includes the term $\gamma_5\, m_0$ in place of the mass then one must also consider such quantities in the presentation of the general theory. One can also form mixed spinors in it, which behave like quantities of the first (second, resp.) kind under space (time, resp.) inversion, and conversely [**2**]. Such mixed spinors correspond precisely to the "anomalous" representation of the reflection group that was treated by GELFAND, ZEITLIN, and SOKOLNIK [**3**].

In the construction of bilinear quantities in DE BROGLIE's spinor theory or the unified one, one can employ various spinors, first, of the second kind, as well as mixed quantities. This possibility leads us to bosons that possess extraordinary reflection properties. Let it be remarked that reasoning that is analogous to that of fusion theory can also be applied to isotopic spin space. Since the "anomalous" representation of the reflection group does not leave the DIRAC equation invariant, in general, but requires the introduction of eight-component spinors, one also comes, in this way, to a possible interpretation of isotopic spin space (cf., also the investigations of FESHBACH, SCHREMP, and WATANABE). Similarly, the anomalous spinors do not seem to obey the PAULI principle, either.

In connection with the theory of fusion, we further remark that according to MIRIANASHVILI, the possibility exists of forming bilinear combinations of spinors that are pure imaginary. That is connected with the fact that each of the spinors that enter into the product can belong to one of four classes, according to its transformation character under inversions of the coordinates and time; in the transformation formulas:

$$\psi' = \rho_s\, \gamma_4\, \psi,$$

the ρ_s can assume the values ± 1, $\pm i$, etc.

The second direction in which a unified description of elementary particles was sought led to the model of bosons that are composite particles formed from two fermions. FERMI and YANG were the first to propose that a π-meson was composed of a nucleon and an anti-nucleon, between which a force of high intensity acted over an exceedingly short distance. Later on, SAKATA and his colleagues (MAKI and others) started with nucleons and Λ-particles; M. MARKOV is also linked with this direction. In connection with this, let us mention the papers of GOLDHABER, in which fermions and bosons are based in isospin space; e.g., κ-mesons and nucleons. A weak point in the original theory of composite bosons and other particles is the arbitrariness in the choice of interaction energy. The analogy between fusion theory and the model of composite particles is conspicuous (cf., also the WATANABE model).

A third group of difficult problems exists in connection with the understanding of the gravitational field, which must be treated as the curvature of the space-time manifold, according to EINSTEIN's theory of relativity. In it, according to FRIEDMANN, the geometric structure of the universe is such that all appearances suggest that there is a tendency towards the expansion of the cosmos that is expressed in the observed receding of spiral nebulae, even though different kinds of oscillating or contracting solutions are also possible. This first yields the problem of connecting the gravitational field with the other forms of matter, and secondly, various questions arise concerning the behavior of matter in the hypothetical singular state of highest pressure at the beginning of the expansion of the universe. Up to now, only questions concerning the formation of elements were investigated, in that regard. However, by means of this theory, one must be able to understand why there would be such a strong preference for ordinary matter over anti-matter in the part of the expanding universe that is known to us. It is possible that there exist other regions of the universe in which anti-particles are present in greater abundance than ordinary matter (cf., DIRAC) or particles with opposite "helicity" or "screw sense" (cf., LEE-YANG), and perhaps, in turn, there exists a connection between expansion and contraction and the preponderance of particles (anti-particles, resp.).

As far as the gravitational field is concerned, there exists a fundamental problem in proving that there is a free gravitational field, and therefore gravitational waves. It is know that such a field cannot be established experimentally, up to now, and there exists disunity amongst the theoreticians regarding its existence. Some authors – among them, INFELD – assert that gravitational waves that transport energy cannot exist; other authors cleave to EINSTEIN's interpretation and lean towards the suggestion that there are gravitational waves (DIRAC, PAPEPETROU, WHEELER, et al.). In any event, one must remark that the feeble intensity of any possible gravitational radiation is to be understood on the grounds that the gravitational waves do not possess a dipole character, but a quadrupole character. This rests upon the fact that the gravitational potential is a tensor of rank two, and not a vector, like the electromagnetic potential. From that, it will be immediately clear that in the known formula for gravitational radiation, which was derived in a complicated way, the square of the third time derivative of the quadrupole moment of the mass distribution $\left|\bar{Q}\right|^2$ appears, and not the square of second derivative of the dipole moment, as in electrodynamics. In the event that gravitational waves exist, they must, by our hypothesis, be capable of conversion into ordinary matter; e.g., into photons, electron-positron pairs, and conversely be produced by the annihilation of them. The probability of such transitions is very small, but increases rapidly with the additional energy [**4**]. The presence of such transitions, which must mean something for cosmological problems, might lead to an even closer relationship between spacetime structure and ordinary matter. It is known that one obtains the gravitons, which possess spin 2 (cf., PAULI and FIERZ), by quantizing the weak gravitational field; their exchange gives NEWTON's law of gravitation (M. BRONSTEIN). In addition, the presence of gravitons must also be considered in the balance of heat, where one ascribes a temperature T to a graviton gas and introduces its energy, which is proportional to T^4, into the stability condition for systems of JEANS type.

The examination of the gravitational field is also important for the fact that we undoubtedly have an exceptional example of a nonlinear field theory before us whose nonlinearity is such that the gravitational field will be generated by all forms of matter, and among, them, the gravitational field itself. The nonlinear character of the field equations for the gravitational field makes it possible to derive the equations of motion for the particles that generate the field from those equations (EINSTEIN, GROMMER, INFELD, HOFFMANN, FOCK). Up to now, such equations of motion have been obtained in the classical NEWTONIAN approximation with certain additional terms; the solution of the problem of deriving the quantized equations of motion from the quantized field equations remains reserved for future endeavors.

It is appropriate to recall that the success of EINSTEIN's theory of gravitation creates the hope that it would be possible to bring about a unified theory that encompasses electrodynamics (WEYL, EDDINGTON, CARTAN, EINSTEIN, et al.) and mesodynamics (SCHRÖDINGER). Except for numerous mathematical generalizations, all attempts in this direction have led to no physical results. Obviously, they were all of a quantum character, which, as we will prove later, do not account for a wide variety of elementary particles. Much less successful was the attempt to relate all matter to the electromagnetic field, although MIE's idea of a nonlinear generalization of electrodynamics was exceptionally fruitful for physics. Let it be stressed that in MIE's nonlinear theory, in which no gauge invariance was present, but which proved to be very useful for mesodynamics, as well as in the BORN-INFELD theory, the possibility of a unified description of fields and elementary particles – in the form of singular points – already exists.

We do not need to stress to what degree the idea of the Eighteenth – and some of the Nineteenth – Century of reducing all phenomena to classical mechanics has proved inadequate. To begin with, the properties of the electromagnetic field – a form of matter with vanishing rest mass – cannot be explained by them. The impossibility of being able to reduce the explanation of physical processes to a non-relativistic, non-quantum theory of mechanics is therefore obvious.

II

ON NONLINEAR SPIN THEORY

If we now direct our attention anew to the problem of a unified description of all known forms of matter – i.e., elementary particles and fields – then, with hindsight of the foregoing brief account of the situation, one must obviously start with a nonlinear generalization of the DIRAC equation for the spinor field. The spinorial character of the fundamental field comes from DE BROGLIE's argument and the model of composite particles, and the nonlinearity follows from the necessity of introducing the interaction of this field with itself, since there is indeed only one form of matter in the unified theory. This gives us a starting point in the form of an equation that was suggested by us in 1938 [**5**]:

$$\gamma_\nu \frac{\partial \psi}{\partial x_\nu} + m_0 \psi + \lambda \psi (\bar{\psi}\psi) = 0 \qquad (\lambda = hcl^2), \qquad \text{(A)}$$

in which l refers to the "smallest length," while the remaining symbols have the usual meaning; the simplest nonlinear generalization then consists of a term of type ψ^3, which is obtained from the invariant $I_1 = (\bar{\psi}\psi)^2$. This and other forms for a nonlinear generalization of the DIRAC equation are based in the modern unified spinor theory of matter, whose development chiefly by HEISENBERG and his co-workers produced the most significant progress.

We emphasize that the assumption of nonlinear terms in electrodynamics, as well as the field equations of gravitation and spinor theory, represent an example of "original" nonlinearity (class A). Nonlinear terms were incorporated into the basic equation for the meson field in an analogous way. The simplest example is the generalized KLEIN-GORDON equation:

$$\Delta\psi - \frac{\ddot{\varphi}}{c^2} - m_0\,\varphi + \lambda\varphi^3 = 0.$$

However, the iteration of the nonlinear DIRAC equation leads to complicated nonlinearities up to terms of type ψ^5 inclusive. Before we turn to the discussion of the consequences of the nonlinear spinor equation, we would like to remark that according to quantum theory nonlinearities are present in all equations, in a well-known way, as a result of the reciprocal conversion processes between fields and particles. As a first example of such an "induced" nonlinearity (class B), one has the nonlinearity in the equations of electrodynamics that comes from a process of type $e_- + e_+ \rightleftharpoons 2\gamma$ (cf., EULER, KOCKEL, HEISENBERG, WEISSKOPF, SCHWINGER), or also processes of type $\pi^- + \pi^+ \rightleftharpoons 2\gamma$ (cf., KURDGELAIDSE). In an analogous way, the virtual processes of the creation of nucleon-anti-nucleon pairs $p + \overline{p} \rightleftharpoons 2\pi$ lead to nonlinearities in mesodynamics (cf., MALENKA, KURDGELAIDSE), which are of type at least φ^3. For the calculation of such nonlinearities, one can employ either the SCHWINGER method of GREEN functions or the method of HEISENBERG and his co-workers, which consists of the comparison of a nonlinear LAGRANGE function of general structure with the results of the calculation of a concrete nonlinear effect; e.g., the scattering of light by light by means of electrons and positrons or the scattering of mesons by mesons by means of nucleons. In connection with that, we refer to the research of M. MIRIANASHVILI, who, in generalizing the earlier results of MALENKA and KURDGELAIDSE, gave the general form of the nonlinear extension, not for the weak case, but for arbitrary interactions, but for slowly-varying scalar, pseudo-scalar, vectorial, and pseudo-vectorial meson fields, in a way that is similar to the way that HEISENBERG and SCHWINGER obtained the general form for the vacuum LAGRANGE function for electrodynamics and scalar meson dynamics [**6**].

In particular, one gets the vacuum LAGRANGE density in the case of a neutral, constant, pseudo-scalar field that interacts with a spinor field:

$$L_{\text{vac.}} = -\tfrac{1}{2}(2\pi)^{-2}(m^2 + g^2\overline{\varphi}^2)^{1/2} e^{-\frac{1}{2}(m^2+g^2\overline{\varphi}^2)s_0} W_{-3/2,-1}[(m^2 + g^2\overline{\varphi}^2)s_0]\,.$$

In this, $W_{-3/2,\,-1}$ is the WHITTAKER function, and φ is the function of the meson field; moreover, g means the coupling constant, m, the mass of the nucleon, and s_0 is the cutoff constant. For a weak field, this yields the previous result of MALENKA and KURDGELAIDSE:

$$L'_{\text{vac.}} = -\frac{m^2}{8\pi^2}\frac{g^2}{\hbar c}\left[-\frac{1}{m^2 s_0} + \ln\frac{1}{\gamma m^2 s_0} + 1\right]\overline{\varphi}^2 - \frac{1}{(4\pi)^2}\frac{g^4}{(\hbar c)^2}\left[\ln\frac{1}{\gamma m^2 s_0}\right]\overline{\varphi}^4,$$

($\gamma = e^C = 1.78\ldots$, C = EULER const), in which the first term was united with the meson mass by renormalization and the second one yields an additional nonlinear term in the KLEIN-GORDON equation. Intuitively speaking, it comes from the virtual creation of a pair of spinor particles (i.e., nucleons) by mesons and the subsequent annihilation of the nucleons with the emission of the mesons. We remark that the structure of the coupling constant $g^4 / \hbar^3 c^2$ for the nonlinear term comes about quite simply when one specifies the distribution of the sources of the meson field according to the THOMAS-FERMI model in the ultra-relativistic domain.

In addition, we would like to remark that a constant pseudo-scalar field yields not only the pseudo-scalar vacuum current, but also a non-zero scalar current that is responsible for the coupling between the fields.

M. MIRIANASHVILI obtained the following normalized expression for the vacuum term in the LAGRANGE density that arises from the coupling to a spinor field for the case of neutral vectorial meson dynamics:

$$L = - F - \mu^2 A_\mu^2 - \frac{1}{8\pi^2}\int_0^\infty ds\, s^{-3} \exp(-m^2 s)\left[(gs)^2 I \frac{\operatorname{Re}\cosh gsx}{\operatorname{Im}\cosh gsx} - 1 + \frac{2}{3}(gs)^3 F\right],$$

in which:

$$F = (1 + C g_0^2)\, F_0 , \qquad I = (1 + C g_0^2)\, I_0 ,$$

$$g^2 = \frac{g_0^2}{1 + C g_0^2}, \qquad \mu^2 = \frac{g_0^2}{1 + C g_0^2}, \qquad C = \frac{1}{12\pi^2}\int_0^\infty ds\, s^{-1} \exp(-ms^2) .$$

This expression generalizes the previous result of HEISENBERG and SCHWINGER for electrodynamics.

We obtain a new closed expression for the vacuum LAGRANGE density of a constant, pseudo-vectorial meson field (wave function $\bar{A}_\mu$) that originates in the coupling of a spinor field (i.e., nucleon), and which is useful for not just a weak field:

$$L_{\text{vac.}} = -\tfrac{1}{2}(2\pi)^{-2}(m^2 - 2g^2\bar{A}_\mu^2)^{1/2} e^{-\frac{1}{2}(m^2 - 2g^2\bar{A}_\mu^2)s_0} W_{-3/2,-1}[(m^2 - 2g^2\bar{A}_\mu^2)s_0] .$$

We now consider the interaction of a DIRAC field $\chi(x)$ with another given DIRAC field $\psi(x)$ that corresponds to a LAGRANGE density of the form:

$$L = - g\, \bar{\chi}(x)\gamma_5\, \chi(x)\bar{\psi}(x)\gamma_5\, \psi(x),$$

for example. (The precise form of the LAGRANGE density is inessential here.) Assuming that $\psi(x)$ is a given field and $\chi(x)$ is found in the vacuum state, we then obtain the following equation for the GREEN function G of the field $\chi(x)$:

$$(\gamma_\nu p_\nu - g\, \gamma_5 \bar{\psi}\, \gamma_5\, \psi + m)\, G = 1,$$

and for the supplementary term in the LAGRANGE density:

$$L_{\text{vac.}}^{s'} = -\tfrac{1}{2}(2\pi)^{-2}\int_0^\infty s^{-3}\exp[-i\{m^2 + g^2(\bar{\psi}\gamma_5\psi)^2\}s]\,ds\,,$$

or, for small values of ($\bar{\psi}\,\gamma_5\,\psi$) (in the usual units):

$$L_{\text{vac.}}^{s'} = -\frac{m^2}{8\pi^2}\frac{g^2}{\hbar c}\left[-\frac{1}{m^2 s_0} + \ln\frac{1}{\gamma m^2 s_0} + 1\right][\bar{\psi}(x)\,\gamma_5\,\psi(x)]^2\,,$$

where s_0 is the square of the minimal proper time, which is introduced as a cutoff parameter in other cases.

We obtain the desired nonlinear spinor equations from this [**7**]:

$$\left(-i\gamma_\mu\frac{\partial}{\partial x_\mu} + m + \lambda\bar{\psi}\psi\right)\psi(x) = 0,$$

where

$$\lambda = \lambda_{\text{vac.}} = -\frac{m^2}{4\pi^2}\frac{g^2}{\hbar c}\left[-\frac{1}{m^2 s_0} + \ln\frac{1}{\gamma m^2 s_0} + 1\right].$$

If we set $\sqrt{s_0} \sim 10^{-14}$ cm then we obtain the value 10^{-48} cm^3 for λ, which approximates the FERMI constant. In the subsequent approximations, one gets nonlinear terms of higher order than ψ^3. In this way, we get supplementary nonlinear terms for the spinor equation as induced by the interaction with the nonlinearity that is required by the vacuum, and not as original ones that are introduced into the equation on the basis of this or that reasoning *a priori*. Naturally, the induced and original nonlinearities must be combined by the renormalization process, and lead to an (in principle) observable nonlinearity constant:

$$\lambda = \lambda_0 + \lambda_{\text{vac.}}$$

In concluding this consideration of induced nonlinearities, we mention the case of the gravitational field [**4**]. For the sake of simplicity, we take the case of scalar (pseudo-scalar, resp.) particles that interact with the gravitational field, and write the LAGRANGE function in the form of an anti-commutator:

$$L = \tfrac{1}{4}[\varphi(x), S\varphi(x)],$$

in which one has:

$$S\varphi = g^{\alpha\beta}\frac{\partial^2\varphi}{\partial x_\alpha\,\partial x_\beta} + \frac{1}{\sqrt{-g}}\frac{\partial\sqrt{-g}\,g^{\alpha\beta}}{\partial x^\alpha}\frac{\partial\varphi}{\partial x^\beta} - m^2\,\varphi.$$

The GREEN function $G(x, x')$ for a particle, which is defined as the vacuum expectation value of the ordered product $[T(\varphi(x), \varphi(x'))]_0$, where T means the WICK (chronological) operator, then satisfies the equation:

$$S\,G(x, x') = \frac{1}{\sqrt{-g}}\,\delta(x - x').$$

One gets the vacuum value of L from this:

$$L_0 = \tfrac{1}{2}\,S\,G(x, x')\,|_{\,x \to x'}\,.$$

Finally, from A. BRODSKI, one has a theorem for the vacuum value of the action function, which is connected with the GREEN function:

$$W_{\text{vac.}} = -\tfrac{1}{2}\,\text{Spur}\,\ln G + \text{const.}$$

This theorem is also true for spin fields when one includes the summation over the spinor indices. For the case in which there is no creation of real pairs, one gets:

$$W_0 = \frac{1}{2}\frac{1}{(2\pi)^2}\int_0^\infty \tau^{-1}d\tau \int \exp[-ikx]\exp[i\sqrt{-g}\,]S\tau\exp[ikx](dx)(dk)\,,$$

where τ is the proper time, or fifth coordinate. If one now goes to the weak-field approximation then one obtains an expression that first includes a term that is proportional to the action function of the free gravitational field with an infinite coefficient. This divergence will be isolated by renormalizing the coordinates and the gravitational charge – i.e., the mass:

$$m \to \left(\frac{1}{1 + \dfrac{\kappa m^3}{2\pi\hbar c}\cdot\dfrac{1}{c}\displaystyle\int_0^\infty s^{-2}\exp(-s)\,ds}\right)^{1/2} m.$$

Here, κ is the gravitational constant. The second term also has an infinite coefficient and includes the D'ALEMBERTIAN operator. Its renormalization would be possible only if the usual EINSTEIN theory included higher derivatives.

If one employs the given expression for the vacuum value of the action function then one can calculate the probability for the creation of a pair by the gravitational field and other analogous effects. For the collision cross-section of the conversion of two scalar bosons into two gravitons, one gets ([1]):

([1]) It is quite instructive to compare this expression with the known formula for the annihilation cross-section of an electron-positron pair: $\sigma = \pi r_0^2\,\frac{c}{v}$ $(v \ll c)$.

$$\sigma = 24p\ r_g^2 \frac{c}{v}\left(\frac{E}{mc^2}\right)^2 ,$$

in which one has the gravitational radius:

$$r_g = \frac{\kappa m}{c^2} .$$

The further development terms in the vacuum value of the LAGRANGE function or the action function necessarily lead to nonlinear supplementary terms in the original linear equations of the weak gravitational field, precisely as in the case of the electromagnetic, meson, or spinor equations. When one starts with the linear equations for the weak gravitational field, whose quantum has rest mass zero and spin 2, one then necessarily arrives at nonlinear generalizations in which the nonlinear terms are generally induced by the vacuum, independently of the equations of the original nonlinearities that are introduced by the general theory of relativity.

We now go somewhat more rigorously into some questions in nonlinear spinor theory, whose starting point, as we have shown, was equation (A). The next problem is the presentation of all possible nonlinear supplementary terms in the spinor equations or corresponding invariants of the form ψ^4 that contain no derivatives. For this, we start with the expression that describes the interaction of arbitrary fields. As a foundation, one can intuitively take the known expression $g_F(\bar{\psi}_N \Omega_i \psi_P)(\bar{\psi}_\nu \Omega_i \psi_e)$ from the theory of β-decay; g_F is the FERMI coupling constant. If one assumes all spinors to be equal here then one gets the desired five types of nonlinear invariants, since one can choose Ω_i to be a scalar, a pseudo-scalar, a vector, a pseudo-vector, or an anti-symmetric tensor of rank two. In addition, one can also introduce invariants that one obtains, intuitively speaking, in such a way that one could identify all spinors with each other (nonlinear "contraction") in the four fermions in the interaction terms, which do not conserve leptons or baryon charges under β-decay. In this way, we obtain the following combination of nonlinear invariants [**4**]:

$$\sum_m \{g_m[(\bar{\psi}\,\Omega_m\psi)^2 + (\psi\,\Omega_m^T\bar{\psi})^2 + g'_m[(\bar{\psi}\,\Omega_m\psi),(\psi\,\Omega_m^T\bar{\psi})]_+ + g''_m[(\psi C\,\Omega_m\psi),(\bar{\psi}\,\Omega_m^T C\,\bar{\psi})]_+\} . \tag{B}$$

In this, Ω_m^T is the quantity that is the transpose of Ω_m . As PH/ LERUSTE has emphasized, there are some relations between the various invariants of the non-quantized theory that were presented by DE BROGLIE, PAULI, KOFFINK, and TAKABAYASI:

$$J_S - J_{PS} = J_V = J_{PV} ; \quad J_S + J_{PS} = J_T ,$$

where:

$$J_S = (\bar{\psi}\psi)^2 ; \quad J_V = (\bar{\psi}\,\gamma_\mu\psi)^2 ,$$

etc.

We shall now briefly pause to examine the further stages in the development of the nonlinear spinor theory and chiefly address the thorough papers of HEISENBERG and his co-workers; in particular, the overview of 1957, the presentation to the congress in Padua-Venice in October 1957, and the contribution to the PLANCK celebration in Berlin and Leipzig in April 1958 [**9**].

Once we have presented a fundamental generalization of the DIRAC spinor equation, the next step consists of a cancelation of the mass term in (A) and the proposing of the thus-altered nonlinear equation:

$$\gamma_\mu \frac{\partial \psi}{\partial x_\mu} + \lambda(\bar{\psi}\psi)\psi = 0 \qquad (\lambda = \hbar c l^2) \tag{C}$$

as the fundamental equation of matter whose quantization should lead to values for the particle masses (HEISENBERG). In order to choose the possible nonlinear supplementary term, one poses further requirements (e.g., isotopy invariance, etc., cf. *infra*) in place of the original pure-scalar term in (A), with the help of (B). The third step consists of an introduction of the new HEISENBERG quantization rules with the help of an indefinite metric in HILBERT space. The modified commutation relations lead to vanishing values of the anti-commutator on the light-cone; that would be in agreement with the solution of the general nonlinear equation. The introduction of an indefinite metric into HILBERT space that was already proposed by DIRAC generally requires additional analysis in order to exclude states with negative norms. In connection with this, we direct our attention to an attempt by GUPTA and BLEULER to impose an additional condition on the vector components in HILBERT space II that would guarantee that these states would make no contribution to the expectation values of observable quantities. On the other hand, when HEISENBERG used real states from HILBERT space I for $t = -\infty$, he showed that the chosen commutation functions guaranteed the presence of only physical real states also for $t = +\infty$. To that end, N. N. BOGLIUBOV and his co-workers have decomposed each state amplitude into a physical and a non-physical part F, the latter of which is determined uniquely from the physical part with the help of the additional condition $F(-\infty) = -F(+\infty)$. This boundary condition has a non-local form. Thus, we arrive at a non-local theory whose elements also appear in HEISENBERG's variant of the theory as the mean in the neighborhood of the light-cone. A similar smearing of the light-cone leads, on the one hand, to the interesting problem of describing the internal structure of elementary particles, and on the other, to the problem of the structure of space and time in very small dimensions.

It is known that HEISENBERG and his co-workers obtained a finite value for the fine structure constant ($\alpha = 1/267$) by approximate integration of the nonlinear spinor equation (C) on the basis of the new, altered commutation relations, as well as finite values for the masses of the free fermions (i.e., nucleons) ($\kappa = 7.426/l$) and the masses of four bosons that represent higher excited states of the world spinor field:

$$\kappa l = 0.33, \qquad 0.95, \quad 1.74, \quad 3.32.$$

In this connection, we would like to refer to some exact solutions of wave type of the nonlinearly generalized KLEIN-GORDON and DIRAC equations that were obtained by KURDGELAIDSE [**10**] with the help of elliptic functions. For example, the equation:

$$-\frac{\partial^2 \varphi}{\partial x_\mu^2} + k_0^2 \varphi + \lambda \varphi^3 = 0$$

has the solution:

$$\varphi = \varphi_0 \,\mathrm{cn}(\sigma + c), \qquad \sigma = k_\mu\, x_\mu\,, \qquad k_\mu = (k_n\,,\, k_4 = i\,\omega),$$

$$-\,k_\mu^2 = -\,k^2 + \omega^2 = k_0^2 + \lambda \varphi_0^2\,.$$

In addition, we have derived radially-symmetric exact solutions for some nonlinear equations. For example, the equation:

$$\psi'' + \frac{3}{s}\,\psi' + \lambda \psi^3 = 0, \qquad \psi = \psi(s), \qquad s = \sqrt{x_\mu^2}$$

has the solution:

$$\psi = \frac{1}{\sqrt{\lambda s_0^2}} \sqrt{\frac{2k_1^2}{2k_1^2 - 1}}\, \mathrm{cn}\left[\frac{1}{\sqrt{2k_1^2 - 1}} \ln \frac{s}{s_0} + K(k_1)\right], \qquad k_1^2 = \frac{1}{2} \frac{\lambda \varphi_0^2}{k_0^2 + \lambda \varphi_0^2}\,,$$

where s_0 is an arbitrary constant, k_1 is the modulus of the elliptic function, and $K(k_1)$ is an elliptic integral of first order.

One can employ the last solution for the construction of characteristic, singular functions in the neighborhood of the light-cone:

$$\Delta(s) = \frac{1}{\sqrt{4\lambda s_0^2}} \left\{1 + \left(\frac{s_0}{s}\right)^2\right\}_{s \to 0}\,.$$

It is interesting that we arrive at exact solutions for the fundamental and very-closely related nonlinear equation that are apparently analogous to the ones that HEISENBERG obtained by approximation methods.

On the other hand, we can use the exact solutions of wave type of the nonlinearly-generalized KLEIN-GORDON equation to express the energy H and impulse of the field as a sum of quantities that are connected with the individual components.

We obtain:

$$H = \frac{1}{2} \int \left\{\left(\frac{\partial \varphi}{\partial x_n}\right)^2 - \frac{1}{c^2}\left(\frac{\partial \varphi}{\partial x_n}\right)^2 + k_0^2 \varphi^2 + \frac{1}{2} \lambda \varphi^4\right\} (dx) = a\, \frac{k^2 + K_0^2}{\omega} \qquad (n = 1, 2, 3),$$

in which:

$$a = \frac{1}{2} \varphi_0^2 l \omega\,, \qquad K_0^2 = \frac{1}{l}\left\{k_0^2 + \frac{1}{2} \lambda \varphi_0^2\right\},$$

$$l = \frac{2}{3}\left\{2-(1-k_1^2)\frac{1}{k_1^2}\left(1-\frac{E(k_1)}{K(k_1)}\right)\right\}.$$

($l = 1$ in the linear case.) $E(k_1)$ is a second-order elliptic integral.

If we now set the energy of the field equal to the mass $\bar{k}_0$ of the bosons for vanishing rest-mass, so $k_0 = 0$, as well as $k = 0$, then it follows that:

$$\varphi_0^2 = \frac{2}{\bar{k}_0}, \qquad \lambda = \bar{k}_0^3 .$$

If one substitutes the spectral development of our wave solutions:

$$\frac{\varphi}{\varphi_0} = \sum_{n=0}^{\infty} a_n \cos\sigma_n , \quad \sigma_n = (2n + 1)\,\beta(\lambda)\,\sigma = k_\mu^{(n)} x_\mu , \qquad n = 0, 1, 2, \ldots$$

$$\beta(\lambda) = \frac{\pi}{2}\frac{1}{K(k_1)}, \qquad a_n = \pi\,\frac{1}{k_1 K(k_1)}\cdot\frac{1}{\cosh\rho_n},$$

$$\rho_n = (2n + 1)\,\rho_0 , \qquad \rho_0 = \frac{\pi}{2}\frac{K'(k_1)}{K(k_1)}$$

then one obtains, after some regrouping:

$$H = \varphi_0^2\left(\frac{K_0}{\bar{k}_0}\right)\sum_{n=0}^{\infty}\frac{1}{2}\left(\frac{a_n}{a_0}\right)^2 M_0^{(n)2} , \qquad M_0^{(n)} = (2n + 1)\,M_0^{(0)} ,$$

$$M_0^{(0)} = \left(\sqrt{3}\,\frac{\beta(\lambda)}{4}\right)^4 \bar{k}_0 \sim 0.36\,\bar{k}_0 ,$$

$$K_0 = (2\sqrt{2}\,a_0)^2 = 7.29\,\bar{k}_0 .$$

If one employs such coarse – so-to-speak, even “semi-classical” – quantization methods then one obtains a mass spectrum for the mesons from them that, according to KURDGELAIDSE, is similar to the spectrum that was given by HEISENBERG:

0.36, 1.08, 1.80, 2.52, 3.24, 3.96, …

The last step in the construction of a unified, nonlinear, spinor theory is connected with the consideration of the isotopy properties of particles. In that, first and foremost, the SALAM-TOUSHEK [**2**] group:

$$\psi' = \psi\, e^{i\alpha\gamma_5}$$

plays a role. According to the theory of E. NOETHER, it does not correspond to the conservation of a vector, like the cyclic group $\psi' = \psi\, e^{i\alpha}$, but a pseudo-vector, and thus a new "charge" appears. In the final analysis, this conservation law is apparently connected with the conservation of baryon number. In addition, one has the PAULI transformation group:

$$\psi' = a\,\psi + b\, i\,\gamma_5\,\psi^c, \qquad |a|^2 + |b|^2 = 1,$$

$$i\,\gamma_5\,\psi^c = -\,b^*\psi + a^*\, i\,\gamma_5\,\psi^c.$$

In this, ψ^c is the wave function that is the charge-conjugate – or better, anti-particle conjugate. As one easily sees, this group is an isomorphic spinor representation of the rotation group, and, as GÜRSEY [**11**] first showed, can be compared to a rotation in isotopy space.

This now raises the problem of choosing from the previously-given nonlinear invariants (B) the ones that not only satisfy the condition of LORENTZ invariance, but which are also invariant under the PAULI and SALAM-TOUSHEK groups. Here, one can add the condition of invariance under spatial reflections and particle-anti-particle conjugation, as well as the condition of the vanishing of the nonlinear supplementary term for the two-component neutrino. As a choice of invariant, we, with A. BRODSKI, introduce the eight-component spinor:

$$\Psi = \begin{pmatrix} \psi \\ i\gamma_5\psi^c \end{pmatrix}.$$

Under LORENTZ transformations, it transforms with the matrices $\gamma_\mu \times E_2$, and under PAULI transformations, with the matrices $E_2 \times i\,\rho_i$. In this, the ρ_i are the usual PAULI matrices, and E_2 and E_4 are unit matrices. We introduce seven anti-commutating matrices Γ_r ($r = 1, 2, \ldots, 7$):

$$\Gamma_{\mu+1} = \gamma_\mu \times E_2 \qquad (\mu = 0, 1, 2, 3), \qquad \Gamma_{i+4} = \overline{\tau}_i = i\,\gamma_5 \times \rho_i\,.$$

In order to be able to compare the PAULI transformations with the rotations in isotopic space, we write down the eight-component spinor in a form in which, from FEYNMANN, exhibits each particle as a two-component semi-spinor [**12**]:

$$\Psi = \frac{1}{2}\begin{vmatrix} (1+i\gamma_5)\psi + (1-i\gamma_5)\psi \\ \\ [(1-i\gamma_5)\psi]^c - [(1+i\gamma_5)\psi]^c \end{vmatrix} = \begin{vmatrix} \psi_p \\ \psi_n \\ \psi_n^c \\ \psi_p^c \end{vmatrix}.$$

The matrices $\Gamma_{i+4} = \overline{\tau}_i = i\,\gamma_5 \times \rho_i$ then refer to a transformation in isotopic space, where the isotopic two-component spinor is composed of a proton-anti-neutron (anti-proton-neutron, resp.). It is interesting to establish that a reality condition is true, namely:

$$\Psi^c = C\,\Psi^* = \Psi.$$

We then prove that is it desirable to examine nonlinear spinor theories from the standpoint of FEYNMANN's two-component theory of fermions. If one starts from the FEYNMANN variant of the theory of β-decay and identifies all spinors with each other then one gets only the square of the vector and the square of the pseudo-vector for the possible nonlinear invariants. If we return to the general case of the choice of nonlinear invariants that are the various restrictions of the four-component, second quantized theory then when we limit ourselves to just the invariance under the LORENTZ group, PAULI group, and SALAM-TOUSHEK group, we obtain just one nonlinear invariant in the form of the square of the pseudo-vector [**7**]. However, if one adds the condition here of the vanishing of the nonlinear supplementary term for the two-component neutrino and the invariance under operations of spatial reflection and charge conjugation (which one can indeed link with reflection in isotopic spin space) then one obtains a sum of squares of vectors, pseudo-vectors, scalars, and pseudo-scalars.

HEISENBERG and PAULI have derived the isomultiplet distribution by considering the isotopy properties of particles. In it, for example, the hyperons were constructed with the help of a representation of the form ψ^3, in close analogy with the model of composite particles in the theory of fusion. In the nonlinear case, however, we have made a certain advance, in that the interaction of free "world-spinors" with themselves enters into the theory.

We further mention the interesting research on the nonlinear theory of elementary particles by DE BROGLIE, who regarded particles as nonlinear structures whose motion obeyed the usual linear equations, as well as the papers of VIGIER, BOHM, and TAKABAYASI, which sought to develop a purely classical relativistic, hydrodynamical spinor theory of matter. However, in my opinion, the strong invariance demands in this variety of theory must, in the final analysis, necessarily lead to the fundamental nonlinear spinor equation that was examined above.

In summary, one can say that the nonlinear spinor theory of matter has already arrived at many interesting results up to now, and today can apparently represent the single unified description of all elementary particles. On the other hand, the nonlinear spinor theory is still very far from its terminus. The values obtained by it for the masses and charges are still far from exhibiting a quantitative agreement with the experiments. We shall not go further here into the objections that were recently raised by PAULI, TOUSHEK, and FIERZ, but mention only that the theory of the description of the gravitational field is one of the most important problems. A possibility for the treatment of gravitation might be the introduction of covariant derivatives of the spinors in place of the usual ones into the kinematical term of the nonlinear spinor equation, which is, in turn, similar to what we did with FOCK in our earlier papers [**13**] on the DIRAC theory. On the other hand, it would be desirable to regard the gravitons as particles of spin 2 in the fundamental nonlinear equations, precisely like the other particles.

However, even after successfully solving the difficult gravitation problem, some important cosmological questions would still remain. Be that as it may, we are eyewitnesses to the start of a new period of development in physics, in which the structure and the interaction of elementary particles will be investigated by the introduction of a new universal constant of the type of a nonlinear self-interaction constant $\lambda = \hbar c l^2$, or a minimal length. On the other hand, the exceptionally fruitful period of physics that began with the discovery of the quantum of action by MAX PLANCK and lasted for almost a half-century is, to some extent, concluding.

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