Ueber die Pfaff'sche Methode, eine gewöhnliche lineäre Differentialgleichung zwischen 2n variabeln durch ein System von n Gleichungen zu integrieren," J. f. Reine u. angew. Math. **2** (1827), 347-357.

On the Pfaffian method of integrating an ordinary linear differential equation in 2n variables by means of a system of n equations.

(By Herrn Prof. C. G. J. Jacobi at Königsberg in Prussia.)

Translated by D. H. Delphenich

First treatise.

1.

In a treatise that was read to the Berlin Academy, along with others, in 1814-1815, Pfaff showed how one could integrate any equation of the form:

$$X_1\,\partial x_1 + X_2\,\partial x_2 + \ldots + X_{2n}\,\partial x_{2n} = 0,$$

where $X_1, X_2, \ldots, X_{2n}$ are arbitrary functions of $x_1, x_2, x_3, \ldots, x_{2n}$, by means of a system of n equations, so the problem of integrating partial differential equations of first order in n variables is only a special case. To that end, he expressed $2n - 1$ of the variables $x_1, x_2, x_3, \ldots, x_{2n}$ in terms of the remaining one x_m and $2n - 1$ new quantities variables $a_1, a_2, a_3, \ldots, a_{2n-1}$, where these $a_1, a_2, a_3, \ldots, a_{2n-1}$ are certain functions of $x_1, x_2, x_3, \ldots, x_{2n}$. After such a substitution, the equation:

$$X_1\,\partial x_1 + X_2\,\partial x_2 + \ldots + X_{2n}\,\partial x_{2n} = 0$$

is always converted into another one of the form:

$$U\,\partial x_m + A_1\,\partial a_1 + A_2\,\partial a_2 + \ldots + A_{2n-1}\,\partial a_{2n-1} = 0,$$

where $U, A_1, A_2, \ldots, A_{2n-1}$ are functions of $x_m, a_1, a_2, a_3, \ldots, a_{2n-1}$. Pfaff determined the functions $a_1, a_2, a_3, \ldots, a_{2n-1}$ in such a way that $U = 0$ and x_m entered into the quantities $A_1, A_2, \ldots, A_{2n-1}$ only in a factor that was common to all of them. If one divides by it then one has converted the given equation into another one that is similar, except that it only involves $2n - 1$ variables $a_1, a_2, a_3, \ldots, a_{2n-1}$. Since this process is possible only for an even number of variables, one cannot further convert them into an equation in only $2n - 2$ variables in the same way. Pfaff then set one of these variables equal to a constant, and then once more converted the equation in the remaining $2n - 3$ variables into another one in only $2n - 3$ variables, one of which he again set equal to a constant, and proceeded in that manner until he came to an equation in only 2 variables, whose integration gives the last n^{th} equation with the n^{th} arbitrary constant.

Pfaff then further showed, in a similar way, how these things happened for partial differential equations such that one could, from such a solution with n arbitrary constants, derive other solutions with arbitrary functions. Namely, one imagines that the n integral equations have been brought into the form:

$$F_1 = C_1, \qquad F_2 = C_2, \qquad \ldots, \qquad F_n = C_n,$$

where $C_1, C_2, \ldots C_n$ are arbitrary constants and $F_1, F_2, \ldots F_n$ no longer include them. If one now thinks of the quantities $C_1, C_2, \ldots C_n$ as variables then one must have that by means of the equations:

$$F_1 = C_1, \qquad F_2 = C_2, \qquad \ldots, \qquad F_n = C_n,$$

the expression:

$$X_1\,\partial x_1 + X_2\,\partial x_2 + \ldots + X_{2n}\,\partial x_{2n}$$

can be converted into another one of the form:

$$K_1\,\partial C_1 + K_2\,\partial C_2 + \ldots + K_n\,\partial C_n,$$

because this expression must vanish when $C_1, C_2, \ldots C_n$ are set equal to constants. In an identical way, one must then have:

$$X_1\,\partial x_1 + X_2\,\partial x_2 + \ldots + X_{2n}\,\partial x_{2n} = K_1\,\partial C_1 + K_2\,\partial C_2 + \ldots + K_n\,\partial C_n.$$

However, this expression does not vanish merely when one sets $F_1, F_2, \ldots, F_n$ equal to constants, but also when one makes m of the quantities $F_1, F_2, \ldots, F_n$ take the form of arbitrary functions of the remaining ones – e.g., $F_1, F_2, \ldots, F_m$ as functions of $F_{m+1}, F_{m+2}, \ldots, F_n$ – from which:

$$K_1\,\partial F_1 + K_2\,\partial F_2 + \ldots + K_n\,\partial F_n = \Pi_1\,\partial F_{m+1} + \Pi_2\,\partial F_{m+2} + \ldots + K_n\,\partial F_n,$$

and then appends the equations:

$$\Pi_1 = 0, \qquad \Pi_2 = 0, \;\ldots, \qquad \Pi_{n-m} = 0.$$

If one sets:

$$\begin{aligned} F_1 &= \psi_1(F_{m+1}, F_{m+2}, \ldots, F_n), \\ F_2 &= \psi_2(F_{m+1}, F_{m+2}, \ldots, F_n), \\ &\ldots \\ F_m &= \psi_m(F_{m+1}, F_{m+2}, \ldots, F_n) \end{aligned}$$

then one obtains:

$$\Pi_1 = K_1\left(\frac{\partial\psi_1}{\partial F_{m+1}}\right) + \cdots + K_m\left(\frac{\partial\psi_m}{\partial F_{m+1}}\right) + K_{m+1},$$
$$\Pi_2 = K_1\left(\frac{\partial\psi_1}{\partial F_{m+2}}\right) + \cdots + K_m\left(\frac{\partial\psi_m}{\partial F_{m+2}}\right) + K_{m+2},$$
$$\cdots$$
$$\Pi_{n-m} = K_1\left(\frac{\partial\psi_1}{\partial F_n}\right) + \cdots + K_m\left(\frac{\partial\psi_m}{\partial F_n}\right) + K_n,$$

and the given equation:

$$X_1\,\partial x_1 + X_2\,\partial x_2 + \ldots + X_{2n}\,\partial x_{2n} = 0,$$

will also be integrated by means of the system of n equations:

$$F_1 = \psi_1(F_{m+1}, F_{m+2}, \ldots, F_n),$$
$$F_2 = \psi_2(F_{m+1}, F_{m+2}, \ldots, F_n),$$
$$\cdots$$
$$F_m = \psi_m(F_{m+1}, F_{m+2}, \ldots, F_n),$$

$$\Pi_1 = 0, \qquad \Pi_2 = 0, \ \ldots, \qquad \Pi_{n-m} = 0.$$

One finally obtains a solution when one sets:

$$K_1 = 0,\ K_2 = 0, \ldots, K_n = 0,$$

which, along with the one in which one sets $F_1, F_2, \ldots, F_m$ equal to arbitrary constants, defines, in a certain sense, the two extreme cases that correspond to the so-called singular and complete solution of the partial differential equations, although the remaining ones correspond to the so-called general solutions. All of these solutions have a well-defined, distinct character when compared to the other ones, and one can never, e.g., obtain the original solution with n arbitrary constants when one chooses the arbitrary functions to be functions with n constants. Pfaff gave only the solution that one obtains when one makes one of the functions $F_1, F_2, \ldots, F_n$ take the form of a function of the remaining ones.

2.

One sees from the foregoing that everything comes down to a general method of determining the functions $a_1, a_2, \ldots, a_{2n-1}$ in every case, which we would like to undertake with Pfaff's guidance.

Therefore, let the equation:

$$0 = X_1\,\partial x_1 + X_2\,\partial x_2 + \ldots + X_p\,\partial x_p$$

be given. Let $a_1, a_2, \ldots, a_p$ be certain functions of $x, x_1, x_2, \ldots, x_p$, and one thinks of them as being expressed in terms of $x_1, x_2, \ldots, x_p$, and x. With that assumption, I will denote the partial differentials with respect to the $x, a_1, a_2, \ldots, a_p$ without brackets, such that, e.g.:

$$\frac{\partial X}{\partial a_1} = \left(\frac{\partial X}{\partial x_1}\right)\cdot\frac{\partial x_1}{\partial a_1} + \left(\frac{\partial X}{\partial x_2}\right)\cdot\frac{\partial x_2}{\partial a_1} + \cdots + \left(\frac{\partial X}{\partial x_p}\right)\cdot\frac{\partial x_p}{\partial a_1}.$$

The given equation:

$$0 = X\,\partial x + X_1\,\partial x_1 + X_2\,\partial x_2 + \ldots + X_p\,\partial x_p$$

is then converted into the following one:

$$0 = U\,\partial x + A_1\,\partial a_1 + A_2\,\partial a_2 + \ldots + A_p\,\partial a_p,$$

where:

$$U = X + \quad X_1\frac{\partial x_1}{\partial x} + X_2\frac{\partial x_2}{\partial x} + \cdots + X_p\frac{\partial x_p}{\partial x},$$

$$A_1 = \qquad X_1\frac{\partial x_1}{\partial a_1} + X_2\frac{\partial x_2}{\partial a_1} + \cdots + X_p\frac{\partial x_p}{\partial a_1},$$

$$A_2 = \qquad X_1\frac{\partial x_1}{\partial a_2} + X_2\frac{\partial x_2}{\partial a_2} + \cdots + X_p\frac{\partial x_p}{\partial a_2},$$

$$\ldots$$

$$A_p = \qquad X_1\frac{\partial x_1}{\partial a_p} + X_2\frac{\partial x_2}{\partial a_p} + \cdots + X_p\frac{\partial x_p}{\partial a_p}.$$

One first sets:

$$U = X + X_1\frac{\partial x_1}{\partial x} + X_2\frac{\partial x_2}{\partial x} + \cdots + X_p\frac{\partial x_p}{\partial x} = 0.$$

In order for x to enter into $A_1, A_2, \ldots, A_p$ only by way of a common factor M, moreover, one must have:

$$\frac{1}{A_1}\cdot\frac{\partial A_1}{\partial x} = \frac{1}{A_2}\cdot\frac{\partial A_2}{\partial x} = \ldots = \frac{1}{A_p}\cdot\frac{\partial A_p}{\partial x} = \frac{1}{U}\cdot\frac{\partial U}{\partial x},$$

or:

$$\frac{\partial \log A_1}{\partial x} = \frac{\partial \log A_2}{\partial x} = \ldots = \frac{\partial \log A_p}{\partial x} = \frac{\partial \log U}{\partial x}.$$

Now, let:

$$A = X_1\frac{\partial x_1}{\partial a} + X_2\frac{\partial x_2}{\partial a} + \cdots + X_p\frac{\partial x_p}{\partial a},$$

an expression from which one obtain the various values of $A_1, A_2, \ldots, A_p$ when one sets a equal to $a_1, a_2, \ldots, a_p$ in sequence, so one has:

$$\frac{\partial A}{\partial x} = \frac{\partial X_1}{\partial x}\cdot\frac{\partial x_1}{\partial a}+\frac{\partial X_2}{\partial x}\cdot\frac{\partial x_2}{\partial a}+\cdots+\frac{\partial X_p}{\partial x}\cdot\frac{\partial x_p}{\partial a} + X_1\frac{\partial^2 x_1}{\partial a\,\partial x}+X_2\frac{\partial^2 x_2}{\partial a\,\partial x}+\cdots+X_p\frac{\partial^2 x_p}{\partial a\,\partial x}.$$

However, it follows from the equation:

$$0 = X + X_1\frac{\partial x_1}{\partial x}+X_2\frac{\partial x_2}{\partial x}+\cdots+X_p\frac{\partial x_p}{\partial x},$$

when one differentiates it with respect to a, that:

$$X_1\frac{\partial^2 x_1}{\partial a\,\partial x}+X_2\frac{\partial^2 x_2}{\partial a\,\partial x}+\cdots+X_p\frac{\partial^2 x_p}{\partial a\,\partial x} = -\left\{\frac{\partial X}{\partial a}+\frac{\partial X_1}{\partial a}\cdot\frac{\partial x_1}{\partial x}+\cdots+\frac{\partial X_p}{\partial a}\cdot\frac{\partial x_p}{\partial x}\right\}.$$

One now has:

$$\begin{aligned}
&\frac{\partial X_1}{\partial x}\cdot\frac{\partial x_1}{\partial a}+\frac{\partial X_2}{\partial x}\cdot\frac{\partial x_2}{\partial a}+\cdots+\frac{\partial X_p}{\partial x}\cdot\frac{\partial x_p}{\partial a}=\\
&+\frac{\partial x_1}{\partial a}\left\{\left(\frac{\partial X_1}{\partial x}\right)+\left(\frac{\partial X_1}{\partial x_1}\right)\cdot\frac{\partial x_1}{\partial x}+\cdots+\left(\frac{\partial X_1}{\partial x_p}\right)\cdot\frac{\partial x_p}{\partial x}\right\}\\
&+\frac{\partial x_2}{\partial a}\left\{\left(\frac{\partial X_2}{\partial x}\right)+\left(\frac{\partial X_2}{\partial x_1}\right)\cdot\frac{\partial x_2}{\partial x}+\cdots+\left(\frac{\partial X_2}{\partial x_p}\right)\cdot\frac{\partial x_p}{\partial x}\right\}\\
&\cdots\\
&+\frac{\partial x_p}{\partial a}\left\{\left(\frac{\partial X}{\partial x_p}\right)+\left(\frac{\partial X_1}{\partial x_p}\right)\cdot\frac{\partial x_1}{\partial x}+\left(\frac{\partial X_2}{\partial x_p}\right)\cdot\frac{\partial x_2}{\partial x}+\cdots+\left(\frac{\partial X_p}{\partial x_p}\right)\cdot\frac{\partial x_p}{\partial x}\right\}.
\end{aligned}$$

The difference of both expressions gives $\frac{\partial A}{\partial x}$. If one sets, for brevity, $\left(\frac{\partial X_\alpha}{\partial x_\beta}\right)-\left(\frac{\partial X_\beta}{\partial x_\alpha}\right) = (\alpha, \beta)$, where one then has $(\alpha, \beta) + (\beta, \alpha) = 0$, and e.g.:

$$(0, 1) = \left(\frac{\partial X}{\partial x_1}\right)-\left(\frac{\partial X_1}{\partial x}\right),$$

then one obtains:

$$\frac{\partial A}{\partial x} =$$

$$\frac{\partial x_1}{\partial a}\left\{(1,0)+\ *\ +(1,2)\frac{\partial x_2}{\partial x}+(1,3)\frac{\partial x_1}{\partial x}+\cdots+(1,p)\frac{\partial x_p}{\partial x}\right\}$$

$$+\frac{\partial x_2}{\partial a}\left\{(2,0)+(2,1)\frac{\partial x_1}{\partial x}+\;*\;+(2,3)\frac{\partial x_3}{\partial x}+\cdots+(2,p)\frac{\partial x_p}{\partial x}\right\}$$
$$+\frac{\partial x_3}{\partial a}\left\{(3,0)+(3,1)\frac{\partial x_1}{\partial x}+(3,2)\frac{\partial x_2}{\partial x}+\;*\;+\cdots+(3,p)\frac{\partial x_p}{\partial x}\right\}$$
$$\cdots$$
$$+\frac{\partial x_p}{\partial a}\left\{(p,0)+(p,1)\frac{\partial x_1}{\partial x}+(p,2)\frac{\partial x_2}{\partial x}+\cdots\cdots\cdots\cdots+\;*\right\}.$$

If one sets a equal to a_1, a_2, …. a_p in sequence then one obtains the various expressions for $\frac{\partial A_1}{\partial x}$, $\frac{\partial A_2}{\partial x}$, …, $\frac{\partial A_p}{\partial x}$. Now, no matter which of the expressions a_1, a_2, …. a_p one sets a equal to, one always has $\frac{\partial A}{\partial x} = \frac{A}{U}\cdot\frac{\partial U}{\partial x} = N\,A$, when one sets $\frac{\partial \log U}{\partial x} = N$, for brevity, or:

$$\frac{\partial A}{\partial x} = \frac{\partial x_1}{\partial a}\cdot NX_1+\frac{\partial x_2}{\partial a}\cdot NX_2+\cdots+\frac{\partial x_p}{\partial a}\cdot NX_p\,,$$

which will be the case whenever the coefficients of $\frac{\partial x_1}{\partial a}$, $\frac{\partial x_2}{\partial a}$, …, $\frac{\partial x_p}{\partial a}$ are equal to both of the expressions that were found for $\frac{\partial A}{\partial x}$, respectively. One thus obtains the equations:

$$N\,X_1 = (1,0)\quad+\quad *\quad+(1,2)\frac{\partial x_2}{\partial x}\quad+\cdots\quad+(1,p)\frac{\partial x_p}{\partial x}\,,$$
$$N\,X_2 = (2,0)\;+(2,1)\frac{\partial x_1}{\partial x}\;+\quad *\quad+\cdots\;+(2,p)\frac{\partial x_p}{\partial x}\,,$$
$$\cdots$$
$$N\,X_p = (p,0)+(p,1)\frac{\partial x_1}{\partial x}+(p,2)\frac{\partial x_2}{\partial x}+\cdots+\quad *$$

If one multiplies these equations by $\frac{\partial x_1}{\partial x}$, $\frac{\partial x_2}{\partial x}$, …, $\frac{\partial x_p}{\partial x}$, resp., and adds them then one obtains:

$$N\left\{X_1\frac{\partial x_1}{\partial x}+X_2\frac{\partial x_2}{\partial x}+\cdots+X_p\frac{\partial x_p}{\partial x}\right\} = (1,0)\frac{\partial x_1}{\partial x}+(2,0)\frac{\partial x_2}{\partial x}+\cdots+(p,0)\frac{\partial x_p}{\partial x}\,,$$

in which one has suppressed the remaining terms, or since:

$$0 = X+\;X_1\frac{\partial x_1}{\partial x}+X_2\frac{\partial x_2}{\partial x}+\cdots+X_p\frac{\partial x_p}{\partial x}\,,$$

one obtains the equation:

$$N\,X = (0,1)\frac{\partial x_1}{\partial x} + (0,2)\frac{\partial x_2}{\partial x} + \cdots + (0,p)\frac{\partial x_p}{\partial x}.$$

In this way, one obtains $p + 1$ linear equations between the $p + 1$ unknown quantities $N, \frac{\partial x_1}{\partial x}, \frac{\partial x_2}{\partial x}, \ldots, \frac{\partial x_p}{\partial x}$.

Therefore, everything has been reduced to merely relations between the derivatives with respect to x, namely, $\frac{\partial x_1}{\partial x}, \frac{\partial x_2}{\partial x}, \ldots, \frac{\partial x_p}{\partial x}$. One now realizes that when one finds the values $\frac{V_1}{V}, \frac{V_2}{V}, \ldots, \frac{V_p}{V}$ from the equations presented for $\frac{\partial x_1}{\partial x}, \frac{\partial x_2}{\partial x}, \ldots, \frac{\partial x_p}{\partial x}$, resp., the desired functions $a_1, a_2, \ldots, a_p$ are the functions that will be set equal to the p arbitrary constants in the integration of the equations:

$$\partial x : \partial x_1 : \partial x_2 : \ldots : \partial x_p = V : V_1 : V_2 : \ldots : V_p ,$$

or:

$$\frac{\partial x_1}{\partial x} = \frac{V_1}{V}, \quad \frac{\partial x_2}{\partial x} = \frac{V_2}{V}, \quad \ldots, \quad \frac{\partial x_p}{\partial x} = \frac{V_p}{V},$$

which was required. If one thus finds the values of $V, V_1, V_2, \ldots, V_p$ from the $p + 1$ equations then the integration of the equations:

$$\partial x : \partial x_1 : \partial x_2 : \ldots : \partial x_p = V : V_1 : V_2 : \ldots : V_p ,$$

give the desired functions $a_1, a_2, \ldots, a_p$.

3.

The equations by which one must look for $\frac{\partial x_1}{\partial x}, \frac{\partial x_2}{\partial x}, \ldots, \frac{\partial x_p}{\partial x}$ are a consequence of the foregoing when one multiplies them by ∂x:

$$\text{A.} \qquad \begin{cases} NX\,\partial x = \quad * \quad + (0,1)\partial x_1 + (0,2)\partial x_2 + \cdots + (0,p)\partial x_p, \\ NX_1\partial x = (1,0)\partial x_1 + \quad * \quad + (1,2)\partial x_2 + \cdots + (1,p)\partial x_p, \\ NX_2\partial x = (2,0)\partial x + (2,1)\partial x_1 + \quad * \quad + \cdots + (2,p)\partial x_p, \\ \quad \cdots \\ NX_p\partial x = (p,0)\partial x + (p,1)\partial x_1 + (p,2)\partial x_2 + \cdots + \quad * \end{cases}$$

If one finds from these equations that:

$$\partial x = \frac{NV\partial x}{\Delta}, \quad \partial x_1 = \frac{NV_1\partial x}{\Delta}, \ \ldots, \ \partial x_p = \frac{NV_p\partial x}{\Delta}$$

then one will have:

$$\partial x : \partial x_1 : \partial x_2 : \ldots : \partial x_p = V : V_1 : V_2 : \ldots : V_p \,.$$

One further obtains $N = \Delta / V$.

Equations (A) have very remarkable properties. It is characteristic of them that the vertical rows of coefficients are precisely the negative of the horizontal rows; therefore, the terms in which the m^{th} horizontal row and the m^{th} vertical row agree will vanish as will clearly be the case for the stars that one finds in the diagonals. From this property, it then follows that $p + 1$ – i.e., the number of variables $x, x_1, x_2, \ldots, x_p$ – must be an even number. It is, in fact, known that for any system of n equations in n unknown variables, one must see whether the common denominator of the values of the unknowns – which Gauss referred to by the name of *determinant* in his *Disquis. Arith.* – can vanish. That would be a sign that the system of n equations cannot exist, provided that perhaps no equation of condition is found between the constants by means of which the n^{th} equation is a consequence of the $n - 1$ remaining equations. Now, this remains unchanged in the known algorithm by which the determinant is constructed when one switches the horizontal rows and vertical rows of the coefficients with each other. Now, for our special case, if we denote the determinant by Δ, then it will follow that $\Delta = (-1)^{p+1}\Delta$, since every term in the determinant is a product of $p + 1$ coefficients, each of which is converted into its negative under the exchange of horizontal and vertical rows. However, this equation $\Delta = (-1)^{p+1}\Delta$ can exist only when $p + 1$ is an even number, provided that one does not have $\Delta = 0$.

I will now develop some special cases.

For $p + 1 = 4$, one obtains:

$$\begin{aligned}
V &= \quad * \quad + (2, 3)\, X_1 + (3, 1)\, X_2 + (1, 2)\, X_3 \,, \\
V_1 &= (3, 2)\, X + \quad * \quad + (0, 3)\, X_2 + (2, 0)\, X_3 \,, \\
V_2 &= (1, 3)\, X + (3, 0)\, X_1 + \quad * \quad + (0, 1)\, X_3 \,, \\
V_3 &= (1, 3)\, X + (3, 0)\, X_1 + (1, 0)\, X_2 + \quad * \quad , \\
\\
\Delta &= (0, 1)\,(3, 2) + (0, 3)\,(2, 1) + (0, 2)(1, 3).
\end{aligned}$$

For $p + 1 = 6$, when one denotes the expression:

$$(1, 2)(3, 4) + (1, 3)(4, 2) + (1, 4)(2, 3)$$

by (1, 2, 3, 4), for brevity, and forms the similar expressions of this type, one obtains:

$$\begin{aligned}
V &= \quad * \quad + (2, 3, 4, 5)\, X_1 + (3, 4, 5, 1)\, X_2 + (4, 5, 1, 2)\, X_3 + (5, 1, 2, 3)\, X_4 + (1, 2, 3, 4)\, X_5 \,, \\
V_1 &= (3, 2, 4, 5)\, X + \quad * \quad + (4, 3, 5, 0)\, X_2 + (5, 4, 0, 2)\, X_3 + (0, 5, 2, 3)\, X_4 + (2, 0, 3, 4)\, X_5 \,, \\
V_2 &= (1, 3, 4, 5)\, X + (3, 4, 5, 0)\, X_1 + \quad * \quad + (4, 5, 0, 1)\, X_3 + (5, 0, 1, 3)\, X_4 + (0, 1, 3, 4)\, X_5 \,, \\
V_3 &= (2, 1, 4, 5)\, X + (4, 2, 5, 0)\, X_1 + (5, 4, 0, 1)\, X_2 + \quad * \quad + (0, 5, 1, 2)\, X_4 + (1, 0, 2, 4)\, X_5 \,, \\
V_4 &= (1, 2, 3, 5)\, X + (2, 3, 5, 0)\, X_1 + (3, 5, 0, 1)\, X_2 + (5, 0, 0, 2)\, X_3 + \quad * \quad + (0, 1, 2, 3)\, X_5 \,, \\
V_5 &= (2, 1, 3, 4)\, X + (3, 2, 4, 0)\, X_1 + (4, 3, 0, 1)\, X_2 + (0, 4, 1, 2)\, X_3 + (1, 0, 2, 3)\, X_4 + \quad * \quad .
\end{aligned}$$

In order to specify the general procedure for these expressions, I will say that one runs through a kind of cycle, in which one sets the numerical elements 0, 1, 2, …, p of which it is composed equal to:

$$\begin{array}{ll} 0, & 1, 2, 3, \ldots, p-1, \; p, \\ 1, & 2, 3, 4, \ldots, p, \quad 0, \\ 2, & 3, 4, 5, \ldots, 0, \quad 1, \\ & \ldots \\ p-1, & p, 0, 1, \ldots, p-3, p-2, \\ p, & 0, 1, 2, \ldots, p-2, p-1, \end{array}$$

in succession.

As one can see from the last example, one thus obtains the expression that is equal to V_m from one of its terms when one runs through the cycle, after one has omitted the number m from the number sequence 0, 1, 2, …, p, where we remark that one also had to do the same thing with the index of X. One thus obtains from the term (3, 2, 4, 5) X_0 in the expression that is found for V_1, the remaining ones when one replaces 0, 2, 3, 4, 5 with 2, 3, 4, 5, 0; 3, 4, 5, 0, 2; 4, 5, 0, 2, 3; 5, 0, 2, 3, 4, in succession. Furthermore, one always obtains from all of the expressions found for V_m the ones that follow for V_{m+1} when one sets 0, 1, 2, 3, …, p equal to 1, 2, 3, …, p, 0, resp., and replaces the first two elements in them with a bracket. When one replaces 0, 1, 2, 3, 4, 5 with 1, 2, 3, 4, 5, 0, resp., one then obtains from the term (1, 0, 2, 4) X_5 in V_3, the term (2, 1, 3, 5) X, and when one replaces the first two elements in (2, 1, 3, 5), the term (1, 2, 3, 5) X, which is the first term in the expression found for V_4.

What still remains for us to do is to give the construction of a typical element, such as (1, 2, 3, 4). If one sets the coefficients of X_1 in V equal to (2, 3, 4, 5,…, p – 1, p) for p + 1 elements then (2, 3, …, p) will consist of $1 \times 3 \times 5 \times \ldots \times p - 2$ terms. The first of them will be:

$$(2, 3) \cdot (4, 5) \cdot (6, 7) \ldots (p - 1, p).$$

From this, one constructs p – 2 more, when one lets the last p – 2 elements 3, 4, …, p run through a cycle. From each of these p – 2 terms, one constructs p – 4 more when one lets the last p – 4 elements 5, 6, …, p run through a cycle, etc., until finally the last three elements p – 2, p – 1, p run through a cycle. In this way, one obtains, e.g.:

$$\begin{aligned} &(2, 3, 4, 5, 6, 7) = \\ &\quad (2, 3) \cdot (4, 5) \cdot (6, 7) + (2, 3) \cdot (4, 6) \cdot (7, 5) + (2, 3) \cdot (4, 7) \cdot (5, 6) \\ &+ (2, 4) \cdot (5, 6) \cdot (7, 3) + (2, 4) \cdot (5, 7) \cdot (3, 6) + (2, 4) \cdot (5, 3) \cdot (6, 7) \\ &+ (2, 5) \cdot (6, 7) \cdot (3, 4) + (2, 5) \cdot (6, 3) \cdot (4, 7) + (2, 5) \cdot (6, 4) \cdot (7, 3) \\ &+ (2, 6) \cdot (7, 3) \cdot (4, 5) + (2, 6) \cdot (7, 4) \cdot (5, 3) + (2, 6) \cdot (7, 5) \cdot (3, 4) \\ &+ (2, 7) \cdot (3, 4) \cdot (5, 6) + (2, 7) \cdot (3, 5) \cdot (6, 4) + (2, 7) \cdot (3, 6) \cdot (4, 5). \end{aligned}$$

If p + 1 is an odd number then we have seen that one must always have an equation of condition in order for the equations (A) to be possible or when one wishes that the equation:

$$0 = X \, \partial x + X_1 \, \partial x_1 + X_2 \, \partial x_2 + \ldots + X_p \, \partial x_p$$

should produce a similar equation in only p variables. For $p + 1 = 3$, this condition equation becomes:

$$X(1, 2) + X_1(2, 0) + X_2(0, 1) = 0,$$

which is the desired *conditio integrabilititatis.*

For $p + 1 = 5$, it becomes:

$$X(1, 2, 3, 4) + X_1(2, 3, 4, 0) + X_2(3, 4, 0, 1) + X_3(4, 0, 1, 2) + X_4(0, 1, 2, 3) = 0.$$

In general, if $p + 1$ is an odd number then it becomes:

$$\sum X(1, 2, 3, \ldots, p) = 0,$$

where one forms all of the terms in the aggregate that is denoted by the $\sum$ from $X(1, 2, 3, \ldots, p)$ when one lets $0, 1, 2, \ldots, p$ run through a cycle. This is then the condition equation for the equation:

$$0 = X\,\partial x + X_1\,\partial x_1 + X_2\,\partial x_2 + \ldots + X_p\,\partial x_p,$$

where p is an even number, to be integrated by means of a system of $p/2$ equations.

The presentation and treatment of equations (A) in the elegant and completely symmetric form that was given here is the peculiarity and the actual purpose of this treatise. Therefore, for the sake of logical continuity, the rest of the Pfaffian method must be briefly presented. These equations have a high degree of similarity with the ones of the known type in which the horizontal rows and vertical rows of coefficients are the same, which one encounters in many analytical investigations – among others, the method of the least square. In the expressions that were found for V, V_1, etc., the horizontal rows and the vertical rows of the coefficients of X, X_1, etc., are again the negatives of each other, just as both rows are again the same in the results that give the solution there. If one applies the algorithm that Gauss gave in his treatise on the elliptic elements of Pallas to our system then one sees how one can always eliminate two quantities at once with great ease, and how the new equations, whose number is less by two, again take on the same form. This makes it possible for one to be able to solve such a system of equations with great rapidity.

Addendum. After the conclusion of this treatise, I remarked that the equations to which Lagrange and Poisson arrived in their celebrated papers on the variation of constants in the problems of mechanics define just such a system as we have discussed here more closely. (See the 15th volume of the polytechnic journal, pp. 288-89.) Since the Pfaffian method likewise rests upon the variation of constants, this system of equations seems to particularly emerge from the method of the variation of constants.

The 14th of August, 1827.
