"Introduction des paramétres relativistes de Cayley-Klein dans la représentation hydrodynamique de l'équation de Dirac," C. R. Acad. Sci. **243** (1956), 234-237.

Introduction of relativistic Cayley-Klein parameters into the hydrodynamical representation of the Dirac equation

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The causal interpretation of quantum theory considers the Dirac wave to be a spinning fluid that is endowed with internal stresses. One studies it by introducing angular variables that describe the motion of proper rotation of the fluid elements.

1. The development of the hydrodynamical representation of the Dirac equation leads one to interpret the Dirac wave as a spinning fluid (Møller-Weyssenhoff fluid) that is endowed with internal stresses [**1**].

It has been shown [**2**], moreover, that a Møller-Weyssenhoff fluid is nothing but an ordinary relativistic fluid whose "molecules" are endowed with a *classical* motion of proper rotation. The spin density is then the macroscopic representation of classical rotations that are performed in domains whose dimensions are considered to be negligible at the scale that one considers. The spin density is then justified when one considers the fluid to be continuous and the "molecules" to be infinitely small.

The hydrodynamical interpretation of Dirac theory has already led to a description of the Dirac fluid in tensorial form [**3**].

Our interpretation of spinning fluids necessitates the use of angular variables that describe the behavior of each "molecule" of fluid.

In the proper system of a fluid element, with respect to which that element exhibits neither rotation nor translation, we write the Dirac spinor in the form:

$$q = \sqrt{D}\, e^{i\sigma_4 A/2} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

in which D is the invariant density of the fluid, and A is such that:

$$\tan A = \frac{\Omega_2}{\Omega_1} = \frac{q^* \alpha_5 q}{q^* \alpha_4 q},$$

where Ω_1 and Ω_2 are the invariants that are coupled with the spinor field [**4**]. One then shows [**5**] that in an arbitrary coordinate system, one will have:

(I)
$$q = \sqrt{D}\, e^{i\sigma_4 A/2} L R \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

with
$$R = e^{i\sigma_3 \psi/2} e^{i\sigma_1 \theta/2} e^{i\sigma_3 \varphi/2},$$

which is a spatial rotation that described with the aid of the Euler angles: φ is the angle of proper rotation, θ is the angle of nutation, and ψ is the angle of precession.

$$L = e^{\alpha_1 \gamma_1/2} e^{\alpha_2 \gamma_2/2} e^{\alpha_3 \gamma_3/2}$$

is a general Lorentz transformation, in which γ_i is the rotation of the i^{th} spatial axis about the time axis. The velocity along that axis is defined by $\tanh \gamma_i = v_i / c$.

One sees that φ, θ, ψ, and the γ_i are the relativistic Euler angles.

2. We then get the Dirac tensorial magnitudes as functions of the Euler angles:

The invariant: $\Omega_1 = q^* \alpha_4\, q = D \cos A,$

The pseudo-invariant: $\Omega_2 = q^* \alpha_5\, q = D \sin A,$

The current: $j_\mu = q^* \alpha_\mu\, q,$

namely:

$$\begin{aligned}
j_1 &= D \sinh \gamma_1 \cosh \gamma_2 \cosh \gamma_3 ,\\
j_2 &= - D \sinh \gamma_2 \cosh \gamma_3 ,\\
j_3 &= D \cosh \gamma_2 \cosh \gamma_2 \cosh \gamma_3 ,\\
j_4 &= D \cosh \gamma_1 \cosh \gamma_2 \cosh \gamma_3 .
\end{aligned}$$

The spin: $s_\mu = q^* \sigma_\mu\, q,$

namely:

$$\begin{aligned}
s_1 &= D\,[\cos\theta \sinh \gamma_1 \cosh \gamma_2 \sinh \gamma_3 - \sin\theta\,(\cosh \gamma_1 \sin\psi + \sinh \gamma_1 \sinh \gamma_2 \cos\psi)],\\
s_2 &= D\,[- \cos\theta \sinh \gamma_1 \cosh \gamma_2 \sinh \gamma_3 + \sin\theta \cosh \gamma_1 \cosh \gamma_2 \cosh \psi],\\
s_3 &= D\,[- \cos\theta \cosh \gamma_1 \cosh \gamma_2 \sinh \gamma_3],\\
s_4 &= D\,[\cos\theta \cosh \gamma_1 \cosh \gamma_2 \sinh \gamma_3 - \sin\theta\,(\sinh \gamma_1 \sin\psi + \cosh \gamma_1 \sinh \gamma_2 \cos\psi)].
\end{aligned}$$

The electromagnetic moment $m_{\mu\nu}$ is given by the Pauli-Kofink relation:

$$m_{\mu\nu} = \frac{\Omega_1}{D^2}\left(\overline{s_\mu j_\nu - s_\nu j_\mu}\right) - \frac{\Omega_2}{D^2}(s_\mu j_\nu - s_\nu j_\mu)\,.$$

The preceding formulas will take on a simpler significance if one notes that:

$$\cosh\gamma_i = \frac{1}{\sqrt{1-\left(\frac{v_i}{c}\right)^2}} \qquad \text{and} \qquad \sinh\gamma_i = \frac{v_i/c}{\sqrt{1-\left(\frac{v_i}{c}\right)^2}},$$

in which $\cosh\gamma_i$ and $\sinh\gamma_i$ are then the components of unit velocity vectors that are directed along the three spatial axes.

3. The Dirac Lagrangian is:

$$\mathcal{L} = \frac{\hbar c}{2i} q^* \alpha^\mu [\partial_\mu]\, q - m_0\, q^* \alpha_4\, q.$$

One easily shows that the Dirac equations can be written with the aid of the Hamiltonian:

$$\text{(II)} \qquad \mathcal{H} = \frac{\hbar c}{2i} q^* \alpha^k [\partial_k]\, q - m_0\, q^* \alpha_4\, q \qquad (k = 1, 2, 3)$$

and the *classical Poisson brackets:*

$$\text{(II}'\text{)} \qquad [q_\alpha(\mathbf{r}, t),\ q^*_\beta(\mathbf{r}', t)] = \frac{i}{\hbar}\,\delta_{\alpha\beta}\,\delta(\mathbf{r} - \mathbf{r}'),$$

in which q_α and q^*_β are the components of the spinor q and its conjugate q^*, $\delta_{\alpha\beta}$ is Kronecker symbol, and $\delta(\mathbf{r} - \mathbf{r}')$ is the Dirac measure.

Since the Lagrangian is zero in the course of motion, one knows that the Hamiltonian will take the value:

$$\mathcal{H} = -T_{44} = -\frac{\hbar c}{2i} q^* [\partial_4]\, q = -\frac{\hbar}{2i}\left(q^*\frac{\partial q}{\partial t} - \frac{\partial q^*}{\partial t} q\right).$$

Now introduce the expression (I) for the spinor. Calculation, which will be simplified by the commutation rules for the Dirac matrices, will then give:

$$\boxed{\begin{aligned}\mathcal{H} = j_4\frac{\partial}{\partial t}\left(-\frac{\varphi}{2}\right) + s_3\cosh\gamma_1\cosh\gamma_2\left(-\frac{\psi}{2}\right) + s_4\frac{\partial}{\partial t}\left(-\frac{A}{2}\right)\\ + (s_2\sinh\gamma_1\cosh\gamma_2 - s_1\sinh\gamma_2)\frac{\partial}{\partial t}\left(-\frac{\gamma_3}{2}\right).\end{aligned}}$$

We can then describe the Dirac fluid with the aid of the independent variables: φ, ψ, A¸ γ_3 . The equations of motion will then be given by the Hamiltonian (II) and the *classical* Poisson brackets.

$$(\mathrm{II}'') \qquad \begin{cases} \left[\dfrac{\varphi}{2}, j_4\right] = \dfrac{1}{\hbar}\delta(\mathbf{r}-\mathbf{r}'), \\ \left[\dfrac{\psi}{2}, \cosh\gamma_1 \cosh\gamma_2\, s_3\right] = \dfrac{1}{\hbar}\delta(\mathbf{r}-\mathbf{r}'), \\ \left[\dfrac{\gamma_3}{2}, s_2 \cosh\gamma_2 \sinh\gamma_1 - s_1 \sinh\gamma_2\right] = \dfrac{1}{\hbar}\delta(\mathbf{r}-\mathbf{r}'), \\ \left[\dfrac{A}{2}, s_4\right] = \dfrac{1}{\hbar}\delta(\mathbf{r}-\mathbf{r}'). \end{cases}$$

4. In the non-relativistic approximation, the parameters γ_i and A will tend to 0 and π, respectively, which will imply the annihilation of their conjugate momenta.

Moreover, j_3 will tend to ρ, which is the density of the Pauli fluid, cosh γ_1 cosh γ_2 s_3 will tend to s_3, which is a component of the Pauli spin, and the Hamiltonian will tend to the Pauli Hamiltonian.

What will then remain is the latter Hamiltonian, with the Poisson brackets:

$$\left[\frac{\varphi}{2}, \rho\right] = \frac{1}{\hbar}\delta(\mathbf{r}-\mathbf{r}'),$$
$$\left[\frac{\psi}{2}, \rho\right] = \frac{1}{\hbar}\delta(\mathbf{r}-\mathbf{r}').$$

This result is well-known [**6**].

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[2] BOHM, LOCHAK, and VIGIER, *loc. cit.*
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[3] TAKABAYASI, *loc. cit.*
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[5] BOHM, LOCHAK, and VIGIER, *loc. cit.* [**1**].
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