"Decomposition en paramétres de Clebsch de l'impulsion de Dirac et la interprétation physique de l'invariance de jauge des équations de la Mécanique ondulatoire," C. R. Acad. Sci. **243** (1956), 357-360.

Decomposition of the Dirac impulse in Clebsch parameters and the physical interpretation of the gauge invariance of the equations of wave mechanics

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The introduction of relativistic Euler angles permits one to write the Clebsch decomposition of the impulse in the context of the hydrodynamical representation of the Dirac wave. One exhibits the role of the angle of proper rotation, which explains the gauge invariance as being the impossibility of defining an absolute proper rotation in a spinning fluid.

1. Decomposition of the impulse of the Dirac fluid in Clebsch parameters. – Utilizing the results of a preceding Note [**1**], recall the expression for the spinor:

(I) $$q = \sqrt{D}\, e^{i\sigma_4 A/2} e^{i\sigma_3 \psi/2} e^{i\sigma_1 \theta/2} e^{i\sigma_3 \varphi/2} e^{\alpha_1 \gamma_1/2} e^{\alpha_2 \gamma_2/2} e^{\alpha_3 \gamma_3/2} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

in which ψ, θ, φ, γ_i are the relativistic Euler angles.

Introduce that expression into the Gordon current:

$$g_\mu = \frac{1}{i}\, q^* \,[\partial_\mu]\, \alpha_4\, q$$

and the Proca current:

$$p_\mu = \frac{1}{i}\, q^* \,[\partial_\mu]\, \alpha_5\, q\,.$$

One gets:

$$g_\mu = \Omega_1\, \partial_\mu \varphi + \Omega_1 \cos\theta\, \partial_\mu \psi + m_{14}\, \partial_\mu \gamma_1 + (m_{24} \operatorname{ch} \gamma_1 - m_{12} \operatorname{sh} \gamma_1)\, \partial_\mu \gamma_2 + \Omega_1 \cos\theta\, \partial_\mu \gamma_3,$$
$$p_\mu = \Omega_2\, \partial_\mu \varphi + \Omega_2 \cos\theta\, \partial_\mu \psi + m_{23}\, \partial_\mu \gamma_1 + (m_{31} \operatorname{ch} \gamma_1 + m_{34} \operatorname{sh} \gamma_1)\, \partial_\mu \gamma_2 - \Omega_1 \cos\theta\, \partial_\mu \gamma_3,$$

in which the $m_{\mu\nu}$ are the components of the Dirac electromagnetic moment.

Those relations seem difficult to interpret. We then apply the same decomposition of the impulse vector as Yvon-Takabayasi [**2**]:

$$k_\mu = \frac{\hbar}{2}\frac{\Omega_1 g_\mu + \Omega_2 p_\mu}{D}.$$

Moreover, that impulse has the same expression as that of the Møller-Weyssenhoff fluid, which will permit us to interpret the Dirac fluid as being a spinning fluid with internal stresses [**3**].

One easily finds that:

$$k_\mu = \frac{\hbar}{2} D\,(\partial_\mu \varphi + \cos\theta \partial_\mu \psi) + \frac{\hbar}{2}(\Omega_1 m_{23} + \Omega_2 m_{14})\,\partial_\mu \gamma_1$$
$$+ \frac{\hbar}{2}[(\Omega_1 m_{24} + \Omega_2 m_{31})\cosh\gamma_1 - (\Omega_1 m_{12} - \Omega_2 m_{34})\sinh\gamma_1]\,\partial_\mu \gamma_2 .$$

Take the impulse "per particle" by dividing this by the invariant density of the fluid D. At the same time, simplify the expression by applying the Pauli-Kofink decomposition of the electromagnetic moment [**4**]. That will give:

(II)
$$\frac{k_\mu}{D} = \frac{\hbar}{2}(\partial_\mu\varphi + \cos\theta\,\partial_\mu\psi) + \frac{\hbar}{2}\frac{1}{D^2}(s_2 j_3 - s_3 j_1)\partial_\mu\gamma_1$$
$$+\frac{\hbar}{2}\frac{1}{D^2}[(s_3 j_1 - s_1 j_3)\cos\gamma_1 - (s_3 j_4 - s_4 j_3)\sinh\gamma_1\partial_\mu\gamma_2].$$

With the obvious definitions, one can then write:

(II′)
$$\frac{k_\mu}{D} = \frac{\hbar}{2}\partial_\mu\varphi + \frac{\hbar}{2}\sum_{\alpha=1}^{4}\xi_\alpha\partial_\mu\eta_\alpha.$$

That decomposition is nothing but the Clebsch decomposition [**5**]. In classical hydrodynamics, that decomposition is applied to the velocity vector, which is collinear with the impulse. For a relativistic spinning fluid, the impulse is no longer collinear with the velocity, and only the impulse will decompose into Clebsch parameters.

In the non-relativistic approximation, sinh γ_i and the spatial components j_i of the current will become negligible. One then sees that the two Clebsch parameters are annulled in (II), and the spatial components of k_μ / D can be written:

(III)
$$m\mathbf{v} = \frac{\hbar}{2}(\nabla\varphi + \cos\theta\,\nabla\psi),$$

which is nothing but Pauli's decomposition of the impulse [**6**], which is then collinear with the convection current.

One sees that the angle of proper rotation φ plays a privileged role in the theory.

Indeed, the no-spin approximation of the Pauli equation consists of annulling the angle of nutation θ and the angle of precession ψ, which is equivalent to making the spins parallel. One of the components of the Pauli spinor is annulled, and the other one is identified with the Schrödinger wave. The phase S of that wave that was introduced by Louis de Broglie is then such that:

$$S = \frac{\hbar}{2}\,\varphi.$$

One then sees that the angle of proper rotation is identified with the Jacobi function [**7**]. If one takes into account the fact that the matrix s_3 is diagonal then the expression (I) for the spinor will show that one can write:

$$\text{(IV)} \qquad q = e^{i\varphi/2}\,\Phi,$$

in which Φ is a spinor that does not depend upon φ [**8**]. One can compare the expression (IV) with the expression for the Dirac spinor:

$$\text{(IV}'\text{)} \qquad q = e^{iS/2}\,\Phi,$$

that was introduced by Louis de Broglie and Pauli in the WKB approximation [**9**] and by Vigier in his interpretation of the Dirac current [**10**]. It then seems that the phase S that was introduced by those authors can be identified with the proper rotation of the particle, up to a factor.

2. On gauge invariance of the second kind. – One knows that one can define a conservative current for a field whose equations are derived from a Lagrangian $L(\psi, \psi^*, \partial_\mu\psi, \partial_\mu\psi^*)$ only if the Lagrangian enjoys gauge invariance, in addition to relativistic invariance; i.e., if one has:

$$L(\psi\, e^{i\alpha}) = L(\psi),$$

in which α is a constant. The wave function must then be able to preserve its phase indeterminacy.

If the field describes a particle with spin then one will know from the theory of fusion [**11**] that ψ is either the Dirac spinor q or a tensor on the space of spinors. The relation (IV) will then show that:

$$\psi = e^{in\varphi/2}\,\chi,$$

in which φ is always the angle of proper rotation.

The phase indeterminacy can be considered to be something that bears upon the angle φ. The gauge invariance then signifies that it is impossible to define an *absolute* rotation at a point of the fluid, and that one can only speak of the differences between states of rotation from one point to another, which is quite natural.

However, all of the equations that are obtained by fusion of the Dirac equations describe spinning fluids. One can apply the considerations of our preceding Note [**12**] to them and say that spin is only the macroscopic appearance of a classical rotation that exists at the microscopic level. It then seems that gauge invariance is nothing but a reflection of the relativistic invariance of the equations that describe the motion of the fluid “molecules.” That result remains to be proved.

(*) Session on 16 July 1956.

[1] G. JAKOBI and G. LOCHAK, Comptes rendus **243** (1956), pp. 276.

[2] YVON, J. Phys. Rad. (8) **1** (1940), pp. 18.
TAKABAYASI, Nuovo Cimento **3** (1956), pp. 233.

[3] BOHM, LOCHAK, and VIGIER (*to appear* in Phys. Rev.).

[4] PETIAU, J. Math. pures et appl. **25** (1946), pp. 335.

[5] LAMB, *Hydrodynamics*, Cambrdige, 1932.

[6] BOHM, TIOMNO, and SCHILLER, Nuovo Cimento (10) supp. 1 (1955), pp. 48.

[7] BOHM, LOCHAK, and VIGIER, *loc. cit.* and *Séminaire Darmois*, 1955.
BOHM, TIOMNO, and SCHILLER, *loc. cit.*

[8] Φ is indeed a spinor, because the expression for k_μ shows that φ is necessarily a relativistic invariant.

[9] L. DE BROGLIE, *La théorie des particules de spin* 1 / 2, Paris, 1952.

[10] VIGIER, *Structure des Microobjets dans l’interprétation Causale de la Théorie des Quanta*, Paris, 1956.

[11] L. DE BROGLIE, *La théorie générales des Particules à spin (Méthode de Fusion)*, Paris, 1943.

[12] G. JAKOBI and G. LOCHAK, *loc. cit.*
