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On the hydrodynamical model for quantum mechanics (*)

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The lectures that I attended on wave mechanics (which was still quite new at the time) belong to the most exciting recollections that I have of my years as a student in Berlin. The wave function ψ, which can represent the various properties of atoms with surprising precision in a wonderfully harmonious way, fills all of us with admiration. Those memories are connected with the lectures of Professor E. SCHRÖDINGER, Professor G. HETTNER, and Dr. F. MÖGLICH that I heard at the time. I hope that Professor HETTNER will not think ill of me if I dedicate an article to him on the occasion of his 70th birthday in which I put down on paper for the first time in the present form the not-entirely-orthodox arguments that I have been concerned with since my student days, but to which I have once more returned in recent years especially.

In the following article, we would like to address the problem of the extent to which it is possible to transform the Schrödinger wave equations mathematically in such a way that the new form of the wave equation will correspond to a classical equation of motion of hydrodynamical type. EHRENFEST (1) already addressed that problem in his well-known article, and later on, MADELUNG (2), TAKABAYASI (3), *et al*. In the following article, we will be required to recall some results that are known already in order to give a complete picture. However, we are anxious to complete and complement the known results.

Here, we would like to draw attention to only identical conversions, and thus equations that emerge from the original equation by essentially a transformation of variables. We would not like to concern ourselves here with the question of whether such a transformation is possible in the most general case, but concentrate on a simple special case for the moment. We would like to go into the problems that are associated with the complicated cases step-by-step in later papers.

(*) For Professor G. HETTNER on his 70th birthday.
(1) EHRENFEST, P.: Zeit. Phys. **45** (1927), 455.
(2) MADELUNG, E.: Zeit. Phys. **40** (1926), 322.
(3) TAKABAYASI, T.: Prog. Theor. Phys. **8** (1952), 143; **9** (1953), 187.

I

§ 1. We consider the Schrödinger equation of the one-body problem in the case for which the force that acts upon the body can be derived from a potential $V(\mathbf{r})$. The wave equation can be written in the usual way as:

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = i\hbar\dot{\psi}\,. \tag{1}$$

We would like to always denote differentiation with respect to time by a dot. If we multiply (1) by ψ^* and subtract the complex conjugate of the equation thus-obtained then the result of that simple conversion will be:

$$\operatorname{div}\mathbf{v}\,\rho + \frac{\partial\rho}{\partial t} = 0 \tag{2}$$

with

$$\rho = \psi^*\psi \tag{3}$$

and

$$\rho\,\mathbf{v} = -\frac{i\hbar}{2m}(\psi^*\operatorname{grad}\psi - \psi\operatorname{grad}\psi^*). \tag{4}$$

Eq. (2) can be regarded as the continuity equation of a fluid whose density is ρ and whose flow velocity is $\mathbf{v}$. Naturally, ρ and $\mathbf{v}$ are functions of position and time, so we have written, more briefly:

$$\rho(\mathbf{r}, t) = \rho, \qquad \mathbf{v}(\mathbf{r}, t) = \mathbf{v}.$$

The question might be posed of whether (3) and (4) are the only expressions that we can define from the wave function ψ that will satisfy an equation of continuity. Obviously, the question must be answered in the negative. We will come to another possible expression for the current density if we, e.g., add the rotation of an arbitrary vector to the expression on the right-hand side of (4); we will come back to this question in a later publication.

§ 2. Eq. (2) describes the motion of a fluid in a purely kinematical way. In order to obtain a dynamical description, we must consider the acceleration of the fluid element. If we differentiate (4) with respect to time then we will obtain:

$$\frac{\partial\,\rho\mathbf{v}}{\partial t} = -\frac{i\hbar}{2m}(\dot{\psi}^*\operatorname{grad}\psi + \psi^*\operatorname{grad}\dot{\psi} - \text{c.c.}). \tag{5}$$

We can express $\dot{\psi}$ and $\dot{\psi}^*$ in terms of ψ and its derivatives with respect to position with the help of (1) and then obtain the equation of motion. Before we write down the equation, we remark that from the hydrodynamical standpoint, the total derivative of $\mathbf{v}$ is what is interesting, namely:

$$\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\ \mathrm{grad})\ \mathbf{v}. \tag{6}$$

With the help of (1), (5), and (6), and a simple calculation, we will obtain the result that:

$$m\rho\ \frac{d\mathbf{v}}{dt} = -\ \rho\ \mathrm{grad}\ (V + Q), \tag{7}$$

with

$$Q = -\frac{\hbar^2}{2m}\frac{\nabla^2 \rho^{1/2}}{\rho^{1/2}}, \tag{8}$$

or also:

$$m\rho\ \frac{d\mathbf{v}}{dt} = -\ \rho\ \mathrm{grad}\ V - \mathrm{div}\ \mathfrak{Q}, \tag{9}$$

in which $\mathfrak{Q}$ means a symmetric tensor whose definition is given by the following equation:

$$\mathfrak{Q} = -\frac{\hbar^2}{2m}(\nabla \circ \nabla)\ \ln \rho. \tag{10}$$

§ 3. Eq. (2), together with eqs. (7) and (8), or together with (9) and (10), will give a complete system of equations of motion. In fact, if we characterize our fluid by an initial condition that might be given by, say:

$$\mathbf{v}(\mathbf{r}, 0) = \mathbf{v}_0\ , \qquad \rho(\mathbf{r}, 0) = \rho_0\ , \tag{11}$$

then we can determine the temporal derivatives of $\mathbf{v}$ and ρ for $t = 0$. Perhaps with the help of step-wise integration, we can then determine the changes in the distributions over time uniquely from the given initial state step-by-step. The equations of motion (7), (8), and (9) then describe the motion of a fluid from the initial state uniquely. The original variable – viz., the wave function ψ – has been eliminated completely from the equations of motion.

In what follows, we shall show that the system of hydrodynamical equations (2), (7), and (8) has a form that satisfies the usual requirements that one poses for classical hydrodynamical equations of motion. Initially, we can also write:

$$m\rho\ \frac{d\mathbf{v}}{dt} = (\mathbf{F}_a + \mathbf{F}_i),$$

with

$$\mathbf{F}_a = -\ \rho\ \mathrm{grad}\ V, \qquad \mathbf{F}_i = -\ \rho\ \mathrm{grad}\ Q \qquad (\text{or} = -\ \mathrm{Div}\ \mathfrak{Q});$$

i.e., the fluid elements experience accelerations that consist of two parts. The first part of the acceleration will be due to an external force that is derivable from the external potential V and whose density is $\mathbf{F}_a$. The second part of the acceleration will be generated by the effect of an internal force that can be represented by the gradient of an internal potential Q. Q might be interpreted as an elastic potential, and in the literature it is known by the name of the *quantum-mechanical potential*.

The potential Q has precisely the properties that one must expect of the internal potential of an elastic body in the classical theory. We emphasize three essential properties:

1. In the case of ρ = constant, one will have $Q = 0$, so the elastic potential will vanish for a constant density distribution. Elastic forces will appear only when the fluid is distributed inhomogeneously. The forces that appear then will be functions of the density ρ and its derivatives with respect to position. The force at a point is determined exclusively from the distribution in the neighborhood of the point.

2. The resultant of the internal force will vanish identically when it is taken over the entire system. Namely, we have:

$$\int \mathbf{F}_i \, d\tau = -\int \rho \, \mathrm{grad} \, Q \, d\tau = -\int \mathrm{Div} \, \mathfrak{Q} \, d\tau = 0, \tag{12}$$

in which we have applied Gauss's law in the usual way, and we have assumed that ρ and $\mathfrak{Q}$ vanish sufficiently quickly at infinity.

3. The total rotational moment that is generated by the internal forces vanishes. We find that the rotational moment $\mathbf{M}_i$ is:

$$\mathbf{M}_i = m \int (\mathbf{r} \times \mathbf{F}_i) \, d\tau = -m \int (\mathbf{r} \times \mathrm{Div} \, \mathfrak{Q}) \, d\tau = -m \, (\mathfrak{Q} - \tilde{\mathfrak{Q}}) \, d\tau,$$

in which $\tilde{\mathfrak{Q}}$ means the transposed tensor to $\mathfrak{Q}$. However, it follows from (10) that $\mathfrak{Q} = \tilde{\mathfrak{Q}}$, and we will then find that:

$$\mathbf{M}_i = 0. \tag{13}$$

§ 4. It further follows from (12) and (13) that:

$$\left.\begin{aligned} \dot{\mathbf{P}} &= \int \mathbf{F}_a \, d\tau, \\ \dot{\mathbf{M}} &= \int (\mathbf{r} \times \mathbf{F}_a) \, d\tau, \end{aligned}\right\} \tag{14}$$

where:

$$\mathbf{P} = m \int \rho \, \mathbf{v} \, d\tau, \qquad \mathbf{M} = m \int (\rho \mathbf{r} \times \mathbf{v}) \, d\tau;$$

i.e., they are the impulse and angular momentum of the system. We then see that the impulse and angular momentum of the system change under the action of the external forces only in a way that corresponds to classical mechanics. The first of eqs. (14) is then known as *Ehrenfest's theorem*.

II

§ 5. In what follows, we shall examine the extent to which eq. (1), on the one hand, and the hydrodynamical equations of motion (2), (7), and (8), on the other, correspond.

In order address the question of the connection between the two systems of equations, we remark that the complex wave function ψ can be represented with the help of two real functions R and S:

$$\psi = R\, e^{iS}. \tag{15}$$

If we demand that:

$$R \geq 0, \qquad 0 \leq S < 2\pi$$

then R and S will be determined uniquely for points where $\psi \neq 0$; for points where $\psi = 0$, $R = 0$, and S is undetermined. If substitute (15) into (3) and (4) then we will find that:

$$\rho = R^2, \qquad \mathbf{v} = \frac{\hbar}{m}\ \text{grad}\ S. \tag{16}$$

It follows from eq. (16) that:

$$\text{rot}\ \mathbf{v} = 0$$

for regular points of the field. That is, the only hydrodynamical configurations ρ, $\mathbf{v}$ that correspond to configurations that can be represented by a wave function are the ones for which the velocity is irrotational (in the regular components of the field).

We will obtain a sharper condition in the following way: It follows from the second of eqs. (16) that:

$$S = S(\mathbf{r}, t) = \frac{m}{\hbar}\int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{v}(\mathbf{r}', t)\ d\mathbf{r}' + S_0(t), \tag{17}$$

in which $\mathbf{r}_0$ means the radius vector of an arbitrary (fixed) point, and $S_0(t)$ means an arbitrary function of time.

As one can also infer directly from (16), it follows from (17) that S can be increased by an arbitrary function of time without affecting $\mathbf{v}$. Furthermore, since the path of integration on the right-hand side of (17) can be chosen arbitrarily, we see that (except for a constant) unique values of S can be obtained from $\mathbf{v}$ only when the integral of $\mathbf{v}$ over a closed path of integration vanishes. If we do not require the single-valuedness of S, but only that of ψ, then, from (15), S can remain undetermined up to an arbitrary whole-number multiple of 2π, so we will find that an essentially single-valued ψ can be found for a given distribution of $\mathbf{v}$ if and only if:

$$\oint \mathbf{v}\, d\mathbf{r} = 2\pi k / m, \qquad k = 0, \pm 1, \pm 2, \ldots \tag{18}$$

is true for an arbitrary path of integration that contains no singularities. The value of k might prove to be different for different paths of integration.

If we take the total temporal derivative of the expression on the right-hand side of (18) then we will get, with the help of the equations of motion:

$$\frac{d}{dt}\oint \mathbf{v}\, d\mathbf{r} = 0, \tag{10}$$

in which we understand the total derivative to mean the change of the integral along a path that moves with the fluid. Eq. (19) expresses Helmholtz's vortex theorem for our fluid. Eq. (18) shows that the strength of the vortex line cannot be arbitrary, but can amount to only whole-number multiples of $2\pi\hbar/m$ when the fluid is capable of only states of motion that correspond to a single-valued ψ.

Moreover, a comparison of (18) and (19) shows that prescribes (18) for the distribution of **v** means merely a restriction on the possible initial configuration.

In connection with that, we see that a given distribution **v**, ρ corresponds to a wave function:

$$\psi = \sqrt{\rho}\exp\left\{\frac{im}{\hbar}\int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{v}\, d\mathbf{r}' + \frac{i}{\hbar} S_0(t)\right\}. \tag{20}$$

We find that when ψ is computed from (20), it will satisfy the wave equation (1), in the event that **v** and ρ satisfy the hydrodynamical equations, and when we set $\mathbf{r}_0 = 0$ and choose $S_0(t)$ suitably (*). With that, the uniquely-invertible association of a representation of the system nu the wave function *y*, on the one hand, and the hydrodynamical variables **v**, *r*, on the other, can be proved, with the following restrictions:

1. ψ can be determined from **v** and ρ only up to a constant phase factor $e^{i\gamma}$ with grad $\gamma = \dot{\gamma} = 0$. However, the restriction above is inessential, since a constant phase factor is regarded as meaningless in the quantum-mechanical treatment of a system.

2. **v** and ρ correspond to a single-valued ψ-function only when the vortex in the fluid satisfies the condition (18).

It must be observed that in the usual discussion of the solutions of (1), one demands that they must be normalized; i.e., that:

$$\int \psi^* \psi\, d\tau = 1. \tag{21}$$

The condition (21) corresponds to the hydrodynamical condition:

$$\int \rho\, d\tau = 1. \tag{22}$$

(*) A simple calculation will yield that we have to set:

$$S_0(t) = \int_0^t E\, dt\,,$$

in which we have set:

$$E = \int (Q + \tfrac{1}{2} mv^2) + V(r))\, \rho\, d\tau\,.$$

E corresponds to the total energy of the system.

Condition (22) expresses a physically-essential state of affairs. Namely, up to now, we have spoken of ρ as a density, *per se*. A closer consideration of the equations of motion will show that:

$$\rho_m = m\,\rho$$

actually plays the role of the mass density in them. Moreover, in the event that we would like to consider the electrical properties of our system, we would have to introduce $\rho_e = e\,\rho$ the electrical charge density. The condition (22) then fixes the total mass (total charge, resp.) of the system.

Stationary states

§ 6. In the usual conception of wave mechanics, the stationary solutions of the form:

$$\psi(\mathbf{r}, t) = \varphi(\mathbf{r})\ e^{-iEt/\hbar} \tag{23}$$

play a special role. The stationary solutions of the wave equations correspond to states in which the density ρ is temporally constant correspond to states; namely, one will have:

$$\rho = \psi^*\ \psi = \varphi^*\ \varphi = \text{independent of } t.$$

The wave equation (1) has stationary solutions with real amplitudes:

$$\varphi(\mathbf{r}) = \varphi^*(\mathbf{r}). \tag{24}$$

Since the wave function can always be multiplied by a constant phase factor $e^{i\gamma}$, we shall consider a state to be a state with a real amplitude when ψ can be brought into the form (23) with real $\varphi(\mathbf{r})$ by multiplying it by a suitable constant phase factor.

If we substitute (23) and (24) into (4) then we will find that:

$$\rho\,\mathbf{v} = 0 \qquad \text{for all values of } \mathbf{r} \text{ and } t.$$

That is, the stationary states with real amplitude corresponds to equilibrium configurations in which the internal quantum-mechanical forces are in equilibrium with the other forces, and in that state, every element of the medium will be at rest.

We remark that φ satisfies the amplitude equation:

$$-\frac{\hbar^2}{2m}\nabla^2\varphi + V\varphi = E\varphi. \tag{25}$$

In the event that (24) is true, we will have $\varphi = \rho^{1/2}$, and we will find with the help of (25) and (8) that:

$$Q + V = E = \text{constant}, \tag{26}$$

so, from (7):

$$\rho \frac{d\mathbf{v}}{dt} = 0.$$

That is, no resultant forces act upon the elements of the medium, and it thus exists in equilibrium, in fact.

In the case of a stationary state with a (non-constant) complex amplitude, we find that:

$$\rho\,\mathbf{v} = -\frac{i\hbar}{2m}(\varphi^* \operatorname{grad} \varphi - \varphi \operatorname{grad} \varphi^*).$$

The latter expression is independent of time, although it generally does not vanish. The stationary state above then corresponds to a motion of the fluid in which:

$$\rho,\ \mathbf{v} = \text{independent of } t.$$

No electromagnetic radiation will be emitted in a stationary state of this kind, since the stationary state with a real amplitude (for which, the fluid is at rest) does not radiate either, so we see that the stationary states of the wave equation represent the non-radiating states of motion of the fluid.

We remark that, in general, $\rho^{1/2} \neq \varphi$ in the case of a stationary state with a complex amplitude, so (26) will not be fulfilled in that case. Therefore:

$$\rho \frac{d\mathbf{v}}{dt} \neq 0,$$

and accelerated motions will appear. Such a state can be described in the following way: The fluid is held in a stationary state, in such a way that the charge and current density are constant at every point of the distribution, so:

$$\frac{\partial\rho}{\partial t} = 0, \qquad \frac{\partial\mathbf{v}}{\partial t} = 0.$$

However, the fluid elements move on curved paths, and therefore experience acceleration. The differences between the internal and external forces have precisely the appropriate values to keep the fluid elements on those curved stationary paths.

Eigen-oscillations of the medium

§ 7. The general solution of the wave equation (1) can be expressed by a superposition of stationary states, so we will have, in the usual notation:

$$\psi = \sum c_k \varphi_k e^{-iE_k t/\hbar}. \tag{27}$$

If we substitute (27) into (3) and (4) then we will find that:

$$\rho = \sum \rho_{kl} \cos(\omega_{kl}\, t + \alpha_{kl}),$$
$$\rho\, \mathbf{v} = \sum \mathbf{i}_{kl} \cos(\omega_{kl}\, t + \beta_{kl}),$$

with

$$\omega_{kl} = (E_k - E_i) / \hbar\, .$$

The coefficients ρ_{kl}, $\mathbf{i}_{kl}$, and the phases α_{kl} and β_{kl} can be expressed in a simple way with the help of the amplitudes φ_k and the coefficients c_k.

We then see that the general state of motion of the medium is a superposition of oscillations with frequencies ω_{kl}. The ω_{kl} can then be considered to be the mechanical *eigen-oscillations* of our medium.

The hydrodynamical model thus-obtained has a lot of similarity with Hertz's model of the elastically-bound electrons. However, whereas one cannot succeed in explaining the manner by which elastically-bound electrons are capable of different frequencies of oscillation in Hertz's model, those frequencies will emerge quite naturally in the hydrodynamical model.

Furthermore, it is easy to show that the optical selection rules can be understood quite simply with the help of the hydrodynamical model. The frequencies of the "forbidden transitions" correspond to forms of oscillation for the medium in which the different parts of the charge oscillate with different phases, such that the emitted radiation would be cancelled by interference. For example, in the case of lines that are forbidden in the first approximation, the state of oscillation will correspond to two dipoles of equal strength and opposite phase; in that case, only quadrupole radiation will be emitted.

III

§ 8. The arguments in the first part can be generalized to the case where the wave equation represents the motion of an electric charge in an electromagnetic field that can be derived from a given vector potential **A** and a given scalar potential **Φ**. In that case, the wave equation can be written:

$$\frac{1}{2m}\left(-i\hbar\nabla - \frac{e}{c}\mathbf{A}\right)^2 \psi + (e\,\Phi + V)\,\psi = i\hbar\dot{\psi}\, . \tag{28}$$

If we again introduce:

$$\rho = \psi^* \psi \tag{29}$$

as a density and:

$$\rho\,\mathbf{v} = -\frac{i\hbar}{2m}(\psi^* \operatorname{grad} \psi - \psi \operatorname{grad} \psi^*) - \frac{e\rho}{mc}\mathbf{A} \tag{30}$$

as the current density then we will find that with the help of (28) that the quantities that are defined by (29) and (30) will satisfy the continuity equation:

$$\operatorname{div} \rho\,\mathbf{v} + \frac{\partial \rho}{\partial t} = 0. \tag{31}$$

If we further differentiate (20) with respect to time then we will find with the help of (28), when we introduce:

$$\mathbf{H} = \text{rot}\,\mathbf{A}, \qquad \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - \text{grad}\,\Phi$$

in the usual way, that:

$$\rho_m \frac{d\mathbf{v}}{dt} = -\rho\,\text{grad}\,(V + Q) + \frac{\rho_e}{c}(\mathbf{v} \times \mathbf{H}) + \rho_e\,\mathbf{E}, \tag{32}$$

in which have set $m\rho = \rho_m$ and $e\rho = \rho_e$, as above, and Q is given by (8). We then see that the wave equation (28), which contains the effect of external fields, can be replaced with hydrodynamical equations in precisely the same as the wave equation (1).

We infer from eq. (32) that an external electromagnetic field will act upon the fluid elements with the Lorentz force density:

$$\mathbf{F} = \rho_e\,[\mathbf{E} + \frac{1}{c}(\mathbf{v} \times \mathbf{H})]$$

but otherwise the motion proceeds as it would in the absence of external fields.

§ 9. We have succeeded in replacing the wave equation (38) with the hydrodynamical eqs. (31) and (32). It remains for us to discuss the question of the extent to which (31) and (32) are equivalent to the wave equation (28).

We see from the outset that (31) and (32), together with an initial condition of the form (11), determine the course of motion uniquely for times $t > 0$; the equations then give a complete description of the motion that is similar to the one that the wave equation (28) provides. However, it must be remarked that the hydrodynamical equations contain only the field strengths **E** and **H**, while the wave equation (28) contains the potentials **A**, Φ explicitly. We then see that the hydrodynamical equations are gauge invariant in a trivial way, while the gauge invariance of the wave equation (28) indeed exists, although cannot be regarded as trivial.

§ 10. We must still discuss the question of the extent to which a state that satisfies a hydrodynamical initial condition (11) corresponds to a single-valued wave equation $\psi(\mathbf{r}, t)$. An argument that is analogous to the argument in § 5 shows that the initial condition (11) must satisfy the auxiliary conditions if the hydrodynamical state is to be represented by a normalized and single-valued wave function:

$$\int \rho\, d\tau = 1, \tag{33}$$

$$\oint \mathbf{v}\; d\mathbf{r} = \frac{2\pi\hbar k}{m} - \frac{e}{mc}\int \mathbf{H}\, d\mathbf{f}\,, \tag{34}$$

in which the second term on the right-hand side means the magnetic flux through the surface that is bounded by the path of integration. Eq. (34) will go to eq. (18) for $\mathbf{H} = 0$; i.e., in the absence of a magnetic field. By the way, like eq. (18), if eq. (34) is true for time $t = 0$ then from the equations of motion it must also stay true in the later course of motion.

Similarly to the argument in § 5, it can also be deduced here that a state ρ, $\mathbf{v}$ that obeys the conditions (33) and (34) at time $t = 0$ will correspond to a state that can be described by a single-valued and normalized wave function. The equivalence of the wave-mechanical way of looking at things with that of the hydrodynamical model is then shown in that way.

In later publications, we would like to discuss the further peculiarities of the hydrodynamical model; in particular, we will show that the hydrodynamical model can be extended to the spinning electron, as it is described by the Pauli equation.