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On the quantum electrodynamics of charge-free fields

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As a continuation of **Dirac**’s theory, in which the electrodynamical field quantities are regarded as non-commuting numbers (q-numbers), commutation relations between the field quantities will be presented here that have a *relativistically-invariant* form, at least in the special case of the absence of charged particles (pure radiation field). It will be shown that these relations can also be formulated without employing the Fourier decomposition of the field. Moreover, a general mathematical method will be given that allows one to reinterpret relations between q-numbers that depend continuously upon space and time coordinates (q-functions) as relations between suitably-chosen operators that are applied to generalized ψ-functions that depend upon the entire field evolution (functionals).

As is known, **Dirac** (*) was the first to succeed in adapting quantum-mechanical methods to the treatment of the electromagnetic fields itself when he regarded the amplitude of the partial waves of the field as “q-numbers” and presented commutation relations for them. Due to the fact that one will arrive at an essential advance in that way, now that an analogous treatment of a simpler problem has already been given (**), namely, the problem of the scalar (one-dimensional) wave equation, it must certainly appear that a known difficulty that **Einstein** found in regard to the energy oscillations of a wave field by a quantum-mechanical treatment of the eigen-oscillations of the field can be solved. In fact, **Dirac** succeeded in presenting a consistent theory of the emission, absorption, and dispersion of radiation. Furthermore, **Jordan** (***) has adapted the **Dirac** method for the quantization of wave-fields to the case of matter waves that correspond to Fermi statistics, and the results of a recent paper by **Jordan** and **Klein** (****) make it seem very promising moreover, that one might be able to attack the still-unsolved problem of a quantum theory of the interaction of particles when one considers the final propagation velocity of the force effects. Such a theory must also treat the electrostatic and radiation effects of the electromagnetic field by means of a unified methodology.

Nonetheless, the topic of the present article shall still not be the general interaction problem, but rather, for the time being, we intend to rectify something that was missing from the formulation of the theory that was achieved in the cited papers, and which was also continually stressed by their authors. Namely, in those papers, the time coordinate

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(*) **P. A. M. Dirac**, Proc. Roy. Soc. London (A) **114** (1927), 243, 710.

(**) **M. Born**, **W. Heisenberg**, and **P. Jordan**, Zeit. Phys. **35** (1926), 557.

(***) **P. Jordan**, *ibidem*, **44** (1927), 473. Added in proof: Cf., also **P. Jordan** and **E. Wigner** (to appear).

(****) **P. Jordan** and **O. Klein**, *ibidem*, **45** (1927), 751.

was always singled out from the spatial coordinates in a peculiar way, so the results were not relativistically invariant. By contrast, the methods that will be employed in the present paper for the quantization of the electromagnetic fields are *relativistically invariant.*

At first, in **§ 1** the standpoint will be assumed that the electromagnetic field strengths can be decomposed into polarized, monochromatic partial waves *à la* **Fourier**, and that their amplitudes will fulfill certain commutation relations as "q-numbers." We will succeed in formulating these relations in such a way that no reference system of the special theory of relativity will be singled out from the other ones, while, at the same time, the previous results for the oscillation properties of the radiation energy will be reproduced correctly by the theory. However, this standpoint can be replaced by a more general one (*) from which a Fourier decomposition of the field is not used explicitly, and the field strengths themselves can be regarded as a continuum of q-numbers that depend continuously upon the space-time coordinates. Such sets of q-numbers might be briefly referred to as "q-functions." In **§ 2** to **4** of the first part of this paper, this more general standpoint will be developed, always while maintaining relativistic invariance. Let it be remarked here that these arguments can also be adapted completely to matter waves of force-free particles and will lead to a relativistically-invariant quantization of those waves in the event that one is dealing with particles of the same type that obey the **Einstein-Bose** statistics. However, since the quantization of matter waves in the other case of particles with Fermi statistics has still not been fully explained (**), we shall not go into that in more detail here in this article. One might probably expect of a still-pending general relativistically-invariant quantum theory of wave fields that, on the one hand, also has to consider electromagnetic fields that correspond to the presence of charged particles, and on the other hand, has to include the influence of electromagnetic fields on matter fields in the calculations, that it will include the commutation relations of the free electromagnetic radiation field that are presented here, along with the matter waves of force-free particles, as special limiting cases.

The second part of this paper will address the question of how the q-functions that are applied to certain "probability amplitudes" ψ can be interpreted. In ordinary quantum mechanics, as is known, one goes from the equations:

$$pq - qp = \frac{h}{2\pi i}$$

and the law of energy:

$$H(p, q) = E,$$

which are initially relations between q-numbers, to a differential equation for the function $\psi_E(q)$, in which one replaces p with the operator $\frac{h}{2\pi i}\frac{\partial}{\partial q}$, q with the multiplication operator, and then writes $H(p, q)$ as an operator that is applied to ψ:

(*) Cf., also **P. Jordan**, Zeit. Phys. **45** (1927), 766.
(**) Remark by the editor: However, one might confer the aforementioned paper of **Jordan** and **Wigner**.

$$H\left(\frac{h}{2\pi i}\frac{\partial}{\partial q}, q\right)\psi_E(q) = E\;\psi_E(q)\,.$$

In the case of a harmonic oscillator, where one can set:

$$H(p, q) = \frac{1}{2m}p^2 + \frac{m}{2}(2\pi\,\nu_0)^2\,q^2,$$

as **Schrödinger** has shown, the associated differential equation for ψ will lead to the eigenvalues:

$$E_n = (n + \tfrac{1}{2})\,h\nu_0 \qquad \text{with} \qquad n = 0, 1, 2, \ldots,$$

while the ψ-functions will be given by the so-called **Hermite** polynomials; in particular, one will have $\psi_0(q) = C\,e^{-\frac{2\pi^2 m\nu_0}{h}q^2}$ for $n = 0$.

Now, this has a complication as a consequence when one treats *infinitely-many* oscillators (which correspond to the infinitely-many degrees of freedom of radiation), such as in the eigen-oscillations of cavity radiation. First of all, the total energy density of the radiation will become infinitely large, since (in the limiting case, a very large cavity) radiation with a frequency between ν and $\nu + d\nu$ for $n = 0$ would itself yield the amount:

$$\frac{8\pi\nu^2}{c^3}\frac{h\nu}{2}\,d\nu$$

for it. Secondly, when only a finite number of eigen-oscillations is excited, the product of the infinitely-many eigen-oscillations will not converge, in general, such that the ψ-function of those infinitely-many amplitudes q_k of the oscillators will not possess a well-defined value at first.

Various considerations seem to suggest that, in contrast to the eigen-oscillations in a crystal lattice (where one can speak of there being theoretical, as well as empirical grounds for the presence of a zero-point energy), the "zero-point energy" of $h\nu / 2$ per degree of freedom has no physical reality for the eigen-oscillations of radiation. Namely, since one deals with strictly-harmonic oscillators for those eigen-oscillations, and since the "zero-point radiation" cannot be absorbed, scattered, or reflected, its existence, including its energy or mass, would seem to destroy any possibility of proving that it has any physical reality. For that reason, it is probably simplest and most satisfying to imagine that such a zero-point radiation does not even exist for the electromagnetic field.

In conjunction with that, it is perhaps interesting to remark that it is possible to also formulate that picture mathematically for an individual harmonic oscillator. Namely, if one introduces the quantities:

$$P = \frac{1}{2\sqrt{\pi\nu_0\,m}}\,p - i\sqrt{\pi\nu_0\,m}\,q\,,$$

$$Q = \frac{1}{2\sqrt{\pi\nu_0 m}}\, p + i\sqrt{\pi\nu_0 m}\, q\,,$$

in place of p and q, then it will follow from:

$$pq - qp = \frac{h}{2\pi i}$$

that one has the relation:

$$PQ - QP = i\,(pq - qp) = \frac{h}{2\pi},$$

and furthermore, one will have:

$$\frac{1}{2m}\, p^2 + \frac{m}{2}(2\pi\,\nu_0)^2\, q^2$$

$$= 2\pi\nu_0 \left(\frac{1}{2\sqrt{\pi\nu_0 m}}\, p + i\sqrt{\pi\nu_0 m}\, q\right)\left(\frac{1}{2\sqrt{\pi\nu_0 m}}\, p - i\sqrt{\pi\nu_0 m}\, q\right) + \pi\,\nu_0\, i\,(pq - qp)$$

$$= 2\pi\,\nu_0\, QP + \frac{h\nu_0}{2}\,.$$

If one introduces a new Hamiltonian function:

$$H\,(P, Q) \equiv 2\pi\,\nu_0\, QP = E,$$

with

$$PQ - QP = \frac{h}{2\pi},$$

then one will come to the eigenvalues:

$$E_n = n\, h\nu_0$$

with no zero-point energy. One can also exhibit eigenfunctions of $\psi_E\,(Q)$ for which the variable Q is generally a complex quantity. One might perhaps hope that the problems with convergence that are connected with the zero-point radiation for infinitely-many oscillators might someday be overcome in that way.

In the second part of the present work, however, a method shall be given for defining ψ-functions of the field and operations with them that are in harmony with the given relations between q-functions without an explicit use of the Fourier decomposition of the field being necessary. Unfortunately, we have not succeeded in carrying out an elimination of the zero-point energy under this condition in a satisfactory way that would be analogous to the consideration above for the individual oscillator. Therefore, what will be done in the second part of this paper will be, to a large extent, improvements that require extension, and will have more to do with the general mathematical methods that will employed there than with the special relations that will be given there.

I. Method of *q*-functions and *q*-numbers

§ 1. Fourier decomposition of the field. Relativistically-invariant commutations relations for the amplitudes of the eigen-oscillations. – We imagine that the electromagnetic radiation field is decomposed into partial monochromatic plane waves, and indeed, we think of travelling waves that then fulfill no special limiting conditions that might suggest, perhaps, impermeable cavity walls. By contrast, it is preferable to employ Fourier series at first, instead of Fourier integrals. Let $\mathfrak{k}_s$ be the propagation vector of a plane partial wave (viz., a vector in the direction of the wave normal with a magnitude that is equal to the wave number), let $|\,\mathfrak{k}_s\,| = k_s$ be its absolute value, and let ν_s be its oscillation number, such that one has:

$$k_s = \frac{\nu_s}{c}\,, \qquad \mathfrak{k}_s^2 = \frac{\nu_s^2}{c^2}\,. \tag{1}$$

The index *s* shall only distinguish between the various eigen-frequencies. At first, the propagation vectors $\mathfrak{k}_s$ that appear in the Fourier decomposition of the field might be assigned a density in the space of ($\mathfrak{k}_x$, $\mathfrak{k}_y$, $\mathfrak{k}_z$) (briefly: "$\mathfrak{k}$-space") that would correspond to the eigen-oscillations of a cubical cavity of edge *L* (so a volume of L^3). That is, we assume that the mean volume of a cell in $\mathfrak{k}$-space upon which (except for the polarization factor that is yet to be discussed) *one* partial wave of the Fourier series falls is equal to:

$$\Delta k_x\,\Delta k_y\,\Delta k_z = \frac{1}{L^3}\,. \tag{2}$$

The field strengths $\mathfrak{E}$ and $\mathfrak{H}$ are now combinations of the field strengths $\mathfrak{E}_s$ and $\mathfrak{H}_s$ of the individual eigen-oscillations, which consist of monochromatic waves:

$$\mathfrak{E} = \sum_s \mathfrak{E}_s\,, \qquad \mathfrak{H} = \sum_s \mathfrak{H}_s\,.$$

Now, we have yet to consider that *two* linearly-independent polarized waves are possible for each $\mathfrak{k}_s$ whose directions of oscillation are perpendicular to $\mathfrak{k}_s$. In order to represent them formally, for each *s*, we introduce an orthogonal coordinate system $(\xi, \eta, \zeta)_s$ whose ζ-axis is parallel to $\mathfrak{k}_s$, and let $\mathfrak{e}_\xi^{(s)}$, $\mathfrak{e}_\eta^{(s)}$, $\mathfrak{e}_\zeta^{(s)}$ be unit vectors in the directions ξ, η, ζ. Let the amplitudes $a_s^{(1)}$ of the electric field strengths of the one linearly-polarized eigen-oscillation (which will be denoted by the index 1) be parallel to the ξ-axis, while the other one (which will be demoted by the index 2) is parallel to the η-axis. If we include the factor $\sqrt{\frac{\nu_s}{L^3}}$, on grounds that will become clear shortly, then we can write:

$$\left.\begin{aligned}
\mathfrak{E}_s &= \sqrt{\frac{\nu_s}{L^3}}\{(\mathfrak{e}_\xi^{(s)} a_s^{(1)} + \mathfrak{e}_\eta^{(s)} a_s^{(2)}) \cos 2\pi[(\mathfrak{k}_s \mathfrak{r}) - |\mathfrak{k}_s|\, ct] \\
&\qquad + (\mathfrak{e}_\xi^{(s)} b_s^{(1)} + \mathfrak{e}_\eta^{(s)} b_s^{(2)}) \sin 2\pi[(\mathfrak{k}_s \mathfrak{r}) - |\mathfrak{k}_s|\, ct]\}, \\
\mathfrak{H}_s &= [\mathfrak{e}_\zeta^{(s)} \mathfrak{E}_s] = \sqrt{\frac{\nu_s}{L^3}}\{(\mathfrak{e}_\eta^{(s)} a_s^{(1)} - \mathfrak{e}_\xi^{(s)} a_s^{(2)}) \cos 2\pi[(\mathfrak{k}_s \mathfrak{r}) - |k_s|\, ct] \\
&\qquad + (\mathfrak{e}_\eta^{(s)} b_s^{(1)} - \mathfrak{e}_\xi^{(s)} b_s^{(2)}) \sin 2\pi[(\mathfrak{k}_s \mathfrak{r}) - |k_s|\, ct]\}.
\end{aligned}\right\} \tag{3}$$

The factor $\sqrt{\frac{\nu_s}{L^3}}$ in (3) is chosen so that the total cavity energy:

$$E_s = \tfrac{1}{2}\int (\mathfrak{E}_s^2 + \mathfrak{H}_s^2)\, dV$$

(to the extent that it originates in a single, linearly-polarized partial wave) will be equal to:

$$E_s = \tfrac{1}{2}\nu_s (a_s^2 + b_s^2), \tag{4}$$

in which either $a_s^{(1)}$, $b_s^{(1)}$ or $a_s^{(2)}$, $b_s^{(2)}$ are substituted for a_s and b_s. (The field strengths in this are measured in **Heaviside** units.)

Since the energy E_s (except for the zero-point energy) must be a multiple of $h\nu_0$ – that is, that:

$$\tfrac{1}{2}(a_s^2 + b_s^2)$$

(in any case, up to an additive constant) must have the characteristic values $N_s = 0, 1, 2, \ldots$ – it is reasonable to set:

$$a_s^{(1)} b_s^{(1)} - b_s^{(1)} a_s^{(1)} = a_s^{(2)} b_s^{(2)} - b_s^{(2)} a_s^{(2)} = ih, \tag{I}$$

in which one naturally has that a_s and $b_{s'}$ commute with each other for $s \neq s'$, just as different a_s or different b_s will. Moreover, it also seems natural to assume that the amplitudes that belong to the various polarization directions commute with each other:

$$\left.\begin{aligned}
a_s^{(1)} a_s^{(2)} - a_s^{(2)} a_s^{(1)} = 0, \quad b_s^{(1)} b_s^{(2)} - b_s^{(2)} b_s^{(1)} = 0, \\
a_s^{(1)} b_s^{(2)} - b_s^{(2)} a_s^{(1)} = 0, \quad a_s^{(1)} b_s^{(2)} - b_s^{(2)} a_s^{(1)} = 0
\end{aligned}\right\} \tag{I$'$}$$

It is easy to see that the commutation relations (abbreviation: "CR") (I) and (I′) are independent of the choice of unit vectors $\mathfrak{e}_\xi^{(s)}$, $\mathfrak{e}_\eta^{(s)}$, as long as they are perpendicular to each other and to $\mathfrak{k}_s$. One can similarly prove the invariance of (I) and (I′) under a change of the zero-point of the coordinate system that was initially distinguished by the Fourier decomposition (3) of the field. Namely, the a_s and b_s transform according to:

$$\begin{aligned} a_s' &= \quad a_s \cos \delta_s + b_s \sin \delta_s , \\ b_s' &= - a_s \sin \delta_s + b_s \cos\delta_s , \end{aligned}$$

for each polarization direction, and one therefore has, in fact:

$$a_s' \, b_s' - b_s' \, a_s' = a_s \, b_s - b_s \, a_s . \tag{5}$$

When one further observes that it is not the precise values of $\mathfrak{k}_s$, but only its density (2) that enters into $\mathfrak{k}$-space, it will be further easy to see, when one considers (1), that the CR (I) fulfill the requirement of *relativistic invariance.*

This will also become especially clear when one *passes from Fourier series to Fourier integrals in the limit.* One will then have:

$$\sum_s a_s^2 \frac{1}{L^3} = \sum_s a_s^2 \, \Delta k_x \, \Delta k_y \, \Delta k_z \;\to\; \int E(\mathfrak{k}) \, dk_x \, dk_y \, dk_z$$

for each polarization direction [we drop the index (1) or (2) for the sake of simplicity], and analogously for $\sum_s b_s^2 \frac{1}{L^3}$. Furthermore, from the definition of E (k_x, k_y, k_z) = E ($\mathfrak{k}$), that will imply:

$$\begin{gathered} \sum_s E_s \frac{1}{L^3} = \sum_s \frac{1}{L^3} \frac{1}{2} \int (\mathfrak{E}_s^2 + \mathfrak{H}_s^2) \, dV \;\to\; \int E(\mathfrak{k}) \, dk_x \, dk_y \, dk_z , \\ E(\mathfrak{k}) = \tfrac{1}{2} \, \nu(\mathfrak{k}) \, [A^2(\mathfrak{k}) + B^2(\mathfrak{k})]. \end{gathered} \tag{6}$$

When we, on the one hand, sum over all eigen-oscillations with $\mathfrak{k}_s$ in a certain domain $\Omega_1(\mathfrak{k})$ in $\mathfrak{k}$-space, and on the other hand, sum over the one with $\mathfrak{k}_s$ in another domain Ω_2 ($\mathfrak{k}$), and let Ω_{12} ($\mathfrak{k}$) denote the volume of the domain in $\mathfrak{k}$-space that is *common* to Ω_1 and Ω_2 , we can further compute:

$$\frac{1}{ih} \frac{1}{L^3} \left(\sum_{\mathfrak{k}_s \text{ in } \Omega_1(\mathfrak{k})} a_s \sum_{\mathfrak{k}_s \text{ in } \Omega_2(\mathfrak{k})} b_s - \sum_{\mathfrak{k}_s \text{ in } \Omega_2(\mathfrak{k})} b_s \sum_{\mathfrak{k}_s \text{ in } \Omega_1(\mathfrak{k})} a_s \right) = \Omega_{12}(\mathfrak{k}).$$

That will then next imply that the value on the left-hand side is equal to the number of eigen-oscillations that are common to both sums, divided by L^3, which agrees with

$\Omega_{12}(\mathfrak{k})$, according to (2). On the other hand, the sums that appear on the left-hand side will be equal, in the limit, to the corresponding integrals that are defined by A ($\mathfrak{k}$) and $B(\mathfrak{k})$, such that we can write:

$$\int_{\Omega_1} A(\mathfrak{k})\,dk_x\,dk_y\,dk_z \int_{\Omega_2} B(\mathfrak{k})\,dk_x\,dk_y\,dk_z - \int_{\Omega_2} B(\mathfrak{k})\,dk_x\,dk_y\,dk_z \int_{\Omega_1} A(\mathfrak{k})\,dk_x\,dk_y\,dk_z = ih\,\Omega_{12}\,, \tag{7}$$

or with the help of the **Dirac** δ-function that will be considered more closely in the following paragraph:

$$A\,(\mathfrak{k})\,B\,(\mathfrak{k}') - B\,(\mathfrak{k}')\,A\,(\mathfrak{k}) = ih\;\delta(\mathfrak{k} - \mathfrak{k}'). \tag{8}$$

What is more important than the passage to the limit from Fourier series to Fourier integral is to abandon the Fourier decomposition of the field entirely, along with its direct conception as a continuum of q-numbers (“q-functions”). In order to do that, it will be necessary to define a new, relativistically-invariant δ-function, which will be done in the following paragraphs.

§ 2. Definition and meaning of the relativistically-invariant Δ-function. – The ordinary **Dirac** δ-function for one variable x is defined by the equation:

$$\int_a^b \delta(x)\,dx = \begin{cases} 1 & \text{when } (a,b) \text{ includes the zero-point,} \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

One will also have:

$$\int_a^b f(x)\,\delta(x)\,dx = \begin{cases} f(0) & \text{when } (a,b) \text{ includes the zero-point,} \\ 0 & \text{otherwise} \end{cases} \tag{10}$$

then.

The “function” $\delta(x)$ can be regarded as an *abbreviation* for a *sequence* of functions $\delta_1(x)$, $\delta_2(x)$, …, $\delta_N(x)$, … for which $\lim_{N\to\infty}\int_a^b \delta_N(x)\,dx$ exists and has the value that is given above. Similarly:

$$\int_a^b f(x)\,\delta(x)\,dx \qquad \text{shall mean} \qquad \lim_{N\to\infty}\int_a^b f(x)\,\delta_N(x)\,dx\,.$$

As such a sequence of functions, one can take, e.g.:

$$\delta_N(x) = \frac{\sin 2\pi N x}{\pi x} = 2\int_0^N \cos 2\pi k\, x\, dk\,, \tag{11}$$

since one will then have:

$$\lim_{N\to\infty}\int_a^b f(x)\,\delta_N(x)\,dx = \int_{a\infty}^{b\infty} f\left(\frac{y}{2\pi N}\right)\frac{\sin y}{\pi y}\,dx = \begin{cases} f(0) & \text{when } a<0, b>0, \\ 0 & \text{when } a>0, b>0. \end{cases}$$

Naturally, (11) is, however, not the only possible sequence $\delta_N(x)$ that satisfies (10) in the limit as $N \to \infty$.

Now, we shall encounter a well-defined sequence of functions $\Delta_N(x, y, z, t)$ in the following paragraphs, and in fact, it is given by:

$$\Delta_N(x, y, z, ct) = \iiint\limits_{\substack{\text{Sphere} \\ |\mathfrak{k}|\le N}} \frac{2}{|\mathfrak{k}|}\sin 2\pi(k_x\, x + k_y\, y + k_z\, z - |k|\, ct)\, dk_x\, dk_y\, dk_z \tag{12}$$

$$(|k| = \sqrt{k_x^2 + k_y^2 + k_z^2}\,).$$

What is essential in this is the coupling of the coefficients of t with those of x, y, z, which says that all partial waves from which (12) is composed advance with the velocity c. $\Delta_N(x, y, z, t)$ is, moreover, relativistically-invariant for a fixed zero-point of the coordinate system, since, as one easily computes, for the case in which:

$$k_x\,,\ k_y\,,\ k_z\,,\ i\,|k|$$

define the components of a four-vector of length zero:

$$\frac{1}{|k|}\ k_x\, k_y\, k_z\,,$$

it is invariant under Lorentz transformations.

We would now like to characterize the sequence $\Delta_N(\ldots)$ as a Δ-function, i.e., by way of:

$$\lim_{N\to\infty}\int_{V_4} f(x, y, z, t)\,\Delta_N(x \ldots t)\,dV_4\,,$$

in which we have integrated over a four-dimensional world-domain, and we have set:

$$dV_4 = dx\, dy\, dz\, c\, dt.$$

We will once more write this limit symbolically as:

$$\int f(x, y, z, t)\,\Delta(x, y, z, t)\,dV_4$$

and consider all sequences Δ_N for which this limit coincides for all f to not be essentially different. Next the integral (12) by be evaluated. Introducing polar coordinates into $\mathfrak{k}$-space with $\measuredangle(\mathfrak{r}, \mathfrak{k}) = \vartheta$, $\cos\vartheta = u$:

$$dk_x\, dk_y\, dk_z = 2\pi |\, k\,|^2\, d\,|\, k\,|\, du$$

yields:

$$\Delta_N(x\ldots t) = 4\pi\int_0^N |\,k\,|\, d\,|\,k\,| \int_{-1}^{+1} \sin 2\pi\,|\,k\,|\,(ru - ct)\, du \qquad (r = +\sqrt{x^2+y^2+z^2}\,),$$

$$= 2\int_0^N d\,|\,k\,|\, \frac{1}{r}[\cos 2\pi |\,k\,|\,(r + ct) - \cos 2\pi |\,k\,|\,(r - ct),$$

or finally:

$$\Delta_N(x\ldots t) = \frac{1}{\pi r}\left[\frac{\sin 2\pi N(r+ct)}{r+ct} - \frac{\sin 2\pi N(r-ct)}{r-ct}\right]. \tag{12′}$$

(Observe that the negative sign in the bracket causes Δ_N to remain finite for $t \neq 0$, $r = 0$!)

In complete analogy with the properties of the function $\delta_N(x)$ at the beginning of this paragraph, we can now also give the $\lim_{N\to\infty}\int f(\ldots)\Delta_N dV_4$. Let V_4 be the domain of integration, and let V_3^+ be its three-dimensional section with the "light-cone" $r + ct = 0$, while V_3^- is the section with the light-cone $r - ct = 0$. One will then have:

$$\int f(x\ldots t)\,\Delta_N(x\ldots t)\, dV_4$$

$$= \int_{V_3^+} f(x, y, z, ct = -r)\frac{1}{r}dx\,dy\,dz - \int_{V_3^-} f(x, y, z, ct = r)\frac{1}{r}dx\,dy\,dz\,, \tag{II}$$

and this equation should now be regarded as the definition of the relativistically-invariant Δ-function (observing the invariance of $\frac{dx\,dy\,dz}{r}$), independently of its *particular* realization by the sequence (12). If one sets $f = 1$ in (II) then one will get the value of $\int_{V_4}\Delta\, dV_4$:

$$\int_{V_4}\Delta\, dV_4 = \int_{V_3^+}\frac{dx\,dy\,dz}{r} - \int_{V_3^-}\frac{dx\,dy\,dz}{r}. \tag{II′}$$

On the basis of (12′), we can say, intuitively, that: The Δ-function that is introduced here is a spatially-isotropic spherical-shell wave that is concentrated on an infinitely-thin shell $r = ct$ in the limit, and that first contracts with the speed of light in order to make $t = 0$ at the zero-point $r = 0$, and then expands again with the speed of light. Moreover, one has:

$$\Delta(-x, -y, -z, -t) = -\Delta(x, y, z, t). \tag{13}$$

It still remains for us to remark that the derivatives of the Δ-function are defined by the limit:

$$\int_{V_4} f\,\frac{\partial\Delta}{\partial x_i}\,dV_4 \equiv \lim_{N\to\infty}\int_{V_4} f\,\frac{\partial\Delta}{\partial x_i}\,dV_4 = \lim_{N\to\infty}\left(-\int \frac{\partial f}{\partial x_i}\,\Delta_N\,dV_4\right)$$
$$= \int_{V_3^+}\left(-\frac{\partial f}{\partial x_i}\right)\frac{dx\,dy\,dz}{r} - \int_{V_3^-}\left(-\frac{\partial f}{\partial x_i}\right)\frac{dx\,dy\,dz}{r},$$

in which it has been assumed that f vanishes on the boundary of the integration region. The higher partial derivatives are defined analogously. It should be remarked that in the sense of this definition, one has:

$$\sum_{a=1}^{4}\frac{\partial^2\Delta}{\partial x_a^2} \equiv 0. \tag{14}$$

§ 3. CR for the electromagnetic field strengths, when considered to be *q*-functions, while eliminating the Fourier decomposition. – We would now like to attempt to characterize the values of the commutation of any components of the electromagnetic field strengths at two different space-time points while maintaining relativistic invariance, without explicitly calling upon the Fourier decomposition in the final result. We shall then deal with ascertaining the expressions:

$$\mathfrak{E}_i(P)\,\mathfrak{E}_k(P') - \mathfrak{E}_k(P)\,\mathfrak{E}_i(P'), \qquad \mathfrak{H}_i(P)\,\mathfrak{H}_k(P') - \mathfrak{H}_k(P)\,\mathfrak{H}_i(P'),$$
$$\mathfrak{E}_i(P)\,\mathfrak{H}_k(P') - \mathfrak{E}_k(P)\,\mathfrak{H}_i(P'),$$

in which P and P' shall be abbreviations for the four coordinates x, y, z, t of P and those x', y', z', t' of P', and in which $i, k = 1, 2, 3$ are indices that characterize the components along the x, y, and z directions, resp. We will also employ the square bracket symbols:

$$[\mathfrak{E}_i(P), \mathfrak{E}_k(P')], \qquad [\mathfrak{H}_i(P), \mathfrak{H}_k(P')], \qquad [\mathfrak{E}_i(P), \mathfrak{H}_k(P')]$$

for the given expressions.

We would like to start our calculation with the expressions (3) for the field strengths:

$$\mathfrak{E}_s = \sqrt{\frac{\nu_s}{L^3}}\{(\mathfrak{e}_\xi\,a_s^{(1)} + \mathfrak{e}_\eta\,a_s^{(2)})\cos 2\pi[(\mathfrak{k}_s\,\mathfrak{r}) - |\,\mathfrak{k}_s\,|\,ct]$$
$$+\ (\mathfrak{e}_\xi\,b_s^{(1)} + \mathfrak{e}_\eta\,b_s^{(2)})\sin 2\pi[(\mathfrak{k}_s\,\mathfrak{r}) - |\,\mathfrak{k}_s\,|\,ct]\},$$

$$\mathfrak{H}_s = \sqrt{\frac{\nu_s}{L^3}}\{(\mathfrak{e}_\eta\,a_s^{(1)} - \mathfrak{e}_\xi\,a_s^{(2)})\cos 2\pi[(\mathfrak{k}_s\,\mathfrak{r}) - |\,\mathfrak{k}_s\,|\,ct]$$
$$+\ (\mathfrak{e}_\eta\,b_s^{(1)} - \mathfrak{e}_\xi\,b_s^{(2)})\sin 2\pi[(\mathfrak{k}_s\,\mathfrak{r}) - |\,\mathfrak{k}_s\,|\,ct]\}.$$

Equations (I) are true individually for $a_s^{(1)}$, $b_s^{(1)}$ and $a_s^{(2)}$, $b_s^{(2)}$, while $a_s^{(1)}$ commutes with $b_s^{(2)}$ and $a_s^{(2)}$ with $b_s^{(2)}$, according to (I′).

We must employ relations of the form:

$$(\mathfrak{e}\xi)_i\,(\mathfrak{e}\xi)_k + (\mathfrak{e}\eta)_i\,(\mathfrak{e}\eta)_k = \delta_{ik} - (\mathfrak{e}\zeta)_i\,(\mathfrak{e}\zeta)_k\,,$$

$$(i, k = x, y, z;\quad \delta_{ik} = 0 \text{ for } i \neq k,\ 1 \text{ for } i = k),$$

$$(\mathfrak{e}\xi)_i\,(\mathfrak{e}\eta)_k - (\mathfrak{e}\eta)_i\,(\mathfrak{e}\xi)_k = (\mathfrak{e}\zeta)_l = \frac{(\mathfrak{k}_s)_l}{|\,\mathfrak{k}_s\,|}$$

$$(i, k, l \text{ is an even permutation of } 1, 2, 3),$$

in which one consider the ζ-axis to be parallel to ($\mathfrak{k}_s$), moreover. For a given meaning of the indices, if we then set:

$$\left.\begin{aligned} \alpha_{ik} &= \alpha_{ki} &&= |\,\mathfrak{k}_s\,|^2\,\delta_{ik} - (\mathfrak{k}_s)_i(\mathfrak{k}_s)_k\,, \\ \beta_{ik} &= -\beta_{ki} &&= |\,\mathfrak{k}_s\,|\cdot(\mathfrak{k}_s)_l \qquad (\beta_{ik} = 0 \text{ for } i = k), \end{aligned}\right\} \tag{15}$$

and further:

$$(P_s) = 2\pi\,[(\mathfrak{k}_s\,\mathfrak{r}) - |\,\mathfrak{k}_s\,|\,ct], \qquad (P'_s) = 2\pi\,[\,(\mathfrak{k}'_s)\ - |\,\mathfrak{k}_s\,|\,ct'\,],$$

then we will get from I) that:

$$[\mathfrak{E}_i\,(P), \mathfrak{E}_k\,(P')] = [\mathfrak{H}_i\,(P), \mathfrak{H}_k\,(P')] = ih\,c\,\frac{1}{L^3}\sum_s \overline{|\,\mathfrak{k}_s\,|}\,\alpha_{ik}^{(s)}[\cos(P_s)\sin(P'_s) - \sin(P_s)\cos(P'_s)]$$

$$= ih\,c\,\frac{1}{L^3}\sum_s \overline{|\,\mathfrak{k}_s\,|}\,\alpha_{ik}^{(s)}\sin(P'_s - P_s)\,,$$

and likewise:

$$[\mathfrak{E}_i\,(P), \mathfrak{H}_k\,(P')] = -\,[\mathfrak{H}_i\,(P), \mathfrak{E}_k\,(P')] = ih\,c\,\frac{1}{L^3}\sum_s \frac{2}{|\,\mathfrak{k}_s\,|}\,\beta_{ik}^{(s)}\sin(P'_s - P_s)$$

[so, in particular, $\mathfrak{E}_i\,(P)$ commutes with $\mathfrak{H}_i\,(P')$].

We now replace $\frac{1}{L^3}\sum_s(\cdots)$ with $\int(\cdots)\,dk_x\,dk_y\,dk_z$, according to (2), but first we would like to integrate over a sphere of radius N in $\mathfrak{k}$-space and only then pass to the limit $N \to \infty$. Furthermore, we employ the fact that when we take the second derivative of $\sin(P'_s - P_s)$ with respect to the spatial coordinates x_i and x_k of P or P', the factor

$-4\pi^2 \mathfrak{k}_i\,\mathfrak{k}_k$ will appear before sin (…), or in the case of the derivatives with respect to x_l and ct, the factor + $4\pi^2 \mid \mathfrak{k}_s \mid \mathfrak{k}_l$. In that way, the factors $\alpha_{ik}^{(s)}$ and $\beta_{ik}^{(s)}$ can be replaced with suitable combinations of such second derivatives and one will get:

$$[\mathfrak{E}_i (P), \mathfrak{E}_k (P')] = [\mathfrak{H}_i (P), \mathfrak{H}_k (P')]$$
$$= \frac{ihc}{8\pi^2}\iiint \frac{2}{\mid\mathfrak{k}\mid}\left(\frac{\partial^2}{\partial x_l\,\partial x_k} - \delta_{ik}\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) \sin (P' - P)_k\, dk_x\, dk_y\, dk_z \,,$$

$$[\mathfrak{E}_i (P), \mathfrak{H}_k (P')] = - [\mathfrak{H}_i (P), \mathfrak{E}_k (P')]$$
$$= \frac{ihc}{8\pi^2}\iiint \frac{2}{\mid\mathfrak{k}\mid}\frac{\partial^2}{c\,\partial t\,\partial x_l} \sin (P' - P)_k\, dk_x\, dk_y\, dk_z \,.$$

Differentiation and integration can be exchanged in this, and the integral in front of the differentiation will give precisely the function Δ_N that appears in (12) when the arguments $x' - x, \ldots, t' - t$ have been replaced. If we denote $\Delta (x' - x, \ldots, t' - t)$ by $\Delta (P' - P)$ and pass to the limit of $N \to \infty$ then we will ultimately come to:

$$\left.\begin{aligned} &[\mathfrak{E}_i(P), \mathfrak{E}_k(P')] = [\mathfrak{H}_i(P), \mathfrak{H}_k(P')] \\ &\qquad\qquad = \frac{ihc}{8\pi^2}\left(\frac{\partial^2}{\partial x_i\,\partial x_k} - \delta_{ij}\frac{\partial^2}{c^2\,\partial t^2}\right)\Delta(P' - P), \\ &[\mathfrak{E}_i(P), \mathfrak{H}_k(P')] = -[\mathfrak{H}_i(P), \mathfrak{E}_k(P')] = \frac{ihc}{8\pi^2}\frac{\partial^2}{c\,\partial t\,\partial x_l}\Delta(P' - P). \end{aligned}\right\} \qquad \text{(III)}$$

(i, k = 1, 2, 3; in the second equation, the right-hand side will be zero for $i = k$ and i, k, l will be an even permutation of 1, 2, 3, for $i \neq k$.)

One should further remember that according to (13), one has:

$$\Delta (P' - P) = - \Delta (P' - P). \qquad (13')$$

With:

$$(F_{41}, F_{42}, F_{43}) = i\,\mathfrak{E}, \quad (F_{23}, F_{31}, F_{12}) = \mathfrak{H},$$
$$(x_1, x_2, x_3, x_4) = (x, y, z, ict),$$

(III) can be summarized in the four-dimensional invariant form:

$$[F_{ik} (P), F_{lm} (P')] = \frac{ihc}{8\pi^2}\Delta_{ik,lm} (P' - P), \qquad \text{(III')}$$

in which $\Delta_{ik,lm}$ is an abbreviation for:

$$\Delta_{ik,lm} = \left(\delta_{kl}\frac{\partial^2}{\partial x_i\,\partial x_m} - \delta_{il}\frac{\partial^2}{\partial x_k\,\partial x_m} + \delta_{im}\frac{\partial^2}{\partial x_k\,\partial x_l} - \delta_{km}\frac{\partial^2}{\partial x_i\,\partial x_l}\right)\Delta\,. \tag{16}$$

When one compares (III) and (III′), one will call into play the property that:

$$\sum_\alpha \frac{\partial^2}{\partial x_\alpha^2}\Delta = 0,$$

which was expressed in equation (14) of the previous paragraph.

§ 4. Simple consequences of the CR for the field strengths. On the relationship between quantum electrodynamics and Maxwell's equations. – The *q*-functions that represent the field strengths in the form of quantum electrodynamics that is founded here are not *arbitrary* functions of space and time, but ones that satisfy **Maxwell**'s vacuum field equations:

$$\left.\begin{aligned} \frac{\partial F_{ik}}{\partial x_j} + \frac{\partial F_{kj}}{\partial x_l} + \frac{\partial F_{ji}}{\partial x_k} &= 0, \\ \sum_\alpha \frac{\partial F_{i\alpha}}{\partial x_\alpha} &= 0. \end{aligned}\right\} \tag{IV}$$

This is already included in our starting point, namely, the decomposition of fields into transverse partial waves that propagate with the speed of light. The charge and current densities are assumed to vanish everywhere in this. That is based upon the assumption that the consideration of that special case is an abstraction that is compatible with the laws of quantum electrodynamics. However, if one accepts that assumption then one can say that the classical field equations (IV) will also enter into quantum electrodynamics explicitly, and indeed as *auxiliary conditions that are imposed upon the q-functions of the field strengths.*

In order for the CR (III) to be compatible with the field equations (IV), the left-hand side of equations (16) must commute with any of the field strength components F_{lm} by means of (III′). Therefore, since that is, in fact, the case, that will already guarantee us the derivation of the CR (III′) from those of the Fourier decomposition of the field. However, one can easily confirm the CR by direct calculation. The consideration that arises from the second of equations (IV) is especially simple, since the operation $\sum_k \frac{\partial}{\partial x_k}$ will give:

$$\left(-\delta_{il}\frac{\partial}{\partial x_m} + \delta_{im}\frac{\partial}{\partial x_l}\right)\sum_\alpha \frac{\partial^2\Delta}{\partial x_\alpha^2}$$

for arbitrary fixed *l*, *m* when it is applied to the right-hand side of (III), and that will be identically zero, from (14). The calculation that relates to the first equations in (IV) will proceed in an analogous, but somewhat lengthier, way. One will reach the goal more rapidly when one introduces the tensor F_{ik}^* that is dual to F_{ik}, whose components are given by:

$$(F_{23}^*, F_{31}^*, F_{12}^*) = -i\,\mathfrak{E}, \qquad (F_{41}^*, F_{42}^*, F_{43}^*) = -\mathfrak{H},$$

and with their help, the first of equations (IV) are known to also be capable of being written:

$$\sum_\alpha \frac{\partial F_{i\alpha}^*}{\partial x_\alpha} = 0. \tag{IV′}$$

Now, the transition to the dual tensor means that one replaces $i\,\mathfrak{E}$ with $-\mathfrak{H}$, and therefore, $\mathfrak{E}$ with $i\,\mathfrak{H}$, and $\mathfrak{H}$ with $-i\,\mathfrak{E}$. As one sees immediately from (III), the values of all bracket expressions will simply change sign in that way. Hence, one also has:

$$[F_{ik}^*(P), F_{lm}^*(P')] = -[F_{ik}(P), F_{lm}(P')] = -\frac{ihc}{8\pi^2}\Delta_{ik,\,lm}(P'-P), \tag{III′}$$

from which, the commutability of (IV′) with F_{lm} likewise follows, just as the commutability of $\sum_\alpha \frac{\partial F_{i\alpha}}{\partial x_\alpha}$ with F_{lm} would follow from (III′).

Moreover, since it is easy to verify that one has $[F_{ik}(P), F_{lm}^*(P')] = -[F_{lm}^*(P), F_{ik}(P')] = 0$, on the basis of (III), along with (17), it will follow further in conjunction with (III″) that the tensors satisfy:

$$E_{ik} = F_{ik} + F_{ik}^*, \qquad E_{ik}^* = F_{ik} - F_{ik}^*, \tag{18a}$$

$$[E_{ik}(P), E_{lm}(P')] = 0, \qquad [E_{ik}^*(P), E_{lm}^*(P')] = 0, \tag{18b}$$

$$[E_{ik}(P), E_{lm}^*(P')] = 2\,[F_{ik}(P), F_{lm}(P')] + 2\,[F_{ik}^*(P), F_{lm}(P')],$$

$$[E_{ik}^*(P), E_{lm}(P')] = 2\,[F_{ik}(P), F_{lm}(P')] - 2\,[F_{ik}^*(P), F_{lm}(P')].$$

For that reason, the relations (18b) are especially remarkable, since they mean that one is allowed to substitute ordinary functions (viz., "c-functions") for the q-functions $E_{ik}(P)$ alone [or for the functions $E_{ik}^*(P)$ alone] in special applications, since their values at different space-time points will always commute. One will also obtain functions with similar properties when one reflects the field strengths $F_{ik}(P)$ relative to an arbitrarily-chosen zero-point:

$$F_{ik}^+(P) = \tfrac{1}{2}[F_{ik}(P) + F_{ik}(-P)], \qquad F_{ik}^-(P) = \tfrac{1}{2}[F_{ik}(P) - F_{ik}(-P)],$$

such that one has:

$$F_{ik}^{+}(P) = F_{ik}^{+}(-P), \qquad F_{ik}^{-}(P) = -F_{ik}^{-}(-P).$$

One easily gets:

$$[F_{ik}^{+}(P), F_{lm}^{+}(P')]$$
$$= \frac{ihc}{8\pi^2}\cdot\frac{1}{4}[\Delta_{iklm}(P'-P) + \Delta_{iklm}(P'+P) + \Delta_{iklm}(-P'-P) + \Delta_{iklm}(-P'+P)].$$

Since one has the symmetry properties for Δ_{iklm} :

$$\Delta_{iklm}(P'-P) = -\Delta_{iklm}(-P'+P),$$
$$\Delta_{iklm}(P'+P) = -\Delta_{iklm}(-P'-P),$$

which are analogous to (13′), the middle two terms, as well as the first and last term in the bracket will cancel each other, and the right-hand side will vanish. One will also find an analogous result for $[F_{ik}^{-}(P), F_{lm}^{-}(P')]$, such that one has:

$$[F_{ik}^{+}(P), F_{lm}^{+}(P')] = [F_{ik}^{-}(P), F_{lm}^{-}(P')] = 0. \tag{19a}$$

By contrast, it easily follows in the same way that:

$$[F_{ik}^{+}(P), F_{lm}^{-}(P')] = \frac{ihc}{16\pi^2}[\Delta_{iklm}(P'-P) + \Delta_{iklm}(P'+P)], \tag{19b}$$

$$[F_{ik}^{-}(P), F_{lm}^{-}(P')] = \frac{ihc}{16\pi^2}[\Delta_{iklm}(P'-P) - \Delta_{iklm}(P'+P)]. \tag{19c}$$

The commutation of the left-hand sides of the **Maxwell** equations with all field strength components can also be formulated by applying the latter equations: *When the* $F_{ik}^{+}(P)$ *are replaced with the right-hand side of* (19b) *for fixed l, m, and P′, they will be solutions of the* ***Maxwell*** *equations* (IV), *and the same thing will also be true when the right-hand side of* (19b) *is substituted for* $F_{lm}^{-}(P')$ *for fixed i, k, and P*. Due to the use of the Δ-function, it is more rigorous to always speak of singular limiting cases of the solutions of the **Maxwell** equations, instead of those solutions.

We will employ the latter property of the relations (19) later on. Here, let us merely remark that no simply-formulated, relativistically-invariant CR exists for the four-potential in which only the Δ-function and its derivatives.

II. Method of functionals and functional operators

§ 1. One-dimensional continuum, treated non-relativistically. – We consider longitudinal standing oscillations in a one-dimensional continuum with the boundary conditions:

$$q(x) = 0 \qquad \text{for} \qquad x = 0 \quad \text{and} \qquad x = l.$$

We can then set:

$$\left.\begin{aligned} &q(x) = \frac{1}{\sqrt{l}} \sum_{s=0}^{\infty} q_s \sin 2\pi k_s x, \quad k_s = s\frac{x}{2l}, \quad s \text{ integer,} \\ &\text{and analogously for the "impulse":} \\ &p(x) = \frac{1}{\sqrt{l}} \sum_{s=0}^{\infty} p_s \sin 2\pi k_s x. \end{aligned}\right\} \tag{20}$$

Let the classical equations of motion be:

$$\dot{p}_s = -2\pi\, \nu_s\, q_s\,, \qquad \dot{q}_s = 2\pi\, \nu_s\, p_s\,, \qquad \nu_s = c\, k_s\,,$$

and let the total energy be:

$$E = \sum_s \tfrac{1}{2}\{p_s^2 + (2\pi\nu_s)^2 q_s^2\} = \frac{1}{2}\int \left[p^2(x) + c^2 \left(\frac{\partial q}{\partial x}\right)^2 \right] dx\,. \tag{21}$$

Quantum-mechanically, the CR:

$$p_s\, q_{s'} - q_{s'}\, p_s = \begin{cases} \dfrac{h}{2\pi i} & \text{when} \quad s = s', \\ 0 & \text{when} \quad s \neq s', \end{cases} \tag{22}$$

are combined with (18). As would follow from a simple calculation (*), these are equivalent to:

$$p\,(x)\, q\,(x') - q\,(x')\, p\,(x) = \frac{h}{2\pi i}\, \delta(x - x')\ [x, x' \text{ in } (0, l)], \tag{23}$$

in which δ means the **Dirac** function (cf., I, § **2**).

Now, as is known, when one introduces a Schrödinger function:

$$\psi(q_1, \ldots, q_s, \ldots),$$

the infinitude of variables $q_1, \ldots, q_s, \ldots$ can also be interpreted as an operator equation when one replaces:

q_s	with the operator:	multiplication by q_s ,
p_s	″ ″ ″	differentiation $\dfrac{h}{2\pi i}\dfrac{\partial}{\partial q_s}$.

This relates to the identity:

(*) Cf., e.g., **P. Jordan** and **O. Klein**, Zeit. Phys. **45** (1927), 751.

$$\frac{\partial}{\partial a_s}(a_s\,\psi) - a_s\frac{\partial\psi}{\partial a_s} = \psi .$$

The law of energy then leads from (21) to the differential equation:

$$\frac{1}{2}\sum_s\left(-\frac{h^2}{4\pi^2}\right)\frac{\partial^2\psi}{\partial a_s^2}+\left(\sum_s\frac{1}{2}(2\pi\nu_s)^2 a_s^2\right)\psi = E\,\psi. \tag{24}$$

The solution of this equation does not generally converge for infinitely-many variables, which is connected with the finite zero-point energy of $h\nu_s / 2$ per eigen-oscillation. This still-completely-unsolved difficulty was discussed thoroughly in the introduction.

Except for that, however, we are forced to pose the following question: What is the analogue of the operator representations of equations (22) and (24) when we start from the function $q\,(x)$, and therefore, a continuum of independently-varying variables, instead of a countable infinitude of variables $q_1, \ldots, q_s, \ldots$? The answer to that can be given with the help of **Volterra**'s functional mathematics. A functional:

$$\Psi\{q\,(x)\}$$

is the association of a *number* to a function $q\,(x)$. It is said to be *differentiable* at the point P when the following limiting value always exists independently of the particular way that is arrived at: One defines a varied function $q\,(x) + \bar{q}(x)$, and lets the interval in which $\bar{q}(x)$ is non-zero contract to the point $x_0 = P$, while at the same time $\int\bar{q}(x)\,dx$ also converges to zero. One then has:

$$\Psi_{q(x);\,P} = \lim \frac{\Psi\{q(x)+\bar{q}(x)\}-\Psi\{q(x)\}}{\int\bar{q}(x)\,dx}.$$

One can also write this as:

$$\Psi_{q(x);\,P} = \lim_{\substack{\bar{q}(x,p')\to\delta(x_{P'}-x_P)\\ \alpha\to 0}}\frac{1}{\alpha}[\Psi\{q(x)+\alpha\,\bar{q}(x)\}-\Psi\{q(x)\}], \tag{25}$$

with the help of the δ-function. The ordinary rule for the differentiation of sums and product remains true. The second derivative is defined analogously by:

$$\Psi_{q(x),q(x);PP_1} = \lim_{\substack{\alpha\to 0\\ \bar{q}(x)\to\delta(x-x_{P_1})}}\frac{1}{\alpha}[\Psi_{q(x);P}\{q(x)+\alpha\,\bar{q}(x)\}-\Psi_{q(x);P}\{q(x)\}]. \tag{25a}$$

A special case of this is the second derivative for $P_1 = P$, which we will denote by the index $q(x)$, $q(x)$; PP.

We shall now look for functional operators for $p\,(x)$ and $q\,(x)$, that is, ones that associate new functionals $\bar{\Psi}$ and $\bar{\bar{\Psi}}$ with Ψ. They can be described by the formulas:

$$\left(\int_J \underline{p(x)}\, dx\right) \cdot \Psi\{q(x)\} \to \bar{\Psi}_J \{q(x)\},$$

$$\left(\int_J \underline{q(x)}\, dx\right) \cdot \Psi\{q(x)\} \to \bar{\bar{\Psi}}_J \{q(x)\},$$

in which one must integrate over an arbitrarily-given interval J of x on the left, and the dependency of the functionals $\bar{\Psi}$ and $\bar{\bar{\Psi}}$ on that interval is expressed by adding an index J to the expression. Now, that association must be chosen in such a way that the relation (23) will be fulfilled when it is regarded as an operator equation. It is clear that:

$$\left.\begin{aligned} \left(\int_{x_1}^{x_2} \underline{p(x)\, dx}\right) \cdot \Psi\{q(x)\} &= \frac{h}{2\pi i} \int_{x_1}^{x_2} \Psi_{q(x);P}\, dx_P, \\ \left(\int_{x_1}^{x_2} \underline{q(x)\, dx}\right) \cdot \Psi\{q(x)\} &= \Psi\{q(x)\} \cdot \int_{x_1}^{x_2} q(x)\, dx \end{aligned}\right\} \tag{26}$$

satisfy that condition.

The law of energy (21) further yields the functional integro-differential equation:

$$\int (-)\left(\frac{h}{4\pi}\right)^2 \Psi_{q(x),q(x);PP}\, dx_P + c^2 \left[\int \left(\frac{\partial q}{\partial x}\right)^2 dx\right] \cdot \Psi = E\,\Psi. \tag{27}$$

In order to exhibit the analogue of the orthogonality condition, one needs the definition of:

$$\int \psi_E\, \psi_{E'}\, \delta\Omega$$

in function space. A closely-related definition might be to divide the line segment $(0, l)$ into N intervals and consider the step-polygons $q(x)$ that might have the constant values q_1 to q_N in the individual intervals. One then passes to the limit $N \to \infty$ to:

$$\int \psi_E\, \psi_{E'}\, \delta\Omega = \lim_{N\to\infty} \int \psi_E(q_1, \ldots, q_N)\, \psi_{E'}(q_1, \ldots, q_N)\, dq_1 \ldots dq_N = \delta(E - E').$$

Nevertheless, the aforementioned convergence difficulty is a hindrance here, for the moment.

§ 2. Relativistically-invariant functional treatment of the case of two canonically-conjugate scalar *q*-functions that satisfy the wave equation. – As a preparation for the problem of vacuum electrodynamics, the following simpler problem shall be treated to begin with. Two scalar state-quantities f and g both satisfy the (four-dimensional) wave equation:

$$\sum_{\alpha=1}^{4} \frac{\partial^2 f}{\partial x_\alpha^2} = 0, \quad \sum_{\alpha=1}^{4} \frac{\partial^2 g}{\partial x_\alpha^2} = 0. \tag{28}$$

Furthermore, when they are regarded as “*q*-functions” of *x*, *y*, *z*, *t*, the CR:

$$f(P)\, q(P') - g(P')\, f(P) = ih\, \Delta(P - P') \tag{29}$$

shall be true for them, in which Δ is the function that was defined in I, § **2**, while the values of *f* at different points commute with each other, and likewise for the values of *g*. One asks how this CR can be interpreted as a relation between functional operators, analogous to the introduction of the operators (26) into (23).

As a result of the fact that *g* (*P*) commutes with *g* (*P*′), one is allowed to consider the functionals:

$$\Psi\, \{g(x_1, \ldots, x_4)\}$$

in which the values of $g(x_1, \ldots, x_4)$ are now ordinary numbers. However, it is essential that $g(x_1, \ldots, x_4)$ can no longer be arbitrary function of $x_1, \ldots, x_4$, but only one that satisfies the wave equation. We must also remain inside of the domain of these special functions when we vary $g(x_i)$. In particular, it is no longer possible then to choose the variation of $g(x_1, \ldots, x_4)$ in such a way that it is non-zero only in the neighborhood of a world-point. The fact that the argument of the function Ψ is subject to the wave equation, or more generally, a linear partial differential equation, as an auxiliary condition, then makes an alteration of **Volterra**’s concept of functional derivation necessary.

Meanwhile, such a thing will emerge automatically when we simply replace the ordinary δ-function in the notation (25) for the **Volterra** derivative with the spherical shell wave Δ-function of I, § **2** when we recall that, from I, § **2**, equation (13), it is a solution of (28).

We shall now define a functional derivative by:

$$\Psi_{g(x_i);P} = \lim_{\substack{\alpha \to 0 \\ \bar{g}[x_i(P')] \to \Delta(P-P')}} \frac{1}{\alpha}[\Psi\{g(x) + \alpha\, \bar{g}(x)\} - \Psi\{g(x)\}] \tag{30}$$

then. Since the rules of differentiation of sums and products still persist for this derivative, as well, it is further immediately clear that (29) will be satisfied by the following operator Ansatz, which is completely analogous to (26):

$$\underbrace{\left[\int g(x_i)\, dx_1 \cdots dx_4\right]}_{\text{operator}} \cdot \Psi\, \{g(x)\} = ih \int_{V_4} \Psi_{g(x_i);P}\, dx_1 \ldots dx_4\, ; \tag{31}$$

likewise, *f* (*x*), as an operator, means simple multiplication by *f* (*x*).

The question that was posed in this paragraph is answered completely in that way, and we can now direct our attention to our actual goal, namely, the functional equation of the quantum electrodynamics of light.

§ 3. Representation of the CR of vacuum electrodynamics as relativistically-invariant relations between functional operators. – When we introduce the Fourier amplitudes $b_1, \ldots, b_s, \ldots$ that are defined by I, equation (3) as independent variables, the functional representation will be clear, and the q-number a_s will correspond to the operator $ih\,\partial / \partial a_s$. The difficulty with convergence that was mentioned in II, § **1** will also assert itself here, and can also appear as problematic to a large extent in what follows.

However, above and beyond that, a difficulty will also arise when we would not like to employ an explicit Fourier decomposition in our functional representation. As we have mentioned already in the previous paragraph, in fact, the only physical field quantities that can be employed as the argument of a functional are the ones that commute with each other at all space-time points when they are regarded as q-functions. By the assumption of the CR (III) for the field strengths F_{ik} that was formulated in Part I, they cannot come under consideration as arguments of a functional, but rather, according to I, § **4**, equation (18b) or (19a), only one of the four systems of quantities $F_{ik}^{+}(P)$, $F_{ik}^{-}(P)$, $E_{ik}(P)$, $E_{ik}^{*}(P)$ that were defined there. Appealing to these functions, and especially the reflected quantities $F_{ik}^{+}(P)$ and $F_{ik}^{-}(P)$ relative to a fixed point, seems to be very artificial, but it is nonetheless impossible for us to avoid it.

The following consideration is carried out for functionals of the skew-symmetric part $F_{lm}^{-}(x_1, \ldots, x_4)$ of the **Maxwell** equations [Part I, equation (IV)] relative to a fixed zero-point, which we can also denote by:

$$\Psi\{\, F_{lm}^{-}(x_1, \ldots, x_4)\,\}.$$

Naturally, one can also switch the roles of F_{ik}^{-} and F_{ik}^{+} in all of the following arguments. Analogous arguments will also be true when one introduces E_{ik} or E_{ik}^{*} as arguments of the functionals.

The problem is entirely similar to the one in the previous paragraph, except that here several (viz., six) functions will appear *simultaneously* as the argument of the functional that will be independent of each other, due to equations (IV), namely, make **Maxwell**'s equations. Hence, one cannot differentiate each of the six components of the field strengths *individually* now, since *one* field-strength component cannot be varied without varying the other ones. Once more, it is the Δ-function that serves to remedy that, and this time with its second derivatives. If we introduce the expression $\Delta_{ik,lm}$ that was defined in Part I, equation (16) then, from I, § **4**, for each index-pair (i, k):

$$F_{lm}^{-}(P') = \Delta_{ik,lm}(P' - P) + \Delta_{ik,lm}(P' + P) = \Delta_{ik,lm}^{-}(P', P) \tag{32}$$

will be *an allowable variation* of the F_{lm}^{-} for a fixed P, since it satisfies **Maxwell**'s equations and also fulfills the symmetry condition [viz., a change of sign when one goes from (P') to ($-P'$)]. We can then define the *following six derivatives* of our functional

$\Psi\{F_{lm}^{-}(P')\}$, *which are characterized by* (i, k) *and skew-symmetry in that index-pair,* in analogy with (30):

$$\Psi'_{ik;P}\{F_{lm}^{-}(P')\} = \lim_{\substack{\alpha\to 0 \\ \delta F_{lm}^{-}(P')\to\Delta_{ik,lm}^{-}(P',P)}} \frac{1}{\alpha}[\Psi\{F_{lm}^{-}(P')+\alpha(\delta F_{lm}^{-})(P')\}-\Psi\{F_{lm}^{-}(P')\}]. \tag{33}$$

It is then also immediately clear that the CR (III) is fulfilled, when it is regarded as an operator equation, when the operator that belongs to $F_{ik}^{+}(P)$ is defined according to:

$$\left(\int_J\cdots\int \underline{F_{ik}^{+}(P)}\,dV_P\right)\cdot\Psi\{F_{lm}^{-}(P')\} = \frac{ihc}{16\pi^3}\int_J\int \Psi'_{ik;P}\{F_{lm}^{-}(P')\}dV_P\,, \tag{34}$$

in which dV_P refers to the volume element of the four-dimensional space of coordinates $x_1, \ldots, x_4$ of P in this, and J refers to an arbitrary, finite, four-dimensional interval, while:

$$\int_J\int F_{lm}^{-}(P)\,dV_P$$

means simple multiplication by that quantity, as an operator.

Here, some brief arguments should be mentioned that are analogous to the ones that led to the exhibition of equation (25) in II, § **1**. At first, it is probably clear how one might define the *second* derivatives of our functional Ψ; we write the most general second derivative as:

$$\Psi'_{ik,rs;PP_1}\,,$$

yet in what follows only the special case $P_1 = P$ will be needed. It is essential to consider that the energy integral must be regarded as equivalent to the impulse integral in a relativistically-invariant theory, such that we will obtain four simultaneous second-order partial functional differential equations from the "eigen"-functional Ψ_{J_k} that depends upon the four total energy-impulse components $J_4 = -E$; $(J_1, J_2, J_3) = ic\,\mathfrak{G}$. It is known that J_k is expressed in terms of the field strengths in classical electrodynamics as follows:

$$J_k = \int_{t=\text{const.}}\left[\sum_{r=1}^{4}F_{kr}\,F_{4r} - \delta_{k4}\cdot\sum_{(rs)}\tfrac{1}{2}(F_{rs})^2\right]dx\,dy\,dz.$$

We can choose the section t = const. to be $t = 0$, in particular; i.e., the one that goes through the zero-point that we employed for the splitting of the field strengths into F_{ik}^{+} and F_{ik}^{-}. Each of the four integrals J_k then splits into two parts that depend upon the F_{ik}^{+} (F_{ik}^{-}, resp.) alone, since the integrals over the mixed terms vanish on the grounds of symmetry. We then get the four simultaneous (corresponding to k = 1 to 4) equations:

$$\left(\frac{ihc}{16\pi^2}\right)^2 \int\int\limits_{\substack{-\infty \\ t=0}}^{+\infty}\int \left[\sum_{r=1}^{4} \Psi''_{kr;4r;PP} - \delta_{k4} \sum_{(rs)} \tfrac{1}{2} \Psi''_{rs;rs;PP}\right] dx_P\, dy_P\, dz_P$$

$$+ \Psi \iiint \left[\sum_{r=1}^{4} F_{kr}^{-} F_{4r}^{-} - \delta_{k4} \tfrac{1}{2} \sum_{rs} (F_{rs}^{-})^2\right] dx_P\, dy_P\, dz_P = J_k\, \Psi, \tag{35}$$

which are formed analogously to (27), and in which Ψ is a function that depends upon the $F_{rs}^{-}(x_k)$, and in turn, the J_k as parameters. These equations play a role for a "closed" radiation field that is analogous to that of the **Schrödinger** differential equation for a certain quantum state of a closed mechanical system.

As we have mentioned already in the Introduction, the equations that were cited in the last paragraphs of this Part II, for which, direct integration methods are still not available, are regarded as provisional to a greater degree than the arguments about *q*-functions that were developed in Part I. However, we regard the introduction of functionals into a consistent quantum-theoretic reinterpretation of the classical field physics to be natural, despite many unsolved problems in regard to the implementation in special cases.