"Le théorème des tourbillons en Thermodynamique," J. math. pures et appl. **7** (1901), 235-257.

The vortex theorem in thermodynamics

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1. Consider a continuous fluid mass that is animated by a motion that does not alter its continuity. Let u, v, w denote the components of the velocity of the molecule along the three axes; they are continuous functions of time t and the coordinates x, y, z of a geometric point (viz., Euler variables). The molecules that are situated on a closed curve C_0 at the time t_0 form a closed curve C at any instant. One can state Helmholtz's theorem by saying that the curvilinear integral:

$$\int_C u\,dx + v\,dy + w\,dz$$

preserves the same value at any instant.

The proof of that theorem that is applicable to fluids that are devoid of viscosity is subordinate to the following restrictions, in addition:

a. The forces that act upon each volume element, whether internal or external, admit a potential.

b. The pressure is a function of only the density.

Duhem has shown ([1]) how thermodynamics permits one to study the motion of fluids for which these restrictions make no sense. We propose to study what happens to Helmholtz's theorem in his theory; we continue to suppose that the viscosity is zero.

One will easily perceive everything that is due to Duhem in the lecture of this note. Not only is his method imprinted upon it, but indeed some of its pages are devoted to reproducing his results without modification. We pray that one will excuse us for the fact that he has already done that himself. It seems to us that this reproduction is useful in bringing to light the inadequacy of the statements (*a*) and (*b*) in hydrodynamics and the possibility of replacing them with other ones that are more general.

([1]) "Le potential thermodynamique et la pression hydrostatique," Annales scientifiques de l'École Normale supériure (3) **10** (1893). *Traité d'Électricité et de Magnétisme*, t. II, 1892.

I.

2. In the fluids that are customarily studied in hydrodynamics, the physical and chemical state at a point is defined completely by the density ρ and the absolute temperature T. Here, we shall suppose that in order to make that definition, one must add a finite number of algebraic parameters to these two variables – one, for example, which we denote by λ – and a finite number of geometric parameters – one, for example, such as the vector M whose projections onto the three axes will be A, B, C. The variables ρ, λ, A, B, C, T will be supposed to be normal. Thermodynamics then will lead one to put the internal thermodynamic potential of a fluid mass into the form:

$$F = \int_E \varphi(\rho, \lambda, A, B, C, T)\, \rho\, d\tau + \Psi. \tag{1}$$

In that expression, $\varphi(\rho, \lambda, A, B, C, T)\, \rho\, d\tau$ is the internal thermodynamic potential of the volume element $d\tau$; the $\int$ sign represents a triple integral that is extended over all of the volume E of the fluid. Ψ is a complementary term that depends upon the relative position of the various fluid elements and the variables that fix the state of each of them *except for the temperature.*

The most natural hypothesis that one can make on Ψ is that it can be written in the form of a sextuple integral ([1]):

$$\Psi = \tfrac{1}{2}\iint \psi(\rho, \lambda, A, B, C, \rho', \lambda', A', B', C', x, y, z, x', y', z')\, \rho\rho'\, d\tau\, d\tau', \tag{2}$$

whose field is the six-dimensional set that is obtained by successively associating each point of E with all of the other points of the same domain. ρ, λ, A, B, C refer to the volume element $d\tau$ with the coordinates x, y, z; ρ', λ', A', B', C' refer to the element $d\tau'$ with the coordinates x', y', z'.

In truth, since a fluid is isotropic, φ depends upon A, B, C only through the intermediary of M. Similarly, A, B, C, A', B', C', x, y, z, x', y', z' do not enter into the function ψ in an arbitrary manner. That function depends upon only the magnitude of the vectors M and M', the angle that they make between them, the ones that they make with the line that joins the elements $d\tau$ and $d\tau'$, and the length of that line. However, these remarks will be of no interest in the context of what we have in mind.

We suppose that the integral Ψ can be calculated by two successive triple integrals, and we set:

$$V(x, y, z, t) = \int_E \psi\rho'\, d\tau'. \tag{3}$$

The point x, y, z can be found inside of E or outside of it. By hypothesis, the function V exists in both domains. We suppose, in addition, that it is continuous in each of these domains and that it will admit first-order partial derivatives with respect to x, y, z that are given by the rule for derivation under the $\int$ sign. These are the hypotheses that were

([1]) DUHEM, "Le potential thermodynamique et la pression hydrostatique."

made by Duhem in his memoir on the *thermodynamic potential and hydrostatic pressure* that we have already cited. In that memoir, one will find the analytical discussion of them, which we shall not duplicate. It permits one to write:

$$\frac{\partial V}{\partial x} = \int \frac{\partial \psi}{\partial x}\rho' d\tau' + \frac{\partial \rho}{\partial x}\int \frac{\partial \psi}{\partial \rho}\rho' d\tau' + \frac{\partial \lambda}{\partial x}\int \frac{\partial \psi}{\partial \lambda}\rho' d\tau' + \frac{\partial A}{\partial x}\int \frac{\partial \psi}{\partial A}\rho' d\tau' + \frac{\partial B}{\partial x}\int \frac{\partial \psi}{\partial B}\rho' d\tau' + \frac{\partial C}{\partial x}\int \frac{\partial \psi}{\partial C}\rho' d\tau',$$

and if we set:

$$(4)\qquad \begin{cases} X_i = -\int \frac{\partial \psi}{\partial x}\rho' d\tau', \quad Y_i = -\int \frac{\partial \psi}{\partial y}\rho' d\tau', \quad Z_i = -\int \frac{\partial \psi}{\partial z}\rho' d\tau', \\ a_i = -\int \frac{\partial \psi}{\partial A}\rho' d\tau', \quad b_i = -\int \frac{\partial \psi}{\partial B}\rho' d\tau', \quad c_i = -\int \frac{\partial \psi}{\partial C}\rho' d\tau', \\ I = -\int \frac{\partial \psi}{\partial \rho}\rho' d\tau', \\ L_i = -\int \frac{\partial \psi}{\partial \lambda}\rho' d\tau' \end{cases}$$

then the value of $\partial V / \partial x$ will become:

$$(5)\qquad \begin{cases} \frac{\partial V}{\partial x} = -X_i - I\frac{\partial \rho}{\partial x} - L_i\frac{\partial \lambda}{\partial x} - a_i\frac{\partial A}{\partial x} - b_i\frac{\partial B}{\partial x} - c_i\frac{\partial C}{\partial x}. \\ \text{One will likewise have :} \\ \frac{\partial V}{\partial y} = -Y_i - I\frac{\partial \rho}{\partial y} - L_i\frac{\partial \lambda}{\partial y} - a_i\frac{\partial A}{\partial y} - b_i\frac{\partial B}{\partial y} - c_i\frac{\partial C}{\partial y}. \\ \frac{\partial V}{\partial z} = -Z_i - I\frac{\partial \rho}{\partial z} - L_i\frac{\partial \lambda}{\partial z} - a_i\frac{\partial A}{\partial z} - b_i\frac{\partial B}{\partial z} - c_i\frac{\partial C}{\partial z}. \end{cases}$$

The actions that the rest of the masses exert upon the element $d\tau$ are composed of a force $\rho\, d\tau \left(\bar{X}_i + \bar{Y}_i + \bar{Z}_i\right)$ and the influences $\rho\, d\tau\, I$, $\rho\, d\tau\, L_i$, $\rho\, d\tau\, a_i$, $\rho\, d\tau\, b_i$, $\rho\, d\tau\, c_i$. Equations (5) show that the force is not derived from a potential; that is why the restriction (*a*) of paragraph 1 will not be stated here.

One remarks that the existence of the influences $\rho\, d\tau\, a_i$, $\rho\, d\tau\, b_i$, $\rho\, d\tau\, c_i$ supposes that a couple acts upon the element $d\tau$: Indeed, the virtual work that it does is non-zero when the element $d\tau$ turns around itself. One knows that similar couples are encountered in the study of magnetic bodies, and if one adopts the ideas of Helmholtz, in that of the interaction between two electric currents.

3. The actions that bodies that are outside of the fluid exert upon it are, in one case, the pressure $P\, d\omega$ that is applied to each element $d\omega$ of the exterior surface, and in the

other, the forces $\rho\left(\overline{X}_i+\overline{Y}_i+\overline{Z}_i\right)\,d\tau$ and the influences $\rho\,a_e\,d\tau$, $\rho\,b_e\,d\tau$, $\rho\,c_e\,d\tau$ that are applied to each volume element $d\tau$. One must add the inertial forces:

$$-\rho\left(\overline{j}_x+\overline{j}_y+\overline{j}_z\right)\,d\tau$$

to the latter, in which j is the acceleration.

Let us express the virtual work that is done by these actions. Imagine a virtual modification for which each material point is displaced by:

$$\overline{\delta x}+\overline{\delta y}+\overline{\delta z}\,.$$

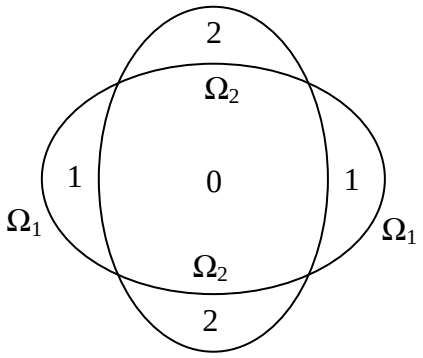

Figure 1

The volume that is occupied by the fluid is deformed. Before the modification, it consisted of the part 0 and the infinitely small part 1, which were divided along the portion Ω_1 of the original surface. After the modification, it will be composed of 0 and the space 2 that is confined to 0 along the portion Ω_2 of the original surface. We agree to denote the variations that the parameters ρ, λ, A, B, C, T experience at each geometric point by the symbol δ and the ones that they experience at each material point by the symbol Δ. The virtual work of the external actions and the inertial forces is:

$$\begin{aligned}\delta\mathcal{T}_e+\delta J=&\int_{\Omega_1+\Omega_2}[P\cos(P,x)\,\delta x+P\cos(P,y)\,\delta y+P\cos(P,z)\,\delta z]\,d\omega\\&+\int_{0+1}\rho[(X_e-j_x)\,\delta x+(Y_e-j_y)\,\delta y+(Z_e-j_x)\,\delta z\\&+L_e\,\Delta\lambda+a_e\,\Delta A+b_e\,\Delta B+c_e\,\Delta C]\,d\tau\,.\end{aligned}$$

One obviously has:

$$\Delta\lambda=\delta\lambda+\frac{\partial\lambda}{\partial x}\delta x+\frac{\partial\lambda}{\partial y}\delta y+\frac{\partial\lambda}{\partial z}\delta z\,,$$

and one can write analogous relations for ΔA, ΔB, ΔC. The expression for $\delta\mathcal{T}_e+\delta J$ can be transformed into:

$$
(6)\qquad \begin{cases} \delta\mathcal{T}_e + \delta J = \displaystyle\sum_{\Omega_1+\Omega_2} [P\cos(P,x)\,\delta x + P\cos(P,y)\,\delta y + P\cos(P,z)\,\delta z]\,d\omega \\ \qquad + \displaystyle\int_0 \rho [L_e \delta\lambda + a_e \delta A + b_e \delta B + c_e \delta C]\,d\tau \\ \qquad + \displaystyle\int_0 \rho \left[X_e - j_x + L_e \frac{\partial\lambda}{\partial x} + a_e \frac{\partial A}{\partial x} + b_e \frac{\partial B}{\partial x} + c_e \frac{\partial C}{\partial x} \right] \delta x\, d\tau \\ \qquad + \displaystyle\int_0 \rho \left[Y_e - j_y + L_e \frac{\partial\lambda}{\partial y} + a_e \frac{\partial A}{\partial y} + b_e \frac{\partial B}{\partial y} + c_e \frac{\partial C}{\partial y} \right] \delta y\, d\tau \\ \qquad + \displaystyle\int_0 \rho \left[Z_e - j_z + L_e \frac{\partial\lambda}{\partial z} + a_e \frac{\partial A}{\partial z} + b_e \frac{\partial B}{\partial z} + c_e \frac{\partial C}{\partial z} \right] \delta z\, d\tau. \end{cases}
$$

4. One will obtain the equations of motion by expressing the idea that:

$$\delta F - \delta\mathcal{T}_e - \delta J = 0$$

for any virtual modification for which the temperature of each molecule remains constant.

In the sequel, the variables *A*, *B*, *C* will play a role that is identical to that of λ, as they have done up to now. One can simplify the notation greatly by now supposing that for the sake of calculations they do not exist. It is easy to re-establish the terms that they contribute at the end by analogy with the ones that are given for λ.

5. The calculation of $\delta\int \varphi\,\rho\,d\tau$ can be performed by following the method that was developed by Duhem in his memoir "Sur l'équilibre et le mouvement des fluides mélanges" ([1]). One sets:

$$
(7)\qquad \begin{cases} H = \varphi + \rho \dfrac{\partial\varphi}{\partial\rho}, \\ ES = -\dfrac{\partial\varphi}{\partial T}. \end{cases}
$$

S will be the entropy of the unit of mass of the fluid, which is regarded as homogeneous. The desired variation will be:

([1]) *Travaux et Mémoires des Facultés de Lille*, t. III, Mémoire no. 11, Chap. VI, pp. 91; 1893.

$$
(8)\qquad \left\{
\begin{aligned}
\delta\int \varphi\,\rho\,d\tau = \;& \int_0 \rho\frac{\partial\varphi}{\partial\lambda}\delta\lambda\,\delta\tau \\
&+\int_0 \rho\left[\left(\frac{\partial H}{\partial x}+ES\frac{\partial T}{\partial x}\right)\delta x+\left(\frac{\partial H}{\partial x}+ES\frac{\partial T}{\partial x}\right)\delta y\right. \\
&\qquad\left.+\left(\frac{\partial H}{\partial x}+ES\frac{\partial T}{\partial x}\right)\delta z\right]d\tau \\
&+\mathop{\mathrm{S}}_{\Omega_1+\Omega_2}\rho^2\frac{\partial\varphi}{\partial\rho}[\delta x\cos(n,x)+\delta y\cos(n,y)+\delta z\cos(n,z)]\,d\omega,
\end{aligned}
\right.
$$

in which *n* denotes the *interior* normal to the fluid.

6. We nonetheless insist upon the calculation of $\delta\Psi$.

Ψ varies for two reasons:

1. Due to the variations $d\rho$, $d\lambda$, $d\rho'$, $d\lambda'$ at any geometric point of the field 0.
2. Due to the annihilation of matter that is contained in 1 and the creation of matter that is contained in 2.

One will write:

$$
(9)\qquad \left\{
\begin{aligned}
2\delta\Psi = \;& \iint_{00}(\psi+\delta\psi)(\rho+\delta\rho)(\rho'+\delta\rho')\,d\tau\,d\tau' \\
&+\iint_{02,02}(\psi+\delta\psi)(\rho+\delta\rho)(\rho'+\delta\rho')\,d\tau\,d\tau' \\
&+\iint_{22}(\psi+\delta\psi)(\rho+\delta\rho)(\rho'+\delta\rho')\,d\tau\,d\tau' \\
&-\left(\iint_{00}\psi\rho\rho'\,d\tau\,d\tau'+\iint_{01,10}\psi\rho\rho'\,d\tau\,d\tau'+\iint_{11}\psi\rho\rho'\,d\tau\,d\tau'\right).
\end{aligned}
\right.
$$

The symbol 00 represents the six-dimensional set that is obtained by successively associating each point of 0 with all of the other points of 0. The symbol 02, 20 represents a set that is composed of two parts, the first of which 02 is obtained by successively associating each point of 0 with all the points of 2, and the second of which 20, by associating each point of 2 with all the points of 0. The symbols 22, 11, and 01, 10 have analogous interpretations.

First consider the two integrals that relate to the region 00 whose difference figures in (9). By reason of symmetry, that difference will be equal to:

$$
2\iint_{00}\rho'\psi\,\delta\rho\,d\tau\,d\tau'+2\iint_{00}\rho'\rho\frac{\partial\psi}{\partial\rho}\delta\rho\,d\tau\,d\tau'+2\iint_{00}\rho\rho'\frac{\partial\psi}{\partial\lambda}\delta\lambda\,d\tau\,d\tau',
$$

or rather:

$$
(9')\qquad 2\int_0 V\,\delta\rho\,d\tau-2\int_0\rho I\,\delta\rho\,d\tau-2\int_0\rho L_i\,\delta\lambda\,d\tau.
$$

Moreover, $\delta\rho$ can be expressed as a function of δx, δy, δz. Indeed:

$$\delta\rho = \Delta\rho - \frac{\partial\rho}{\partial x}\delta x - \frac{\partial\rho}{\partial y}\delta y - \frac{\partial\rho}{\partial z}\delta z \quad \text{and} \quad \Delta\rho = -\rho\left(\frac{\partial\delta x}{\partial x} + \frac{\partial\delta y}{\partial y} + \frac{\partial\delta z}{\partial z}\right).$$

Hence:

$$\delta\rho = -\left[\frac{\partial(\rho\,\delta x)}{\partial x} + \frac{\partial(\rho\,\delta y)}{\partial y} + \frac{\partial(\rho\,\delta z)}{\partial z}\right].$$

An integration by parts then transforms the sum (9′) into:

$$(9'') \quad \begin{cases} 2\int_0 \rho\left[\frac{\partial(V-\rho I)}{\partial x}\delta x + \frac{\partial(V-\rho I)}{\partial y}\delta y + \frac{\partial(V-\rho I)}{\partial z}\delta z\right]d\tau - 2\int_0 \rho L_i\,\delta\lambda\,d\tau \\ +2\underset{\Omega_1+\Omega_2}{S}(\rho V - \rho^2 I)[\delta x\cos(n,x) + \delta y\cos(n,y) + \delta z\cos(n,z)]\,d\omega. \end{cases}$$

Now, take the integrals that relate to 01, 10, and 11 in (9). One can write them as:

$$-\int_1 \rho\,d\tau\int_{1+0}\psi\,\rho'\,d\tau' - \int_1 \rho\,d\tau\int_0 \psi\,\rho'\,d\tau',$$

namely:

$$-2\int_1 V\,\rho\,d\tau.$$

Similarly, the ones that are taken over 02, 20, and 22 give:

$$2\int_2 V\,\rho\,d\tau.$$

In regard to the latter term, we remark that the $\delta\rho$ and the $\delta\lambda$ are not infinitely small in the region 2. However, the integrals that relate to them are: One calculates them by giving the values to ρ, λ, V that those quantities have at the points of 0 that are infinitely close to the surface Ω_2 before the modification.

The difference $2\int_1 V\,\rho\,d\tau - 2\int_2 V\,\rho\,d\tau$ is written:

$$(9''') \qquad -2\underset{\Omega_1+\Omega_2}{S}\rho\,V\,[\delta x\cos(n,x) + \delta y\cos(n,y) + \delta z\cos(n,z)]\,d\omega.$$

In this calculation, one supposes that:

$$\int_1 \rho\,d\tau\int_{1+0}\psi\,\rho'\,d\tau' \quad \text{and} \quad \int_1 \rho\,d\tau\int_0 \psi\,\rho'\,d\tau'$$

are equal, up to second order. Indeed, that is what will happen, in general, with the hypotheses that Duhem stated for the function ψ. However, it will be false if one

supposes, for example, as one does in the theory of capillarity, that this function has a meaningful value only when the points (x, y, z), (x', y', z') are very close to each other. We leave that case aside, and remark simply that if one encounters it then the only things that will be modified will be the double integrals of $\delta\Psi$, but not the triple integrals. Now, the latter suffice for the purpose that we have in mind.

By virtue of (9), (9′), (9″), we then write:

$$(10)\quad \begin{cases} \delta\Psi = \int_0 \rho\left[\dfrac{\partial(V-\rho I)}{\partial x}\delta x + \dfrac{\partial(V-\rho I)}{\partial y}\delta y + \dfrac{\partial(V-\rho I)}{\partial z}\delta z\right] d\tau - \int_0 \rho L_i \delta\lambda\, d\tau \\ \qquad - \underset{\Omega_1+\Omega_2}{\mathrm{S}} \rho^2 I\,[\delta x\cos(n,x) + \delta y\cos(n,y) + \delta z\cos(n,z)]\, d\omega. \end{cases}$$

7. Formulas (6), (8), (10) give $\delta F - \delta\mathcal{T}_e - \delta I$. That expression is composed of a sum of volume integrals and surface integrals. The consideration of the volume integrals will give the equations of motion for the interior of a fluid mass. We write them by re-establishing the terms that are due to A, B, C:

$$(11)\quad \begin{cases} \dfrac{\partial\varphi}{\partial\lambda} = L_e + L_i, \\ \dfrac{\partial\varphi}{\partial A} = a_e + a_i, \\ \dfrac{\partial\varphi}{\partial B} = b_e + b_i, \\ \dfrac{\partial\varphi}{\partial C} = c_e + c_i, \end{cases}$$

$$(12)\quad \begin{cases} \dfrac{\partial H}{\partial x} + ES\dfrac{\partial T}{\partial x} + \dfrac{\partial(V-\rho I)}{\partial x} = X_e - j_x + L_e\dfrac{\partial\lambda}{\partial x} + a_e\dfrac{\partial A}{\partial x} + b_e\dfrac{\partial B}{\partial x} + c_e\dfrac{\partial C}{\partial x}, \\ \dfrac{\partial H}{\partial y} + ES\dfrac{\partial T}{\partial y} + \dfrac{\partial(V-\rho I)}{\partial y} = Y_e - j_y + L_e\dfrac{\partial\lambda}{\partial y} + a_e\dfrac{\partial A}{\partial y} + b_e\dfrac{\partial B}{\partial y} + c_e\dfrac{\partial C}{\partial y}, \\ \dfrac{\partial H}{\partial z} + ES\dfrac{\partial T}{\partial z} + \dfrac{\partial(V-\rho I)}{\partial z} = Z_e - j_z + L_e\dfrac{\partial\lambda}{\partial z} + a_e\dfrac{\partial A}{\partial z} + b_e\dfrac{\partial B}{\partial z} + c_e\dfrac{\partial C}{\partial z}. \end{cases}$$

Suppose that:

$$X_e\,dx + Y_e\,dy + Z_e\,dz + L_e\,d\lambda + a_e\,dA + b_e\,dB + c_e\,dC - ES\,dT$$

is the exact differential of a function – $R(x, y, z, t)$ with respect to x, y, z at each instant. Equations (12) will be written:

$$(12') \qquad \begin{cases} \dfrac{\partial(H+V-\rho I+R)}{\partial x} = -j_x, \\ \dfrac{\partial(H+V-\rho I+R)}{\partial y} = -j_y, \\ \dfrac{\partial(H+V-\rho I+R)}{\partial z} = -j_z, \end{cases}$$

which are formulas that can be inferred from Helmholtz's theorem in a known way [*see* POINCARÉ, *Théorie des tourbillons*, Chap. I. Equations (12′) have the form of equations (7), pp. 10 in that book.]

The function R will exist, in particular, whenever the following conditions are satisfied:

a′. The external actions admit a potential; i.e., $X_e\,dx + Y_e\,dy + Z_e\,dz + L_e\,d\lambda + a_e\,dA + b_e\,dB + c_e\,dC$ is the exact differential of a function $-\Omega\,(x, y, z, \lambda, A, B, C)$.

b′. At any instant, there exists a relation:

$$(13) \qquad K\,(S, T) = 0$$

between temperature and entropy per unit mass in the entire fluid mass.

The latter condition will be, in turn, certainly fulfilled if, upon starting with a state in which the relation (13) is true, each element of matter experiences only transformations for which that relation does not cease to be verified. In particular, it will be fulfilled if, upon starting at an instant at which all of the mass is homogeneous, each material element experiences only *isothermal* or *adiabatic* transformations.

8. A consideration of double integrals of $\delta F - \delta \mathcal{T}_e - \delta J$ will give the boundary conditions. They are:

$$(14) \qquad \begin{cases} P\cos(P,x) = \left[\rho^2 \dfrac{\partial\varphi}{\partial\rho} - \rho^2 I\right]\cos(n,x), \\ P\cos(P,y) = \left[\rho^2 \dfrac{\partial\varphi}{\partial\rho} - \rho^2 I\right]\cos(n,y), \\ P\cos(P,z) = \left[\rho^2 \dfrac{\partial\varphi}{\partial\rho} - \rho^2 I\right]\cos(n,z). \end{cases}$$

These equations show, first of all, that P has the same direction as n (the pressure must then be normal to the surface); it then shows that P must have the value $\rho^2 \dfrac{\partial\varphi}{\partial\rho} - \rho^2 I$.

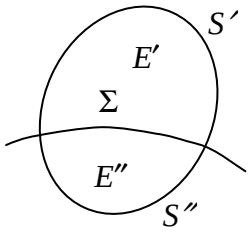

Figure 2.

In order to define the pressure that is exerted on an interior element $d\sigma$, pass a surface Σ through $d\sigma$ that divides the fluid into two parts E' and E''. Remove E' without suppressing the action that it exerts upon the volume elements of E'', and which will have:

$$\int_{E''} \rho\, d\tau \int_{E'} \psi\rho'\, d\tau'$$

for a potential. In order to maintain d'Alembert's fictitious equilibrium in E', one must then apply an exterior pressure Π to each elements $d\sigma$ of Σ. By definition, Π will be the pressure inside of the fluid. We shall now calculate it.

The thermodynamic potential in E'' is:

$$\int_{E''} \varphi\rho\, d\tau + \tfrac{1}{2}\iint_{E''E''} \psi\rho\rho'\, d\tau\, d\tau' .$$

By virtue of formulas (8) and (10), its variation under a virtual modification will be:

$$\int_{E''} \rho \frac{\partial\varphi}{\partial\lambda}\delta\lambda\, d\tau + \int_{E''} \rho \left[\left(\frac{\partial H}{\partial x} + ES\frac{\partial T}{\partial x} \right)\delta x + \left(\frac{\partial H}{\partial y} + ES\frac{\partial T}{\partial y} \right)\delta y + \left(\frac{\partial H}{\partial z} + ES\frac{\partial T}{\partial z} \right)\delta z \right] d\tau$$

$$+ \mathop{\mathrm{S}}_{S''+\Sigma} \rho^2 \frac{\partial\varphi}{\partial\rho}[\delta x \cos(n, x) + \delta y \cos(n, y) + \delta z \cos(n, z)]\, d\omega$$

$$- \int_{E''} \rho L_i''\delta\lambda\, d\tau + \int_{E''} \rho \left[\frac{\partial(V'' - \rho I'')}{\partial x}\delta x + \frac{\partial(V'' - \rho I'')}{\partial y}\delta y + \frac{\partial(V'' - \rho I'')}{\partial z}\delta z \right] d\tau$$

$$- \mathop{\mathrm{S}}_{S''+\Sigma} \rho^2 I''\ [\delta x \cos(n, x) + \delta y \cos(n, y) + \delta z \cos(n, z)]\, d\omega .$$

V'', I'', L_i'' are the integrals V, I, L_i, when they are taken in only the region E''.

The work that is done by the action that is exerted by E' on E'', with the opposite sign, has the expression:

$$\int_{E''} [\Delta\rho \int_{E'} \frac{\partial\psi}{\partial\rho}\rho'\, d\tau' + \Delta\lambda \int_{E'} \frac{\partial\psi}{\partial\lambda}\rho'\, d\tau'$$

$$+ \delta x \int_{E'} \frac{\partial \psi}{\partial x} \rho' d\tau' + \delta\lambda \int_{E'} \frac{\partial \psi}{\partial y} \rho' d\tau' + \delta z \int_{E'} \frac{\partial \psi}{\partial z} \rho' d\tau' \Big] \rho \, d\tau ,$$

namely:

$$\int_{E'} \left(\frac{\partial V'}{\partial x} \delta x + \frac{\partial V'}{\partial y} \delta y + \frac{\partial V'}{\partial z} \delta z \right) \rho \, d\tau - \int_{E'} (I' \delta\rho + L_i' \delta\lambda) \, \rho \, d\tau ,$$

or finally:

$$- \int_{E''} \rho L_i' \delta\lambda \, d\tau + \int_{E''} \rho \left[\frac{\partial (V' - \rho' I')}{\partial x} \delta x + \frac{\partial (V' - \rho' I')}{\partial y} \delta y + \frac{\partial (V' - \rho' I')}{\partial z} \delta z \right] d\tau$$
$$- \underset{S''+\Sigma}{S} \rho^2 I' [\delta x \cos (n, x) + \delta y \cos (n, y) + \delta z \cos (n, z)] \, d\omega$$

V', I', L_i' are the integrals V, I, L_i, when they are taken in only the region E'.

Finally, the virtual work that is done by the other external actions is, with the opposite sign, the expression:

$$- \int_{E''} \rho \Big[(X_e - j_x) \delta x + (Y_e - j_y) \delta y + (Z_e - j_z) \delta z \Big] d\tau$$
$$- \underset{S'}{S} P [\cos (P, x) \, \delta x + \cos (P, y) \, \delta y + \cos (P, z) \, \delta z] \, d\omega$$
$$- \underset{\Sigma}{S} \Pi [\cos (\Pi, x) \, \delta x + \cos (\Pi, y) \, \delta y + \cos (\Pi, z) \, \delta z] \, d\omega .$$

Upon equating the sum of the three expressions above to zero, and remarking that:

$$V = V' + V'',$$
$$I = I' + I'',$$
$$L_i = L_i' + L_i'' ,$$

one will obtain the equations of motion for the mass E'' in the form (11), (12), and the value of Π in the form:

(15) $$\Pi = \rho^2 \frac{\partial \varphi}{\partial \rho} - \rho^2 I,$$

and one verifies that Π is normal to $d\sigma$.

The relation (15) shows that the pressure on the element $d\sigma$ depends upon the state of the *total* mass by the intermediary of I. One sees how difficult that it would be to introduce the restriction (*b*) of no. **1** here.

9. We recall that an example of a fluid to which the preceding considerations will apply is given by the ones whose elements obey the law that was proposed by Faye in his studies on the tails of comets.

II.

10. The hypotheses that were made in no. **2** on the complementary term Ψ are not necessary. Weakly-magnetic perfect fluids offer an example in which they are not true. The action of a magnetic mass on an interior element is not defined. That implies the impossibility of putting Ψ into the form (2). That also implies the impossibility of stating the restriction (*a*) of no. **1** for these bodies. However, if one adopts the ideas of Duhem ([1]), one can write the internal thermodynamic potential of a system that is composed of permanent and immobile magnet *P* and a weakly-magnetic perfect fluid *E* as follows:

$$F = \int_{P+E} \varphi(\rho, M, T)\, \rho\, d\tau - \tfrac{1}{2} \int_{P+E} (A\alpha + B\beta + C\gamma)\, d\tau . \tag{16}$$

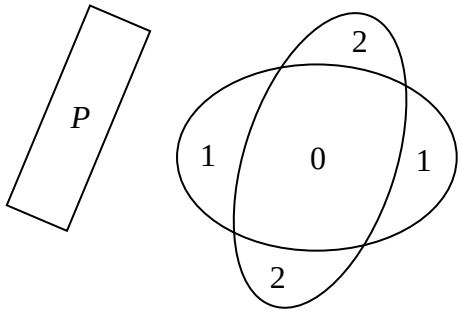

Figure 3.

Here, $M = \sqrt{A^2 + B^2 + C^2}$ is the magnetic moment of the element $d\tau$. As for α, β, γ, they are the components of the magnetic field that is inside the magnetic mass along the three axes.

We assume, moreover, that the virtual work that is done by the inertial forces and external actions and act upon the fluid has the form:

$$\left\{ \begin{array}{c} \delta \mathcal{T}_e + \delta J = \mathop{\mathrm{S}}_{\Omega_1 + \Omega_2} [P\cos(P,x)\,\delta x + P\cos(P,y)\,\delta y + P\cos(P,z)\,\delta z]\, d\omega \\ + \int_0 \rho [(X_e - j_x)\,\delta x + (Y_e - j_y)\,\delta y + (Z_e - j_z)\,\delta z]\, d\tau, \end{array} \right. \tag{17}$$

which is only a particular case of the form (6).

The variation of $\int \varphi\rho\, d\tau$ can be calculated by formula (8): It will be:

([1]) *Traité d'Électricité et de Magnetisme*, t. II, pp. 159 and 405.

$$
(18)\quad \left\{
\begin{aligned}
\delta\int \varphi\rho\, d\tau = {} & \int_0 \frac{\rho}{M}\frac{\partial\varphi}{\partial M}(A\,\delta A + B\,\delta B + C\,\delta C)\,d\tau \\
& + \int_0 \rho\left[\left(\frac{\partial H}{\partial x} + ES\frac{\partial T}{\partial x}\right)\delta x\right. \\
& \qquad \left. + \left(\frac{\partial H}{\partial y} + ES\frac{\partial T}{\partial y}\right)\delta y + \left(\frac{\partial H}{\partial z} + ES\frac{\partial T}{\partial z}\right)\delta z\right] d\tau \\
& + \mathop{\mathrm{S}}_{\Omega_1+\Omega_2} \rho^2 \frac{\partial\varphi}{\partial\rho}[\delta x\cos(n,x) + \delta y\cos(n,y) + \delta z\cos(n,z)]\,d\omega,
\end{aligned}
\right.
$$

in which H and S are defined by the equalities (7), as always.

However, the variation of $\Psi = -\frac{1}{2}\int_{P+E}(A\alpha + B\beta + C\gamma)\,d\tau$ cannot be obtained by applying the preceding results. It can nonetheless be calculated by method that is completely analogous to the one that we discussed above. It was Liénard who first gave the exact value ([1]). We content ourselves by pointing out the result of his calculations, while returning to his work in the work that follows.

$$
(19)\quad \left\{
\begin{aligned}
\delta\Psi = {} & -\int_0 (\alpha\,\delta A + \beta\,\delta B + \gamma\,\delta C)\,d\tau \\
& + \mathop{\mathrm{S}}_{\Omega_1+\Omega_2} [\alpha A + \beta B + \gamma C + 2\pi M^2\cos^2(M,n)] \\
& \quad \times[\delta x\cos(n,x) + \delta y\cos(n,y) + z\cos(n,z)]\,d\omega.
\end{aligned}
\right.
$$

11. The expression $\delta F - \delta\mathcal{T}_e - \delta J$ is a sum of triple and double integrals. A consideration of the former will give the equations of motion inside of the fluid:

$$
(20)\quad \left\{
\begin{aligned}
A &= \alpha\frac{M}{\rho\dfrac{\partial\varphi}{\partial M}}, \\
B &= \beta\frac{M}{\rho\dfrac{\partial\varphi}{\partial M}}, \\
C &= \gamma\frac{M}{\rho\dfrac{\partial\varphi}{\partial M}},
\end{aligned}
\right.
$$

([1]) "Pressions à l'intérieur des aimantes et des diélectriques," *Lumière électrique*, t. II, pp. 7; 1984.

$$(21)\qquad \begin{cases} \dfrac{\partial H}{\partial x} + ES\dfrac{\partial T}{\partial x} = X_e - j_x, \\ \dfrac{\partial H}{\partial y} + ES\dfrac{\partial T}{\partial y} = Y_e - j_y, \\ \dfrac{\partial H}{\partial z} + ES\dfrac{\partial T}{\partial z} = Z_e - j_z. \end{cases}$$

Imagine that $X_e\,dx + Y_e\,dy + Z_e - ES\,dT$ is an exact differential with respect to x, y, z of a function – R (x, y, z, t) at any instant. Equations (21) then take the form:

$$(21')\qquad \begin{cases} \dfrac{\partial (H+R)}{\partial x} = -j_x, \\ \dfrac{\partial (H+R)}{\partial y} = -j_y, \\ \dfrac{\partial (H+R)}{\partial z} = -j_z, \end{cases}$$

which leads to Helmholtz's theorem.

The function R will exist, in particular, whenever conditions (a') and (b') of no. **7** are verified.

12. The boundary conditions are given by considering the double integrals of $\delta F - \delta T_e - \delta J$. Upon taking (20) into account in order to transform the term $\alpha A + \beta B + \gamma C$, and upon setting:

$$(22)\qquad \Pi = \rho^2 \frac{\partial \varphi}{\partial \rho} + Mr\frac{\partial \varphi}{\partial M} + 2\pi M^2 \cos^2(M, n),$$

one will have:

$$(23)\qquad \begin{cases} P\cos(P, x) = \Pi\cos(n, x), \\ P\cos(P, y) = \Pi\cos(n, y), \\ P\cos(P, z) = \Pi\cos(n, z). \end{cases}$$

As in no. **8**, one confirms that the pressure on an element $d\sigma$ that is interior to the fluid is precisely Π, and that the pressure is normal to $d\sigma$, but it depends upon its orientation ([1]), even though the pressure at a point undefined, and the restriction (b) of no. **1** will make no more sense here than the restriction (a).

13. In truth, when a magnetic fluid is in motion, either conduction currents or displacement currents will be produced in it. The study of that motion thus belongs to

([1]) LIÉNARD, *loc. cit.*

electrodynamics and goes beyond the usual methods of thermodynamics (Duhem has insisted upon this point); Helmholtz's methods will permit one to address it in all of its complexity. The case that we have treated is an ideal case, namely, the case of a fictitious fluid that is neither a conductor nor one that is susceptible to dielectric polarization. It thus presents very little of interest in the theory of electricity. Here, we shall assume the viewpoint of energetics, and we would simply like to show by that example that is is possible for material systems to exist for which the form (2) of the term Ψ is no longer exact, and for which the notion of internal force, which is paramount in mechanics, fails completely, but can be effectively replaced by that of energy.

III.

14. We have thus given a new statement (a'), (b') to the restrictions (a), (b) to which the proof of Helmholtz's theorem is subordinate that has the advantage of applying to the cases in which the usual statements have no meaning. The conditions (a'), (b') present the greatest analogy with the ones that Duhem found while seeking the case in which the equations of motion admit a *vis viva* integral ([1]) for systems that depend upon a finite number of variables. The following proof of Helmholtz's theorem will perhaps bring to light the reasons for that parallelism better than the preceding developments did.

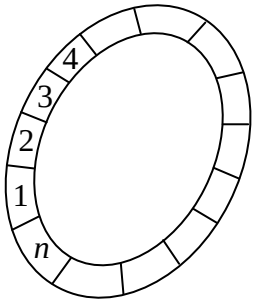

Figure 4

In order to show that $\int_e u\,dx + v\,dy + w\,dz$ is constant, it is necessary and sufficient to show that $\int_e j_x\,dx + j_y\,dy + j_z\,dz$ is zero (see POINCARÉ, *Théorie des tourbillons*, pp. 12). Consider a fluid ring along C with an infinitely small section that we divide by sections that are normal to C into elements 1, 2, …, n that are infinitely small in their three dimensions and that all contain the same mass dm. Since a material system is defined by normal variables, one knows that under an arbitrary virtual displacement (isothermal or not) the variation of the internal thermodynamic potential *minus the terms that are due to the variation of the temperature* will be equal to the force that is done by the external actions plus the forces of inertia. We write down that equality for the ring C,

([1]) "L'intégrale des forces vives en Thermodynamique," Journal de Mathématiques pures et appliqués (5) **4** (1898), 5.

and for the modification of the location and physical and chemical state that each element must experience, 1 will replace 1, …, *n* will replace 1.

The thermodynamic potential of the ring has the form:

$$\sum_1^n \varphi(\rho, \lambda, T)\, dm + \zeta,$$

in which ζ does not depend upon the temperature of the elements 1, 2, …, *n*, and will once more take the same value when the total mass of the fluid returns to the same state. The left-hand side of our equality will then be:

$$-\sum_1^n \frac{\partial \varphi}{\partial T}\, dT\, dm.$$

The external actions consist of:

Firstly, the ones that are due to bodies that are foreign to the fluid. The virtual work that they do is, by hypothesis, of the form:

$$\sum_1^n \left[X_e \delta x + Y_e \delta y + Z_e \delta z + L_e \left(\frac{\partial \lambda}{\partial x} \delta x + \frac{\partial \lambda}{\partial y} \delta y + \frac{\partial \lambda}{\partial z} \delta z \right) \right] dm.$$

One then has the actions that are exerted on the ring by the rest of the fluid. The work that they do $\delta\eta$ will be zero, since all of the mass will return to the same state.

Finally, one has the pressures that are applied to the surface of the ring. The work that they do is zero, since the fluid is inviscid, so they will be normal to the elements they press upon; i.e., to the path that is traversed.

We must then write:

$$-\sum_1^n \frac{\partial \varphi}{\partial T} dT\, dm = \sum_1^n \left[X_e \delta x + Y_e \delta y + Z_e \delta z + L_e \left(\frac{\partial \lambda}{\partial x} \delta x + \frac{\partial \lambda}{\partial y} \delta y + \frac{\partial \lambda}{\partial z} \delta z \right) \right] dm$$
$$-\sum_1^n (j_x\, dx + j_y\, dy + j_z\, dz)\, dm\,.$$

In order to calculate each of the terms in that equation, one can obviously no longer imagine each element substituting for the one that follows, but only that one of them – 1, for example – makes a complete circuit around *C*. One will then get:

$$-\int_e \frac{\partial \varphi}{\partial T} dT = \int_e (X_e dx + Y_e dy + Z_e dz + L_e d\lambda) - \int_e (j_x dx + j_y dy + j_z dz)\,.$$

Upon expressing the idea that $\int_e (j_x dx + j_y dy + j_z dz)$ is zero, one will recover the sufficient conditions (a') and (b') of nos. **7** and **11**. If they are satisfied then one can say that there exists a *vis viva* integral under the virtual displacement that we have considered for the element 1.

15. That proof of Helmholtz's theorem is only a generalization of a method that was pointed out by Lagrange for finding the equilibrium conditions for liquids ([1]). It presents two inconvenient aspects. It leaves the virtual modification that the ring C experiences somewhat obscure; it is only by having a very confused notion of the nature of a fluid that one can glimpse the possibility that it will exist. Moreover, it supposes (with no explanation) that the pressure is normal to the element that it presses upon. From these viewpoints, it therefore cannot replace the developments of nos. **1** through **12**. They have shown us that the normal direction of the pressure will result from the fact that the thermodynamic potential $\varphi \rho \, d\tau$ depends upon the distribution of matter around the point x, y, z by the intermediary of only its density. However, by contrast, it has the advantage of being independent of the form of the term Ψ in no. **2**. In fact, the hypotheses that were made on $\delta\zeta$ and $\delta\eta$ (which replace $\delta\Psi$, here) follow directly from the principles of thermodynamics, and it seems difficult to ignore that fact.

IV.

16. In the motion of a fluid mixture, Duhem has shown that Helmholtz's theorem applies to any fluid, in particular, when the external actions admit a potential, and if the temperature of the mass is uniform at any instant ([2]). Its proof supposes that the various elements of the fluid masses exert no action upon each other. It is easy to remove that restriction. For example, take two fluids that we distinguish by the indices 1 and 2. Let ρ be the density of the mixture, let ρ_1, ρ_2 be the partial densities ($\rho = \rho_1 + \rho_2$), let λ_1, λ_2 be two arbitrary parameters, and let T be the temperature. We write the thermodynamic potential:

$$F = \int \varphi(\rho_1, \rho_2, \lambda_1, \lambda_2, T)\, \rho \, d\tau + \tfrac{1}{2} \iint \psi(\rho_1, \rho_2, \lambda_1, \lambda_2, \rho_1', \rho_2', \lambda_1', \lambda_2', x, y, z, x', y', z')\, \rho\rho' \, d\tau \, d\tau' .$$

One supposes that the virtual work that is done by the external actions has the form:

$$\begin{aligned} \delta\mathcal{T}_e &= \int (X_{e1}\delta x_1 + Y_{e1}\delta y_1 + Z_{e1}\delta z_1 + L_{e1}\Delta\lambda_1)\, \rho_1 \, d\tau \\ &+ \int (X_{e2}\delta x_2 + Y_{e2}\delta y_2 + Z_{e2}\delta z_2 + L_{e2}\Delta\lambda_2)\, \rho_2 \, d\tau \\ &+ \mathrm{S}\, P\, [\cos(n, x)\, \delta x + \cos(n, y)\, \delta y + \cos(n, z)\, \delta z]\, d\omega, \end{aligned}$$

([1]) *Mécanique analytique*, Part I, section VII, art. 7.
([2]) *Équilibre et mouvement des fluides mélanges*, pp. 101.

δx_1, δy_1, δz_1 refer to the fluid 1, while δx_2, δy_2, δz_2 refer to fluid 2.

On the surface, the expression:

$$\cos(n, x)\,\delta x + \cos(n, y)\,\delta y + \cos(n, z)\,\delta z$$

will be the same for both fluids ([1]).

We set:

$$H_1 = \varphi + \rho \frac{\partial \varphi}{\partial \rho_1},$$

$$H_2 = \varphi + \rho \frac{\partial \varphi}{\partial \rho_2},$$

$$I_1 = -\int \frac{\partial \psi}{\partial \rho_1} \rho' d\tau',$$

$$I_2 = -\int \frac{\partial \psi}{\partial \rho_1} \rho' d\tau',$$

$$L_{i1} = -\int \frac{\partial \psi}{\partial \lambda_1} \rho' d\tau',$$

$$L_{i2} = -\int \frac{\partial \psi}{\partial \lambda_2} \rho' d\tau',$$

The equations of motion are obtained by a method that is completely analogous to the one that we applied above. One will thus arrive at the following formulas for fluid 1:

$$\frac{\partial \varphi}{\partial \lambda_1} = L_{e1} + L_{i1},$$

$$\frac{\partial H_1}{\partial x} - \left(\frac{\partial \varphi}{\partial T} + \frac{\omega_1}{T}\right)\frac{\partial T}{\partial x} + \frac{\partial (V - \rho I_1)}{\partial x} = X_{e1} + L_{e1}\frac{\partial \lambda_1}{\partial x} - j_{1x},$$

$$\frac{\partial H_1}{\partial y} - \left(\frac{\partial \varphi}{\partial T} + \frac{\omega_1}{T}\right)\frac{\partial T}{\partial y} + \frac{\partial (V - \rho I_1)}{\partial y} = Y_{e1} + L_{e1}\frac{\partial \lambda_1}{\partial y} - j_{1y},$$

$$\frac{\partial H_1}{\partial z} - \left(\frac{\partial \varphi}{\partial T} + \frac{\omega_1}{T}\right)\frac{\partial T}{\partial z} + \frac{\partial (V - \rho I_1)}{\partial z} = Z_{e1} + L_{e1}\frac{\partial \lambda_1}{\partial z} - j_{1z},$$

and some analogous formulas for fluid 2. ω_1 and ω_2 denote some functions that verify the equality $\omega_1 - \omega_2 = 0$ and that thermodynamics is powerless to determine.

If the external actions admit a potential, and if the temperature is uniform in the mass *at any instant* then Helmholtz's theorem will be true. It is further true that $\frac{\partial \varphi}{\partial T} + \frac{\omega_1}{\rho_1}$ is a function of only T. Unfortunately, it is difficult to see what sort of physical reality would correspond to that statement. In particular, it is difficult to find the case here that

([1]) DUHEM, *Équilibre et mouvement des fluides mélanges,* pp. 37.

corresponds to the adiabatic motions of isolated fluids. There is, moreover, no reason to emphasize that the quantity of heat that is released by a fluid element 1 that is mixed with another fluid 2 will not be a well-defined quantity.
