"Sur les équations de l'électromagnetisme," C. R. Acad. Sci. Paris **185** (1927), 341-343.

On the equations of electromagnetism

Note (1) by **F. GONSETH** and **G. JUVET**
Communicated by J. Hadamard

Translated by D. H. Delphenich (†)

In order to afford more clarity to the attempt that was made to reduce the theory of the electromagnetic field to relativity in five dimensions (2), it seems good for us to specify that the usual expression for the ponderomotive force according to Lorentz leads to a theory of that type.

In the Minkowski universe E_4, let ξ^i be the components of the world-impulse of the moving point considered, let m is its rest mass, and let e be its charge. Furthermore, let u^i be the components of its world-velocity, let F^{ik} be those of the tensor that represents the electromagnetic field, let s^i be those of the current, and finally, let p^i be those of the force. One will have:

$$d\xi^i = p^i\, ds, \qquad \xi^i = m\, u^i,$$

and $\qquad\qquad (i \text{ and } k = 0, 1, 2, 3)$

$$p^i = -F^{ik} s_k, \qquad s^k = e\, u^k,$$

and consequently:

$$d\xi^i = -\frac{e}{m} F^{i\cdot}_{\cdot,k}\, \xi^k ds . \tag{1}$$

These equations cannot be interpreted as the Levi-Civita transport of the impulse vector $\boldsymbol{\Xi}$ in E_4, but these can be interpreted in that way in a space E_5 in which one sets the fifth coordinate x^4 equal to:

$$dx^4 = \frac{e}{m} ds . \tag{2}$$

The components of the corresponding affine connection will then be equal to either 0 or $F^{i\cdot}_{\cdot,k}$. In the space of general relativity, one will have:

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(1) Session on 11 July 1927.

(†) Translator's note: The Erratum that was corrected on pp. 483 of this same volume has been incorporated into the translation.

(2) Cf., KALUZA, Sitzungsber. d. preuss. Akad. d. Wiss. (II) (1921), 966-972.

(3) $$m\left(\frac{d^2x^i}{ds^2}+\Gamma^i_{kl}\frac{dx^k}{ds}\frac{dx^i}{ds}\right)=p^i \qquad (i = 0, 1, 2, 3),$$

and here again, one has:

$$p_i = -F_{ik}\,s^k, \qquad F_{ik}=\frac{\partial\varphi_i}{\partial x_k}-\frac{\partial\varphi_k}{\partial x_i}.$$

Equations (3) can then be written:

(4) $$d\xi^i+\Gamma^i_{kl}\,\xi^k dx^l+\frac{e}{m}F^{\,l}_{\cdot k}\,\xi^k ds=0,$$

moreover, or rather, upon again setting:

(2) $$dx^4=\frac{e}{m}ds$$

and then taking the components $G^{\gamma}_{\alpha\beta}$ of the affine connection in the space E_5 of (x_0, x_1, x_2, x_3, x_4) to be:

(5) $$\begin{cases} G^{lk}_{i}=\Gamma^l_{ik}, \\ G^l_{4i}=G^l_{i4}=F^{\;l}_{\cdot\,i}, \\ G^i_{44}=G^4_{4k}=G^4_{ik}=G^4_{44}, \end{cases}$$

one will have:

(6) $$d\xi^i+G^{\,i}_{\alpha\beta}\xi^\alpha dx^\beta=0 \qquad (i = 0, 1, 2, 3),$$

in which the Greek indices takes the values (0, 1, 2, 3, 4), while the Latin indices take only the values (0, 1, 2, 3). The fifth equation:

$$d\xi^4+G^{\,4}_{\alpha\beta}\xi^\alpha dx^\beta=0$$

will be satisfied identically when one assumes the invariability of charge, because:

$$\xi^4=m\frac{dx^4}{ds}=e.$$

The object of this note is to formulate a theory of relativity in five dimensions whose equations provide the laws of gravitational field, the electromagnetic field, the laws of motion of a charged material point, and the Schrödinger wave equation. We will then have a context that will include the laws of gravitation and electromagnetism, and in which it will be possible to also include quantum theory.

The preceding considerations, and on the other hand, the fact that we do not know of any phenomena in which a variation of the charge occurs ([1]) oblige us to consider only changes of variable of the form:

$$(x_0, x_1, x_2, x_3, x_4) \qquad \text{into} \qquad (\bar{x}_0, \bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4),$$

such that:

$$(7) \qquad \begin{cases} (\bar{x}_0, \bar{x}_1, \bar{x}_2, \bar{x}_3) & \text{are functions of only } (x_0, x_1, x_2, x_3), \\ \text{and } \bar{x}_4 & \text{is a function of only } x_4. \end{cases}$$

Moreover, we shall assume that the functions that enter into our argument do not depend upon the variable x^4. Under those conditions, equations (5) will be invariant under the changes of variable (7). If one makes a transformation (7) then one will have:

$$\bar{G}^l_{ik} = \bar{\Gamma}^l_{ik}, \qquad \bar{G}^l_{4i} = \bar{G}^l_{i4} = \bar{F}^l_{\cdot i},$$

$$\bar{G}^k_{ik} = \bar{G}^4_{4k} = \bar{G}^i_{44} = \bar{G}^4_{44} = 0$$

for the components of the affine connection in the new coordinate system.

In order to summarize this note, we shall say that equation (6) expresses the idea that:

If one attaches a charged material point that moves in an electromagnetic field and a gravitational field to the vector whose components five-dimensional space are:

$$(8) \qquad m\frac{dx^0}{ds}, \quad m\frac{dx^1}{ds}, \quad m\frac{dx^2}{ds}, \quad m\frac{dx^3}{ds}, \quad e$$

then that vector will displace parallel to itself.

That will permit one to define inertial systems for both gravitation and electromagnetism at the same time. They will be the systems that are *geodesics* in E_5 through the point $(x_0, x_1, x_2, x_3, x_4)$ in the sense Weyl ([2]).

([1]) One might employ some suggestions that were made by L. de Broglie (cf., his book: *Ondes et Mouvements*, pp. 62)

([2]) Cf., WEYL, *Temps, Espace, Matière*, pp. 97. In the expression in (8), it is intended that the ds is the one in E_4 .

"Sur la métrique de l'espace à 5 dimensions de l'électromagnetique and de la gravitation," C. R. Acad. Sci. Paris **185** (1927), 412-413.

On the metric of the five-dimensional space of electromagnetism and gravitation

Note [1] by **F. GONSETH** and **G. JUVET**
Communicated by Hadamard

Translated by D. H. Delphenich

The components of the affine connection in the five-dimensional universe E_5 that we wrote out in a previous note [2] correspond to a metric that we shall specify first of all. In that E_5, the coordinates (x_0, x_1, x_2, x_3) are once more separate from x_4, and we will have, in particular, that:

$$G_{ik}^l = \Gamma_{ik}^l ,$$

in which the Γ_{ik}^l are the Christoffel indices of the second kind for the ds^2 on the E_4 of all (x_0, x_1, x_2, x_3). Under those conditions, the law of transformation of the G_{ik}^l in E_5 demands simply that the Γ_{ik}^l must transform like the components of an affine connection, and the $F_{\cdot k}^{i}$, like those of a tensor in E_4.

The connection thus-determined possesses a fifteen-parameter fundamental group that is the Lorentz group, just as the group of Galilean kinematics is the group of displacements in an E_3. In particular, it is not a group that can be characterized by a non-degenerate ds^2 in an E_5.

Meanwhile, we shall determine a ds^2 that will provide us with almost the same connection. It is clear that we will abandon the land of classical electromagnetism in doing so (and also modify the expression for the force that Lorentz gave).

Suppose, first of all, that for $i, k = 0, 1, 2, 3$, the g_{ik} are those Einstein's universe, which satisfy our equations:

$$G_{ik}^l = \Gamma_{ik}^l .$$

The equations:

$$G_{4i}^l = G_{i4}^l = F_{\cdot i}^{l}$$

will then give:

$$G_{4i,\,l} + G_{i\,4,\,l} = F_{li} , \tag{1}$$

because $G_{i4}^4 = 0$, and $F_{\cdot i}^{4}$ has no meaning in itself.

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[1] Session on 1 August 1927.
[2] C. R. Acad. Sci. Paris, **185** (1927), 341-343.

As for the functions that we consider to not depend upon x_4 , equations (1) can be written:

$$\frac{\partial g_{4l}}{\partial x_i} - \frac{\partial g_{4i}}{\partial x_l} = 2\left(\frac{\partial \varphi_l}{\partial x_i} - \frac{\partial \varphi_i}{\partial x_l}\right);$$

we solve them by setting ([1]):

$$g_{4l} = 2\varphi_l \, .$$

One will further have:

$$G_{i\,4,\,4} = g_{4l}\, G^l_{i4} + g_{44}\, G^4_{i4} = 2\varphi_i\, F^{\;l}_{\cdot\, l} \, ,$$

$$G_{44,\,i} = g_{il}\, G^l_{44} + g_{i4}\, G^4_{44} = 0 \; ;$$

hence, on the one hand:

$$\frac{\partial g_{44}}{\partial x_i} = 4\varphi_l\, F^{\;l}_{\cdot\, i} \, ,$$

and on the other hand:

$$\frac{\partial g_{44}}{\partial x_i} = 0.$$

We can then remain faithful to the connection that we introduced to begin with as a consequence of classical electromagnetism. We then set:

$$g_{44} = \psi^2,$$

in which ψ is a function of the (x_0, x_1, x_2, x_3) .

Here is the ds^2 for E_5 that we propose to put at the roots of a new form of electromagnetism:

$$-\, ds^2 = g_{ik}\, dx^i\, dx^k + 2\varphi_i\, dx^i\, dx^4 + \psi^2\, dx^4\, dx^4 \, ,$$

in which the g_{ik} $(i, k = 0, 1, 2, 3)$ are functions of x_0 , x_1 , x_2 , x_3 that reduce to the coefficients of the Einstein ds^2 that relates to the given gravitational field in the absence of an electromagnetic field.

The $g_{i4} = 2\varphi_i$ $(i = 0, 1, 2, 3)$ are the components of the electromagnetic potential, up to a factor of 2.

Finally, $\psi = \sqrt{g_{44}}$ is a function of x_0 , x_1 , x_2 , x_3 .

The trajectories of a charged material point are the geodesics of that ds^2, with $\dfrac{dx^4}{ds} = \dfrac{e}{m}$, while e is the charge of the point.

([1]) One can get a more general solution by setting $g_{4i} = 2\varphi_i + \partial\lambda / \partial x_l$, in which λ is an arbitrary function of x_0 , x_1 , x_2 , x_3 .

"Sur l'équation de M. Schrödinger," C. R. Acad. Sci. Paris **185** (1927), 448-450.

On the Schrödinger equation

Note ([1]) by **F. GONSETH** and **G. JUVET**
Communicated by Hadamard

Translated by D. H. Delphenich ([†])

Schrödinger, while pursuing the ideas that L. de Broglie set down in his own beautiful theory, formulated a theory that that one could rightfully call "wave mechanics." It then permitted one to give an interpretation of the quantum conditions that did not represent an essential discontinuity in dynamical phenomena ([2]).

Schrödinger's ideas can be expressed very simply if one uses the theory of characteristics and bicharacteristics of second-order equation. That is what one will perceive upon approaching the beautiful papers of E. Vessiot ([3]) on the propagation of waves by means of the remarkable analysis that Hadamard presented in his two classic treatises on second-order equations ([4]).

Here are the principles that we have found to be useful: When one is given second-order partial differential equation (*O*) that is linear in the second derivatives, one can define some multiplicities that are called "characteristic" by means of a first-order partial differential equation (*J*) and some curves that are called "bicharacteristics" of the (*O*), which are the characteristics of (*J*). If one identifies (*J*) with the Jacobi equation of motion of a material point then the trajectories of that material point will be the characteristics of certain equations (*O*), among which one finds the Schrödinger equation, which is easy to attach to (*J*) by invariance in many cases, moreover.

Consider the universe E_5 of our preceding notes then ([5]), but suppose that the φ_i in them are all zero. The coefficients of ds^2 for E_5 are then given by the following table:

([1]) Session on 8 August 1927.

([†]) Translator's note: The Errata that were cited on pp. 624 of this volume were incorporated into the translation.

([2]) Cf., Ann. Phys. (Leipzig) **79** (1926), 361-376 and 489-527.

([3]) "Essai sur la propagation des ondes," Ann. Ec. Norm. sup. (3) **26** (1909), 403-449, and above all: "Sur l'interprétation mécanique des transformations de contact infinitesimales," Bull. Soc. Math. de France **34** (1906), 265-269. At the end of that paper, one will already find a discussion of the entire of theory of waves attached to a moving material point for which the *vis viva* and the force function do not depend upon time. On the advice of Vessiot, we have resorted to general relativity for the generalizations that we have in mind for the case of the impermanent regime. We take this opportunity to point out that his friendly advice has taken us much further than we had originally imagined possible.

([4]) *Leçons sur la propagation des ondes*, Hermann, Paris, 1903, pages 263 *et seq*. and *Lectures on Cauchy's Problem in linear partial differential equations*, Yale University Press, 1923, pp. 75, 76, 83, 192.

([5]) C. R. Acad. Sci. Paris **185** (1927), pp. 341 and 412.

$$(1) \qquad \begin{vmatrix} g_{00} & g_{01} & g_{02} & g_{03} & 0 \\ g_{10} & g_{11} & g_{12} & g_{13} & 0 \\ g_{20} & g_{21} & g_{22} & g_{23} & 0 \\ g_{30} & g_{31} & g_{32} & g_{33} & 0 \\ 0 & 0 & 0 & 0 & \psi^2 \end{vmatrix} .$$

They are determined by Einstein's equations, except for ψ^2. We generalize them by supposing that the g_{ik}, as well as ψ^2, are determined by the Einstein equations in E_4; i.e., outside of the masses, we set:

$$R_{\alpha\beta} = 0 \qquad (\alpha, \beta = 0, 1, 2, 3, 4).$$

It is clear that one can first of all recover the Einstein equations of E_4, since we have modified the connection of E_5, which includes that space. However, the modification that we have referred to can be assumed to be very small initially, which amounts to assuming that ψ and its derivatives $\psi_i = \partial\psi / \partial x_i$ are very small and that all of our functions, including ψ, have derivatives with respect to x_4 that are also very small. Under those conditions, one will find that:

1. $$R_{ik} = 0 \qquad (i, k = 0, 1, 2, 3)$$

are precisely the Einstein equations that determine the:

$$g_{ik} \qquad (i, k = 0, 1, 2, 3)$$

of the ds^2 of Einstein's E_4.

2. The equations:

$$R_{i4} = 0 \qquad (i = 0, 1, 2, 3)$$

are satisfied identically.

3. $$R_{44} = 0$$

can be written:

$$\psi \frac{\partial(g^{ki}\psi_i)}{\partial x_h} + \Gamma^h_{ih}\psi\, g^{ik}\psi_k = 0,$$

or

$$(O) \qquad g^{ki}\frac{\partial^2\psi_i}{\partial x_i\,\partial x_h} + \left(\Gamma^h_{lh}g^{li} + \frac{\partial g^{hi}}{\partial x_h}\right)\frac{\partial\psi}{\partial x_i} = 0.$$

Now, if one seeks the bicharacteristics of (*O*) then one will find the geodesics of ds^2 for E_4 precisely.

If one considers an Einsteinian universe E_4 *to be a section* x_4 = const. *of a five-dimensional universe* E_5 *of* $(x_0, x_1, x_2, x_3, x_4)$ *whose* ds^2 *has the functions in the table* (1) *for its coefficients then the equations of gravitation will be the equations* $R_{\alpha\rho} = 0$ (*in which* $\alpha, \rho = 0, 1, 2, 3$), *and the trajectories of a material point* ([1]) *in* E_4 *will be the bicharacteristics of the equation* $R_{44} = 0$ *that determines the function* ψ *when the* g_{ik} *are known. That equation governs the propagation of waves; it can be identified with the Schrödinger equation.*

One will then see that the fiction of a five-dimensional universe will permit one to give a deep reason for the Schrödinger equation. It is clear that this artifice will become necessary when phenomena oblige physicists to believe in the variability of charge.

We remark that the equations that Schrödinger proposed in order to interpret the spectral lines involved an electric field in a great number of cases, although we have left it out here.

([1]) Of course, without modifying the field appreciably.

"Les équations de l'électromagnetisme et l'équation de M. Schrödinger dans l'universe à cinq dimensions," C. R. Acad. Sci. Paris **185** (1927), 535-538.

The equations of electromagnetism and the Schrödinger equation in a five-dimensional universe

Note (1) by **F. GONSETH** and **G. JUVET**
Communicated by J. Hadamard

Translated by D. H. Delphenich

Take an ds^2 for an E_5 whose coefficients are given by the table (2):

$$\text{(1)} \qquad \begin{vmatrix} g_{00} & g_{01} & g_{02} & g_{03} & 2\varphi_0 \\ g_{10} & g_{11} & g_{12} & g_{13} & 2\varphi_1 \\ g_{20} & g_{21} & g_{22} & g_{00} & 2\varphi_2 \\ g_{30} & g_{31} & g_{32} & g_{33} & 2\varphi_3 \\ 2\varphi_0 & 2\varphi_1 & 2\varphi_2 & 2\varphi_3 & \psi^2 \end{vmatrix}$$

and look for what the equations will become that we have proposed in order to define the coefficients in the space E_5. We further suppose that all of the functions that enter into play here have derivatives with respect to x_4 that are negligible in comparison to their derivatives with respect to the other variables. Finally, if the $\psi_i = \partial \psi / \partial x_i$ are small quantities then we suppose that the φ_i and the F_{ik} are negligible in comparison to ψ. One will then find the following approximate expressions for the components of the affine connection:

$$G_{ik}^{l} = \Gamma_{ik}^{l} \qquad (i, k = 0, 1, 2, 3),$$

in which the Γ_{ik}^{l} are the Christoffel symbols of the second kind that are attached to the ds^2 of the Einsteinian E_4 of all (x_0, x_1, x_2, x_3) :

$$G_{4i}^{l} = G_{i4}^{l} = F_{\cdot\, i}^{\; l},$$

and the index was studied by means of the g^{ik} of E_4 :

$$G_{ik}^{4} = 0,$$

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(1) Session on 22 August 1927.
(2) Cf., our preceding Notes [C. R. Acad. Sci. Paris **185** (1927), pp. 341, 412, 448].

$$G_{i4}^{4} = \frac{\psi_i}{\psi_4} G_{44}^{i} = -\,\psi\,\psi^{i} \qquad \text{(same remark)}$$

$$G_{44}^{4} = 0.$$

We calculate the $R_{\alpha\beta}$, moreover:

1. The R_{ik} (i, k = 0, 1, 2, 3) are those of Einstein.

2. The R_{i4} (i = 0, 1, 2, 3) are the divergences of the electromagnetic field; i.e., one has ([1]):

$$R_{i4} = \frac{\partial F_{\cdot i}^{h}}{\partial x_h} - \Gamma_{ih}^{l} F_{\cdot l}^{h} + \Gamma_{hr}^{r} F_{\cdot i}^{h} = F_{\cdot i|h}^{h}.$$

3. Finally:

$$R_{44} = -\,\psi\left[\frac{\partial(g^{ri}\psi_i)}{\partial x_r} + \Gamma_{ih}^{h}\psi^{i}\right].$$

In order to obtain the equations of the electromagneto-gravitational field in E_4, it will suffice to take our inspiration from what we know in E_4. Now in general, we will have:

(2) $$R_{ik} = \kappa\left(T_{ik} - \tfrac{1}{2} g_{ik} T\right) \qquad (i, k = 0, 1, 2, 3)$$

in E_4, in which T_{ik} is the energy-quantity of motion tensor, and κ is a constant of order 10^{-47} CGS. Now:

$$T_{ik} = m\,u_i\,u_k \quad \text{and} \quad T = m\,u_i\,u^{i} \qquad \left(u^{i} = \frac{dx^{i}}{ds}\right).$$

We can then take the general equations in E_5 to be:

(3) $$R_{\alpha\beta} = \kappa\left(T_{\alpha\beta} - \tfrac{1}{2} g_{\alpha\beta} T\right) \qquad (\alpha, \beta = 0, 1, 2, 3, 4),$$

with $T_{\alpha\beta} = m\,u_\alpha\,u_\beta$ and $T = m\,u_\alpha\,u^{\alpha}$.

For α, ρ = 0, 1, 2, 3, we will have the Einstein equations in the approximation that we have assumed; for α = 0, 1, 2, 3, and β = 4, we will have:

$$F_{\cdot i|h}^{h} = \kappa\left(m u_i\,u_4 - \tfrac{1}{2} g_{4i} T\right) \qquad \text{or} \qquad F_{\cdot i|h}^{h} = \kappa m\,(u_i\,u_4 - \varphi_i\,u_\alpha\,u^{\alpha}),$$

which can be written:

([1]) Cf., KALUZA, Sitzungsber. d. preuss. Akad. d. Wiss. **2** (1921), 966-972.

(4) $$F^{h}_{\cdot\, i|h} = \kappa\, \psi^2\, s_i$$

with our approximations, in which the s_i are the components of the current. Now, if one compares these equations with the second group of Maxwell equations then one will see that if $\kappa\ \psi^2$ is close to unity then our general equations will imply precisely the equations that allow one to find the potentials when one knows the charges and their velocities ([1]).

Now, if $\kappa\, \psi^2$ has order unity then that equation will become:

(5) $$\frac{\partial(g^{ri}\, \psi_i)}{\partial x_r} + \Gamma^h_{ih}\psi^i + \frac{1}{2}\frac{e^2}{m}\psi = 0.$$

Finally, the equation for $\alpha = \rho = 4$ is written:

$$\frac{\partial(g^{ri}\, \psi_i)}{\partial x_r} + \Gamma^h_{ih}\psi^i = -\frac{1}{2}\kappa\psi^2\,\frac{e^2}{m}.$$

That equation will once more admit the geodesics of the ds^2 for E_4 as bicharacteristics, and due to our approximations, the geodesics of the ds^2 for our E_5 .

Remarks. – It is clear that our approximations will no longer be valid when the φ_i are appreciable, which is the case in the theory of spectra. One must then seek what our equations will become in the case where one neglects the derivatives of the g_{ik} in comparison to $g_{i\,4} = 2\varphi_i$.

Our results lead back to the macroscopic wave mechanics that L. de Broglie called *geometrical*, as opposed to what he called *physical*, in the same way that geometrical optics is opposed to physical optics.

If one writes equations (3) rigorously for a ds^2 of the type (1) then the equation for $\alpha = \rho = 4$ will remain linear in the second derivatives of ψ, and the same relationship will exist between that equation and the geodesics of E_5 that exists between a wave equation and its bicharacteristics.

([1]) Cf., KALUZA, *loc. cit.*; that author showed that the first group of Maxwell equations will result from an identity between the Christoffel symbols for E_5 .