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DIRECTOR

Henri **VILLAT**

Member of the Institute
Professor at the Sorbonne
Director of the "Journal de Mathématiques pures et appliquées"

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Analytical mechanics and wave mechanics

By GUSTAVE JUVET
Professor at the University of Lausanne

Translated by D. H. Delphenich

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NOTICE

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ANALYTICAL MECHANICS

AND

WAVE MECHANICS

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INTRODUCTION

The wave mechanics that was created by L. de Broglie and E. Schrödinger has deep roots in analytical mechanics. Its brilliant founders have shown how their new concepts belong to the ideas that were sketched out a century ago by Hamilton [**27**, **28**]. While interpreting Hamilton's principle and that of de Maupertuis, Hamilton showed how the notion of wave could be juxtaposed with that of trajectory in classical mechanics. The prodigious advances of these discoverers in the new domains that they introduced into science have extended the principles that guided their first attempts considerably. Wave mechanics has taken on an aspect that is quite different from that of analytical mechanics so that despite the presentations that have been made in order to exhibit the solidarity of the Jacobi-Hamilton theory with that of de Broglie waves and the Schrödinger equation (and here we are thinking of the beautiful *Introduction à l'étude de la Mécanique ondulatoire* by de Broglie himself), the neophyte will always be struck by the differences more than the similarities, and he will be seduced by the successes of the new theories in quantum physics more than by their illustrious origins.

Meanwhile, if one utilizes not only the works of Hamilton, but also those of Delassus, Beudon, and Hadamard, on the characteristics of second-order partial differential equations then one can show that even before the birth of quantum physics itself, it was possible for it to arouse some beautiful mathematical forms in a sophisticated mind.

It is easy, they say, to prophesy after the fact. Meanwhile, there is a very strong temptation to show the continuity of the efforts of a spirit that is immanent in the world of learning after a discontinuity in the advance of genius.

Even if one denies that continuity, if one would like to see only some manifestation of a spirit of escalation in the concept of history then one would, of course, have to recognize the utility (which can be called pedagogical) of the attempts that have the search for that continuity as their objective. In truth, the origin of this fascicle was precisely a course in mathematical physics in which we sought to show the attendees that were very familiar with analytical mechanics and physical optics how it is possible to arrive smoothly at the threshold of the new mechanics itself. We recall the beautiful essay in which Levi-Civita has recently shown [**21**] how one can pass from Newton's mechanics to that of Einstein by some elegantly-arranged approximations. Inspired by that example, we have, in turn, sought to show how it is possible to pass to the brilliant and audacious wave mechanics without leaving the time-worn paths of analytical mechanics.

It was Vessiot who gave a perfectly rigorous and elegant presentation of the interpretation of Jacobi-Hamilton theory by means of the concepts of wave theory. In the first two chapters, we shall follow the two papers [**39**, **40**] that Vessiot write in 1906 and 1909, and the second one, in particular, for which we shall give an extended summary. The principle of enveloping waves is presented there with a rigor that one will hardly find in the physics treatises, and its importance for the integration of partial differential equations is clearly brought to light there. We have omitted the consequences that relate to the principle of least action that were inferred by Vessiot from his own principles, as they are hardly useful for our purposes: However, if we have overlooked that principle then it would be easy to show, in passing, its kinship to Fermat's principle and the principle of de Maupertuis. Nevertheless, we shall insist upon the problem of geodesics, whose importance is great in gravitation, and we shall recall the Jacobian form of the equations of motion for the electron.

The third chapter relates to the discoveries of Beudon, Delassus, and Hadamard on second-order partial differential equations. We introduce the notions of characteristic and bicharacteristic, and we shall show that characteristics define the surfaces of equal phase for the propagation of periodic waves whose frequency is infinite.

Having thus attached, on the one hand, a propagation of waves with any motion that is defined by analytical dynamics, *à la* Vessiot, and on the other hand attached trajectories to it in the approximation that is called the geometrical optics approximation, *à la* Hadamard, it will be legitimate to make a wave-like syntheses of mechanics, and to that end we shall recall some attempts to base electromagnetism and gravitation upon a unified theory, along with that of matter waves, as well. We quickly abandon those speculations in order to exploit the theory of Hadamard by using the idea of periodicity. Thanks to the simplest second-order equation that one can attach to the motion of an electron in an electromagnetic field, we can see, on the one hand, how the notion of probability can be introduced into those classical theories thanks to an idea that goes back to L. de Broglie, namely, a fictitious fluid that is defined by a *class* of motions, and on the other hand how one can smoothly introduce Planck's constant, which links the hypothetical frequency of the matter wave to the very real energy of the particle by de Broglie's principle. Finally, we shall review the notion of *group velocity*, which will show that dispersive waves transport energy with a velocity that is not equal to the phase velocity, but with a velocity that is equal to precisely that of the material point in motion to which they are attached. We therefore conclude the latter ramifications in a place where a somewhat clever historian or a sophisticated pedagogue can perceive one of the main roots of analytical mechanics, which mingle and connect with those of the new mechanics.

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CHAPTER I

THE PROPAGATION OF WAVES AND THE INTEGRATION OF FIRST-ORDER PARTIAL DIFFERENTIAL EQUATIONS ([1]).

Principle of enveloping waves.

1. – Let E_n be a Euclidian space, so it is a set of points P (x_1, …, x_n), or more simply P(x), that are referred to a rectangular system of axes. E_n is a *medium* in which certain disturbances can propagate by waves. That signifies that the points of E_n can instantaneously acquire a property: If that property is manifested at the instant t at all points of a multiplicity $\mathcal{M}$ then it will cease to belong to the points of $\mathcal{M}$ at the following instants, and it will be manifested by the points of another multiplicity $\mathcal{M}'$ at $t + \Delta t$. The appearance of the property at a point (x) will be called a *disturbance;* any multiplicity that is the locus of disturbed points at the same instant will be a *wave.*

We propose the following principle:

The multiplicity $\mathcal{M}'$ is determined by the nature of the medium (relative to the property in question), the instant t, the interval Δt, and the multiplicity $\mathcal{M}$.

2. – One defines the *nature* of the medium by giving the system of *derived waves* (or elementary waves) that have their origins at the various points of a medium at the instant t. Let P (x) be the only disturbed point at the instant t, and at $t + \Delta t$, the locus of disturbed points will be a multiplicity M (x | t, Δt) that one says has P for its *origin*, or that it *issues* from P. One takes the homothety that relates to P with the ratio 1 / Δt, and one makes Δt tend to zero. The limiting multiplicity (if it exists, which we assume to be the case) is properly the *derived wave* that has P for its origin at the instant t.

The homothetic image of the derived wave relative to P with the ratio dt is the *elementary wave* that has P for its origin and corresponds to the instant t.

In general, the system of derived waves depends upon t (*variable regime*); however, it can happen that it is *independent* (*permanent regime*).

We suppose that each derived wave has ∞^{n-1} points, since that is the most common case in applications.

3. – Propagation is governed by the following law, which is called the "law of enveloping waves:"

([1]) For a discussion of the various aspects of the notion of wave, one should refer to the excellent article by Levi-Civita and Amaldi [**20**], and I am grateful to them for sending me the first proofs.

Let $\mathcal{M}$ be an arbitrary wave at the instant t ; let $\mathcal{M}'$ be the wave that it produces at the instant $t + dt$. If each point P (x) of $\mathcal{M}$ is disturbed at only the instant t then it will emit an elementary wave M (x | t, dt) in the course of time dt. The envelope $\mathcal{M}''$ of all elementary waves that issue at the instant t from all points of $\mathcal{M}$ represents $\mathcal{M}'$, up to infinitesimals of order higher than dt, which is considered to be the principal infinitesimal.

We shall insist upon neither the difference between $\mathcal{M}'$ and $\mathcal{M}''$ nor the lack of contradiction in the principle of enveloping waves. One can establish with full rigor, as Vessiot did, that the representation of $\mathcal{M}'$ by $\mathcal{M}''$ is a true identification that implies no contradiction. The reader is advised to reread that after the following calculations.

4. – Let (2):

$$u_i\ \xi_i - 1 = 0$$

be the equation of a plane that is referred to a system of rectangular axes that have P (x) for their origin and are parallel to the axes of E_n . The ξ_i are the running point coordinates.

The equation:

$$H\ (t \mid x_1, \ldots, x_n \mid u_1, \ldots, u_n) = 0$$

or

$$H\ (t \mid u) = 0$$

is the tangential equation of the derived wave that issues from P (x), when referred precisely to the system of axes that has P for its origin.

The tangent plane to the elementary wave is:

$$u_i\ \xi_i - dt = 0,$$

and as a result, the tangential equation of the elementary wave is:

$$H\ (t \mid x \mid u\ dt) = 0.$$

Upon taking homogeneous coordinates $U_1, U_2, \ldots, U_n, U_{n+1}$ such that:

$$u_i = \frac{U_i}{U_{n+1}},$$

once one sets $U_{n+1} = 1$, and furthermore $U_i = u_i$, the tangential equation can be written:

$$\pi\ (t \mid x \mid u) = 1;$$

(2) When two indices in a monomial are equal, one intends that they are to be summed from 1 to n with respect to that common index, unless stated to the contrary.

π is homogeneous of degree 1 in the u_i , which amounts to defining that function by the identity:

$$H\left(t \mid x \mid \frac{u}{\pi}\right) = 0.$$

The elementary wave will have the equation:

$$\pi (t \mid x \mid u)\, dt = 1.$$

The coordinates of the contact point of the plane $\pi (u)$ with the derived wave are:

$$\xi_i = \frac{\partial \pi}{\partial u_i},$$

and with the elementary wave:

$$\xi_i = \frac{\partial \pi}{\partial u_i}\, dt .$$

Only the ratios of the u_i enter into these expressions, because they have degree zero in the u_i ; they will then give the coordinates of a point of contact of a tangent plane that is parallel to a given one. In general, there will be several functions π, which represent the various sheets of the derived wave, which are separated in such a manner that each of them will have only a tangent plane that is parallel to a given plane.

When referred to the original system of axes, the coordinates of the contact point of the tangent plane to the elementary wave will be:

$$X_i = x_i + \frac{\partial \pi (t \mid x \mid q)}{\partial q_i}\, dt,$$

in which the equation of that tangent plane is:

$$u_i\, (X_i - x_i) - 1 = 0$$

or

$$q_i\, X_i - 1 = 0,$$

and one will see that the tangential equation of the elementary wave is:

$$\pi (t \mid x \mid q)\, dt + q_i\, x_i = 1. \tag{1}$$

5. – One must find the envelope of all the elementary waves that are represented by the last equation when P (x) describes $\mathcal{M}$. Let $p_1, \ldots, p_n$ be the direction parameters of the normal to the tangent plane to $\mathcal{M}$ at P. For a displacement $\delta \mathbf{P}$ on $\mathcal{M}$, one will have:

$$p_i\ \delta x_i = 0,$$

but one will also have:

$$\frac{\partial \pi(t \mid x \mid q)}{\partial x_i}\ \delta x_i\ dt + q_i\ \delta x_i = 0,$$

and that equation must be a consequence of the preceding one for all δx_i that satisfy it. Therefore:

$$\frac{\partial \pi(t \mid x \mid q)}{\partial x_i}\ dt + q_i = m\ p_i\ ,$$

in which m is a factor that will be determined when one takes (1) into account.

It is easy, moreover, to see that among the contact elements that are common to the elementary wave (1) and to all infinitely-close waves, there is one and only one of them that tends to the contact element $(x_1, \ldots, x_n \mid p_1, \ldots, p_n)$ of the wave $\mathcal{M}$ when dt tends to zero.

If one restricts the p_i, which are defined by their ratios, to ones that verify the equation:

(2) $$\pi\,(t \mid x \mid p) = 1$$

then they will be defined perfectly. If $(x' \mid p')$ are the coordinates of the contact element that tends to $(x \mid p)$ when dt tends to zero then one will further set:

$$\pi\,(t' \mid x' \mid p') = 1$$

in order to define the p', and if one lets dx_i denote the principal part of $x'_i - x_i$, while dp_i denotes that of $p'_i - p_i$, then it will be easy to see that:

$$dx_i = \frac{\partial \pi(t \mid x \mid p)}{\partial p_i}\,dt,$$

$$dp_i = -\frac{\partial \pi(t \mid x \mid p)}{\partial x_i}\,dt + p_i\ d\mu,$$

in which $d\mu$ is infinitely small and is determined by taking into account the fact that:

$$\frac{\partial \pi}{\partial t}dt + \frac{\partial \pi}{\partial x_i}dx_i + \frac{\partial \pi}{\partial p_i}dp_i = 0;$$

one will find that:

$$d\mu = -\,\frac{\partial \pi}{\partial t}\,dt.$$

Moreover:

Each contact element $(x \mid p)$ of a wave $\mathcal{M}$ that is considered at the instant t will correspond to a new contact element on the infinitely-close wave that results after a time dt, and it will be given, up to second-order infinitesimals, by the formulas:

(3) $$dx_i = \frac{\partial \pi(t \mid x \mid p)}{\partial p_i} dt,$$

(4) $$dp_i = -\left[\frac{\partial \pi(t \mid x \mid p)}{\partial x_i} + p_i \frac{\partial \pi(t \mid x \mid p)}{\partial t}\right] dt,$$

while supposing that one always has:

$$\pi(t \mid x \mid p) = 1.$$

If one returns to the inhomogeneous equation:

$$H(t \mid x \mid p) = 0$$

then upon setting:

$$w_i = \frac{p_i}{\pi},$$

one will have the identities:

$$\frac{\partial H(t \mid x \mid w)}{\partial t} - \frac{w_i}{\pi} \frac{\partial H(t \mid x \mid w)}{\partial w_i} \frac{\partial \pi}{\partial t} = 0,$$

$$\frac{\partial H(t \mid x \mid w)}{\partial x_k} - \frac{w_i}{\pi} \frac{\partial H(t \mid x \mid w)}{\partial w_i} \frac{\partial \pi}{\partial x_k} = 0,$$

$$\frac{\partial H(t \mid x \mid w)}{\partial w_k} - \frac{w_i}{\pi} \frac{\partial H(t \mid x \mid w)}{\partial w_i} \frac{\partial \pi}{\partial p_k} = 0,$$

which will reduce to some simple forms that we shall write out when:

$$\pi(t \mid x \mid p) = 1,$$

and the equations that define the dx_i and the dp_i will take the forms ([3]):

(3′), (4′) $$\frac{dx_i}{\dfrac{\partial H}{\partial p_i}} = \frac{dp_i}{-\left(\dfrac{\partial H}{\partial x_i} + p_i \dfrac{\partial H}{\partial t}\right)} = \frac{dt}{\sum_{k=1}^{n} p_k \dfrac{\partial H}{\partial p_k}}.$$

([3]) The i's are not summed over in these expressions.

Integration. Theory of characteristics.

6. – Knowing an original wave $\mathcal{M}_0$ that is given at the instant t_0 , find the wave $\mathcal{M}$ that results at the instant t. That is the general problem of the propagation of waves. One imagines that $\mathcal{M}$ is deduced from $\mathcal{M}_0$ by integrating the differential equations that were obtained above, such that the given $\mathcal{M}_0$ defines the initial conditions. What will make the integration a little complicated is the fact that the differential equations (3) and (4) are accompanied by equation (2).

One sees immediately that if (2) is verified by the initial conditions then one will always have $\pi(t \mid x \mid p) = 1$ by virtue of the differential equations themselves for each instant t, because:

$$\begin{aligned} d(\pi - 1) &= \frac{\partial \pi}{\partial t} dt + \frac{\partial \pi}{\partial x_i} dx_i + \frac{\partial \pi}{\partial p_i} dp_i \\ &= \frac{\partial \pi}{\partial t} \left(1 - p_i \frac{\partial \pi}{\partial p_i} \right) dt \\ &= \frac{\partial \pi}{\partial t} (1 - \pi)\, dt. \end{aligned}$$

Moreover, the function $\pi - 1$ satisfies the homogeneous equation:

$$\frac{d(\pi - 1)}{dt} + \pi_1 (\pi - 1) = 0 \qquad \left(\pi_1 = \frac{\partial \pi}{\partial t} \right),$$

and one sees that if $(\pi - 1)_0 = 0$ then one will always have $\pi - 1$ for any t.

The equation:

$$\pi(t \mid x \mid p) = 1$$

is invariant under the transformation:

(5) $$x_i = A_i (t \mid x^0 \mid p^0 \mid t_0),$$

(6) $$p_i = B_i (t \mid x^0 \mid p^0 \mid t_0),$$

which defines the general integral of (3) and (4), i.e., the integral that reduces to:

$$x_i = x_i^0, \qquad p_i = p_i^0$$

for $t = t_0$.

We remark, in passing, that the functions A_i and the ratios of the functions B_i are homogeneous of degree zero with respect to $p_1^0, \ldots, p_n^0$. One proves that by substituting A_i for x_i and $m\, B_i$ for p_i in (3) and (4); one first sees that:

$$dm = m (1 - m)\, \pi_1\, dt.$$

One determines the integral M of that equation that reduces to the constant m_0 for $t = t_0$. The functions:

$$x_i = A_i\,, \qquad p_i = M\,B_i$$

constitute the solution of (3) and (5) that is defined by the initial conditions:

$$x_i = x_i^0\,, \qquad p_i = m_0\; p_i^0\,;$$

however, that solution is obviously given by the equations:

$$x_i = A_i\,(t \mid x^0 \mid m_0\,p^0 \mid t_0),$$
$$p_i = B_i\,(t \mid x^0 \mid m_0\,p^0 \mid t_0),$$

so

$$A_i\,(t \mid x^0 \mid p^0 \mid t_0) = A_i\,(t \mid x^0 \mid m_0\,p^0 \mid t_0),$$
$$M\,B_i\,(t \mid x^0 \mid p^0 \mid t_0) = B_i\,(t \mid x^0 \mid m_0\,p^0 \mid t_0),$$

which proves our assertion.

7. – The x_i^0 and the p_i^0 constitute the multiplicity $\mathcal{M}_0$; we shall show that formulas (5), (6), which define the integral of (3), (4), will define a multiplicity $\mathcal{M}$ at each instant. It suffices to prove that the equation:

$$p_i\;\delta x_i = 0$$

is invariant under the transformation (5), (6), in which δ is a differentiation symbol that is independent of d; in particular, $d\;\delta x_i = \delta\,dx_i$.

One immediately sees that:

$$\frac{d}{dt}(p_i\;\delta x_i) + \pi_1\,(p_i\;\delta x_i) = 0,$$

by virtue of the differential equations (3) and (4) themselves. Therefore, if the x_i^0 and the p_i^0 are functions of $n - 1$ parameters $\alpha_1, \ldots, \alpha_{n-1}$ that verify the relation:

$$p_i^0\;\delta x_i^0 = 0$$

then the functions:

$$x_i = A_i\,, \qquad p_i = B_i$$

will be functions of the same parameters that verify the relation:

$$p_i\;\delta x_i = 0,$$

in which δ is a differentiation symbol that produces variations of only the α.

The transformation (5) and (6), in which t and t_0 are arbitrary constants, changes any multiplicity into a multiplicity. It is a *contact transformation*.

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7. – One says *trajectory* or *ray* to mean the locus of points whose coordinates are:

$$x_i = A_i(t \mid x_1^0, \ldots, x_n^0 \mid p_1^0, \ldots, p_n^0 \mid t_0)$$

when only t varies, while the quantities x_i^0, p_i^0, t_0 are constant. Each point of the trajectory corresponds to an instant t, but that point is considered only at the instant to which it corresponds.

A contact element passes through each point of the trajectory, and it is defined by the equations:

$$p_i = B_i(t \mid x_1^0, \ldots, x_n^0 \mid p_1^0, \ldots, p_n^0 \mid t_0).$$

One has:

$$\frac{dx_i}{dt} = \frac{\partial \pi}{\partial p_i},$$

moreover, which shows that the point P (x) of a trajectory, which exists at the instant t, is the origin of a derived wave at that instant, and the direction of the trajectory at P (x) is the one that goes from P to the contact point of the tangent plane to the derived wave that is parallel to the contact element that is carried by the point P (x) of the trajectory at the instant t.

The set that consists of a trajectory and all of the contact elements that are carried by its points is a *characteristic.* Equations (5) and (6) will define a characteristic when only t varies.

One will easily understand the following statement, moreover:

A multiplicity M at an instant t will result from the simultaneous transport of the contact elements of a multiplicity $\mathcal{M}_0$ that is given at the instant t_0. That transport is defined spatially and temporally by the characteristics that have the contact elements of $\mathcal{M}_0$ for their elements at the instant t_0.

8. – The family of multiplicities $\mathcal{M}'$ that results from the multiplicity $\mathcal{M}_0$ by the transport along characteristics and the family of waves $\mathcal{M}'$ that issue from $\mathcal{M}_0$ under the mode of propagation envisioned are such that one passes from one multiplicity in each family to the infinitely-close one by means of the variation that was defined in equations (3) and (4), in which (2) is always satisfied, and one will then see that the principle of enveloping waves implies no contradiction. It remains to be shown that these two families are identical.

One can prove that fact by remarking that the family of $\mathcal{M}'$ is the only one that is defined by the variation (3) and (4), while (2) is satisfied. Indeed, imagine that a family of multiplicities is defined by the equation:

(7) $$F(x_1, \ldots, x_n) = t,$$

in which t is the parameter that varies from one multiplicity to the other. Let:

$$\frac{\partial F}{\partial x_i} = P_i$$

and

$$\pi(F \mid x_1, \ldots, x_n \mid P_1, \ldots, P_n) = \bar{\pi}.$$

For a contact element of (7), set:

$$p_i = \frac{P_i}{\bar{\pi}}, \tag{8}$$

so (2) will be verified for those values. Moreover, upon taking (3) into account, one infers from (7):

$$dt = P_i\, dx_i = P_i \frac{\partial \pi}{\partial p_i} dt = P_i \frac{\partial \bar{\pi}}{\partial P_i} dt = \bar{\pi}\, dt,$$

because the derivatives $\partial\pi / \partial p_i$ depend upon only the relationship between the p_i, which is the same as the relationship between the P_i, and consequently, they will be equal to the derivatives $\partial\pi / \partial p_i$; moreover, $\bar{\pi}$ is homogeneous of degree 1 in the P_i. One will then see that:

$$\bar{\pi} = 1. \tag{9}$$

As a result:

$$p_i = P_i .$$

One finds that by virtue of (4):

$$dP_i = -\left(\frac{\partial \pi}{\partial x_i} + p_i \frac{\partial \pi}{\partial t}\right) dt = -\left(\frac{\partial \bar{\pi}}{\partial x_i} + P_i \frac{\partial \bar{\pi}}{\partial F}\right) dt$$

or

$$\frac{\partial^2 F}{\partial x_i\, \partial x_k} dx_k + \left(\frac{\partial \bar{\pi}}{\partial x_i} + \frac{\partial \bar{\pi}}{\partial F}\frac{\partial F}{\partial x_i}\right) dt = 0;$$

i.e.:

$$\frac{\partial \bar{\pi}}{\partial P_k}\frac{\partial P_k}{\partial x_i} + \frac{\partial \bar{\pi}}{\partial x_i} + \frac{\partial \bar{\pi}}{\partial F}\frac{\partial F}{\partial x_i} = 0.$$

Now, when these equations are written, while taking (3) and (4) into account, they will result from (9) by differentiation with respect to x_i. One has then proved that the relations (3) and (4) result from (7) and (8) by differentiation.

The family (7) satisfies the partial differential equation:

$$\pi\left(F \mid x_1, \ldots, x_n \mid \frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n}\right) = 1, \tag{10}$$

which is only equation (2) in which one has set:

$$p_i = \frac{\partial t}{\partial x_i}.$$

Since the most general first-order partial differential equation:

$$H\,(t \mid x_1, \ldots, x_n \mid p_1, \ldots, p_n) = 0 \qquad \left(p_i = \frac{\partial t}{\partial x_i}\right) \tag{11}$$

reduces to the form (10), one will see that the theory of characteristics will permit one to construct a solution of (11) that takes the given value t_0 at all points of an arbitrarily-chosen multiplicity $\mathcal{M}_0$ upon integrating (3) and (4) or (3′), (4′).

The fact that there is only one will result from the following analysis:

Let the family $\mathcal{M}$ that is defined (7) satisfy (9), in which $P_i = \partial F / \partial x_i$. One and only one $\mathcal{M}$ that corresponds to one value of t passes through each point P (x) of E_n. Construct the derived wave that issues from P at that instant, draw the plane that is tangent to it and parallel to the tangent plane to $\mathcal{M}$ at P, and draw the line that joins P to the contact point Q. One has a direction D at each point of E_n; there exists a family of tangent curves to each of its points in the corresponding direction. Each point also corresponds to a value of t and a contact element, which is the contact element to the multiplicity $\mathcal{M}$ that passes through it, so I say that one has defined the characteristics in that way, and therefore any family (7) that satisfies (9) is provided by the construction of paragraph **6**.

The curves that we just discussed are indeed integral curves of the system:

$$dx_i = \frac{\partial \bar{\pi}}{\partial P_i}\,dt,$$

and one infers, by virtue of (8), that:

$$dF = P_i\,dx_i = \bar{\pi}\,dt = dt,$$

and therefore (7), provided that $x_1^0, \ldots, x_n^0, t_0$ satisfy (7).

The contact elements are defined by the equations:

$$p_i = P_i\,. \tag{12}$$

I say that they imply equations (4); i.e., that one must have the equations:

$$P_i = \frac{\partial F}{\partial x_i},$$

by virtue of (7), (9), and:

$$dP_i = -\left(\frac{\partial \bar{\pi}}{\partial x_i} + P_i \frac{\partial \bar{\pi}}{\partial F}\right) dt$$

from (12), or:

$$\frac{\partial P_i}{\partial x_k}\frac{\partial \bar{\pi}}{\partial P_k} + \frac{\partial \bar{\pi}}{\partial x_i} + P_i \frac{\partial \bar{\pi}}{\partial F} = 0,$$

or rather:

$$\frac{\partial \bar{\pi}}{\partial F}\frac{\partial F}{\partial x_i} + \frac{\partial \bar{\pi}}{\partial x_i} + \frac{\partial \bar{\pi}}{\partial P_k}\frac{\partial P_k}{\partial x_i} = 0,$$

because:

$$\frac{\partial P_i}{\partial x_k} = \frac{\partial P_k}{\partial x_i}.$$

However, that will result from (9) when one differentiates it with respect to x_i, and (9) is verified identically, by hypothesis.

Any solution of (9) that takes the value t_0 at the various points of an $\mathcal{M}_0$ is obtained by the construction in paragraph **6** by means of characteristics. As a result, there is only one solution that satisfies that initial condition.

Any first-order partial differential equation corresponds to a propagation of waves, and conversely.

Jacobi's theorem

9. Jacobi's theorem. – If one remarks that the transformation (5), (6) acts upon the contact element $(x^0 \mid p^0)$ without one having to specify that it belongs to $\mathcal{M}_0$ then one can confirm that if two original waves have a common contact element then the waves that it produces at an arbitrary instant will also have a common contact element, which is the transform of the preceding one, and one will see effortlessly, moreover, that the principle of the enveloping wave is verified rigorously for a finite variation of time. The wave $\mathcal{M}$ at time t is then the envelope of the waves that are emitted at t_0 and considered at the instant t for all points of $\mathcal{M}_0$; if one then knows the latter then one can know $\mathcal{M}$ without integration.

There is more: Imagine that one knows the propagation of ∞^n arbitrary original waves. It will be obvious when one is given an integral:

$$t = G\,(x_1, \ldots, x_n \mid a_1, \ldots, a_n)$$

of the partial differential equation (10) that depends upon n arbitrary constants which will serve to define ∞^n waves, if they are essential, and consequently G will be a *complete integral* of (10).

There are ∞^{2n-1} contact elements in space. One can determine all of them for $t = t_0$: Each of them is defined to be common to the multiplicity:

(13) $$t_0 = G(x \mid a),$$

and all of them that result from infinitely-small variations of the a_i will be coupled by just one relation:

$$b_i\, \delta a_i = 0.$$

One will then have $2n + 1$ conditions:

$$G = t_0 , \qquad \frac{\partial G}{\partial a_i} = m\, b_i , \qquad p_i = h \frac{\partial G}{\partial x_i},$$

which will reduce to $2n - 1$ conditions that define just one contact element when the ratios of the b_i are given; conversely, if the contact element is given then one can infer the a_i and the ratios of the b_i .

At the instant t, that element will correspond to an element that is common to the multiplicities that issue from the first one and that one can obtain by giving the value t to G and fixing the a_i and the b_i as we discussed. Moreover, the equations:

(14) $$G(x \mid a) = t, \quad \frac{\partial G}{\partial a_i} = m\, b_i , \quad p_i = h \frac{\partial G}{\partial x_i}$$

define that element, and also as a result, if one considers the a_i and the ratios of the b_i to be $2n - 1$ arbitrary constants then they will represent the general equations of ∞^{2n-1} possible characteristics. They give the general integral of the differential equations of the characteristics and are equivalent to equations (5) and (6). Those propositions constitute *Jacobi's theorem* on the interpretation of the characteristic equations.

If $\mathcal{M}_0$ is given as the envelope of ∞^{n-1} multiplicities (12) then the a_i will verify the equation:

$$\Phi(a_1, \ldots, a_n) = 0,$$

so the wave $\mathcal{M}$ at the instant t will be the envelope of ∞^{n-1} multiplicities:

$$t = G(x \mid a),$$

with $\Phi(a) = 0$.

10. – In the case of the permanent regime, the equations:

$$H(x_1, \ldots, x_n \mid p_1, \ldots, p_n) = 0$$

or

$$\pi(x_1, \ldots, x_n \mid p_1, \ldots, p_n) = 1$$

do not contain t. We leave to the reader the task of writing out the characteristic equations, and we remark that their general integral will have the form:

$$x_i = \mathcal{A}_i\,(t - t_0 \mid x^0 \mid p^0),$$
$$p_i = \mathcal{B}_i\,(t - t_0 \mid x^0 \mid p^0).$$

The wave that is emitted by $\mathcal{M}_0$ at the instant t depends upon only the interval $t - t_0$, and not upon t_0. A contact element is always transported by the same trajectory, no matter what the initial position.

The family of contact transformations that gives the law of propagation then forms a group with one parameter, namely, $t - t_0$. (Cf., Lie [**25** and **26**].)

Trajectories

11. Trajectories. – One can define the trajectories independently of the contact elements that transport them. It will suffice to eliminate the p_i from equations (3), (4), and (2).

One will easily arrive at that upon starting with the point-like equation of the wave that issues from P (x). Let:

(15) $$\Omega\,(t \mid x_1, \ldots, x_n \mid \xi_1, \ldots, \xi_n) = 1$$

be that equation, when written in a form that is homogeneous of degree 1 in the ξ_i. The coordinates of the tangent place to that wave in $\xi_1, \ldots, \xi_n$ are:

$$u_i = \frac{\partial \Omega(t \mid x \mid \xi)}{\partial \xi_i},$$

just as:

$$\xi_i = \frac{\partial \pi(t \mid x \mid u)}{\partial u_i}$$

are the coordinates of the contact point of the tangent plane whose coordinates are u_i.

Rewrite equations (3) and (4), while denoting the derivatives with respect to t by primes:

(3′) $$x_i' = \frac{\partial \pi}{\partial p_i},$$

(4′) $$p_i' = -\frac{\partial \pi}{\partial x_i} - p_i \frac{\partial \pi}{\partial t}.$$

Equations (3′) and (2) are equivalent to the system:

(16) $$\Omega\,(t \mid x \mid x') = 1,$$

$$p_i = \frac{\partial \Omega(t \mid x \mid x')}{\partial x'_i}.$$

On the other hand, the identity:

$$\Omega\left(t \mid x \mid \frac{\partial \pi}{\partial p}\right) = 1$$

will imply that:

$$\frac{d\Omega}{dt} + \frac{\partial \Omega}{\partial x'_k}\frac{\partial^2 \pi}{\partial p_k\, \partial t} = 0$$

or

$$\frac{\partial \Omega}{\partial t} + p_k\, \frac{\partial^2 \pi}{\partial p_k\, \partial t} = 0,$$

but $\partial \pi / \partial t$ is homogeneous of degree 1 in the p_i, so:

$$\frac{\partial \Omega}{\partial t} + \frac{\partial \pi}{\partial t} = 0,$$

moreover; hence:

$$\frac{\partial \Omega}{\partial x_i} + \frac{\partial \pi}{\partial x_i} = 0$$

as well, and equations (4′) will become:

$$\frac{d\dfrac{\partial \Omega}{\partial x'_i}}{dt} - \frac{\partial \Omega}{\partial t}\frac{\partial \Omega}{\partial x'_i} - \frac{\partial \Omega}{\partial x_i} = 0. \tag{17}$$

The trajectories are then defined by the system (16) and (17), which is over-determined. In order to simplify it, we remark that since Ω is homogeneous in the x'_i:

$$\frac{\partial \Omega(t \mid x \mid x')}{\partial x'_i} = \frac{\partial \Omega(t \mid x \mid dx)}{\partial\, dx_i},$$

$$\frac{\partial \Omega(t \mid x \mid x')}{\partial x_i} = \frac{\partial \Omega(t \mid x \mid dx)}{\partial x_i}\frac{1}{dt},$$

$$\frac{\partial \Omega(t \mid x \mid x')}{\partial t} = \frac{\partial \Omega(t \mid x \mid dx)}{\partial t}\frac{1}{dt},$$

and if one sets:

$$\bar{\Omega} = \Omega\,(t \mid x \mid dx)$$

then one will have:

(18) $$d\frac{\partial\bar{\Omega}}{\partial\, dx_i} - \frac{\partial\bar{\Omega}}{\partial t}\frac{\partial\bar{\Omega}}{\partial\, dx_i} - \frac{\partial\bar{\Omega}}{\partial x_i} = 0,$$

instead of (17). Upon multiplying this by dx_i and summing over i, one will get:

$$\frac{\partial\bar{\Omega}}{\partial t}(dt - \bar{\Omega}) = 0.$$

Hence, in the variable regime, the trajectories are defined by the system (18), not only by their form, but also by the law that describes them.

In the permanent regime, it is the system:

$$d\frac{\partial\bar{\Omega}}{\partial\, dx_i} - \frac{\partial\bar{\Omega}}{\partial x_i} = 0$$

that defines their form; it reduces to $n - 1$ equations. The law by which the trajectories are traversed is given by the equation:

$$dt = \bar{\Omega}.$$

One calls a trajectory a *ray* when one considers only its form. If a ray is given then one will find the corresponding characteristic upon remarking that:

$$p_i = \frac{\partial\bar{\Omega}}{\partial\, dx_i}.$$

12. – The Eulerian form of the equations that we have obtained (at least, in the case of the permanent regime) suggests the idea of investigating whether there are maximum or minimum properties with respect to the propagation of waves. We shall not appeal to such properties in what follows, so we shall confine ourselves to pointing out that the problem was treated by Vessiot with full rigor.

13. – It is not necessary to insist upon the theory of integration for partial differential equations that one can infer from the preceding considerations. In truth, that theory coincides with the usual theory, up to the language that one adopts in order to illustrate the use of characteristics and complete integrals and to make it more intuitive. One can solve the Cauchy problem with no difficulty, which consists of finding an integral surface that passes through a given curve that is not a characteristic.

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CHAPTER II

APPLICATIONS OF THE PRECEDING THEORY TO ANALYTICAL MECHANICS AND DIFFERENTIAL GEOMETRY

Dynamics of holonomic systems.

14. – Let a holonomic system have $n - 1$ degrees of freedom. Let $x_1, \ldots, x_{n-1}$ be the Lagrangian parameters that fix the position, and let x_n be time, which is included as a Lagrangian parameter in analytical mechanics. Assume that there exists a force function U, and let $2T$ denote the *vis viva*:

$$2T = \sum_{i=1}^{n-1}\sum_{k=1}^{n-1} a_{ik}\dot{x}_i\,\dot{x}_k + 2\sum_{i=1}^{n-1} b_i\,\dot{x}_i + c \qquad \left(\dot{x}_i = \frac{dx_i}{dx_n}\right).$$

Let τ be an arbitrary parameter by means of which one represents the motion of the system by equations of the form:

$$x_i = x_i\,(\tau) \qquad (i = 1, \ldots, n).$$

The Lagrange equations are then written:

$$\text{(1)} \qquad \frac{d}{d\tau}\left(\frac{\partial L}{\partial x_i'}\right) - \frac{\partial L}{\partial x_i} = 0,$$

in which:

$$x_i' = \frac{dx_i}{d\tau}$$

and

$$L = (T + U)\; x_n'.$$

Set:

$$\bar{\Omega}\,(x_1, \ldots, x_n \mid dx_1, \ldots, dx_n) = (T + U)\, dx_n$$

$$= \frac{1}{2}\sum_{i=1}^{n-1}\sum_{k=1}^{n-1} a_{ik}\,\frac{dx_i}{dx_n}\,dx_n + \sum_{i=1}^{n-1} b_i\,dx_i + \left(U + \frac{c}{2}\right)dx_n\,.$$

$\bar{\Omega}$ is homogeneous of degree 1 in the dx_i. Moreover, set:

$$u_i = \frac{\partial\bar{\Omega}}{\partial x_i} = \frac{\sum_{k=1}^{n-1} a_{ik}\,dx_k}{dx_n} + b_i \qquad (i = 1, \ldots, n - 1),$$

$$u_n = \frac{\partial\bar{\Omega}}{\partial x_i} = -\frac{1}{2}\sum_{i=1}^{n-1}\sum_{k=1}^{n-1} a_{ik}\frac{dx_i\,dx_k}{dx_n^2} + U + \frac{c}{2}.$$

Upon eliminating the ratios dx_i / dx_n from this, one will infer that:

$$u_n = -\frac{1}{2}\sum_{j=1}^{n-1}\sum_{k=1}^{n-1} A_{kk}\,(u_j - b_j)(u_k - b_k) + U + \frac{c}{2}, \tag{2}$$

in which the A_{jk} are the minors of the determinant $|a_{ik}|$, divided by that determinant itself.

The preceding equation is written:

$$u_n + H(x_1, .., x_{n-1}, x_n ; u_1, \ldots, u_{n-1}) = 0, \tag{3}$$

in which H is the Hamiltonian function of the system considered, and the $u_1, \ldots, u_{n-1}$ are conjugate variables of the $x_1, .., x_{n-1}$. Consider that equation to be the tangential equation (in inhomogeneous form) of the derived wave that issues from the point $(x_1, \ldots, x_n)$ in the n-dimensional space that can be legitimately attached to a system of $n - 1$ degrees of freedom when one adds the coordinate $t = x_n$. Any dynamical problem (at least when the system considered is holonomic and there exists a force function) will then correspond to a problem of wave propagation in the permanent regime. Time, as it relates to the propagation of waves, is, in effect, the variable S that is defined by the equation:

$$dS = \bar{\Omega} = (T + U)\,dx_n ; \tag{4}$$

it is the *Hamiltonian action*, which does not figure explicitly in the equations of motion, any more than it does in the wave equation. The wave surfaces that are attached to the dynamical problem in question are then the surfaces of equal Hamiltonian action.

15. – The Lagrange equations are the equations of the trajectories along which propagate the contact elements $(x \mid u)$. In order to obtain the characteristic equations in the usual form, one can make the equation $u_n + H = 0$ homogeneous, but for our present purposes, it will suffice to take the form (3′), (4′) of Chapter I. Here, the function $H(t \mid x \mid p)$ is:

$$u_n + H(x_1, .., x_{n-1}, x_n ; u_1, \ldots, u_{n-1}) = 0$$

$$(p_i = u_i,\ i = 1, \ldots, n),$$

and one will find the differential system for the characteristics immediately in the form:

$$\frac{dx_i}{\dfrac{\partial H}{\partial u_i}} = \frac{du_i}{-\dfrac{\partial H}{\partial x_i}} = \frac{dx_n}{1} = \frac{du_n}{-\dfrac{\partial H}{\partial x_n}} = \frac{dS}{u_n + \sum_{i=1}^{n-1} u_i\dfrac{\partial H}{\partial u_i}}. \tag{5}$$

The equations:

(6) $$\frac{dx_i}{dx_n} = \frac{\partial H}{\partial u_i}, \qquad \frac{du_i}{dx_n} = -\frac{\partial H}{\partial x_i} \qquad (i = 1, \ldots, n-1)$$

are the canonical equations of mechanics, so they are then one part of a system that defines the characteristics of the propagation of waves that are attached to the dynamical problem.

16. – Thanks to the propagation in question, it will be very easy to find the general integral of the canonical equations. The complete integral, which was at issue in paragraph **9**, relates to the homogeneous equation $\pi = 1$, but it is also a complete integral to the equation $H = 0$, in which one has set:

$$u_i = \frac{\partial S}{\partial x_i} \qquad (i = 1, \ldots, n-1).$$

Therefore, let:

$$G(x_1, .., x_{n-1}, x_n ; a_1, \ldots, a_{n-1}) + a_n$$

be a complete integral of the equation:

(7) $$\frac{\partial S}{\partial x_n} + H\left(x_1, \ldots, x_{n-1}, x_n; \frac{\partial S}{\partial x_1}, \ldots, \frac{\partial S}{\partial x_{n-1}}\right) = 0.$$

The contact elements in the propagation in space $(x_1, .., x_n)$ are defined by equations (14) of Chapter I. If:

$$S = G(x_1, .., x_{n-1}, x_n ; a_1, \ldots, a_{n-1}) + a_n$$

is the complete integral then let:

(8) $$\begin{cases} \dfrac{\partial S}{\partial a_n} = mb_n = 1, \quad h\dfrac{\partial G}{\partial x_n} = u_n = -H = -hH \quad (h = 1), \\ \dfrac{\partial G}{\partial a_i} = mb_i = c_i, \quad \dfrac{\partial G}{\partial x_i} = u_i \quad (i = 1, \ldots, n-1). \end{cases}$$

Equations (8), in which c_i are arbitrary, thus define the general integral of the canonical equations (6) by means of a complete integral of (7). One will then obtain *Jacobi's theorem* by a very simple route.

17 – In the case where the constraints are independent of x_n and U contains only $x_1, \ldots, x_{n-1}$, one knows that the equations of motion admit the first integral:

$$T = U + h.$$

Consider the motions when h is fixed; upon changing U, which is defined only up to a constant, one can write:

$$T = U,$$

and in this case T, by a convenient choice of parameters, can be reduced to a quadratic form in the x_i ($i = 1, \ldots, n - 1$):

$$2T = \sum_{i=1}^{n-1}\sum_{k=1}^{n-1} a_{ik}\, \dot{x}_i\, \dot{x}_k\ .$$

Set:

$$\bar{T} = T\, dx_n^2\,.$$

One easily sees that if:

$$\bar{\Omega} = 2\sqrt{U\,\bar{T}}$$

then the Lagrange equations can be written:

$$d\,\frac{\partial\bar{\Omega}}{\partial\, dx_i} - \frac{\partial\bar{\Omega}}{\partial x_i} = 0 \qquad\qquad (i = 1, \ldots, n - 1),$$

and that one can make any motion of the system correspond to the propagation of a wave by taking the time of the propagation to be the variable W that is defined by the equation:

$$dW = \bar{\Omega} = 2\sqrt{U\,\bar{T}}\ .$$

That variable is the *Maupertuisian action*, and the tangential equations for the derived waves can be immediately put into the homogeneous form:

$$\pi\,(x_1\,, .., x_{n-1}\,; u_1, \ldots, u_{n-1}) = 1,$$

with

(9) $$\pi = \sqrt{\frac{\mathcal{T}}{U}}\,,$$

in which $\mathcal{T}$ is the form that is adjoint to $\bar{T}$.

We remark that the surfaces of equal Maupertuisian action are also surfaces of equal Hamiltonian action, because:

$$dW = 2\sqrt{U\,T}\, dx_n\,, \qquad dS = (U + T)\, dx_n\,,$$

and with the hypothesis that conforms to the definition of W:

$$T = U,$$

one will then have:

$$dW = dS.$$

Upon replacing U with $U + h$, one will come down to the case in which the *vis viva* constant has an arbitrary value.

One knows that one can obtain a complete integral of (7) in the case where H does not contain x_n and has the form:

$$S = -\, a_n\, x_n + V\,(x_1\,,\, ..,\, x_{n-1}\,;\, a_1,\, \ldots,\, a_n),$$

in which no constant is additive. One sees that with x_n = const., the multiplicities S = const. will determine the multiplicities V = const. in the space E_{n-1} (x_1 , .., x_{n-1}) precisely, and conversely. As a result, for an observer that has decomposed the space E_n into a "space" and a "time," the propagation that appears in E_{n-1} to be the motion of a multiplicity V = const. will again satisfy the principle of enveloping waves, in which the time by which that motion is framed will be the time x_n itself, which is coupled to the time S of the propagation in E_n by a simple linear relation.

If one makes the decomposition into a space E_{n-1} and a time x_n in the case where H contains x_n then the traces of the multiplicities S = const. in the "space" x_n = const. (i.e., in the space x_1 , .., x_{n-1}) will no longer propagate *à la* Huygens (i.e., according to the principle of enveloping waves).

Dynamics of special relativity.

18. – We have yet to speak of Einsteinian mechanics, but we shall do that in what follows, at least as far as general relativity is concerned. Meanwhile, it is convenient to give some precise indications in regard to the dynamics of special relativity that have great utility for the study of phenomena that do not involve intense gravitational fields.

Let m_0 be the rest mass of a material point. Let s denote its *proper time* and recall that one has:

$$c^2\, ds^2 = c^2\, dt^2 - dx^2 - dy^2 - dz^2 \tag{10}$$

in a Minkowskian system that serves to frame the universe, in which c is the speed of light *in vacuo*. By hypothesis, let U (x, y, z, t) be the force function from which one derives the field that acts upon the point considered, i.e., the function such that:

$$\frac{\partial U}{\partial x},\ \frac{\partial U}{\partial y},\ \frac{\partial U}{\partial z},\ \frac{\partial U}{\partial t}$$

are the components of a quadrivector whose first three components represent the force. If one sets:

$$\beta^2 = \frac{\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2 + \left(\dfrac{dz}{dt}\right)^2}{c^2}$$

at a point on the world-line of the particle m_0 then the equations of motion will be:

$$\frac{d}{dt}\left[\frac{m_0}{\sqrt{1-\beta^2}}\frac{dx}{dt}\right] = \frac{\partial U}{\partial x}$$

if one takes t to be the independent variable, and some similar ones for y and z. One can put them into canonical form by setting:

$$q_1 = x, \qquad q_2 = y, \qquad q_3 = z,$$

$$p_1 = \frac{m_0}{\sqrt{1-\beta^2}}\frac{dx}{dt},$$

$$p_2 = \frac{m_0}{\sqrt{1-\beta^2}}\frac{dy}{dt},$$

$$p_3 = \frac{m_0}{\sqrt{1-\beta^2}}\frac{dz}{dt},$$

and will one see effortlessly that the Hamiltonian function is:

$$\begin{aligned} H &= \frac{m_0\, c^2}{\sqrt{1-\beta^2}} - U(q_1, q_2, q_3, t) \\ &= c\sqrt{m_0^2\, c^2 + p_1^2 + p_2^2 + p_3^2} - U(q_1, q_2, q_3, t). \end{aligned}$$

The Jacobi equation will be:

$$\frac{\partial S}{\partial t} + c\sqrt{m_0^2\, c^2 + \left(\frac{\partial S}{\partial q_1}\right)^2 + \left(\frac{\partial S}{\partial q_2}\right)^2 + \left(\frac{\partial S}{\partial q_3}\right)^2} - U(q_1, q_2, q_3, t) = 0,$$

or rather:

$$\frac{1}{c^2}\left(\frac{\partial S}{\partial t} - 1\right)^2 - \left(\frac{\partial S}{\partial q_1}\right)^2 - \left(\frac{\partial S}{\partial q_2}\right)^2 - \left(\frac{\partial S}{\partial q_3}\right)^2 = m_0^2\, c^2. \tag{11}$$

One can make this equation homogeneous (§ **4**) and interpret the result as the tangential equation of the derived wave for propagation in the universe (x, y, z, t), in which the time of the propagation is S and the regime is permanent. Upon setting:

$$p_1 = \frac{\pi_1}{\pi_5}, \qquad p_2 = \frac{\pi_2}{\pi_5}, \qquad p_3 = \frac{\pi_3}{\pi_5}, \qquad p_4 = \frac{\partial S}{\partial t} = \frac{\pi_4}{\pi_5},$$

equation (11) can be written:

$$\pi_1^2 + \pi_2^2 + \pi_3^2 - \frac{1}{c^2}\pi_4^2 + \frac{2U}{c^2}\pi_1\pi_5 - \left(\frac{U^2}{c^2} - m_0^2 c^2\right)\pi_5^2 = 0. \tag{12}$$

That manner of proceeding, which permits one to pass from Newtonian mechanics to the mechanics of special relativity, is not sufficient, since equation (11) is not invariant under the Lorentz transformations, although it is quite useful in the simplest problems ([4]). The presence of the scalar U is precisely what disturbs that invariance. One knows that electromagnetism leads to some much more symmetric equations. Thanks to the introduction of the quadri-vector potential, whose fourth component is the ordinary potential, and the other three of which are the vector potential from Maxwell's theory, the motion of an electrified material point of mass m_0 and charge ε is given by the following equations, whose proof one will find in the great treatise of de Donder [**9**], for example.

Jacobi equation:

$$\frac{1}{c^2}\left(\frac{\partial S}{\partial x_4} - \varepsilon A_4\right)^2 - \sum_{i=1}^{3}\left(\frac{\partial S}{\partial x_i} - \varepsilon A_i\right)^2 = m_0^2 c^2, \tag{13}$$

in which A_1, A_2, A_3, A_4 are the covariant components of the world-potential.

General integral of the equations of motion:

$$\frac{\partial S}{\partial x_i} = p_i, \qquad \frac{\partial S}{\partial \alpha_i} = \beta_i,$$

in which S (x_1, x_2, x_3, x_4, α_1, α_2, α_3, α_4) is a complete integral of (13). p_1, p_2, p_3, p_4 are the covariant components of the quantity of motion-energy quadrivector. The contravariant components are expressions of the form [cf., pp. 90 ([†])]:

$$p_i = \varepsilon A_i - \frac{m_0}{c}\frac{dx_i}{ds}, \qquad p_4 = \varepsilon A_4 + m_0 c\frac{dx_4}{ds}, \tag{14}$$

and:

$$\frac{\partial S}{\partial x_i} - \varepsilon A_i = -\frac{m_0}{c}\frac{dx_i}{ds} \qquad (i = 1, 2, 3),$$

$$\frac{\partial S}{\partial x_4} - \varepsilon A_4 = m_0 c\frac{dx_4}{ds},$$

([4]) One knows that this defines one of the obstacles that held back the theoreticians who sought to introduce gravitation into special relativity between 1905 and 1912.

([†]) Translator's note: There was no pp. 90 in the original text of this monograph. Perhaps he was referring to the reference to de Donder.

in which the right-hand sides are the covariant components, divided by c, of the quadrivector **V**, whose contravariant components are:

$$m_0 \frac{dx_i}{ds} \qquad (i = 1, 2, 3, 4).$$

In the space E_5 (x_1, x_2, x_3, x_4, x_5), one can attach the motions in question to a wave propagation whose equation is obviously:

$$(16) \qquad \sum_{i=1}^{3} \pi_i^2 - \frac{1}{c^2}\pi_4^2 - 2\varepsilon\left(\sum_{i=1}^{3} A_i\,\pi_i - \frac{1}{c^2}A_i\pi_4\right)\pi_5 + \left[\varepsilon^2\left(\sum_{i=1}^{3} A_i\,\pi_i - \frac{1}{c^2}A_4^2\right) + m_0^2c^2\right]\pi_5^2 = 0.$$

We shall employ the wave equation in that form in paragraph **41**. The fact that its left-hand side is homogeneous of second-degree will make it particularly useful in the theory of second-order equations. The form $\pi = 1$ of paragraph **4** has the same utility in the context of that problem. Meanwhile, it is by means of that fact and the form $\Omega = 1$ of paragraph **11** that one can go on to justify equations (14) ([5]).

Geodesics in a Riemannian space.

19. – The determination of geodesics of a Riemannian space whose ds^2 is given in the form:

$$(17) \qquad ds^2 = g_{ik}\,dx_i\,dx_k$$

will also lead to the consideration of the propagation of waves in the space (x_1, …, x_n). As we have established no metric relation in regard to the theory of waves, it is legitimate to consider the wave surfaces:

$$F(x_1, \ldots, x_n) = \text{const.}$$

to be traced out in Riemannian space itself or in an auxiliary Euclidian space with the rectangular coordinates (x_1, …, x_n). Of course, it will be more natural and simpler to imagine things in the Riemannian space itself.

The regime is permanent, moreover, and the time of propagation is the variable s itself. One makes:

$$\bar{\Omega} = \sqrt{g_{ik}dx_i\,dx_k} = ds,$$

and the tangential equation of the desired waves will be:

$$(18) \qquad \pi(x \mid p) = 1,$$

([5]) Recall that the fastest way to write down the equations of a point is to start from the principle of least action for electromagnetism:

$$\delta S\; m_0\, d\sigma + \sum_{i=1}^{4} A_i\,dx_i = 0.$$

with

$$\pi(x \mid p) = \sqrt{g^{ik} p_i \, p_k} \; ,$$

in which the g^{ik} are coefficients of the form that is adjoint to ds^2; the absolute differential calculus has made them known to all.

Jacobi's theorem permits one to determine all of the geodesics of the given manifold when one knows a complete integral of equation (18), in which $p_i = \partial S / \partial x_i$. It is easy to see then that the geodesics that issue from a point are normal to the wave surfaces that issue from that point, where perpendicularity is intended in the sense of the Riemannian metric (17) here.

20. – Some difficulties will arise when the form (17) is not defined. That is what happens in relativity when one seeks the null-length geodesics that represent the motion of photons. $ds = 0$ on those geodesics, and one does not see the significance of wave surfaces immediately.

One arrives at the notion of wave in the following manner: The geodesics have an invariant significance, and on the other hand, the ds^2 can always be put into the following form by a change of variables when one knows that geodesics that issue from a point:

$$ds^2 = \sum_{\alpha=1}^{n-1} \sum_{\beta=1}^{n-1} g_{\alpha\beta} \, dx_\alpha \, dx_\beta - dx_n^2 \, , \tag{19}$$

so it will suffice to consider the form (19) and to write down the geodesic equations that depend upon it.

If $ds^2 \neq 0$ then those equations will be obtained by eliminating the p_i from the equations:

$$dx_i = g^{ij} \, p_j \, d\lambda \, ,$$

$$dp_k = -\frac{1}{2} \frac{\partial g^{ij}}{\partial x_k} p_i \, p_j \, d\lambda \, ,$$

which one can write upon starting from equation (18). If $ds^2 = 0$ then one will take x_n to be the representative parameter, and the geodesic equations will take the form:

$$\begin{cases} \dfrac{dx_i}{dx_n} = \dfrac{g^{ij} \, p_j}{g^{nn} \, p_n} & (i = 1, \ldots, n-1), \\[2ex] \dfrac{dp_k}{dx_n} = -\dfrac{\frac{1}{2} \dfrac{\partial g^{ij}}{\partial x_k} p_i \, p_j}{g^{nn} \, p_n} & (k = 1, \ldots, n-1), \end{cases} \tag{20}$$

which will make sense in all cases.

Now, it is obvious that these equations are the characteristic equations of propagation in the variable regime for waves in the space E_{n-1} $(x_1, \ldots, x_{n-1})$ when one writes the tangential equation of the wave surfaces in the form:

$$p \equiv \sqrt{\sum_{\alpha=1}^{n-1}\sum_{\beta=1}^{n-1} g^{\alpha\beta} P_\alpha P_\beta} = 1.$$

Indeed, it will suffice to set $p_\alpha / p_n = - P_\alpha$ in (18) in order to verify that assertion.

Hence: The null-length geodesics correspond to wave propagation in the permanent regime in E_n for which the time of propagation is s (which is proper time in special relativity). The null-length geodesics correspond to wave propagation in the variable regime, in general, in a space E_{n-1} $(x_1, \ldots, x_{n-1})$ that is defined by the form (19) for ds^2, and the time of propagation is x_n . If the $g_{\alpha\beta}$ in (19) do not depend upon the x_n then the propagation will be in the permanent regime in E_{n-1} , and one will see that the null-length geodesics of the manifold E_n project onto E_{n-1} along geodesics of E_{n-1} itself. One can recognize one of the classical propositions for the ds^2 of a static universe in this ([6]).

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([6]) Cf., for example, CHAZY [**7**].

CHAPTER III

SECOND-ORDER PARTIAL DIFFERENTIAL EQUATIONS AND THE WAVE PROPAGATIONS THAT ARE ATTACHED TO THEM.

Indeterminacy in the Cauchy-Kowalewsky problem.

21. – It is convenient to recall some useful terms. If a function z of n variables $x_1, \ldots, x_n$ admits partial derivatives up to order $p + 1$ then its derivatives will verify some relations of the form:

$$d\,\frac{\partial^k z}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}} = \sum_{i=1}^{n} \frac{\partial^{n+1} z}{\partial x_1^{\alpha_1}\cdots\partial x_i^{\alpha_i+1}\cdots\partial x_n^{\alpha_n}}\,dx_i \qquad (\alpha_1 + \ldots + \alpha_n = k) \tag{1}$$

for $k = 0, 1, \ldots, p$.

When one fixes the values of the variables:

$$x_1, \ldots, x_n, \qquad z, \qquad \frac{\partial z}{\partial x_1}, \ldots, \qquad \frac{\partial^k z}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}}, \qquad \ldots,$$

when they are considered to be independent, one says that they determine *a contact element of order p in the n + 1-dimensional space E_{n+1}* $(z, x_1, \ldots, x_n)$.

Two infinitely-close elements that verify all of relations (1) are called *united.*

Any system of equations between the coordinates of an element that verifies equations (1) defines a *multiplicity* M^p; the *support* of M^p is defined by the equations of the system, which couple only the variables $x_1, \ldots, x_n, z$ to each other.

If q is the number of dimensions of the support of M^p, which is chosen such that each point of the aforementioned support corresponds to only one contact element of order p then the multiplicity will be denoted by M_q^p.

22. – Consider the second-order equation, which is linear in the derivatives of order 2:

$$\Phi \equiv A_{ik}\, p_{ik} + \varphi = 0 \qquad \left(p_i = \frac{\partial z}{\partial x_i},\; p_{ik} = \frac{\partial^2 z}{\partial x_i\,\partial x_k}\right), \tag{2}$$

in which the A_{ik} and φ are given functions of $x_1, \ldots, x_n, z, p_1, \ldots, p_n$.

If one solves equation (2) for p_{nn}, and if certain conditions of holomorphy are realized (upon which we shall not dwell) then there will exist one and only one integral of (2) that reduces to a given function $\varphi(x_1, \ldots, x_{n-1})$ when $x_n = x_n^0$, while dz / dx_n reduces to another given function $\psi(x_1, \ldots, x_{n-1})$. The determination of that integral, which is assured by the Cauchy-Kowalewsky theorem, comes about by means of a Taylor development, and the latter theorem asserts both its uniqueness and its convergence.

Geometrically, that problem amounts to finding an integral multiplicity M_n that contains the multiplicity M_{n-1}^1 that is defined by the equations:

$$x_n = x_0^n, \qquad z = \varphi(x_1, \ldots, x_{n-1}), \qquad p_n = \psi(x_1, \ldots, x_{n-1}).$$

One can solve (1) for p_{nn} and thus put the proposed equation into *normal form*. One can demand to know if it is possible to give an M_{n-1}^1 by some less-specialized conditions. The role that is played by the plane $x_n = x_0^n$ will be played by a surface Σ:

$$x_n = f(x_1, \ldots, x_{n-1}),$$

or rather:

$$F(x_1, \ldots, x_n) = 0.$$

One seeks to get back to the preceding case by a change of variables:

$$(x_1, \ldots, x_n) \Leftrightarrow (y_1, \ldots, y_n),$$

such that the surface Σ will have the equation:

$$y_n = \text{const.}$$

φ and ψ will then become known functions of $y_1, \ldots, y_{n-1}$ on Σ. Now, in order for one to get back to the preceding case, it is necessary that one must be able to solve the equation that results from the transformation of (1) with respect to $\partial^2 z / \partial y_n^2$, which supposes that the coefficient of $\partial^2 z / \partial y_n^2$ is not zero, and as a result that F has not been chosen too poorly. We shall pass over that method; it was developed masterfully by Levi-Civita in his beautiful work [**23**], but we shall present the method of Beudon [**1**], to which the work of Hadamard [**16**, **17**] gave great importance.

23. – We then seek the integral of (1) that contains an arbitrarily-given multiplicity M_{n-1}^1. Let:

$$\frac{\partial z}{\partial x_i} = p_i + p_n \frac{\partial x_n}{\partial x_i} \qquad (i = 1, \ldots, n-1) \tag{3}$$

be the equations that represent M_{n-1}^1, in which z, x_n, and p_n are the arbitrary functions of $x_1, \ldots, x_{n-1}$.

In order to find the integral multiplicity that contains M_{n-1}^1, it is necessary that one must be able to determine all of the partial derivatives of z on the support Σ of M_{n-1}^1.

They enter into the Taylor development of the solution. We shall not concern ourselves with the case in which the solution is determined by the givens, but rather with the case in which it is indeterminate. The conditions of determinacy characterize certain multiplicities M_{n-1}^1 whose importance for applications is considerable.

One calculates the second derivatives by means of the following formulas:

$$\frac{\partial p_\rho}{\partial x_i} = p_{\rho i} + p_{\rho n}\frac{\partial x_n}{\partial x_i} \qquad (\rho = 1, \ldots, n;\ i = 1, \ldots, n-1), \tag{4}$$

which gives:

$$p_{\rho n} = \frac{\partial p_n}{\partial x_\rho} - p_{nn}\frac{\partial x_n}{\partial x_\rho}, \tag{5}$$

$$p_{\rho i} = \frac{\partial p_\rho}{\partial x_i} - \frac{\partial x_n}{\partial x_i}\frac{\partial x_n}{\partial x_\rho} + p_{nn}\frac{\partial x_n}{\partial x_i}\frac{\partial x_n}{\partial x_\rho}. \tag{6}$$

One substitutes these expressions in (2), and one must then obtain p_{nn} , by means of which all of the second derivatives are calculated. One gets:

$$\left(\sum_{\rho=1}^{n-1}\sum_{i=1}^{n-1} A_{\rho i}\frac{\partial x_n}{\partial x_i}\frac{\partial x_n}{\partial x_\rho} - 2\sum_{\rho=1}^{n-1} A_{\rho n}\frac{\partial x_n}{\partial x_\rho} + A_{nn}\right) p_{nn}$$

$$+ \sum_{\rho=1}^{n-1}\sum_{i=1}^{n-1} A_{\rho i}\left(\frac{\partial p_n}{\partial x_i} - \frac{\partial x_n}{\partial x_i}\frac{\partial p_n}{\partial x_\rho}\right) + 2\sum_{\rho=1}^{n-1} A_{\rho n}\frac{\partial p_n}{\partial x_\rho} + \varphi = 0.$$

It is impossible to solve the Cauchy problem if the coefficient of p_{nn} is zero without the second line on the left-hand side also being zero. In order for it to be indeterminate, it is necessary that the aforementioned coefficient and the second line in question must both be zero. Hence, the first necessary conditions for indeterminacy are written:

$$\sum_{\rho=1}^{n-1}\sum_{i=1}^{n-1} A_{\rho i}\frac{\partial x_n}{\partial x_i}\frac{\partial x_n}{\partial x_\rho} - 2\sum_{\rho=1}^{n-1} A_{\rho n}\frac{\partial x_n}{\partial x_\rho} + A_{nn} = 0, \tag{7}$$

$$\sum_{\rho=1}^{n-1}\sum_{i=1}^{n-1} A_{\rho i}\left(\frac{\partial p_n}{\partial x_i} - \frac{\partial x_n}{\partial x_i}\frac{\partial p_n}{\partial x_\rho}\right) + 2\sum_{\rho=1}^{n-1} A_{\rho n}\frac{\partial p_n}{\partial x_\rho} + \varphi = 0. \tag{8}$$

The multiplicities M_{n-1}^1 that are defined by equations (3), (7), and (8), in which z, x_n, and p_n are functions of x_1 , …, x_{n+1} are the *characteristic multiplicities*, or simply the *characteristics* of equation (2).

Characteristics and bicharacteristics.

24. – If the multiplicity M_{n-1}^1 that figures in the given of the Cauchy problem is a characteristic then the second-order elements in the Taylor development of the solution to the stated problem will be indeterminate. One can give p_{nn} arbitrarily on M_{n-1}^1, while the other p_{ik} are deduced from equations (5) and (6). Upon supposing that one can continue to determine the third-order elements on the integral multiplicity and the elements of arbitrary order, and if the Taylor development converges, then one will say that the integral multiplicity *contains* the characteristic M_{n-1}^1.

However, that fact is hardly important for us. For us, it will suffice to study the characteristics that are placed on a given integral multiplicity.

Now, the calculation that we made proves that we can find an infinitude of characteristics M_{n-1}^1 on an integral. When z is known as a function of $x_1, \ldots, x_{n-1}$, equation (7) will define x_n as a function of $x_1, \ldots, x_{n-1}$. We know that this is possible in an infinitude of ways. An integral of (7) will always pass through a multiplicity that is defined by the functions:

$$x_{n-1} = x_{n-1}^0,$$
$$x_n = f(x_1, \ldots, x_{n-1}),$$

in which f is arbitrary. However, if (2) is verified by z then equation (8) will be a consequence of equation (7), and our assertion will have been proved.

One can pursue the study of the indeterminacy of the Cauchy problem by passing to third-order elements; that is what Beudon did in the cited paper, and he easily defined multiplicities M_{n-1}^2, M_{n-1}^3, …, which were characteristics of order 2, 3, …, resp.

25. – We remark that the characteristic equations are generally determined only for given integrals z because the coefficients A_{ik} depend upon z and the p_i. Meanwhile, it can happen that those characteristics are independent of any particular integral of (1) when the A_{ik} depend upon only $x_1, \ldots, x_n$. That is what happens for the linear equations to which we shall devote the balance of our study.

Any linear second-order partial differential equation corresponds to a first-order equation (7) that obviously defines a wave propagation for which x_n is the time of propagation. The space in which the waves propagate is the space $x_1, \ldots, x_n$, and the wave surfaces are defined by the equations:

$$x_n = x_n(x_1, \ldots, x_{n-1}),$$

in which $x_n(x_1, \ldots, x_{n-1})$ is an arbitrary solution of (7). One replaces n with $n - 1$ in the theory of the first chapter, and t with x_n. The tangential equation of the derived waves is (7). In order to put it into the homogeneous form $p = 1$, one sets:

$$\frac{\partial x_n}{\partial x_i} = \frac{P_i}{\pi} \qquad (i = 1, \ldots, n-1)$$

in (7) and one solves it for π.

Hadamard (*loc. cit*) called the characteristics of the propagation the *bicharacteristics* of equation (1). They are defined by the equations:

$$\text{(9)} \qquad \frac{dx_i}{2\left(\sum_{\rho=1}^{n-1} A_{\rho i} P_\rho - A_{in}\right)} = \frac{dP_i}{-\left(\frac{\partial H}{\partial x_i} + P_i \frac{\partial H}{\partial x_n}\right)} = \frac{dx_n}{2\left(\sum_{\rho=1}^{n-1} A_{\rho n} P_\rho - A_{nn}\right)} \quad (i = 1, \ldots, n-1).$$

26. – It is useful to put equation (7) into a different form. Instead of supposing that the wave equations are given in the form:

$$x_n = x_n (x_1, \ldots, x_{n-1}),$$

one can imagine that they are given by the equation:

$$F (x_1, \ldots, x_n) = 0,$$

and (7) will take the form:

$$\text{(7}'\text{)} \qquad \sum_{i=1}^{n} \sum_{k=1}^{n} A_{ik} \frac{\partial F}{\partial x_i} \frac{\partial F}{\partial x_k} = 0,$$

moreover, and as a simple calculation will show, the bicharacteristics will be defined by the equations:

$$\text{(9}'\text{)} \qquad \frac{dx_i}{\sum_{k=1}^{n} A_{ik} \pi_k} = \frac{d\pi_j}{\frac{1}{2} \sum_{i=1}^{n} \sum_{k=1}^{n} \frac{\partial A_{ik}}{\partial x_j} \pi_i \pi_k} \qquad (i, j = 1, \ldots, n)$$

if $\pi_i = \partial F / \partial x_i$.

We remark that past the point at which one confines oneself to considering only the characteristics on the integral of (2), equation (8) will play no further role, since it is satisfied whenever (7) or (7′) are.

Systems of partial differential equations.

27. – As Hadamard has shown [**16**], the notion of characteristic can be extended to a system of second-order partial differential equations ([7]).

It will also be possible to attach a wave propagation to a system that defines k unknown functions $z_1, \ldots, z_k$ of n independent $x_1, \ldots, x_n$, and that propagation will take place in the space $x_1, \ldots, x_{n-1}$, while time will once more be x_n .

Indeed, let:

([7]) Cf., also JANET [**29**].

(10) $$\sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}^{(ml)} p_{ij}^{(l)} + \varphi^{(m)} = 0 \qquad (m = 1, \ldots, k)$$

be the proposed system; one has:

$$p_i^{(l)} = \frac{\partial z_l}{\partial x_i}, \qquad p_{ik}^{(l)} = \frac{\partial^2 z_l}{\partial x_i\, \partial x_k},$$

and the coefficients $A_{ij}^{(ml)}$, as well as the $\varphi^{(m)}$, will be functions of the x, z, and the $p_{ij}^{(l)}$. One considers a multiplicity M_{n-1}^1:

$$x_n = x_n\,(x_1, \ldots, x_{n-1}),$$

and one proposes to solve the Cauchy problem for the given system. One supposes that the first-order elements are given for each z_l, and one seeks the second-order elements $p_{ij}^{(l)}$. As one will easily see, everything comes down to determining the $p_{nn}^{(l)}$ by means of the system:

$$\sum_{l=1}^{k} H^{(ml)} p_{nn}^{(l)} + K^{(m)} = 0 \qquad (m = 1, \ldots, k),$$

in which:

$$H^{(ml)} = \sum_{i=1}^{n}\sum_{k=1}^{n} A_{ij}^{(ml)} \frac{\partial x_n}{\partial x_i}\frac{\partial x_n}{\partial x_k} - 2\sum_{i=1}^{n-1} A_{in}^{(ml)} \frac{\partial x_n}{\partial x_i} + A_{nn}^{(ml)},$$

and in which $K^{(m)}$ are certain expressions whose explicit form is irrelevant; they contain only the $p_{ij}^{(m)}$.

If the determinant Δ of the $H^{(ml)}$ is zero then the Cauchy problem will be impossible or indeterminate. The necessary conditions for the indeterminacy are very complicated to write down; they involve the rank of the matrix of the $H^{(ml)}$ and they couple the $K^{(m)}$ with each other. If one confines oneself to the study of manifolds that are defined by the equation:

(11) $$\Delta = 0$$

and are found on the integral multiplicities of the system then the conditions in question will be verified identically.

The multiplicities of M_{n-1} that are defined by (11) are once more called the *characteristics* of the system (10).

The equation $\Delta = 0$ is indeed defined in the space $x_1, \ldots, x_{n-1}$ of wave propagation, in which x_n is the time, which is independent of the solutions z_l considered if the $A_{ij}^{(lm)}$ depend upon only the $x_1, \ldots, x_n$. We shall suppose that this is true unless stated to the contrary.

In that case, one can further define the *bicharacteristics* of system (10); they are the characteristics of the first-order partial differential equation.

Physical interpretation of second-order equations. Discontinuity of solutions.

28. – Under the conditions that we have adopted, one can say that being given an integral of (2) or (10) amounts to being given one or k functions z or $z_1, \ldots, z_k$, resp., of $x_1, \ldots, x_n$, whose variation one can follow on the consecutive wave surfaces while traversing each bicharacteristic. Hence, a bicharacteristic, when considered to be a trajectory, serves to transport the contact elements of the wave surfaces and to transport the values of certain functions $z_1, \ldots, z_k$, which can characterize the intensities of certain disturbances. (Which will be concomitants if $k > 1$, as in the Maxwell's equations, in which the disturbances are the perturbations of the ether that are caused by electric and magnetic fields; viz., the fields themselves.)

The theory of the first chapter is, indeed, powerless to represent all of the circumstances that relate to propagation. In particular, it gives no details about the *intensity* of the disturbance that propagates. One must appeal to the second-order equations in order to compensate for that, and the theory that we have sketched out will show neatly the manner by which the kinematic theory of the first chapter enters into the theory that one can call "dynamic" of the second-order equations.

We remark that, when the functions z_l are considered to be functions of x_n along a bicharacteristic, they will have a non-zero value only when x_n has attained the value that corresponds to the point considered on the bicharacteristic. At that moment, the disturbance will reach the point, and the z_l will take on appreciable values there, while they were zero beforehand.

More precisely, the $n - 2$-dimensional multiplicities that are the wave surfaces in the space $x_1, \ldots, x_{n-1}$ are the *discontinuity* multiplicities (for $n = 4$, they will be *discontinuity surfaces*) of the $n - 1$-dimensional physical medium in which the disturbances propagate.

The study of the discontinuities, and above all, the study of the circumstances under which those discontinuities are compatible, has been made by numerous geometers of the first order, among which, one must include Riemann, Christoffel, Hugoniot, and Hadamard [**16**] (in which one finds the biography that relates to the first three cited authors).

They started with the equations of mathematical physics and showed that the characteristic equation of one or the other of them expresses the idea that the discontinuities are compatible on the characteristics, in effect.

We shall limit ourselves to those remarks. The compatibility conditions are outside the scope of our study, so it will suffice to recall only that they also lead to the characteristics. (Cf., also the book by VAN MIEGHEM [**33**]).

29. – Meanwhile, there are some other discontinuities that one should examine more carefully, namely, the ones that affect the integrals z_l in the regions where it becomes infinite. More precisely, following Delassus [**8**], Le Roux [**18**], and Hadamard [**16**], we seek the circumstances under which a *linear equation* ([8]):

([8]) The summation is not notated; it goes from 1 to n.

$$P(z) \equiv A_{ik}\, p_{ik} + A_i\, p_i + A\, z = 0 \tag{12}$$

can possess an integral of the form:

$$z = Z\, F(\pi),$$

in which Z and π are finite functions that are continuous and at least twice differentiable, but in which F, which is considered to be a function of π, is singular for $\pi = 0$, and that singularity is such that if π is infinitely small then $F(\pi)$ will be infinitely large in comparison to $F'(\pi)$, and $F''(\pi)$ will be infinitely large in comparison to $F'(\pi)$; for example:

$$\pi^{p/q}, \quad \frac{1}{\pi^m}, \quad \log \pi, \quad e^{1/\pi}, \quad \ldots$$

Upon substituting this in the proposed equation, it will become:

$$(A_{ik}\, \pi_i\, \pi_k)\, Z\, F''(\pi) + \left\{ 2\frac{\partial Z}{\partial x_i} A_{ij} \pi_j + A[P(\pi) - A\pi] \right\} F'(\pi) + P(Z)\, F(\pi) = 0, \tag{13}$$

when one sets $\partial \pi / \partial x_i = \pi_i$. Now, under the present circumstances, if $\pi = 0$ then the coefficient of $F(\pi)$ must be zero, so:

$$A_{ik}\, \pi_i\, \pi_k = 0.$$

In other words, if a solution of indicated type exists then the multiplicity $\pi = 0$ will be a characteristic of the proposed equation. The same thing will be true if one seeks z in the form:

$$z = Z\, F(\pi) + \zeta, \tag{14}$$

in which ζ is regular. Hadamard (*loc. cit.*, [**16**], pp. 333) has shown that if $\pi = 0$ is a characteristic then one can effectively find all of the solutions of the form (14); we shall have no need for that result in the rest of our study.

Periodic waves. Geometrical optics approximation.

30. – Hadamard (loc. cit., [**16**], pp. 345) made a very important remark along the same lines. It related to *periodic waves*, and it touched upon the fundamental problem of the approximation that is called geometrical optics, which one can make in the study of certain phenomena.

In physics, in a great number of cases, it is convenient to restrict oneself to periodic waves. They are represented by functions z of the form:

$$z = \sin \mu\pi \qquad (\mu = \text{const.}),$$

in which π is a function $x_1, \ldots, x_n$ that is linear with respect to x_n as many times as necessary.

It can also have the form:

$$z = Z \sin \mu\pi + \zeta, \tag{15}$$

in which Z and ζ are regular functions of $x_1, \ldots, x_n$, and one then imagines that the phenomenon that the variable z represents is due to the superposition of two disturbances, one of which is ζ, while the other one is $Z \sin \mu\pi$, which is periodic in x_n if Z does not depend upon x_n and π is linear in x_n, or if Z does depend upon x_n, but the modulation can sometimes be considered to be periodic over a short time interval.

When μ has a very large value, the function $\sin \mu\pi$ will pass from the value + 1 to the value – 1 for small variations of π. In practice, it behaves like a singular function whose derivative has the order of μ, whose second derivatives has the order of μ^2, etc. The physicist that utilizes such functions will often neglect the terms in μ in his calculations, in comparison to the terms in μ^2, and as a result, z in comparison to $\partial z / \partial x_n$, and $\partial z / \partial x_n$ in comparison to $\partial^2 z / \partial x_n^2$.

More precisely, and without making any initial hypothesis about the form of π in regard to x_n, we seek a solution to (12) that has the form (15) in the case where μ is very large. Upon substituting and reordering, one will find that:

$$-\mu^2 Z \sin \mu\pi \left(A_{ik} \frac{\partial \pi}{\partial x_i} \frac{\partial \pi}{\partial x_k} \right) + \mu \cos \mu\pi \left(2A_{ik} Z \frac{\partial Z}{\partial x_i} \frac{\partial \pi}{\partial x_k} + A_i Z \frac{\partial \pi}{\partial x_i} + A_{ik} Z \frac{\partial^2 \Pi}{\partial x_i \partial x_k} \right) + P(Z) \sin \mu\pi + P(\zeta) = 0. \tag{16}$$

Annulling the term in μ^2 will give:

$$A_{ik} \frac{\partial \pi}{\partial x_i} \frac{\partial \pi}{\partial x_k} = 0. \tag{17}$$

Hence, the equation:

$$\pi = \text{const.}$$

will again represent a characteristic.

The surfaces of equal phase, as one calls them for a periodic propagation, or the multiplicities on which the argument of the periodic function has a given constant value for a modulation, will then be characteristic multiplicities when the parameter μ is very large, and one can then neglect μ in comparison to its square.

31. – Choose μ in that fashion. Since the term in μ^2 is zero, annul the term in μ. Now, on a characteristic, one can consider the bicharacteristics that are defined by the equations:

$$\frac{dx_i}{2A_{ik} \dfrac{\partial \pi}{\partial x_l}} = du \qquad (i = 1, \ldots, n),$$

and one can calculate Z on each bicharacteristic by the equation:

$$\frac{dZ}{du} + M\,Z = 0,$$

in which:

$$M = A_i \frac{\partial \pi}{\partial x_i} + A_{ik} \frac{\partial^2 \pi}{\partial x_i\,\partial x_k};$$

this is a known function. One will then find that:

$$Z = Z_0 \exp \int_{u_0}^{u} M\,du,$$

in which Z_0 is an arbitrary function of the variable point that is located on a multiplicity M_{n-2} in each characteristic M_{n-1} ([9]) that propagates from M_{n-1} to M_{n-1} :

$$\pi = \text{const.};$$

i.e., if π is linear as a function of x_n , which we shall suppose from now on, it will have the same equation:

$$\psi(x_1, \ldots, x_n) = 0.$$

Z will then be determined in all of the space $x_1, \ldots, x_n$.

One will finally determine ζ by the condition:

$$P(\zeta) = -P(Z) \sin \mu\pi,$$

in which $P(Z)$ is a known function of $x_1, \ldots, x_n$; in a great number of cases, ζ is negligible (cf., Hadamard, *loc. cit.*, pp. 346).

We remark that Z_0 is the fixed value of Z on a bicharacteristic. If one considers a *pencil* of bicharacteristics that issue from a set of points, for example, and they are chosen in such a way that they traverse a certain region of space $x_1, \ldots, x_n$ then one can suppose that Z_0 is null everywhere except on the set in question, which can correspond to $u = u_0$ for each bicharacteristic; Z will be non-zero only on the pencil, moreover. Now, the choice of the set of points considered can be made physically by means of a screen that is pierced with a hole (if $n = 4$), and one will see that the *periodic* phenomena will propagate only along the pencil as long as the approximation that consists of neglecting μ in comparison to μ^2 is legitimate. Such an approximation is the one that will be permitted in optics when one restricts oneself to the consideration of rays (viz., geometrical optics).

The preceding remarks, which are due to Hadamard, cast a bright light upon the physical prolegomena to wave mechanics. One will see some applications of this in the book by L. de Broglie [**4**]; we shall not need to return to it.

([9]) If $n = 3$, the M_{n-2} must not be characteristics; for arbitrary n, they must not contain them.

32. – One can effortlessly extend the theory of singularities and that of periodic waves to a system of partial differential equations. The reader will indeed see the form in which that extension can be made. That will not be necessary for us in what follows, because we do not propose to enter into the theory of systems ([10]) any further than we did in paragraph **27**, since the scope of this small volume will not permit that. Meanwhile, we would like to cite the work of Levi-Civita [**22** and **23**] and that of Racah [**35**], who have treated, on the one hand, the equations of Einsteinian gravitation, and on the other hand, those of Dirac that relate to the photon and the electron, from that standpoint.

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([10]) The general theory of characteristics of systems has recently given rise to some important research, and we cite those of Cartan [**6**], Thomas and Titt [**38**].

CHAPTER IV

RETURN TO THE WAVE INTERPRETATION OF ANALYTICAL MECHANICS. WORLD GEODESICS. PROBABILISTIC CONSIDERATIONS. DE BROGLIE WAVES.

Second-order equations and mechanics.

33. – In the second chapter, we saw how Vessiot attached a wave propagation to any motion of a holonomic system with a force function by extending a beautiful idea of Hamilton. Up until the work of Louis de Broglie, which has transformed mechanics, one could only produce those waves physically, such that their introduction into analytical dynamics might seem to be more of a refinement of its elegance than a true enrichment of that field, and furthermore, in the simplest case of a material point, the velocity of wave propagation is not the same as the velocity of the material point, so that elegance itself might seem illusory.

While pursuing the succession of articles by Vessiot in light of the ideas of Beudon, and above all, Hadamard, it is interesting to look for the second-order equations that one can propose in such a manner that the propagation of waves that corresponds to them, in the sense of the preceding chapter, is precisely the one that is attached to the motion that the problem of analytical dynamics considers to have been defined.

Meanwhile, it is obvious that if one necessarily passes from equation (2) in Chapter III to equation (7) or equation (7′) then one cannot necessarily pass from a first-order equation that defines the waves to *a single* second-order equation. If the equation of propagation has degree two in the derivatives then one can pass to an equation of type (2), but all of the terms that one has grouped under the notation φ will be indeterminate. One will need some new principles in order to insure the uniqueness of the second-order equation that one proposes to find in order to realize the program that we just discussed in broad terms.

34. – Consider the system with $n - 1$ degrees of freedom that was defined in Chapter II, paragraph **14**. The equation of the derived waves is the Jacobi equation:

$$\text{(J)} \qquad u_n + H(x_1, \ldots, x_{n-1}, x_n ; u_1, \ldots, u_n) = 0,$$

in which the "time" of the propagation is the action S, and one sets:

$$u_k = \frac{\partial S}{\partial x_k} \qquad (k = 1, \ldots, n)$$

in (J).

The Jacobi equation can be identified with equation (7), which defines the characteristics of (2) only if one writes (2) and (7) with $n + 1$ independent variables in the form:

$$(1) \qquad \sum_{i=1}^{n+1}\sum_{k=1}^{n+1} G_{ik}\, p_{ik} + \varphi = 0$$

and

$$(2) \qquad \sum_{\rho=1}^{n}\sum_{i=1}^{n} G_{\rho i}\frac{\partial S}{\partial x_\rho}\frac{\partial S}{\partial x_i} - 2\sum_{\rho=1}^{n} G_{\rho,n+1}\frac{\partial S}{\partial x_\rho} + G_{n+1,\,n+1} = 0,$$

resp., upon setting $S = x_{n+1}$.

Having said that, one should recall that H has degree precisely two in the u_i ($i = 1, \ldots, n - 1$). One can write equation (J) in the form:

$$\frac{\partial S}{\partial x_n} + \sum_{i=1}^{n-1}\sum_{k=1}^{n-1} B_{ik}\frac{\partial S}{\partial x_i}\frac{\partial S}{\partial x_k} + 2\sum_{i=1}^{n-1} B_i\frac{\partial S}{\partial x_i} + B = 0,$$

in which [cf., equations (2), § **14**]:

$$B_{ik} = \tfrac{1}{2}A_{ik}, \qquad B_i = -\sum_{k=1}^{n-1} A_{ik}\, b_k \qquad (i = 1, \ldots, n - 1),$$

$$B = \sum_{j=1}^{n-1}\sum_{k=1}^{n-1} A_{jk}\, b_j\, b_k - U - \frac{c}{2}.$$

Identifying this with equation (2) will give:

$$\begin{array}{lll} G_{\rho i} & = B_{\rho i} & (\rho, i = 1, \ldots, n-1), \\ G_{ni} & = G_{in} = G_{nn} = 0 & (i = 1, \ldots, n-1), \\ G_{i,\,n+1} & = -B_i\,, & (i = 1, \ldots, n-1), \\ G_{n,\,n+1} & = -\tfrac{1}{2}\,, & \\ G_{n+1,\,n+1} & = B. & \end{array}$$

One can formulate the following theorem:

One can make any holonomic system with a force function that has $n - 1$ degrees of freedom and can be described by means of $n - 1$ parameters $x_1, \ldots, x_{n-1}$ correspond to an $n + 1$-dimensional space E_{n+1} ($x_1, \ldots, x_{n-1}, x_n$, S), in which x_n is time and S is the Hamiltonian action. The Jacobi equation of the system defines a wave propagation in the permanent regime in the space E_n ($x_1, \ldots, x_n$). The wave surfaces are the characteristics of certain second-order partial differential equations that are linear in the unknown function z and are determined perfectly. The remaining terms constitute an arbitrary function of z, $\partial z / \partial x_i$, $\partial z / \partial S$, of $x_1, \ldots, x_{n-1}, x_n$, S. The Lagrange equations of the system are the bicharacteristics of that second-order equation, in the sense of Hadamard.

In brief, the equation:

$$B\frac{\partial^2 z}{\partial S^2}-\frac{1}{2}\frac{\partial^2 z}{\partial S\,\partial x_n}-\sum_{i=1}^{n-1}B_i\frac{\partial^2 z}{\partial S\,\partial x_i}+\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}B_{ij}\frac{\partial^2 z}{\partial x_i\,\partial x_j}+\varphi=0, \tag{3}$$

has characteristics:

$$S = S(x_1, \ldots, x_{n-1}, x_n),$$

which are defined by the Jacobi equation:

$$\frac{\partial S}{\partial x_n}+H=0,$$

and the bicharacteristics, which are defined by the Lagrange equations or those of Hamilton.

In the particular case of a material point of mass m and Cartesian coordinates x, y, z that is subject to a force that is derived from a potential U, the second-order equation can be written:

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}-\frac{\partial^2\psi}{\partial S\,\partial t}-2U\frac{\partial^2\psi}{\partial S^2}+\varphi=0,$$

in which ψ denotes the unknown function and φ is an arbitrary function of:

$$x,\ y,\ z,\ t,\ S,\ \psi,\ \frac{\partial\psi}{\partial x},\ \frac{\partial\psi}{\partial y},\ \frac{\partial\psi}{\partial z},\ \frac{\partial\psi}{\partial t},\ \frac{\partial\psi}{\partial S}.$$

35. – If the constraints on the holonomic system considered are independent of x_n, and if U contains only the variables $x_1, \ldots, x_{n-1}$ then the problem will simplify. One will then know that the surfaces S = const. or W = const. agree with the surfaces x_n = const., in such a way that it will suffice to consider a space that contains at least one dimension. Indeed, one knows that the equation of propagation is:

$$\sqrt{\frac{\mathcal{T}}{U}}=1,$$

in which $\mathcal{T}$ is a quadratic form in the $p_1, \ldots, p_{n-1}$. The preceding equation is written:

$$\sum_{i=1}^{n-1}\sum_{k=1}^{n-1}B_{ik}\frac{\partial W}{\partial x_i}\frac{\partial W}{\partial x_k}-2U=0. \tag{4}$$

Now, if one sets:

$$2U=2U\left(\frac{\partial W}{\partial W}\right)^2$$

then one will see that (4) is the equation for the characteristics of Chapter III in the form (7′):

$$W(x_1, \ldots, x_{n-1}) - W = 0$$

for the second-order equation:

$$\sum_{i=1}^{n-1}\sum_{k=1}^{n-1} B_{ik}\frac{\partial^2 z}{\partial x_i\,\partial x_k} - 2U\frac{\partial^2 z}{\partial W_n^2} + \varphi = 0, \tag{5}$$

in which the B_{ik} are the coefficients of the adjoint form $\mathcal{T}$ or T.

Now, neither U nor B_{ik} depend upon x_n, so one can suppose that φ no longer depends upon it either, and one can simplify the preceding equation by searching for solutions of the form:

$$z = \lambda(W)\,\zeta(x_1, \ldots, x_{n-1}).$$

Suppose that φ is linear in z, and to simplify even further, that:

$$\varphi = \varphi(x_1, \ldots, x_{n-1})\,z.$$

One will then have:

$$\sum_{i=1}^{n-1}\sum_{k=1}^{n-1} B_{ik}\frac{\partial^2 \zeta}{\partial x_i\,\partial x_k} + \rho - 2U\frac{\lambda''(W)}{\lambda(W)}\zeta = 0,$$

which is possible only if:

$$\frac{\lambda''(W)}{\lambda(W)} = \text{const.}$$

In these very simple cases, one sees solutions z appear that are exponential functions of W, so under certain circumstances, there will be ones that are *periodic* in W.

The variable x_n plays no role in those considerations. Meanwhile, one knows that:

$$dW = 2\sqrt{UT}\,dx_n,$$

i.e., that W = const. if x_n = const., which will permit one to eliminate W in order to have only x_n to worry about. We shall not belabor that because what follows will permit us to specify the role of action and that of time in the most interesting case of wave mechanics.

Invariance conditions.

36. – Now consider the motion of a material point from the standpoint of general relativity. The world-line is a certain geodesic of a ds^2:

$$ds^2 = g_{ik}\,dx_i\,dx_k. \tag{6}$$

One can attach (cf., § **19**) a wave propagation to it whose equation is:

(7) $$\pi(x \mid p) = \sqrt{g^{ik} p_i \, p_k} = 1;$$

the regime is permanent, and the time of propagation is the variable *s* itself.

Consider a five-dimensional multiplicity M_5 ($x_1, \ldots, x_4, x_5$) whose ds^2 is:

$$ds_5^2 - g_{ik} \, dx_i \, dx_k .$$

The null-length geodesics of that M_5 project onto the universe $x_1, \ldots, x_4$ along geodesics of (6). Those null-length geodesics in M_5 correspond to a wave propagation that is precisely the one that is attached to the world-line of the material point considered in the universe $x_1, \ldots, x_4$. The regime is permanent, and the time of propagation is x_5 – i.e., the *proper time* of the material point.

The tangential equation of the derived waves is equation (7) in the universe, but when one sets $\pi_i / \pi_5 = - p_i$, it will become:

(8) $$g^{\alpha\beta} \, \pi_\alpha \, \pi_\beta - \pi_5 \, \pi_5 = 0$$

in M_5, which corresponds to some second-order partial differential equations, the simplest of which is:

(9) $$g^{\alpha\beta} \frac{\partial^2 z}{\partial x_\alpha \, \partial x_\beta} - \frac{\partial^2 z}{\partial x_5^2} = 0.$$

37. – Meanwhile, it is convenient to add a few new conditions here. In the calculations that we have developed, it is basically irrelevant whether the variables represent rectangular coordinates. The theory of propagation is a theory of contact, and if the variables represent arbitrary curvilinear coordinates in the space of propagation then that would bring about no great alteration of the developments in the preceding chapters. Only the relations in Ω and π (Chap. I, § **11**) will be transformed when one takes into account the metric that is defined by the ds^2 in the space of propagation.

It is still preferable to do that, nonetheless; instead of considering the universe, consider M_5 whose ds^2 is:

(10) $$\sum_{i=1}^{5} \sum_{k=1}^{5} \gamma_{ik} \, dx_i \, dx_k .$$

Equation (8), which relates to null-length geodesics in M_5, is written, with the well-known notations of the absolute differential calculus:

(11) $$\sum_{i=1}^{5} \sum_{k=1}^{5} \gamma_{ik} \, dx_i \, dx_k = 0$$

relative to the form (10), but equation (9) must be transformed if one desires that its relationship to (11) should be invariant when one changes coordinates in M_5, since the function *z* is an invariant. The simplest invariant form is obviously:

(12) $$\frac{1}{\sqrt{\gamma}}\sum_{i=1}^{5}\frac{\partial}{\partial x_i}\left(\sqrt{\gamma}\,\gamma^{ik}\,\frac{\partial z}{\partial x_k}\right)=0,$$

in which γ is the determinant of the γ_{ik}. That equation, which is written in terms of the covariant derivative as:

$$z^r|_r = 0,$$

in which $z^r = g^{ri}\,\partial z / \partial x_i$, is indeed invariant.

If the ds^2 has the particular form:

$$dx_5^2 - g_{ik}\,dx_i\,dx_k$$

then one will have:

$$\sqrt{\gamma} = \sqrt{g}\,, \quad \gamma^{ik} = -\,g^{ik} \qquad (i, k = 1, 2, 3, 4)$$

$$\gamma^{5k} = \gamma^{k5} = 0, \quad \gamma^{55} = 1,$$

upon utilizing the fundamental form (6), and as a result (12) will be written:

$$\frac{1}{\sqrt{g}}\frac{\partial}{\partial x_i}\left(\sqrt{g}\,g^{ik}\,\frac{\partial z}{\partial x_k}\right)-\frac{\partial^2 z}{\partial x_5^2}=0,$$

or rather:

(13) $$g^{ik}\left(\frac{\partial^2 z}{\partial x_i\,\partial x_k}-\Gamma^l_{ik}\,\frac{\partial z}{\partial x_l}\right)-\frac{\partial^2 z}{\partial x_5^2}=0,$$

because:

$$\frac{1}{\sqrt{g}}\frac{\partial(\sqrt{g}\,g^{ik})}{\partial x_k}=-\,g^{rs}\,\Gamma^i_{rs},$$

in which Γ^i_{rs} is the notation for the Christoffel symbol of the second kind for the ds^2 of the universe – namely, (6). Equation (13) will then be replaced with equation (9), for reasons of invariance that we have pointed out before ([11]).

Electromagnetism and gravitation.

38. – It is possible to make some other suggestions, as well. The multiplicity M_5 that was introduced naturally into our calculations can be considered from a more advanced viewpoint.

Originally, x_5 is a parameter that is constant when the action is constant. More precisely, if one examines all the possible motions of a material point that passes through

([11]) Cf., KLEIN [**32**].

the same point at the same instant ([12]) then the various trajectories will define values for the action W. The locus of points W = const. corresponds to the points in which x_5 = const., while the parameter x_5 is the proper time s on each world-line relative to each of the motions considered. The idea of making x_5 an *independent variable* with the same status as x_1, x_2, x_3, x_4 will naturally upset the foundations of mechanics profoundly.

One can remark, first of all, that this upheaval can come about non-violently. One considers a new ds^2 for that universe that has the form:

$$ds^2 = dx_5^2 - g_{ik}\, dx_i\, dx_k\,,$$

in which the g_{ik} do not depend upon x_5. That is exactly what was done above.

Let us generalize this by one degree and introduce a form in which the coefficient of dx_5^2 is no longer a constant, but a function of x_1, x_2, x_3, x_4, while the other ones g_{ik} (i, k = 1, 2, 3, 4) no longer depend upon the x_5. One writes ([13]):

$$g_{55} = \psi^2$$

and

(14) $$ds^2 = -\psi^2 dx_5^2 - g_{ik}\, dx_i\, dx_k\,.$$

We propose to generalize Einstein's theory, moreover, and to find the equations that define the g_{ik} and ψ^2. The simplest idea, at least for regions of the universe that are devoid of matter, is to write:

(15) $$R_{\alpha\beta} = 0 \qquad (\alpha, \beta = 1, 2, 3, 4, 5),$$

in which the $R_{\alpha\beta}$ are the components of the contracted Riemann tensor that relate to the ds^2 that is defined by (14).

Now, for α, β = 1, 2, 3, 4, equations (15) are precisely those of Einstein for a ds^2 that goes with a four-dimensional universe whose coefficients are g_{ik} (i, k = 1, 2, 3, 4), provided that one neglects the derivatives $\partial\psi / \partial x_i$ in comparison to the $\partial g_{ik} / \partial x_l$; i.e., provided that ψ^2 is "not too variable."

The equations:

$$R_{\alpha 5} = 0 \qquad (\alpha = 1, 2, 3, 4)$$

are verified identically, and the equation:

$$R_{55} = 0$$

is written:

(16) $$g^{hi}\frac{\partial^2\psi}{\partial x^i\,\partial x^h} + \left(\Gamma^h_{ih}\, g^{li} + \frac{\partial g^{hi}}{\partial x^h}\right)\frac{\partial\psi}{\partial x^i} = 0,$$

([12]) It is necessary to specify the class of those motions, moreover, but we shall limit ourselves to a suggestion.

([13]) Cf., [**13** and **14**].

in which the Γ_{li}^k are the Christoffel symbols of the form $g_{ik}\, dx_i\, dx_k$, and the g^{ik} relate to that same form.

One can formulate the following theorem, moreover, which attaches a wave propagation to the motion of a material point by an invariant process that is different from the one that we saw in the preceding paragraphs:

If one considers an Einsteinian universe to be a section x_5 = const. of a five-dimensional universe (x_1, x_2, x_3, x_4, x_5) whose ds^2 has the form (14) then the equations of gravitation *in vacuo* will be the equations $R_{ik} = 0$ (i, k = 1, 2, 3, 4) (if the derivatives $\partial\psi^2 / \partial x_i$ are negligible in comparison to the derivatives $\partial g_{ik} / \partial x_l$) that relate to that ds^2, and the world-lines of a material point in such a field will be the bicharacteristics of the equation $R_{44} = 0$, which will determine ψ when the g_{ik} are known.

The equation $R_{44} = 0$ can play the role of the Schrödinger equation ([14]). If one no longer neglects the $\partial\psi^2 / \partial x_i$ in comparison to the $\partial g_{ik} / \partial x_l$ then one can assume that the terms that were omitted in $R_{ik} = 0$ (i, k = 1, 2, 3, 4) will translate into the influence of the wave field that the equation $R_{55} = 0$ defines on the gravitational field. One can then say that the material point first perturbs the field that is defined by the g_{ik} (i, k = 1, 2, 3, 4) by way of the waves that are attached to it before creating it by its mass.

39. – In the spirit of discovery, one can immediate proceed to examine the equations that one will be led to when one considers a ds^2 on the multiplicity M_5 whose g_{i5} are no longer zero. One will then recover the unitary theory of Kaluza [**30**], and the extension that Gonseth and Juvet gave to it [**13** and **14**]. If one sets:

$$ds^2 = g_{\alpha\beta}\, dx_\alpha\, dx_\beta \qquad (\alpha, \beta = 1, 2, 3, 4, 5)$$

with $g_{55} = \psi^2$ or $\psi\bar{\psi}$ then one will be led to first assume that the g_{5i} (i = 1, 2, 3, 4) are very small, and one can write down equations that generalize the Einstein equations:

$$R_{\alpha\beta} = k\left(T_{\alpha\beta} - \tfrac{1}{2} g_{\alpha\beta} T\right) \qquad (\alpha, \beta = 1, 2, 3, 4, 5), \tag{17}$$

the first ten of which (α = 1, 2, 3, 4) are the ordinary equations of gravitation, while the following four (α = 5, β = 1, 2, 3, 4) are Maxwell's equations that define the potentials φ_i as functions of the current, provided that one sets:

$$g_{5i} = \tau\, \varphi_i\,,$$

in which τ is a constant. Finally, if $\alpha = \beta = 5$ then one will recover a second-order equation for ψ. That supposes that one neglects the derivatives with respect to x_5 in comparison to the other derivatives. Those conclusions imply that one must define the

([14]) In that case, it will be preferable to replace ψ^2 with $\psi\bar{\psi}$, in which ψ is a complex function and $\bar{\psi}$ is is the conjugate quantity.

energy-matter tensor $T_{\alpha\beta}$ in such a manner that it will generalize the one that Einstein introduced into his theory. One will succeed in doing that when one sets:

$$dx_5 = \frac{\sigma}{\rho}ds, \tag{18}$$

which defines the fifth direction parameter in M_5 of the world-line of a point in a continuous medium whose mass density is ρ and whose electric density σ.

The latter equation will be imposed, moreover, when one starts with the equations of motion of electricity, when they are written in terms of the Lorentz force, and one seeks to interpret them as the ones that define a parallel displacement in M_5 in the sense Levi-Civita. The notion of force is thus banned from this conception of things. L. de Broglie made an analogous remark [**3**].

What one can keep from that unitary theory is the fact that it is possible to take the wave fields that are created by the motion of matter into account in the equations of gravitation. From the instant at which these fields are no longer a simple formal artifice, but one can assign a true physical significance to them, their influence on the gravitational fields can no longer be negligible, and the preceding remarks show how one can take that into account.

When equation (18) is applied to an electron, it will suggest that the ratio of the charge to mass can vary. Physically, there is a great difficulty associated with that, but that difficulty will disappear when one remains within the scope of a theory of continuous media that is properly the scope of a theory of fields.

Finally, if one establishes an invariant theory that is based upon the ds^2 of M_5, while one considers the derivatives with respect to x_5 to have the same status as the derivatives with respect to the other variables, the simplicity of the relationship between equation (17), for which $\alpha = \beta = 5$, and the geodesics of ds^2, which must then define the world-lines of the motion of a point that is endowed with charge, will then disappear. Those geodesics will no longer be the bicharacteristics of the equation with indices 55. However, in that case, the electromagnetic field will predominate, and the geometrical optics approximation will probably be no longer admissible.

Periodicity of waves in mechanics.

40. – Despite the very formal attempts at a unification of the field theories ([15]), one can try to adopt another viewpoint that conforms better to the history of wave mechanics, and is a better way of comprehending its present state. One knows that the essential idea of L. de Broglie resides in the very bold postulate of attaching a frequency, or if one prefers, a *periodicity* to any particle in motion.

Moreover, one should examine the consequences that one can infer from the hypothesis that makes the Hamilton-Vessiot waves themselves periodic. First of all, just as in geometrical optics, where the *ray* is the main notion that the theory hangs upon, it will then become clear that that notion will be insufficient for phenomena at the small

([15]) One will see similar attempts presented in a fascicle in this collection that is due to de Donder [**10**].

scale, and that a periodic wave will become necessary, similarly, one can think that the *trajectory* or *world-line* of a material point are notions that are insufficient for the phenomena that unfold at the atomic scale, and that the notion of *wave* will serve to put things in order in quantum physics when it is suitably introduced into dynamics.

The problem will preserve a very formal aspect for us. In this little book, it will not be possible to color in the very minimal sketch that the preceding considerations permitted us to make with any physical touches. One must simply say that the wavelengths that are attached to the motions that are studied in classical mechanics must be extremely small: The "physical" effects of those waves must be weak enough that one can immediately understand why the mechanician was unable to observe them, any more than the astronomer.

Our goal is then to introduce the notion of periodic wave as a prolegomena to a type of mechanics that will admit analytical mechanics as a first approximation, just as physical optics admits geometrical optics as a first approximation.

41. – Naturally, just as it is by way of d'Alembert's equation that one can complete optics when one passes (thanks to Kirchhoff) from the "geometric" level to the "physical" level, which is more advanced, it is by way of a second-order equation that one succeeds in completing analytical mechanics.

In order to specify all of the circumstances under which such a second-order equation has been employed to good use, the best way is to adopt the viewpoint of special relativity. The Jacobi equation, under the hypothesis of a field that is derived from a potential, was recalled in paragraph **18**. Equation (13) of that paragraph must be the characteristic equation of a second-order equation. The simplest equation of propagation that one can propose is obviously ([16]):

$$(19)\qquad \sum_{i=1}^{3}\frac{\partial^2 z}{\partial x_i^2}-\frac{1}{c^2}\frac{\partial^2 z}{\partial x_4^2}-2\varepsilon\left(\sum A_i\frac{\partial^2 z}{\partial x_i\,\partial x_k}-\frac{1}{c^2}A_4\frac{\partial^2 z}{\partial x_i\,\partial x_5}\right)$$
$$-\left[\varepsilon^2\left(\sum A_i^2-\frac{1}{c^2}A_4^2\right)-m_0^2c^2\right]\frac{\partial^2 z}{\partial x_5^2}=0.$$

The variable x_5 is due to the fact that the time of propagation is the variable S with the interpretation that was pointed out in paragraph **18**. We then call it x_5 and remember that the equations of the wave surfaces have the form:

$$S = f(x_1, x_2, x_3, x_4),$$

since the regime is permanent, or rather:

$$x_5 + \varphi(x_1, x_2, x_3, x_4) = \text{const.},$$

([16]) Compare this with de Donder [**10**] and Géhéniau [**11** and **12**].

Now, let us introduce the idea of periodic waves into the debate and seek the solutions to (19) that are functions z that contain a sine as a factor, such that function that the sine applies to must be constant on the wave surfaces, which amounts to setting:

$$z = a \sin \mu \, [x_5 + \varphi(x_1, x_2, x_3, x_4)],$$

in which a is a function of x_1, x_2, x_3, and x_4 . Indeed, it is natural to assume that it does not depend upon x_5 due to the permanence of the regime of propagation. The factor μ in the *phase* is a constant.

It is, moreover, preferable, to write:

$$z = e^{i\mu x_4} \; \psi(x_1, x_2, x_3, x_4),$$

and one will effortlessly see that ψ verifies the following equation, from which one can derive all of the *equations that are called Schrödinger equations:*

$$\text{(20)} \qquad \sum_{i=1}^{3} \frac{\partial^2 \psi}{\partial x_i^2} - \frac{1}{c^2}\frac{\partial^2 \psi}{\partial x_4^2} - 2\varepsilon i\mu \left(\sum_{i=1}^{3} A_i \frac{\partial \psi}{\partial x_i} - \frac{A_4}{c^2}\frac{\partial \psi}{\partial x_4} \right) + \mu^2 \left[\varepsilon^2 \left(\sum_{i=1}^{3} A_i^2 - \frac{1}{c^2} A_4^2 \right) - m_0^2 c^2 \right] \psi = 0,$$

which no longer contains x_5. (Cf., Gordon [**15**], who was the first to give a relativistic Schrödinger equation.)

One must then further seek the solutions ψ of the form:

$$\text{(21)} \qquad \psi = a\,(x_1, x_2, x_3, x_4)\; e^{i\mu\varphi(x_1, x_2, x_3, x_4)},$$

but in order to not introduce wave lengths that are too large, we suppose that μ is a very large number. Upon recalling the calculations of a paragraph **30** and changing the sine into an exponential, one will find that if one can neglect μ in comparison to μ^2 and the terms that are independent of μ in comparison to the terms in μ ([17]) then φ must verify the equation:

$$\text{(22)} \qquad \frac{1}{c^2}\left(\frac{\partial \varphi}{\partial x_4} - \varepsilon A_4 \right)^2 - \sum_{i=1}^{3} \left(\frac{\partial \varphi}{\partial x_i} - \varepsilon A_i \right)^2 = m_0^2 c^4,$$

which is found to be the Jacobi equation (13) of paragraph **18** ([18]). The surfaces of equal phase will be the characteristics of equation (20) for the propagation of periodic waves whose frequency in $x_5 = S$ (namely, $\mu / 2\pi$) is very large. To the same approximation, the

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([17]) One can remark that certain coefficients have the same order as μ in the present problem.

([18]) One should mention that Debye has made a very interesting remark regarding the relationship between the equation of propagation and that of geometrical optics. He already pointed out that the latter resulted from the former by passing to the limit when the wavelength of light considered tends to zero. That remark was mentioned by Sommerfeld and Runge [**37**].

bicharacteristics of (22) are the world-lines of a point of mass m_0 and charge ε that displaces in the field of the potential **A**. One will then indeed see that if one assumes that the second-order equation (20) rules the dynamics of a material point in a type of mechanics where waves plays an essential role then the notion of a ray – i.e., of a trajectory – will preserve a useful meaning in the first approximation.

Probabilistic interpretation.

42. – It is natural to look for the meaning of the equations that one will obtain upon annulling the term in μ and the term that is independent of μ by substituting (21) in (20).

L. de Broglie gave a very elegant interpretation for the case in which one confines oneself to the Newtonian approximation to dynamics [**4**, pp. 85]. It is possible to extend it to the case of special relativity. The equation that is provided by the term in μ, which is, moreover, also the one that one obtains upon annulling the term in i after substituting (21) in (20), is:

$$(23)\qquad \sum_{i=1}^{3}\frac{\partial a}{\partial x_i}\frac{\partial \varphi}{\partial x_i}-\frac{1}{c^2}\frac{\partial a}{\partial x_4}\frac{\partial \varphi}{\partial x_4}+\frac{a}{2}\left[\sum_{i=1}^{3}\frac{\partial^2 \varphi}{\partial x_i^2}-\frac{1}{c^2}\frac{\partial^2 \varphi}{\partial x_4^2}\right]-\varepsilon\left[\sum_{i=1}^{3}A_i\frac{\partial a}{\partial x_i}-\frac{1}{c^2}\frac{\partial a}{\partial x_4}\right]=0,$$

which will be written:

$$(24)\qquad \sum_{i=1}^{3}\frac{\partial}{\partial x_i}\left[a^2\left(\frac{\partial \varphi}{\partial x_i}-\varepsilon A_i\right)\right]-\frac{1}{c^2}\frac{\partial}{\partial x_4}\left[a^2\left(\frac{\partial \varphi}{\partial x_4}-\varepsilon A_4\right)\right]=0$$

after multiplying by $2a$, because from the Lorentz relation between the potentials, one will have ([19]):

$$\sum_{i=1}^{3}\frac{\partial A_i}{\partial x_i}-\frac{1}{c^2}\frac{\partial A_4}{\partial x_4}=0.$$

Consider the quadrivector **W** whose covariant components are:

$$(25)\qquad W_i=\frac{\partial \varphi}{\partial x_i}-\varepsilon A_i \qquad (i = 1, 2, 3, 4).$$

Equation (24) can be written:

$$\sum_{i=1}^{4}\frac{\partial (a^2 W^i)}{\partial x_i}=0.$$

Let:

$$(26)\qquad \operatorname{Div} a^2\,\mathbf{W}=0,$$

([19]) That relation is written $\sum_{i=1}^{3}\frac{\partial A^i}{\partial u_i}$ upon utilizing the usual notations of tensorial calculus. Now, $A_i = -A^i$ (i = 1, 2, 3) and $A_4 = c^2 A^4$, because the fundamental form is $ds^2 = c^2\,dx_4^2 - dx_1^2 - dx_2^2 - dx_3^2$.

in which Div is the symbol of the world-divergence, because:

$$W^i = -W_i \qquad (i = 1, 2, 3)$$

and

$$W^4 = \frac{1}{c^2}\, W_4 \,.$$

One then has a fictitious fluid whose definition one must find and whose quadri-dimensional quantity of motion is the vector $a^2\,\mathbf{W}$. Equation (26) is the equation of continuity of that fluid.

43. – Let us define some *classes of motions* that relate to an arbitrary Jacobi equation. Let:

$$S\,(q_1, \ldots, q_n\,;\, a_1\,, \ldots, a_n)$$

be a complete integral of a Jacobi equation. We know that we require n arbitrary constants in order to define the general equation of the equations of motion. We say that all of the motions for which $a_1\,, \ldots, a_n$ are fixed belong to the same class. We will then consider some identical corpuscles whose motions all have the same class. There are ∞^n of them (corresponding to the values of the n constants that do not enter into the complete integral). Those material points form a fluid that fills up a region of the space of q_i . It is the velocity (which comes from the quantity of motion) of the points of that fluid that we shall occupy ourselves with in the particular case of the preceding problem. We remark that the fictitious fluid is composed of material particles that exert no effect upon each other.

If one is given the constants a_i in the complete integral then that will define the values of the p_i at each point in the space of the q_i that relate to the particle that one finds there ([20]). Now, the p_i are the components of a field in the fluid that serves to characterize the quantity of motion. One can then say that one will know the velocity field of a certain fluid when one is given a class of motions. Furthermore, if one knows the class to which the motion of a material point belongs then one will know nothing about its position. One will know only that it is a particle in a certain fluid that is defined by the Euler variables.

Now, suppose that the density of that fluid is very large in a small region and very weak elsewhere. Since we do not know anything about the position of the material point in question, can we say that there is any chance that we can find it precisely in the regions where the density of the fluid is great?

It is then natural to consider the density of the fictitious fluid, which is, in some way, the *collective effect* of the motions of an entire class, to be a *relative probability*, a *probability density*, or even as a measure of the *probability of presence* of our particle at a point at an instant.

([20]) In the case of a stationary regime, one of the q_i is ordinary time; the time of propagation has been eliminated. Upon decomposing E_n into "space" and time, one will determine the state of a quantity at a point of "space" at an instant by way of the p_i .

One will then see that it is upon posing a very natural question that one will be led to introduce the calculus of probabilities into analytical mechanics, or rather the wave conception of analytical mechanics.

A more systematic study of what one calls the *Eulerian method* for the study of force fields was recently carried out by J. Ullmo [**42**].

44. – Recall the calculations of paragraph **42** and seek to interpret the quadrivector **W**. In order to do that, one must refer to paragraph **18**. One sees that:

$$\mathbf{W} = \frac{\mathbf{V}}{c}, \tag{27}$$

and as a result, the continuity equation (26) will be written:

$$\sum_{i=1}^{4} \frac{\partial}{\partial x_i}\left(a^2 m_0 \frac{dx_i}{ds}\right) = 0.$$

Now, the world-vector whose contravariant components are $m_0\, dx / ds$ is the generalized quantity of motion of the point that we consider. The fictitious fluid that we consider to have a density that is equal to $a^2\, m_0$, and for which we can say that $a^2 = \psi\bar{\psi}$ is a measure of the probability that the material point of mass m_0 will be found at the point (x_1, x_2, x_3) when the only thing that we know about it is that its motion has a class that is determined by the choice of constants that enter into the complete integral φ of the Jacobi equation (22).

We remark that function a is determined by the equation:

$$\sum_{i=1}^{3} \frac{\partial^2 a}{\partial x_i^2} - \frac{1}{c^2}\frac{\partial^2 a}{\partial x_4^2} = 0 \tag{28}$$

that one obtains upon writing that the term in (20), in which one has made the substitution (21), that is independent of μ is zero.

Since a is determined up to a constant factor, it is natural to normalize it in such a manner that the total probability of finding the point considered at some location at an instant x_4 will be precisely unity. Upon integrating over all space, one needs only to write:

$$\iiint \psi\bar{\psi}\, d\tau = 1.$$

However, that equation is not invariant with respect to Lorentz transformations. One can also write:

$$\iiint_U\!\!\int \psi\bar{\psi}\, dx_1\, dx_2\, dx_2\, dx_3 = 1,$$

in which the domain of integration is the entire universe, but that integral will diverge, in general.

That is a very grave difficulty, along with some others, as well, and it has led to a new theory of wave mechanics that is due to Dirac, and in which the second-order equation (20) is replaced by a system of first-order equations that will lead to (20) when it is iterated. As Racah [**35**] has shown, there is a relation between the new system and the Jacobi equation that defines the characteristics of the system precisely. It would be impossible to present Dirac's theory within the scope of this monograph. One knows its importance and its fecundity. L. de Broglie has shown that Dirac's theory leads, in turn, to some difficulties that come about from the asymmetric role that must be played by space and time, despite the origins of the theory [**5**].

45. – One must remark that if one abandons the approximation of geometrical optics then it will no longer be possible to determine a and φ from the two equations in one unknown (22) and (28). Meanwhile, equation (23) will persist because it is produced by the annihilation of the term in i, as well as that of the term in μ, in the substitution of (21) in (20) ([21]).

That can always be interpreted by appealing to (26), but **W** would then have a different meaning, since one can no longer express it in terms of a solution to the Jacobi equation. The relationship between a and φ that one obtains upon annulling the real term after substituting (21) into (20) is written:

$$\mu^2\left[\frac{1}{c^2}\left(\frac{\partial\varphi}{\partial x_4}-\varepsilon A_4\right)^2-\sum_{i=1}^{3}\left(\frac{\partial\varphi}{\partial x_i}-\varepsilon A_i\right)^2-m_0^2\,c^2\right]-\frac{1}{a}\left[\frac{1}{c^2}\frac{\partial^2 a}{\partial x_4^2}-\sum_{i=1}^{3}\frac{\partial^2 a}{\partial x_i^2}\right]=0.$$

In the treatises, **W** is given a form that is different from the one that we gave it above (27). One starts from equation (20) and changes i into $-i$ and ψ into $\bar{\psi}$, while the A_k, φ, and a remain unchanged. One will obtain an equation (20′), but there is no point in writing it out. One multiplies the two sides of (20) by $\bar{\psi}$, those of (20′) by ψ, and subtracts the corresponding sides of the equations that are obtained. After taking the Lorentz relation into account, one will find effortlessly [**5**, pp. 93] that:

$$\sum_{i=1}^{4}\frac{\partial C^i}{\partial x^i}=0. \tag{29}$$

The covariant components of **C** are:

$$C_k = 2\mu\varepsilon\, i\, A_k\, \psi\bar{\psi}-\left(\bar{\psi}\frac{\partial\psi}{\partial x_k}-\psi\frac{\partial\bar{\psi}}{\partial x_k}\right) \qquad (k = 1, 2, 3),$$

([21]) Of course, that is after simplifying it by $e^{i\mu\varphi}$.

$$C_4 = 2\mu\varepsilon\, i\, A_4\ \psi\bar{\psi} - \left(\bar{\psi}\frac{\partial\psi}{\partial x_4} - \psi\frac{\partial\bar{\psi}}{\partial x_4}\right).$$

One will recover (26) upon replacing ψ with $a\,e^{i\varphi}$ and $\bar{\psi}$ with $a\,e^{-i\varphi}$ in (29).

The vector **C** is sometimes called the (world) *probability current.*

46. – The reader will effortlessly see how the equations of the preceding paragraphs simplify when one limits oneself to the Newtonian approximation ([22]). In particular, the equation of continuity will become:

$$\frac{\partial a^2}{\partial t} + \operatorname{div} a^2\, \mathbf{v} = 0,$$

in which $a^2 = \psi\bar{\psi}$ and **v** is the ordinary velocity of the point in the fictitious fluid. One normalizes a here with no difficulties in regard to invariance by way of:

$$\iiint_E a^2\, dx\, dy\, dz = 1,$$

in which the domain of integration is all of space. We do not wish to go further into wave mechanics, but we can remark that the second-order equation by means of which one determines $\psi = a\,e^{i\mu\varphi}$ is the *Schrödinger equation.* ψ must be a solution that is *finite and uniform* in all of space, and is such that:

$$\iiint \psi\bar{\psi}\, dv = 1.$$

Those conditions will determine the fundamental functions of the Schrödinger equation after one has determined the values of the dynamical parameters that enter into it in order for the equation to have solutions of the indicated type [**36**].

Plane wave and Planck constant. De Broglie's principle.

47. – It remains for us to say a few words about the number μ, which is assumed to be very large, and which we introduce in order to pass from the second-order equation to the Jacobi equation – i.e., in order to easily obtain the "geometric" approximation to wave mechanics.

Since the entire evolution of quantum physics has shown that when one can neglect Planck's constant h in this or that relation that expresses a quantum law, one will recover a law of classical mechanics, one might think that μ must be a function of h, when it is considered to be a parameter, that will become infinite when h tends to zero.

([22]) In that case, one would start with equation (12) of paragraph **18** and make the usual simplifications when one passes from special relativity to Newtonian mechanics.

In order to make these remarks more precise, we shall address the case of a material point of mass m_0 that displaces in the universe under the hypothesis that the potential **A** that is zero.

The equation of propagation is then:

$$\sum_{i=1}^{3} \frac{\partial^2\psi}{\partial x_i^2} - \frac{1}{c^2}\frac{\partial^2\psi}{\partial x_i^2} - \mu^2 m_0^2 c^2 \psi = 0. \tag{30}$$

The simplest solutions are:

$$\psi = a\, e^{i\mu(a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4)}, \tag{31}$$

in which a and the a_i are constants. The wave that is represented by that function will be a plane wave ([23]). On the other hand, one knows that the rays of propagation are lines in the geometric approximation. In the absence of a field, the trajectories of the material point are indeed rectilinear, and they will be traversed by a uniform motion. The world-lines are then rectilinear. If one substitutes (31) into (30) then one will find:

$$\frac{a_4^2}{c^2} - a_1^2 - a_2^2 - a_3^2 = m_0^2 c^2 . \tag{32}$$

However, the numbers a_1, a_2, a_3, a_4 are covariant parameters of the normal to the plane wave, so they will be proportional to the covariant parameters of the generalized quantity of motion. If k is an undetermined factor then one will have:

$$a_i = -\, k\, m_0 \frac{dx_i}{ds}, \qquad a_4 = k\, m_0\, c^2 \frac{dx_4}{ds}.$$

Upon substituting this in (32), one will find that $k^2 = 1$.

On the other hand, if the frequency of the wave considered relative to time x_4 is v then one must have ([24]):

$$\frac{\partial\psi}{\partial x_4} = 2\pi\, i\, v\, \psi.$$

Now, one has:

([23]) If $m_0 = 0$ then one will have the equation of optics. In regard to that equation, one should cite the very penetrating study of Le Roux [**19**], whose considerations belong to the ones that we made here.

([24]) Here, there might be some uncertainty as to which path one should follow. We remarked above that ψ would have to be periodic in x_5 if one would like to utilize the theory of Delassus and Hadamard. When one treats the simple problems, the function φ will be linear in x_4, so the periodicity relative to x_5 will imply the periodicity relative to x_4. It would be interesting to take periodicity in x_5 in the most general case, and to not append the hypothesis of periodicity in x_4. In the ultimate development of quantum mechanics, that would amount to not making energy play a privileged role, and perhaps we would, in addition, arrive at a theory in which space and time do not play very different roles, which is much-desired today. There is even some chance that the significance of the constant h might be clarified by the essential role that is played by x_4; i.e., by the *action*.

$$\frac{\partial\psi}{\partial x_4} = i\,\mu\,a_4\,\psi = i\,\mu\,m_0\,c^2\frac{dx_4}{ds}\,\psi = i\,\mu\,\frac{m_0c^2}{\sqrt{1-\beta^2}}\,\psi,$$

in which $\beta = c / v$, where v is the velocity of the particle (in space). One can then set:

$$2\pi\,\nu = \mu\,\frac{m_0c^2}{\sqrt{1-\beta^2}}\,. \tag{33}$$

The quantity $\frac{m_0c^2}{\sqrt{1-\beta^2}}$ represents the *energy* E of the material point. In a very large number of phenomena – in particular, the ones that are concerned with radiation – one will be led to attach a frequency to the elements of the energy that seem to be emitted or absorbed by discontinuous matter. As one knows [**2**], L. de Broglie thinks that one should extend the quantum relation that Planck, Einstein, and Bohr have made such fruitful use of. He proposed to make any phenomenon in which the quantity of energy E is brought into play correspond to a frequency ν by the condition ([25]):

$$E = h\nu. \tag{34}$$

Moreover, if one assumes precisely that the wave of analytical mechanics has the frequency that is indicated by de Broglie's principle then the relation (33) can be written:

$$2\pi\,\nu = \mu\,E = \mu\,h\,\nu,$$

and that will show that:

$$\mu = \frac{2\pi}{h}.$$

Group velocity.

48. – Meanwhile, a very grave difficulty presents itself from the outset that L. de Broglie brought to light brilliantly. We have pointed it out for some time already, moreover. It presents itself at the instant when one introduces the motion of waves into analytical mechanics (cf., Chap. VI, § **33**). *The velocity of propagation of a wave that is attached to the motion of a material point is not equal to the velocity of the material point.*

One can account for this immediately with the example that we just treated by calculating the velocity of the plane wave. The simplest way is to remark that for that plane wave:

([25]) Here, one should cite a paper by Persico [**34**]. That author has shown that if one assumes that the energy is a function of the frequency then a theory of propagation that is entirely similar to the one that we have developed above will finally lead to the relation $E = h\nu$.

$$\frac{\partial^2\psi}{\partial x_4^2} = -4\pi^2 \nu^2 \psi.$$

Furthermore, equation (30) can be written:

$$\sum_{i=1}^{3}\frac{\partial^2\psi}{\partial x_i^2} - \frac{n^2}{c^2}\frac{\partial^2\psi}{\partial x_4^2} = 0, \tag{35}$$

with

$$n^2 = \frac{4\pi^2\nu^2 - \mu^2 m_0^2 c^4}{4\pi^2\nu^2} = 1 - \frac{m_0^2 c^4}{h^2\nu^2} = 1 - \frac{\nu_0^2}{\nu^2},$$

if one sets $m_0 c^2 = h\nu_0$, in which ν_0 is the frequency that is attached to the material point for an observer that is at rest with respect to it. Indeed, in that case, the mass m_0 is equivalent to the energy $E_0 = m_0 c^2$ ([26]).

Moreover, from (35), the velocity of the wave is:

$$V = \frac{c}{n};$$

i.e., it is greater than c because $n < 1$. On the other hand, one has:

$$1 - \frac{\nu_0^2}{\nu^2} = 1 - \frac{E_0^2}{E^2} = 1 - \frac{m_0^2 c^4}{m_0^2 c^4/(1-\beta^2)} = \beta^2 = \frac{v^2}{c^2};$$

hence:

$$V = \frac{c^2}{v}.$$

The product of the velocity of the material point with the velocity of the plane wave that is associated with it is equal to the square of the speed of light.

In optics, it has already been known for some time that the phase velocity in a dispersive medium is not equal to the velocity of energy transport. Since Lord Rayleigh, it has been recognized that if n – viz., the *index of refraction* of the medium in which the waves propagate – is a function of the frequency then the velocity U, which is called the *group velocity*, and is the velocity at which energy propagates, will be given by the formula:

$$\frac{1}{U} = \frac{1}{c}\frac{d(n\nu)}{d\nu}.$$

([26]) As L. de Broglie showed, the relation $W = h\nu$ for energy is naturally implied by covariance from the other ones for the components of the quantity of motion. Moreover, we shall confine ourselves here to the very simple case of a plane wave. The generalization to the case in which the force field is arbitrary was also pursued successfully by the founder of wave mechanics. Levi-Civita introduced the notion of a wave in the local sense in the case of an Einsteinian field [**24**].

Now, in our problem we see that:

$$U = v.$$

In the simplest case that we have addressed, the velocity of the material point is equal to the group velocity of the plane waves that are attached to the motion of the point.

This very general principle, which is placed at the roots of wave mechanics, along with the principle of the frequency, was studied exhaustively by L. de Broglie in his *Introduction à l'étude de la Mécanique ondulatoire;* we refer the reader to it. For us, it will have to suffice that we have shown the beginning of the road that leads to wave mechanics when one starts from analytical mechanics. That new path, which was presented by Hamilton and prepared by Vessiot and Hadamard, and along which L. de Broglie and Schrödinger have made bold advances, has penetrated into the heart of a new province of natural philosophy. One can explore it more completely by other paths, but it seems to us that none of them are more attractive than the one that we have described in the course of this attempt.

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