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Dirac operators and Maxwell equations

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Introduction. In his remarkable memoirs and in his book *The Principles of Quantum Mechanics* [1]), P. A. M. Dirac – in order to obtain equations for the electron that conformed to the principles of quantum mechanics more than the one ones that one obtains by starting with d'Alembert's equation – proposed to find a differential operator ∇ of the first order, whose iterate ∇^2 is the d'Alembert operator $\square$:

$$\nabla^2 = \square \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} = \sum_{j=1}^{4} \frac{\partial^2}{\partial x_j^2},$$

This amounts to finding four "numbers" $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$ such that:

$$\left(\sum_{j=1}^{4} \Gamma_j \frac{\partial}{\partial x_j} \right)^2 = \sum_{j=1}^{4} \frac{\partial^2}{\partial x_j^2};$$

i.e., such that one has:

$$\Gamma_j \Gamma_k + \Gamma_k \Gamma_j = 0, \qquad (j \neq k), \qquad\qquad \Gamma_j^2 = 1.$$

These numbers may be considered to be operators that representable by matrices or by hypercomplex numbers. In the representation by matrices, one may take:

$$\Gamma_1 = \begin{Vmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \end{Vmatrix}, \quad \Gamma_2 = \begin{Vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{Vmatrix}, \quad \Gamma_3 = \begin{Vmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{Vmatrix}, \quad \Gamma_4 = \begin{Vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{Vmatrix};$$

however, the function that the operator ∇ acts upon is no longer a scalar. For H. Weyl [2]), it is a matrix with four rows and one column:

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[1]) Oxford University Press, 1930; pp. 238 *et seq.*
[2]) *Gruppentheorie und Quantenmechanik*, pp. 112.

$$\begin{Vmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{Vmatrix},$$

and one has the following four components of $\left\| \sum_{k=1}^{4} \nabla_{jk} \psi_k \right\|$ for the matrix $\| \nabla \psi \|$; i.e. [3]):

$$\begin{aligned} & i\frac{\partial\psi_4}{\partial x_1}+\frac{\partial\psi_4}{\partial x_2}+i\frac{\partial\psi_3}{\partial x_3}+\frac{\partial\psi_3}{\partial x_4}, \\ & i\frac{\partial\psi_3}{\partial x_1}-\frac{\partial\psi_3}{\partial x_2}-i\frac{\partial\psi_4}{\partial x_3}+\frac{\partial\psi_4}{\partial x_4}, \\ -\ & i\frac{\partial\psi_2}{\partial x_1}-\frac{\partial\psi_2}{\partial x_2}-i\frac{\partial\psi_1}{\partial x_3}+\frac{\partial\psi_1}{\partial x_4}, \\ -\ & i\frac{\partial\psi_1}{\partial x_1}+\frac{\partial\psi_1}{\partial x_2}+i\frac{\partial\psi_2}{\partial x_3}+\frac{\partial\psi_2}{\partial x_4}. \end{aligned}$$

A. Proca, who used the representation by hypercomplex numbers, has proposed, in a bold and elegant generalization, to replace the wave function by a hypercomplex number and one obtains a system of 16 first-order partial differential equations for the Dirac equations, from which one hopes to deduce the better part of the dynamics of the electron [4]).

We have recovered the hypercomplex numbers in question by seeking to resolve a problem that was posed by C. Lanczos [5]). It is possible to give the Maxwell equations and the Dirac equations such a form that the one approaches the other by using the calculus of quaternions, and this may permit us to couple the Dirac equations with a field theory. The memoirs of Lanczos are possessed of an amazing ingenuity, and the elegance of their concepts is to be emphasized [5]), although along the way he exhibited difficulties concerning the covariance that led the author to remark: "This situation very much seems to suggest that the Dirac equation should not be regarded as a closed system in itself, but as a component of a larger system."

We confirm that if one uses for one's system of hypercomplex numbers, not the quaternions, but another system that was employed by Proca, then one is led to a very simple formulation of the Maxwell equations. We believe that it is upon reconciling these results with the ones that Proca obtained that one may obtain a field theory for the Dirac equation. For the moment, we confine ourselves to the Maxwell equations.

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[3]) *Gruppentheorie und Quantenmechanik*, pp. 171.

[4]) C. R. Paris, t. 190, session on 16 June 1930, pp. 1377, and t. 191, session on 7 July 1930, pp. 26; finally, J. de Phys. (VII), t. 1, July 1930, pp. 235-248.

[5]) Cf., in particular, the first in a series of three memoirs that was entitled: "Die tensoranalytischen Beziehungen der Diracschen Gleichung," Z. f. Phys., Bd. **57**, pp. 447-473.

1. The hypercomplex numbers that we are concerned with here are the numbers that were considered quite recently by several authors, the first of which in time was Clifford [6]). It serves to represent the linear substitutions that leave invariant the form:

$$x_1^2 + x_2^2 + x_3^2 + x_4^2$$

(where x_4 is ict); they are the rotations of Euclidian space E_4 . One proves that these linear substitutions $x \mapsto x'$ are obtained in the following manner:

One defines a system of hypercomplex numbers with associative multiplication by means of 16 units that are formed from 5 fundamental units:

$$1,\ \Gamma_1,\ \Gamma_2,\ \Gamma_3,\ \Gamma_4,$$

whose pair-wise products satisfy the following equations:

$$\left\{\begin{aligned} &1\cdot\Gamma_j = \Gamma_j, \qquad \Gamma_j^2 = 1, \\ &\Gamma_j\Gamma_k + \Gamma_k\Gamma_j = 0, \ (j \neq k). \end{aligned}\right. \tag{1}$$

The other units of the system are products:

$$\begin{aligned} &\Gamma_1\,\Gamma_2,\ \Gamma_1\,\Gamma_3,\ \Gamma_1\,\Gamma_4,\ \Gamma_2\,\Gamma_3,\ \Gamma_2\,\Gamma_4,\ \Gamma_3\,\Gamma_4\,; \\ &\Gamma_1\,\Gamma_2\,\Gamma_3,\ \Gamma_1\,\Gamma_2\,\Gamma_4,\ \Gamma_1\,\Gamma_3\,\Gamma_4,\ \Gamma_2\,\Gamma_3\,\Gamma_4\,; \\ &\Gamma_1\,\Gamma_2\,\Gamma_3\,\Gamma_4\,. \end{aligned}$$

These 16 units are linearly independent, and every product of an arbitrary number of them reduces to one of them, thanks to (1). A hypercomplex number of our system will have the following form:

$$\begin{aligned} C = c_0 &+ c_1\,\Gamma_1 + c_2\,\Gamma_2 + c_3\,\Gamma_3 + c_4\,\Gamma_4\,, \\ &+ c_{12}\,\Gamma_1\,\Gamma_2 + c_{13}\,\Gamma_1\,\Gamma_3 + c_{14}\,\Gamma_1\,\Gamma_4 + c_{23}\,\Gamma_2\,\Gamma_3 + c_{24}\,\Gamma_2\,\Gamma_4 + c_{34}\,\Gamma_3\,\Gamma_4 \\ &+ c_{123}\,\Gamma_1\,\Gamma_2\,\Gamma_3 + c_{124}\,\Gamma_1\,\Gamma_2\,\Gamma_4 + c_{134}\,\Gamma_1\,\Gamma_3\,\Gamma_4 + c_{234}\,\Gamma_2\,\Gamma_3\,\Gamma_4 \\ &+ c_{1234}\,\Gamma_1\,\Gamma_2\,\Gamma_3\,\Gamma_4\,. \end{aligned}$$

The c_λ are ordinary numbers (which may be Gauss numbers). Such a number will be called a *Lorentz number* [7]).

One will represent a point (x_1, x_2, x_3, x_4) of E_4 or a vector with components (x_1, x_2, x_3, x_4) in the same E_4 , when referred to a system of rectangular by a Lorentz number,12 of whose coordinates are null:

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[6]) For everything that concerns the hypercomplex numbers, one can refer to the Encyclopédie des Sciences mathématiques, Tome I, volume I, fascicule 3 (1903), where *Cartan* and *Study* have summarized, in a very original fashion, the results that were obtained at the time in that branch of mathematics; in particular, for Clifford numbers, cf., pp. 463-466.

[7]) It seems preferable to employ the name of Lorentz number, rather than that of "quadri-quaternion." The former term is more precise about the utility of these numbers.

$$V = x_1\,\Gamma_1 + x_2\,\Gamma_2 + x_3\,\Gamma_3 + x_4\,\Gamma_4\,.$$

2. A Lorentz transformation of E_4 transforms V into:

$$V' = x_1'\Gamma_1 + x_2'\Gamma_2 + x_3'\Gamma_3 + x_4'\Gamma_4\,,$$

and V' is defined in the following fashion, which shows precisely the importance of the Lorentz number system. Let A denote a Lorentz number that is a vector or product of vectors that are not divisors of zero – i.e., none of them can be a factor in a product that equals zero. A will have a well-defined inverse A^{-1} ($AA^{-1} = A^{-1}A = 1$). One passes from V to V' by the relation:

(2) $$V' = A\,V\,A^{-1}.$$

It is clear that the coordinates of A must satisfy certain reality conditions in order for the Lorentz transformation (2) to be real, in its own right. One sees that the x_j' are expressed linearly as functions of the x_j, such that the parameters of the transformation enter the formulas bilinearly, and they are super-abundant if A has more than 6 non-zero independent coordinates. The transformation (2) does not consist in making a change of units Γ_j, as one does in vector algebra if the Γ_j are considered to be basis vectors. It is the rules of algebra that the units Γ_j obey that give equation (2) its Lorentzian character. Proca made a change of the Γ_j that is not in the spirit of the method that was imagined by the inventors of the number system thus used; this does not diminish the interest of the results thus obtained by that physicist, moreover.

3. Proca [8]) first made a remark that amounts to saying that the Lorentz number:

$$\begin{aligned} W &= c_{234}\,\Gamma_2\,\Gamma_3\,\Gamma_4 - c_{134}\,\Gamma_1\,\Gamma_3\,\Gamma_4 + c_{124}\,\Gamma_1\,\Gamma_2\,\Gamma_4 - c_{123}\,\Gamma_1\,\Gamma_2\,\Gamma_3 \\ &= (c_{234}\,\Gamma_1 + c_{134}\,\Gamma_2 + c_{124}\,\Gamma_3 + c_{123}\,\Gamma_4)\,\Gamma_1\,\Gamma_2\,\Gamma_3\,\Gamma_4 \end{aligned}$$

may represent a vector with components:

$$y_1 = c_{234}\,, \qquad y_2 = c_{134}\,, \qquad y_3 = c_{124}\,, \qquad y_4 = c_{123}\,.$$

Indeed, if one writes:

$$W \equiv V\,\Gamma_1\,\Gamma_2\,\Gamma_3\,\Gamma_4$$

then the transformation:

(3) $$W' = A\,W\,A^{-1}$$

amounts to:

$$W' = A\,V\,\Gamma_1\,\Gamma_2\,\Gamma_3\,\Gamma_4\;A^{-1} = A\,V\,A^{-1}\,A\,\Gamma_1\,\Gamma_2\,\Gamma_3\,\Gamma_4\;A^{-1} = -\,V'\Gamma_1\,\Gamma_2\,\Gamma_3\,\Gamma_4\,.$$

[8]) J. de Phys., *loc. cit.*, pp. 241.

Therefore, the multiplication of a vector V by $\Gamma_1 \Gamma_2 \Gamma_3 \Gamma_4$ preserves the vectorial character of the number V; however, the transformation (3) is the Lorentz transformation (2) that has a symmetry with respect to the origin.

4. Another remark of Proca [9]) amounts to saying that the Lorentz number:

$$T = c_{12} \Gamma_1 \Gamma_2 + c_{23} \Gamma_2 \Gamma_3 + c_{31} \Gamma_3 \Gamma_1 + c_{14} \Gamma_1 \Gamma_4 + c_{24} \Gamma_2 \Gamma_4 + c_{34} \Gamma_3 \Gamma_4$$

transforms as a skew-symmetric tensor of second order under the transformation:

$$T' = A\, T\, A^{-1}. \tag{4}$$

This is obvious if one remarks that T may be regarded as the product of two numbers:

$$G = g_1 \Gamma_1 + g_2 \Gamma_2 + g_3 \Gamma_3 + g_4 \Gamma_4 ,$$
$$H = h_1 \Gamma_1 + h_2 \Gamma_2 + h_3 \Gamma_3 + h_4 \Gamma_4 ,$$

such that $g_1 h_1 + g_2 h_2 + g_3 h_3 + g_4 h_4 = 0$. Then $GH + HG = 0$ and $c_{jk} = g_j h_k - g_k h_j$.

The transformation (4) is then written:

$$T' = A\, G\, H\, A^{-1} = A\, G\, A^{-1}\, A\, H\, A^{-1} = G' H',$$

and since $G' H' + H' G' = 0$, T' is indeed the transform of T by the Lorentz transformation whose operator has the symbol A.

5. We finally remark that the Lorentz number:

$$C = c_0$$

is unaltered by the transformation $A\, C\, A^{-1}$, and that:

$$C = c_{1234} \Gamma_1 \Gamma_2 \Gamma_3 \Gamma_4$$

is transformed into $- C$.

6. In summary, a hypercomplex number C may be written in the form:

$$C = I + V + T + W + J,$$

and by the distributive transformation relative to addition:

$$C' = A\, C\, A^{-1} = I' + V' + T' + W' + J',$$

$$I = I', \qquad J = - J',$$

[9]) J. de Phys., *loc. cit.*, pp. 242.

$V' = A\,V A^{-1}$ (V' = vector that is the transform of the vector V by the Lorentz transformation $\mathcal{L}$ (2))

$W' = A\,W A^{-1}$ (W' = vector that is the transform of the vector W by the Lorentz transformation $\mathcal{L}$)

$T' = A\,T A^{-1}$ (T' = vector that is the transform of the vector T by the Lorentz transformation $\mathcal{L}$).

It is, moreover, preferable to write:

$$C = I_1 + V_1 + T + (V_2 + T_2)\,\Gamma_1\,\Gamma_2\,\Gamma_3\,\Gamma_4\,,$$

and then I_1 and I_2 are two invariants, V_1 and V_2 are two vectors, and T is a skew-symmetric tensor.

7. Now consider the differential operator ∇, which we define by the hypercomplex number:

$$\nabla = \Gamma_1 \frac{\partial}{\partial x_1} + \Gamma_2 \frac{\partial}{\partial x_2} + \Gamma_3 \frac{\partial}{\partial x_3} + \Gamma_4 \frac{\partial}{\partial x_4}\,.$$

From the most elementary rules of vector analysis, it is clear that a transformation (2) that acts on the space E_4 (x_1, x_2, x_3, x_4) will transform ∇ into:

$$\nabla = A\nabla A^{-1} = \Gamma_1 \frac{\partial}{\partial x_1'} + \Gamma_2 \frac{\partial}{\partial x_2'} + \Gamma_3 \frac{\partial}{\partial x_3'} + \Gamma_4 \frac{\partial}{\partial x_4'}\,,$$

since from (x_1, x_2, x_3, x_4) $\mapsto (x_1', x_2', x_3', x_4')$, one infers that:

$$\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4}\right) \mapsto \left(\frac{\partial}{\partial x_1'}, \frac{\partial}{\partial x_2'}, \frac{\partial}{\partial x_3'}, \frac{\partial}{\partial x_4'}\right).$$

8. Now, let C be a field of Lorentz numbers; i.e., a set of Lorentz numbers that bijectively correspond to all of the points of a region of E_4 (x_1, x_2, x_3, x_4); the coordinates of C are functions of x_1, x_2, x_3, x_4 . One may define a derived field ∇C, and this in only one manner; i.e., independently of the Lorentz frame (x_1, x_2, x_3, x_4) considered. We shall perform the calculation for each part of C.

9. The application of ∇ to c_0 gives the vector:

$$\nabla I = \Gamma_1 \frac{\partial c_0}{\partial x_1} + \Gamma_2 \frac{\partial c_0}{\partial x_2} + \Gamma_3 \frac{\partial c_0}{\partial x_3} + \Gamma_4 \frac{\partial c_0}{\partial x_4}\,,$$

which is nothing but the gradient of the scalar I_1 . Likewise, the application to:

$$J = c_{1234}\,\Gamma_1\,\Gamma_2\,\Gamma_3\,\Gamma_4$$

gives:

$$\begin{aligned}\nabla J &= \frac{\partial c_{1234}}{\partial x_1}\Gamma_2\Gamma_3\Gamma_4 - \frac{\partial c_{1234}}{\partial x_2}\Gamma_1\Gamma_3\Gamma_4 + \frac{\partial c_{1234}}{\partial x_3}\Gamma_1\Gamma_2\Gamma_4 - \frac{\partial c_{1234}}{\partial x_4}\Gamma_1\Gamma_2\Gamma_3 \\ &= \left(\frac{\partial c_{1234}}{\partial x_1}\Gamma_1 + \frac{\partial c_{1234}}{\partial x_2}\Gamma_2 + \frac{\partial c_{1234}}{\partial x_3}\Gamma_3 + \frac{\partial c_{1234}}{\partial x_4}\Gamma_4\right)\Gamma_1\Gamma_2\Gamma_3\Gamma_4 .\end{aligned}$$

One thus obtains an expression that may be written:

$$\text{grad}\, I_2 \cdot \Gamma_1\,\Gamma_2\,\Gamma_3\,\Gamma_4 .$$

10. If one takes the part $V = c_1\,\Gamma_1 + c_2\,\Gamma_2 + c_3\,\Gamma_3 + c_4\,\Gamma_4$ then one has:

$$\begin{aligned}\nabla V &= \left(\Gamma_1\frac{\partial}{\partial x_1} + \Gamma_2\frac{\partial}{\partial x_2} + \Gamma_3\frac{\partial}{\partial x_3} + \Gamma_4\frac{\partial}{\partial x_4}\right)(c_1\,\Gamma_1 + c_2\,\Gamma_2 + c_3\,\Gamma_3 + c_4\,\Gamma_4) \\ &= \frac{\partial c_1}{\partial x_1} + \frac{\partial c_2}{\partial x_2} + \frac{\partial c_3}{\partial x_3} + \frac{\partial c_4}{\partial x_4} \\ &+ \left(\frac{\partial c_2}{\partial x_1} - \frac{\partial c_1}{\partial x_2}\right)\Gamma_1\Gamma_2 + \left(\frac{\partial c_3}{\partial x_1} - \frac{\partial c_1}{\partial x_3}\right)\Gamma_1\Gamma_3 + \cdots + \left(\frac{\partial c_4}{\partial x_3} - \frac{\partial c_3}{\partial x_4}\right)\Gamma_3\Gamma_4 ,\end{aligned}$$

which shows that ∇V is the sum of an invariant and a skew-symmetric tensor that one might call the divergence and the curl of V:

$$\nabla V = \text{div}\, V + \text{curl}\, V.$$

The application of ∇ to W will give, if one writes:

$$W = (w_1\,\Gamma_1 + w\,\Gamma_2 + w_3\,\Gamma_3 + w_4\,\Gamma_4)\,\Gamma_1\,\Gamma_2\,\Gamma_3\,\Gamma_4 = V_2\,\Gamma_1\,\Gamma_2\,\Gamma_3\,\Gamma_4 ,$$

$$\nabla W = (\text{div}\, V_2 + \text{curl}\, V_2)\,\Gamma_1\,\Gamma_2\,\Gamma_3\,\Gamma_4 .$$

11. Finally, the application to T will give the following formula [10]):

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[10]) In these formulas , $\sum_{j,k}$ is a double sum, where one must consider only the combinations.

$$\nabla T = \left(\sum \Gamma_j \frac{\partial}{\partial x_j}\right)\left(\sum_{k,l} \Gamma_k \Gamma_l\right) =$$

$$\left(\frac{\partial c_{21}}{\partial x_2}+\frac{\partial c_{31}}{\partial x_3}+\frac{\partial c_{41}}{\partial x_4}\right)\Gamma_1+\left(\frac{\partial c_{12}}{\partial x_1}+\frac{\partial c_{32}}{\partial x_3}+\frac{\partial c_{42}}{\partial x_4}\right)\Gamma_2$$

$$+\left(\frac{\partial c_{13}}{\partial x_1}+\frac{\partial c_{23}}{\partial x_2}+\frac{\partial c_{43}}{\partial x_4}\right)\Gamma_3+\left(\frac{\partial c_{14}}{\partial x_1}+\frac{\partial c_{24}}{\partial x_2}+\frac{\partial c_{34}}{\partial x_3}\right)\Gamma_4$$

$$-\left[\left(\frac{\partial c_{43}}{\partial x_2}+\frac{\partial c_{32}}{\partial x_4}+\frac{\partial c_{24}}{\partial x_3}\right)\Gamma_1+\left(\frac{\partial c_{34}}{\partial x_1}+\frac{\partial c_{41}}{\partial x_3}+\frac{\partial c_{13}}{\partial x_4}\right)\Gamma_2\right.$$

$$\left.+\left(\frac{\partial c_{14}}{\partial x_2}+\frac{\partial c_{42}}{\partial x_1}+\frac{\partial c_{21}}{\partial x_4}\right)\Gamma_3+\left(\frac{\partial c_{23}}{\partial x_1}+\frac{\partial c_{12}}{\partial x_3}+\frac{\partial c_{31}}{\partial x_2}\right)\Gamma_4\right]\Gamma_1\Gamma_2\Gamma_3\Gamma_4 .$$

∇T is then the sum of two terms, one of which has the type V and the other of which has the type W, or, if one prefers:

$$\nabla T = T_1 - V_2\,\Gamma_1\,\Gamma_2\,\Gamma_3\,\Gamma_4 ,$$

T_1 and T_2 being two vectors. One sees that T_1 is the vector divergence of the tensor T (with components of the type $\sum_j \frac{\partial c_{jk}}{\partial x_j}$); T_2 is another vector whose component m is $\frac{\partial c_{jk}}{\partial x_l}+\frac{\partial c_{kl}}{\partial x_j}+\frac{\partial c_{lj}}{\partial x_k}$, $(jklm)$ being an even permutation of the four numerals 1, 2, 3, 4.

One may abbreviate this by writing:

$$\nabla T = \text{DIV}\, T - \max T \cdot \Gamma_1\,\Gamma_2\,\Gamma_3\,\Gamma_4 ,$$

DIV T being the vector-divergence of T, and max T being a vector that we shall call the *Maxwellian* of the tensor T.

11. In summary, if one writes:

$$C = I_1 + V_1 + T + (V_2 + I_2)\,\Gamma_1\,\Gamma_2\,\Gamma_3\,\Gamma_4$$

then one will have:

$$\nabla C = \text{div}\, V_1 + (\text{grad}\, I_1 + \text{DIV}\, T) + \text{curl}\, V_1 + \text{curl}\, V_2 \cdot \Gamma_1\,\Gamma_2\,\Gamma_3\,\Gamma_4$$
$$+ (\text{grad}\, I_2 - \max T)\,\Gamma_1\,\Gamma_2\,\Gamma_3\,\Gamma_4 + \text{div}\, V_2 \cdot \Gamma_1\,\Gamma_2\,\Gamma_3\,\Gamma_4 .$$

12. This being the case, one may write the Maxwell equations in a very elegant form that allows one to see their invariance vis-à-vis Lorentz transformations immediately.

Recall the form in which Weyl wrote them – for example [11]).

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[11]) For example, *Raum, Zeit, Materie*, 5th ed., pp. 154-155, and also Frenkel, *Lehrbuch der Elektrodynamik*, Bd. 1, passim.

Since the electromagnetic potential is a vector $\Phi(\varphi_1, \varphi_2, \varphi_3, \varphi_4)$ in the space E_4, the electromagnetic field F is a skew-symmetric tensor with components:

(5) $$F_{ik} = \frac{\partial \varphi_k}{\partial x_i} - \frac{\partial \varphi_i}{\partial x_k},$$

which implies that:

(6) $$\frac{\partial F_{ik}}{\partial x_l} + \frac{\partial F_{kl}}{\partial x_i} + \frac{\partial F_{li}}{\partial x_k} = 0.$$

If one further introduces the current vector C with the components c_1, c_2, c_3, c_4 then one has:

(7) $$\sum_{k=1}^{4} \frac{\partial F_{ik}}{\partial x_k} = c_i .$$

Finally:

(8) $$\frac{\partial \varphi_1}{\partial x_1} + \frac{\partial \varphi_2}{\partial x_2} + \frac{\partial \varphi_3}{\partial x_3} + \frac{\partial \varphi_4}{\partial x_4} = \text{div}\, \Phi = 0,$$

and

(9) $$\frac{\partial c_1}{\partial x_1} + \frac{\partial c_2}{\partial x_2} + \frac{\partial c_3}{\partial x_3} + \frac{\partial c_4}{\partial x_4} = \text{div}\, C = 0.$$

13. We represent the potential and current as two Lorentz numbers:

$$\Phi = \varphi_1 \Gamma_1 + \varphi_2 \Gamma_2 + \varphi_3 \Gamma_3 + \varphi_4 \Gamma_4$$
$$C = c_1 \Gamma_1 + c_2 \Gamma_2 + c_3 \Gamma_3 + c_4 \Gamma_4 .$$

One has:

$$\nabla\Phi = \text{div}\, \Phi + \text{curl}\, \Phi,$$

but if one writes the electromagnetic field F by the Lorentz number:

$$F = \sum_{i,k} F_{ik} \Gamma_i \Gamma_k ,$$

then upon writing:

$$\nabla\Phi = F,$$

one has, at the same stroke, equations (5) and equation (8).

Now form:

$$\nabla F = \text{DIV}\, F = \max F \cdot \Gamma_1 \Gamma_2 \Gamma_3 \Gamma_4 .$$

If one sets:

$$\nabla F = - C$$

then one writes both equations (7) and equations (6) at the same time. One immediately infers from this that:

$$\nabla\nabla\Phi = - C;$$

i.e.:

$$\Box\Phi = -C,$$

while, on the other hand, one must have identically:

$$\nabla\nabla F = \Box F = \sum_i \frac{\partial^2}{\partial x_i^2}\left[\sum_{j,k} F_{jk}\Gamma_j\Gamma_k\right] = \sum_{j,k}\Box F_{jk}\Gamma_j\Gamma_k ,$$

but:

$$\Box F = -\nabla C = -\operatorname{div} C - \operatorname{curl} C,$$

so:

$$\operatorname{div} C = 0;$$

i.e., (9).

The problem that we have posed is then solved completely. One might propose to give an analogous form to the equations of energy, which we shall do in a later memoir.

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