The Frenet formulas for a Weyl space

By

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Blaschke recently gave (¹) the Frenet formulas for a curve that is traced in a space \( R_n \) with a Riemannian metric in which one defines parallel displacement as Levi-Civitá did (²). Upon employing the same calculation procedures that Blaschke did, we have obtained the Frenet formulas for a curve that is traced in a space \( W_n \) with a Weyl metric.

An \( n \)-dimensional Weyl (³) space \( W_n \) is an \( n \)-dimensional multiplicity in which the metric is defined by two forms (one quadratic and one linear):

\[
ds^2 = \sum_{i,k=1}^{n} g_{ik} \, dx_i \, dx_k , \quad d\varphi = \sum_{i=1}^{n} \varphi_i \, dx_i .
\]

\( d\varphi \) is an invariant for any continuous transformation \((T)\) of the form:

\[
x_i = \psi_i (y_1, \ldots, y_n) \quad (i = 1, 2, \ldots, n).
\]

Moreover, if one makes a change of calibration – i.e., if one takes a unit of length that is \( \sqrt{\lambda} \) times smaller (\( \lambda = \) continuous function of \( x_1, \ldots, x_n \)) – then the two forms will become:

\[
ds'^2 = \sum_{i,k=1}^{n} \lambda g_{ik} \, dx_i \, dx_k , \quad d\varphi' = d\varphi - \frac{d\lambda}{\lambda}.
\]

The laws of geometry must be satisfied under the following two conditions:

1. They are expressed by formulas that are invariant under any transformation \((T)\).

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¹ Math. Zeit. 6 (1919).
² Rendiconti del Circolo mat. di Palermo 42 (1917).
³ See WEYL, Raum, Zeit, Materie, 4th edition, § 16.
2. Those formulas remain invariant if one changes $g_{ik}$ into $\lambda \ g_{ik}$ and $\varphi_i$ into $\varphi_i - \frac{1}{\lambda} \frac{\partial \lambda}{\partial x_i}$.

Weyl define parallel displacement in relation to this new concept. Let a vector with components $(\xi^1, \ldots, \xi^n)$ be attached to the point $P(x_1, \ldots, x_n)$. We say that its measure is:

$$m = \sum_{i,k=1}^{n} g_{ik} \xi^i \xi^k.$$ 

Upon displacing $P$ to $P'(x_i + dx_i)$ by congruence, its components will become $\xi^i + d\xi^i$, with:

$$d\xi^i = -\frac{1}{2} \sum_{k,r=1}^{n} g_{ik} \left[ \frac{\partial g_{kr}}{\partial x_r} \xi^k + \frac{\partial g_{ik}}{\partial x_r} \xi^r + g_{ik} \varphi_r + g_{ik} \varphi_r - g_{kr} \varphi_i - g_{rk} \varphi_i \right] \xi^r dx_i.$$

Let $C$ be a curve whose parametric equations are $x_i = x_i(s)$. Imagine that we have fixed a vector $\Xi$ at each point $P(s)$ whose components $(\xi^i)$ are continuous functions of $s$ according to a continuous law. Let $P(s)$ and $P'(s + ds)$ be two neighboring points, so they will then correspond to the two vectors $\Xi$ and $\Xi'$. Displace $\Xi$ from $P$ to $P'$ by congruence, so one will get a vector $\Xi^*$ at $P$ that is generally different from $\Xi'$. The difference $\Xi' - \Xi^*$ is an infinitely small vector that is attached to the arc $PP'$. Form:

$$\theta(\Xi) = \frac{\Xi' - \Xi^*}{ds}.$$

We will then get a new vector $\theta(\Xi)$ that is attached to the point $P(s)$ of the curve $C$ and depends upon the field $\Xi$ in an invariant manner (1). Then let $\Xi = \Xi_1$ with the components $\xi^i_{(1)} = dx_i / ds$. Then set:

$$\theta(\Xi_1) = \Xi_2 \quad \text{with components } \xi^i_{(2)},$$

$$\theta(\Xi_2) = \Xi_3 \quad \text{" } \text{" } \xi^i_{(3)},$$

$$\theta(\Xi_{n-1}) = \Xi_n \quad \text{with components } \xi^i_{(n)}.$$ 

The $n$-hedron $\Xi_1, \Xi_2, \ldots, \Xi_n$ is not orthogonal, in general. Orthogonalize it using Schmidt’s method (2) by defining an $n$-hedron that is composed of the $n$ vectors:

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(1) Weyl, loc. cit., pp. 103.
\[ H_p = \begin{vmatrix} (1,1) & (1,2) & \cdots & (1,p-1) & \Xi_i \\ (2,1) & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ (p,1) & \cdots & \cdots & (p,p-1) & \Xi_p \end{vmatrix} \quad (p = 1, 2, \ldots, n), \]

in which one sets:

\[ (\alpha, \beta) = \sum_{i,k=1}^{n} g_{ik} \xi_{(\alpha)}^{i} \xi_{(\beta)}^{k}, \quad D_p = \begin{vmatrix} (1,1) & \cdots & (1,p) \\ \vdots & \ddots & \vdots \\ (p,1) & \cdots & (p,p) \end{vmatrix}, \quad D_0 = 1. \]

The \( n \)-hedron \( (N), H_1, \ldots, H_n \) is orthogonal and normalized; i.e.:

\[ \sum_{i,k=1}^{n} g_{ik} \eta_{i(\alpha)}^{i} \eta_{k(\beta)}^{k} = \delta_{pq} = \begin{cases} 1 & \text{is } p = q \\ 0 & \text{is } p \neq q. \end{cases} \]

The Frenet formulas for the curve \( C \) are the formulas that give the values of \( s \):

\[ \theta(H_p) = \frac{W'_p - W_p}{ds}. \]

One finds, by some simple calculations, that:

\[ \begin{cases} \theta \eta_{i(1)}^i = \frac{1}{2} \frac{d\varphi}{ds} \eta_{i(1)}^i + \frac{1}{\rho_1} \eta_{i(2)}^i, \\
\theta \eta_{i(2)}^i = -\frac{1}{\rho_1} \eta_{i(1)}^i + \frac{1}{2} \frac{d\varphi}{ds} \eta_{i(2)}^i + \frac{1}{\rho_2} \eta_{i(3)}^i, \\
\vdots \end{cases} \quad (p = 2, \ldots, n-1), \]

\[ \begin{cases} \theta \eta_{i(p)}^i = -\frac{1}{\rho_{p-1}} \eta_{i(p-1)}^i + \frac{1}{2} \frac{d\varphi}{ds} \eta_{i(p)}^i + \frac{1}{\rho_p} \eta_{i(p+1)}^i, \\
\vdots \end{cases} \quad \]

\[ \begin{cases} \theta \eta_{i(n)}^i = -\frac{1}{\rho_{n-1}} \eta_{i(n-1)}^i + \frac{1}{2} \frac{d\varphi}{ds} \eta_{i(n)}^i, \end{cases} \]

in which:

\[ \rho_k = \frac{D_k}{\sqrt{D_{k-1}D_{k+1}}}. \]

The \( \theta \eta_{i(k)}^i \) are then homogeneous linear functions of the \( \eta_{i(q)}^i \). The determinant of those functions is skew-symmetric; the \( \rho_i \) are the \( (n-1) \) radii of curvature of the curve. That determinant
possesses a principal diagonal whose terms are all \( \text{equal to } \frac{1}{2} \frac{d\phi}{ds} \). For a space \((R^n)\), the formulas \((F)\) will be the same as the ones that we just found, except that all of the terms in the principal diagonal will be equal to zero. (The \(\rho_i\) will not have the same value, since they depend upon the \(\phi_i\).) If one regards the trihedron \((N)\) as moving along the curve \(C\) then one can say that one passes from one of its position to the neighboring position by displacing by congruence and then subjecting it to a rotation that is defined by the curvatures \(1/\rho_i\) of \(C\), and finally deforming it with a homothety of ratio \(1 + \frac{d\phi}{2}\).