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On the compatibility conditions in a *Cosserat* continuum

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Summary: The conditions of compatibility and the jump conditions of Weingarten and Volterra are derived for a multiply-connected Cosserat continuum. The equations of compatibility are shown to be natural conditions from Castigliano's principle.

1. Introduction

If one interprets the defining equations of the deformation tensor:

$$\tfrac{1}{2}(\partial_i u_k + \partial_k u_i) = \varepsilon_{ik}$$

in the classical linear theory of elasticity as a system of differential equations for the displacement vector field u_k then the integration of those differential equations will produce a vector field u_k that is single-valued in a region G when the tensor field ε_{ik} in satisfies certain compatibility conditions in G. In a simply-connected region G, those conditions are known to read:

$$e_{ilp}\, e_{kmq}\, \partial_l\, \partial_m\, \varepsilon_{pq} = 0 .$$

If G is multiply-connected then the deformation tensor field ε_{ik} must satisfy additional integral conditions. One can either derive all of those conditions from a simple kinematical argument [**1**], [**2**], or obtain them from *Castigliano's* variational principle as natural conditions [**3**].

If one drops the requirement of single-valuedness for the displacement vector field in a multiply-connected region G, but still assumes continuity and twice-differentiability of the tensor field ε_{ik} in G, as well as the fulfillment of the differential compatibility conditions, then one will get the *Weingarten-Volterra* jump conditions for the displacement vector field, which are meaningful in the theory of dislocations [**4**], [**5**], [**6**]. The goal of the present study is to derive the compatibility and jump conditions for the kinematics of the *Cosserat* continuum.

2. The kinematics of the *Cosserat* continuum

In a *Cosserat* continuum, any point will have six degrees of freedom: viz., three translational degrees of freedom and three rotational degrees of freedom. In order to be able to speak of the rotation of a point, we think of each point as being the carrier of a rigid dreibein. By the rotation of the point, we then mean the rotation of the dreibein relative to a spatially-fixed, Cartesian reference system. In the initial, undeformed state, all of the dreibeins shall be directed parallel to the spatially-fixed coordinate system. Under displacement and rotation of the points, we will take the continuum to a deformed state, in which we would like to assume that the gradients of the displacement vector components are $\partial_i\, u_k\, (x_1,\, x_2,\, x_3) \ll 1$, and the rotations of the rigid dreibeins can be described by rotation vectors $\Phi_k\, (x_1,\, x_2,\, x_3)$. The deformation state of the *Cosserat* continuum is then characterized by the following tensor field:

$$\varepsilon_{ik} = \tfrac{1}{2}(\partial_i\, u_k + \partial_k\, u_i), \tag{1}$$

$$\varphi_i = \Phi_i - \tfrac{1}{2}\, e_{ikl}\, \partial_i\, u_k\,, \tag{2}$$

$$\kappa_{ik} = \partial_i\, \Phi_k\,. \tag{3}$$

ε_{ik} is the symmetric displacement deformation tensor, φ_i is the rotational deformation vector, which yields the difference between the absolute rotation Φ_i and the displacement vector field, and κ_{ik} is the curvature tensor.

We shall calculate in Cartesian coordinates and write ∂_i as an abbreviation for the gradient operator $\partial / \partial x_i$. e_{ikl} is the completely-antisymmetric permutation tensor (e_{123} = + 1, e_{132} = – 1, e_{123} = 0, etc.), and we shall employ the *Einstein* summation rule: Indices that occur twice shall be summed over from 1 to 3.

We construct the asymmetric deformation tensor from (1) and (2):

$$\gamma_{ik} = \varepsilon_{ik} - e_{ikl}\, \varphi_l\,. \tag{4}$$

We then get:

$$\partial_i\, u_k = \gamma_{ik} + e_{ikl}\, \Phi_l \tag{5}$$

for the gradient tensor of the displacement vector field. If we now regard (3) and (5) as differential equations for Φ_k and u_k then the solutions for these differential equations can be written [**7**]:

$$\Phi_k\, (P) = \Phi_k\, (P_0) + \int_{P_0}^{P} \kappa_{mk}\, d\xi_m\,, \tag{6}$$

$$u_k\, (P) = u_k\, (P_0) + e_{ikl}\, \Phi_l\, (P_0)\, [x_m\, (P) - x_m\, (P_0)] + \int_{P_0}^{P} \{\gamma_{mk} + e_{ikl}[x_i(P) - \xi_i]\,\kappa_{ml}\}\, d\xi_m\,. \tag{7}$$

$u_k\, (P_0)$ and $\Phi_k\, (P_0)$ are the displacement vector and rotation vector, resp., at a point P_0 with the coordinates $x_k\, (P_0)$; ξ_i is a point on the path of integration from P_0 to P. If the line integrals in (6) and (7) are path-independent in a region G then we will get continuous and single-valued vector fields Φ_k and u_k . That will then be the case when the given tensor fields κ_{ik} and γ_{ik} satisfy the compatibility conditions.

3. Compatibility conditions in a simply-connected body

In a simply-connected body *V*, let the tensor fields γ_{ik} and κ_{ik} be given as once-continuously-differentiable functions (γ_{ik} , $\kappa_{ik} \in C^1$ in *V*). If we assume that the line integrals in (6) and (7) are path-independent then those integrals with have the value zero for arbitrary closed paths in *V*:

$$\oint_K \kappa_{ml}\, d\xi_m = 0, \tag{8}$$

$$\oint_K \{\gamma_{mk} + e_{kli}[x_i(P) - \xi_i]\,\kappa_{ml}\}\, d\xi_m = 0. \tag{9}$$

If we take equations (8) into account in (9) then we will get:

$$\oint_K [\gamma_{mk} - e_{kli}\xi_i\,\kappa_{ml}]\, d\xi_m = 0. \tag{10}$$

Any closed curve *K* in *V* can be regarded as the boundary curve of a surface *f* that lies completely within *V*. With the help of *Stokes's* theorem, one can now convert the integrals (8) and (10) into surface integrals over one such surface *f*:

$$\iint_f e_{ipq}\partial_p \kappa_{ql}\, n_i\, df = 0, \tag{11}$$

$$\iint_f e_{ipq}\partial_p (\gamma_{qk} - e_{kli} x_i\, \kappa_{ql}) n_i\, df = 0, \tag{12}$$

in which n_i is the unit normal vector to a surface element. Since the surface *f* is arbitrary, (11) and (12) will imply the eighteen (in total) differential compatibility conditions:

$$e_{ipq}\,\partial_i\,\kappa_{qk} = 0, \tag{13}$$

in *V*

$$e_{ipq}\,\partial_i\,\gamma_{qk} + \delta_{ik}\,\kappa_{qq} - \kappa_{kl} = 0. \tag{14}$$

Those conditions are necessary and sufficient for the existence of single-valued displacement and rotation vector fields in *V*.

4. Compatibility conditions in a doubly-connected body.

Let γ_{ik} and $\kappa_{ik} \in C^1$ be given in a doubly-connected body *V*, and assume that the compatibility conditions (13) and (14) are fulfilled. However, due to the doubleconnectedness, there will now be closed curves K in V that do not bound any surface that lies completely with V, and we cannot conclude from (13) and (14) that the line integral over K will vanish.

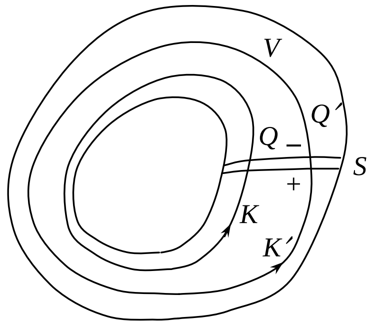

Figure 1.

The simply-connected body V' will arise when one makes a cut S in V. We shall call the values of u_k and Φ_k on the edges of the cut u_k^+, Φ_k^+ and u_k^-, Φ_k^- resp. We denote the jumps in those vector fields when one crosses the cut surface S at the location Q by:

$$[u_k](Q) \equiv u_k^+(Q) - u_k^-(Q), \tag{15}$$

$$[\Phi_k](Q) \equiv \Phi_k^+(Q) - \Phi_k^-(Q), \tag{16}$$

and we will arrive at:

$$[u_k] = 0, \tag{17}$$

$$[\Phi_k] = 0 \tag{18}$$

on S.

We get from (6) and (7) that:

$$[u_k](Q) = \int_K \{\gamma_{mk} + e_{ikl}[x_i(P) - \xi_i]\,\kappa_{ml}\}\,d\xi_m = 0, \tag{19}$$

$$[\Phi_k](Q) = \int_K \kappa_{lk}\,d\xi_l = 0, \tag{20}$$

and for an arbitrary point Q' on S:

$$[u_k](Q') = \int_{K'} \{\gamma_{mk} + e_{ikl}[x_i(P) - \xi_i]\,\kappa_{ml}\}\,d\xi_m = 0, \tag{19a}$$

$$[\Phi_k](Q') = \int_{K'} \kappa_{lk}\,d\xi_l = 0. \tag{20a}$$

The first integral condition for the compatibility of the deformation tensor field follows from (20) and (20a):

$$\oint_K \kappa_{lk}\,d\xi_l = 0, \tag{21}$$

for any curve K that surrounds the hole.

With (21), (19) and (19a) will yield the second integral condition:

$$\oint_K [\gamma_{mk} - e_{kli}\xi_i \, \kappa_{ml}] \, d\xi_m = 0, \tag{22}$$

for every curve K that surrounds the hole. Since the differential compatibility conditions are fulfilled in V, by assumption, when these integral conditions are fulfilled on any curve K, it will suffice that V cannot be contracted to a point.

In an n-fold-connected body V, γ_{ik} and κ_{ik} will yield single-valued, continuous vector fields u_k and Φ_k in V when $2\,(n-1)$ integral conditions of the type that was given above are fulfilled along with the differential compatibility conditions.

5. The *Weingarten-Volterra* jump conditions

We shall again consider a doubly-connected body V that is cut into a simply-connected body V' by a surface S (Fig. 1). Let γ_{ik} and $\kappa_{ik} \in C^1$, which satisfy the differential compatibility conditions, be given in V. Under that assumption, we shall answer the question of whether discontinuities in the rotational vector field Φ_k and the displacement vector field u_k are possible on S.

With (6) and (7), one will have:

$$[u_k](Q) = \int_K \{\gamma_{mk} + e_{kli}[x_i(P) - \xi_i]\,\kappa_{ml}\}\, d\xi_m = 0, \tag{23}$$

$$[\Phi_k](Q) = \int_K \kappa_{mk}\, d\xi_m = 0, \tag{24}$$

and

$$[u_k](Q') = \int_{K'} \{\gamma_{mk} + e_{kli}[x_i(P) - \xi_i]\,\kappa_{ml}\}\, d\xi_m = 0, \tag{23a}$$

$$[\Phi_k](Q') = \int_{K'} \kappa_{mk}\, d\xi_m = 0. \tag{24a}$$

Due to the continuity of κ_{ik} on the cut-surface S, one will have:

$$\int_{Q'}^{Q} \kappa_{mk}^{-}\, d\xi_m = -\int_{Q}^{Q'} \kappa_{mk}^{+}\, d\xi_m \,, \tag{25}$$

and due to the validity of (13) in V', one will finally have:

$$\oint_{K'} \kappa_{mk}\, d\xi_m = \oint_{K} \kappa_{mk}\, d\xi_m \,, \tag{26}$$

such that we will get a constant vector a_k on S for the rotational jump $[\Phi_k]$:

$$[\Phi_k] = a_k \,. \tag{27}$$

With that result, one will have:

$$[u_k](Q) = \oint_K [\gamma_{mk} - e_{kli}\xi_i\, \kappa_{ml}]\, d\xi_m + e_{kli}\, a_l\, x_i\,(Q) \tag{28}$$

and

$$[u_k](Q') = \oint_{K'} [\gamma_{mk} - e_{kli}\xi_i\, \kappa_{ml}]\, d\xi_m + e_{kli}\, a_l\, x_i\,(Q'). \tag{28a}$$

Since the compatibility conditions (14) are true in V', and γ_{ik} is also continuous on S, we will then immediately get:

$$[u_k](Q') = [u_k](Q) + e_{kli}\, a_l\, [x_i\,(Q') - x_i\,(Q)] \tag{29}$$

for the displacement jump across S. If we denote the displacement jump at the fixed position Q by b_k :

$$[u_k](Q) = b_k\,, \tag{30}$$

then

$$[u_k](Q') = b_k + e_{kli}\, a_l\, [x_i\,(Q') - x_i\,(Q)] \tag{29a}$$

will mean that the two edges of the cut by S can be moved apart by that infinitesimal rigid motion if one would like to obtain an everywhere single-valued and continuous deformation state on V from the "welding" of V' to V (viz., a proper stress state).

In addition, that will show that a rotational jump is not compatible with only the assumptions on γ_{ik} and κ_{ik} that were given above.

If the constants a_k and b_k , which are characteristic of the discontinuities in the displacement and rotational vector fields, are given in a doubly-connected body then the deformation tensor fields γ_{ik} and κ_{ik} that will have to satisfy the integral conditions:

$$\oint_K \kappa_{mk}\, d\xi_m = a_k\,, \tag{31}$$

$$\oint_{K_Q} [\gamma_{mk} - e_{kli}\xi_i\, \kappa_{ml}]\, d\xi_m = b_k - e_{kli}\, a_l\, x_i\,(Q), \tag{32}$$

in which K_Q is a curve through the point Q that surround the hole, in addition to the differential conditions (13) and (14).

Equations (27) and (29a) agree formally with the *Weingarten* and *Volterra* [**2**], [**6**] that are true in a classical continuum. One calls the displacement jump (29a) a *Volterra* distortion.

6. Jump conditions in a classical continuum

In order to conclude the kinematical arguments, we treat the special case of the classical continuum, for which one sets:

$$\varphi_k \equiv 0. \tag{33}$$

That will imply:

$$\Phi_k = \tfrac{1}{2} e_{klm}\, \partial_l\, u_m\,, \tag{34}$$

$$\gamma_{mk} = \varepsilon_{mk} = \varepsilon_{(mk)}\,, \tag{35}$$

and

$$\kappa_{mk} = \partial_m\, \Phi_l = e_{lpq}\, \partial_p\, \varepsilon_{qm}\,, \tag{36}$$

so:

$$\partial_m\, \Phi_l = \tfrac{1}{2} e_{lpq}\, \partial_p\, \omega_{qm}\,,$$

$$\omega_{pq} = \tfrac{1}{4} e_{pqi}\, e_{ikl}\, (\partial_k\, u_l - \partial_l\, u_k),$$

$$\partial_m\, \omega_{pq} = \tfrac{1}{4} e_{pql}\, e_{ikl}\, (\partial_k\, \varepsilon_{ml} - \partial_l\, \varepsilon_{mk}).$$

(31) and (32) then imply the well-known *Weingarten* formulas:

$$\oint_K e_{lpq} \partial_p \varepsilon_{qm}\, d\xi_m = a_l\,, \tag{37}$$

$$\oint_{K_Q} (\varepsilon_{mk} - e_{kli}\, \xi_i\, e_{lpq}\, \partial_p \varepsilon_{qm})\, d\xi_m = b_k - e_{kli}\, a_l\, x_i\, (Q). \tag{38}$$

7. Derivation of the compatibility conditions from *Castigliano's* principle

In the classical theory of elasticity, differential and integral compatibility conditions will follow from *Castigliano's* variational principle as natural conditions, as *Stickforth* has shown [**3**].

A variational principle of the *Castigliano* type can also be formulated in the theory of elasticity for *Cosserat* continua, and we will now show that the differential, as well as the integral, compatibility conditions will follow as natural conditions from the minimal principle in a multiply-connected, simply bounded body.

For the sake of simplicity, we shall consider a doubly-connected body V that is composed of an elastic *Cosserat* material that is loaded with volume forces X_i and volume moments Y_i in V and with force-stresses p_i and moment-stresses q_i on the outer surface F. Let the elastic energy density of the material be $W(\gamma_{ik}\,,\ \kappa_{ik})$. The principle of minimum potential energy will then read [**8**]:

$$\delta\, \{\int_V [W(\gamma_{ik}, \kappa_{ik}) - X_i\, u_i - Y_i\, \Phi_i]\, dV - \int_F (p_i\, u_i + q_i\, \Phi_i)\, dF\} = 0, \tag{39}$$

when one observes the constraint conditions:

$$\gamma_{ik} - \partial_i\, u_k + e_{ikl}\, \Phi_l = 0, \tag{40}$$

in V

$$\kappa_{ik} - \partial_i\, \Phi_k = 0. \tag{41}$$

We restrict ourselves to bodies for which no kinematical constraint conditions are prescribed on F.

(39) are associated with the natural conditions:

$$\sigma_{ik} = \frac{\partial W}{\partial \gamma_{ik}}, \tag{42}$$

$$\mu_{ik} = \frac{\partial W}{\partial \kappa_{ik}}, \tag{43}$$

which represent the definitions of the force and moment stresses:

$$\partial_i\ \sigma_{ik} = X_k\ , \tag{44}$$

in V

$$\partial_i\ \mu_{ik} + e_{klm}\ \sigma_{lm} = Y_k\ , \tag{45}$$

are the equilibrium conditions, and finally one has the outer surface conditions:

$$n_i\ \sigma_{ik} = p_k\ , \tag{46}$$

on F

$$n_i\ \mu_{ik} = q_k\ . \tag{47}$$

n_i is the external unit normal vector of a surface element dF. A *Friedrich* transformation [**9**] will now give us the *Castigliano* variational principle for the *Cosserat* continuum:

In the case of equilibrium, one has:

$$\delta \int_V W^*(\sigma_{ik}, \mu_{ik})\, dV = 0, \tag{48}$$

with the constraint conditions (44) to (47). $W^*(\sigma_{ik}, \mu_{ik})$ arises from $W(\gamma_{ik}, \kappa_{ik})$ from a *Legendre* transformation:

$$W^*(\sigma_{ik}, \mu_{ik}) = \gamma_{ik}\,\sigma_{ik} + \kappa_{ik}\,\mu_{ik} - W(\gamma_{ik}, \kappa_{ik})\ . \tag{49}$$

Correspondingly, one has:

$$\frac{\partial W^*}{\partial \sigma_{ik}} = \gamma_{ik} \tag{50}$$

and

$$\frac{\partial W^*}{\partial \mu_{ik}} = \kappa_{ik}\ . \tag{51}$$

The homogeneous equilibrium conditions (44) and (45) will be satisfied identically when one makes the stress function Ansatz [**7**]:

$$\sigma_{ik}^{(H)} = e_{ipq}\,\partial_p\,F_{qk}\ , \tag{52}$$

$$\mu_{ik}^{(H)} = e_{ipk}\,\partial_p\,G_{qk} + \delta_{ik}\,F_{pp} - F_{ki}\ , \tag{53}$$

with the asymmetric stress-function tensors F_{ik} and G_{ik} . If we let $\sigma_{ik}^{(P)}$ and $\mu_{ik}^{(P)}$ denote particular solutions of the inhomogeneous equilibrium conditions then:

$$\sigma_{ik} = \sigma_{ik}^{(P)} + \sigma_{ik}^{(H)} \tag{54}$$

and

$$\mu_{ik} = \mu_{ik}^{(P)} + \mu_{ik}^{(H)} \tag{55}$$

will be the general solutions of equations (44) to (47). We imagine that the deformation tensors in:

$$\int_V (\gamma_{ik}\delta\sigma_{ik} + \kappa_{ik}\delta\mu_{ik})\, dV = 0 \tag{56}$$

are expressed in terms of stress functions and fulfill the conditions:

$$\partial_i\ \delta\sigma_{ik} = 0, \qquad \partial_i\ \delta\sigma_{ik} + e_{klm}\ \delta\sigma_{lm} = 0, \qquad \text{in } V, \tag{57}$$

$$n_i\ \delta\sigma_{ik} = 0, \qquad n_i\ \delta\mu_{ik} = 0, \qquad \text{on } F, \tag{58}$$

for $\delta\sigma_{ik}$ and $\delta\mu_{ik}$ by means of the Ansätze:

$$\delta\sigma_{ik} = e_{ipq}\ \partial_p\ \delta F_{qk}, \tag{59}$$

in V

$$\delta\mu_{ik} = e_{ipq}\ \partial_p\ \delta G_{qk} + \delta_{ik}\ \delta F_{qq} - \delta F_{kl}\ , \tag{60}$$

with

$$n_i\ e_{ipq}\ \partial_p\ \delta F_{qk} = 0, \tag{61}$$

on F,

$$n_i\ (e_{ipq}\ \partial_p\ \delta G_{qk} + \delta_{ik}\ \delta F_{qq} - \delta F_{kl}) = 0. \tag{62}$$

Equation (56) then reads:

$$\int_V \{\gamma_{ik}\ e_{ipk}\partial_p\delta F_{qk} + \kappa_{ik}(e_{ipq}\partial_p\delta G_{qk} + \delta_{ik}\delta F_{pp} - \delta F_{ki})\} dV = 0, \tag{63}$$

and a partial integration of this and an application of *Gauss's* theorem will yield:

$$\int_V \{[e_{ipk}\partial_p\gamma_{ik} + \delta_{ik}\kappa_{pp} - \kappa_{ik}]\delta F_{qk} + [e_{ipq}\partial_p\kappa_{qk}]\delta G_{ik}\} dV + \int_F n_l\ e_{lim}(\gamma_{mk}\delta F_{ik} + \kappa_{mk}\delta G_{ik})\, dF = 0. \tag{64}$$

Since the variations δF_{ik} and δG_{ik} are arbitrary in V, we will get the differential compatibility conditions:

$$e_{ipq}\ \partial_p\ \kappa_{qk} = 0 \tag{13}$$

and

$$e_{ipq}\ \partial_p\ \gamma_{qk} + \delta_{ik}\ \kappa_{qq} - \kappa_{kl} = 0 \tag{14}$$

in V. The stress function variations in the surface integral must fulfill the homogeneous equilibrium conditions. We will achieve that when we write δF_{ik} and δG_{ik} as null stress functions [**7**]:

$$\delta F_{ik} = \partial_i f_k , \tag{65}$$
$$\delta G_{ik} = \partial_i g_k - e_{ikl} f_k \tag{66}$$

in a neighborhood of F [**10**], [**11**]. We assume that δF_{ik} and δG_{ik} are single-valued C^1 functions in a neighborhood of F.

With (65) and (66), the surface integral in (63) will become:

$$\int_F n_l\, e_{lim} (\gamma_{mk} \partial_i f_k + \kappa_{mk}[\partial_i g_k - e_{ikl} f_l])\, dF = 0. \tag{67}$$

After partial integration, we will get:

$$\int_F n_l\, e_{lim} \{\partial_i(\gamma_{mk} f_k) + \partial_i(\kappa_{mk} g_k) - (\partial_i \gamma_{mk} + \kappa_{mk} e_{ilk}) f_k - \partial_i \kappa_{mk} g_k\}\, dF = 0. \tag{68}$$

Due to the arbitrariness of f_k and g_k in a neighborhood of F, the last two terms in the integrand will yield the continuation of the differential compatibility conditions to F. The first two terms can be transformed into a line integral by using *Stokes's* theorem; in that, the outer surface F should be cut up canonically into a simply-connected surface F'[**10**].

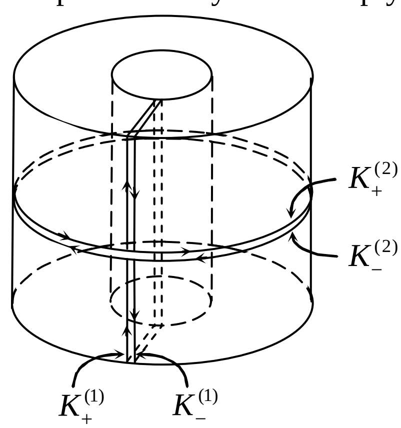

Figure 2.

The boundary curve K of the surface F is: $K = K_+^{(1)} + K_-^{(2)} + K_-^{(1)} + K_+^{(2)}$. We then get the line integral:

$$\oint_K (\gamma_{mk} f_k + \kappa_{mk} g_k)\, d\xi_m = 0. \tag{69}$$

Since γ_{mk} and κ_{mk} are continuous on F, the integral:

$$\oint_{K_+^{(1)}} \gamma_{mk} f_k d\xi_m + \oint_{K_-^{(1)}} \gamma_{mk} f_k\, d\xi_m ,$$

for example, can be combined into an integral over $K^{(1)}$, in which the jump function $[f_k]$ will appear in the integrand:

$$\oint \gamma_{mk}[f_k]\,d\xi_m .$$

For that reason, we can write the line integral (69) as a sum of two line integrals:

$$\oint_{K_+^{(1)}} \{\gamma_{mk}[f_k]+\kappa_{mk}[g_k]\}d\xi_m + \oint_{K_+^{(2)}} \{\gamma_{mk}[f_k]+\kappa_{mk}[g_k]\}d\xi_m = 0. \tag{70}$$

Since have assumed that the variations δF_{ik} and δG_{ik} are continuous in a neighborhood of F, the *Weingarten-Volterra* jump conditions (27) and (29) will imply that:

$$[f_k] = a_k , \tag{71}$$
$$[g_k] = c_k + e_{kli}\, a_l\, \xi_i , \tag{72}$$

with arbitrary infinitesimal constant vectors a_k and c_k ; ξ_i is a point on the cut curve K.

Equation (70) then reads:

$$a_k\{\oint_{K_+^{(1)}} (\gamma_{mk}+\kappa_{mk}e_{lki}\,\xi_i)\,d\xi_m\}+c_k\{\oint_{K_+^{(1)}} \kappa_{mk}\,d\xi_m\}$$
$$+\ a_k\{\oint_{K_+^{(2)}} (\gamma_{mk}+\kappa_{mk}e_{lki}\,\xi_i)\,d\xi_m\}+c_k\{\oint_{K_+^{(2)}} \kappa_{mk}\,d\xi_m\} = 0 . \tag{73}$$

The line integrals over $K^{(1)}$ are zero, since any surface V that is bounded by $K^{(1)}$ will lie entirely within V, and the differential compatibility conditions are fulfilled in V.

Since a_k and c_k are arbitrary, (73) will finally imply the integral compatibility conditions:

$$\oint_{K_+^{(2)}} (\gamma_{mk}-\kappa_{mk}e_{lki}\,\xi_i)\,d\xi_m = 0 \tag{32}$$

and

$$\oint_{K_+^{(2)}} \kappa_{mk}\,d\xi_m = 0. \tag{31}$$

The extension of the results that were derived in the section to n-fold-connected bodies ($n > 2$) will raise no difficulties.

We have then shown that all conditions for the compatibility of a *Cosserat* deformation state are natural conditions for *Castigliano's* variational principle, which is a result that was not to be expected from our experience with the classical theory of elasticity.

References

[1] A. E. H. Love, *A Treatise on the Mathematical Theory of Elasticity*, Cambridge University Press, 1952.

[2] B. A. Boley and J. H. Weiner, *Theory of Thermal Stresses*, New York, 1960.

[3] J. Stickforth, "Zur Anwendung des Castiglianoschen Prinzips und der Beltramischen Spannungsfunktionen bei mehrfach zusammenhängenden elastischen Körper unter Berücksichtigung von Eigenspannungen," Tech. Mitt. Krupp, Forsch. Ber. **22** (1964).

[4] F. R. N. Nabarro, "The Mathematical Theory of Stationary Dislocations," Adv. Phys. **1** (1952).

[5] G. Weingarten, "Sulle superficie di discontinuità nella teoria della elasticità dei corpi solidi," Rend. Accad. Lincei (5) **10** (1901).

[6] V. Volterra, "Sur l'équilibre des corps élastiques multiplement connexes," Ann. École Norm. (3) **24** (1907).

[7] W. Günther, "Zur Statik und Kinematik des Cosseratkontinuums," Abh. Braunsch. Wiss. Ges. **10** (1958).

[8] S. Kessel, "Lineare Elastizitätstheorie des anisotropien Cosserat-Kontinuum," Abh. Braunsch. Wiss. Ges. **16** (1964).

[9] R. Courant and D. Hilbert, *Methods of Mathematical Physics*, Interscience Publishers, 1960.

[10] G. Rieder, "Topologische Fragen in der Theorie der Spannungsfunktionen," Abh. Braunsch. Wiss. Ges. **12** (1960).

[11] H. Schaefer, "Die Spannungsfunktionen des dreidimensionalen Kontinuums; statische Deutung und Randwerte," Ing. Arch. **28** (1959).