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Stress fields of a screw dislocation and an edge dislocation in a Cosserat continuum

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The force- and couple-stresses of a screw dislocation and an edge dislocation in an infinitely extended, linear elastic-isotropic Cosserat continuum are calculated by means of stress functions.

Notations and conventions

We calculate in Cartesian coordinates and employ the summation convention that Greek indices are summed from 1 to 3. e_{ikl} is the permutation symbol that is skew-symmetric in all indices.

Statement of the problem and path of solution

We imagine that a singular VOLTERRA distortion of type 1 has been produced in an infinitely-extended, isotropically elastic COSSERAT continuum by the following cutting process: The continuum is cut in the *xz*-half plane that is bounded by the *z*-axis of a Cartesian coordinate system, and the negative lip of the cut is rigidly displaced with respect to the positive one through an infinitesimal BURGERS vector **b**. If **b** lies in the cut plane then we can weld the two lips of the cut together after the displacement and thus generate a singular screw displacement in the continuum when $\mathbf{b} = b\mathbf{e}_z$ and a singular edge displacement $\mathbf{b} = b\mathbf{e}_x$. As was shown in [**1**], the deformation state of the continuum may be described by continuous deformation tensor fields ε_{ik} and κ_{ik} . In the continuum minus the *z*-axis, they satisfy the conditions:

$$e_{i\alpha\beta}\,\partial_\alpha\,\kappa_{\beta k} = 0, \qquad e_{i\alpha\beta}\,\partial_\alpha\,\varepsilon_{\beta k} + e_{i\alpha\beta}\,e_{k\alpha\gamma}\,\kappa_{\beta\gamma} = 0, \tag{1}$$

and for every closed integration curve that goes around the *z*-axis once, one has:

$$\oint \kappa_{\alpha k}\,ds_\alpha = 0, \qquad \oint (\varepsilon_{\alpha k} - \varepsilon_{k\beta\gamma}\,s_\gamma\,\kappa_{\alpha\beta})\,ds_\alpha = b_k\,. \tag{2}$$

The equations (1) and (2) can be summarized in the field equations:

$$e_{i\alpha\beta}\,\partial_\alpha\,\kappa_{\beta k} = 0, \qquad e_{i\alpha\beta}\,\partial_\alpha\,\varepsilon_{\beta k} + e_{i\alpha\beta}\,e_{k\alpha\gamma}\,\kappa_{\beta\gamma} = \delta_{i3}\,b_k\,\delta(x)\,\delta(y), \tag{3}$$

in which $\delta(x)\,\delta(y)$ is the DIRAC delta function that is singular for $x = y = 0$.

We can calculate the force-stresses $\tilde{\sigma}_{ik}$ and moment-stresses $\tilde{\mu}_{ik}$ that correspond to these deformations for the material law of the isotropically elastic COSSERAT continuum [**3**]:

$$\tilde{\sigma}_{ik} = 2G\left[\left(\frac{1}{2}+\frac{c_1}{4}\right)\varepsilon_{ik}+\left(\frac{1}{2}-\frac{c_1}{4}\right)\varepsilon_{ki}+\frac{\nu}{1-2\nu}\delta_{ik}\varepsilon_{\alpha\alpha}\right],$$

$$\tilde{\mu}_{ik} = 2GL^2\left[\left(\frac{1}{2}+\frac{c_2}{4}\right)\kappa_{ik}+\left(\frac{1}{2}-\frac{c_2}{4}\right)\kappa_{ki}+c_3\delta_{ik}\kappa_{\alpha\alpha}\right]. \tag{9}$$

In order to fulfill the equilibrium conditions:

$$\partial_\alpha\,\sigma_{\alpha k} = -\,X_k\,, \qquad \partial_\alpha\,\mu_{\alpha k} + e_{k\alpha\beta}\,\sigma_{\alpha\beta} = -\,Y_k\,, \tag{10}$$

the continuum must be loaded with volume forces:

$$\tilde{X}_k = -\,\partial_\alpha\tilde{\sigma}_{\alpha k} \tag{11a}$$

and volume moments:

$$\tilde{Y}_k = -\,\partial_\alpha\tilde{\mu}_{\alpha k} - e_{k\alpha\beta}\tilde{\sigma}_{\alpha\beta}\,. \tag{11b}$$

We now begin the second part of the solution process, which is essentially more difficult. The stress fields $\tilde{\sigma}_{ik}$ and $\tilde{\mu}_{ik}$ must be overlaid with compatible stress fields $\bar{\sigma}_{ik}$ and $\bar{\mu}_{ik}$ that arise from volume loads $-\tilde{X}_k$ and $-\tilde{Y}_k$ such that the continuum becomes free of the undesired volume loads, and the stress fields:

$$\sigma_{ik} = \tilde{\sigma}_{ik} + \bar{\sigma}_{ik}\,, \qquad \mu_{ik} = \tilde{\mu}_{ik} + \bar{\mu}_{ik} \tag{12}$$

are ultimately the desired stress fields of the singular dislocations.

In order to determine $\bar{\sigma}_{ik}$ and $\bar{\mu}_{ik}$, we employ the method of stress functions that was described in [**4**] and [**5**].

The equilibrium conditions and the compatibility conditions are fulfilled when we set:

$$\bar{\sigma}_{ik} = \partial_i\,\partial_\alpha\,F_{\alpha k} - \Delta F_{ik} + \partial_i\,S_k\,,$$

$$\bar{\mu}_{ik} = \partial_i\,\partial_\alpha\,G_{\alpha k} - \Delta G_{ik} + e_{ik\beta}\partial_i\,T_k + \partial_\alpha\,F_{\alpha\beta} + e_{ik\alpha}\,S_\alpha\,, \tag{13}$$

in which:

$$F_{ik} = F_{(ik)} + e_{ik\alpha}\,F_\alpha = \delta_{ik}\,f + e_{ik\alpha}\,F_\alpha\,, \quad G_{ik} = G_{(ik)} + e_{ik\alpha}\,G_\alpha = \delta_{ik}\,g + e_{ik\alpha}\,G_\alpha\,, \tag{14}$$

$$F_i = -\frac{1+c_3}{2c_3}\partial_i g - \frac{1}{2}T_i + e_{ik\alpha}\partial_\alpha A_\beta ,$$

$$G_i = 2A_i + L^2 \frac{2-c_1}{2c_1}S_i - L^2 \frac{2+c_1}{2c_1}\left(\Delta A_i + \frac{1}{2}e_{i\alpha\beta}\partial_\alpha T_\beta\right) + \partial_i \rho , \qquad (15)$$

and the following differential equations are valid for the stress functions:

$$\Delta S_i = \tilde{X}_i , \qquad \Delta T_i = \tilde{Y}_i ; \qquad (16)$$

$$\Delta f = -\frac{\nu}{1-\nu}\partial_\alpha S_\alpha , \qquad \Delta g - \frac{1}{l_2^2} g = -\frac{c_3}{1+c_3}\partial_\alpha T_\alpha , \qquad l_2^2 = \frac{L^2(1+c_3)}{c_1} , \qquad (17)$$

$$\Delta(A_i - \Delta A_i) = S_i - \frac{2-c_1}{2+c_1} l_2^2 \Delta S_i + \frac{1}{2} e_{i\alpha\beta}(\partial_\alpha T_\beta + l_2^2 \Delta \partial_\alpha T_\beta) , \qquad .$$

$$l_1^2 = \frac{L^2(2+c_1)(2+c_2)}{8c_1} , \qquad (18)$$

$$\Delta\rho = f - \partial_\alpha A_\alpha + L^2 \frac{(2+c_1)(2-c_2)}{8c_1}\partial_\alpha \Delta A_\alpha + L^2 \frac{(2-c_1)(2-c_2)}{8c_1}\partial_\alpha S_\alpha . \qquad (19)$$

We now carry out the calculations for the screw dislocation and the edge dislocation separately.

The stress fields of a screw dislocation

The BURGERS vector **b** has the three components:

$$b_1 = 0, \qquad b_2 = 0, \qquad b_3 = b.$$

It then follows from (8) and (9) that:

$$\tilde{\mu}_{ik} = 0, \qquad \tilde{\sigma}_{11} = \tilde{\sigma}_{12} = \tilde{\sigma}_{21} = \tilde{\sigma}_{22} = \tilde{\sigma}_{33} = 0,$$

$$\tilde{\sigma}_{13} = -G\left(1+\frac{c_1}{2}\right)\frac{b}{2\pi}\partial_y \ln r , \qquad \tilde{\sigma}_{31} = -G\left(1-\frac{c_1}{2}\right)\frac{b}{2\pi}\partial_y \ln r , \qquad (20)$$

$$\tilde{\sigma}_{23} = G\left(1+\frac{c_1}{2}\right)\frac{b}{2\pi}\partial_x \ln r , \qquad \tilde{\sigma}_{32} = G\left(1-\frac{c_1}{2}\right)\frac{b}{2\pi}\partial_x \ln r ,$$

and from (11a, b):

$$\tilde{X}_i = 0,$$

$$\tilde{Y}_1 = -\beta\partial_x \ln r, \qquad \tilde{Y}_2 = -\beta\partial_y \ln r, \qquad \tilde{Y}_3 = -\frac{\beta}{2}\, y \ln r, \qquad \beta = \frac{Gc_1 b}{2\pi}. \tag{21}$$

Equations (16) then have the following solutions:

$$S_i = 0, \qquad T_1 = -\frac{\beta}{2}\, x \ln r, \qquad T_2 = -\frac{\beta}{2}\, y \ln r, \qquad T_3 = 0. \tag{22}$$

Because boundary conditions for the stresses have not been posed, we concern ourselves with only the particular solutions of the inhomogeneous partial differential equations. For that reason, we fulfill equation (17) with:

$$f = 0, \tag{23}$$

and for equation $(17)_2$:

$$\Delta g - \frac{1}{l_2^2} g = \frac{c_3\beta}{1+c_3}\left(\ln r + \frac{1}{2}\right), \tag{24}$$

we make the solution Ansatz:

$$g = g_{(1)} - \frac{c_3\beta l_2^2}{1+c_3}\left(\ln r + \frac{1}{2}\right), \tag{25}$$

which leads to the equation:

$$\Delta g_{(1)} - \frac{1}{l_2^2}\, g_{(1)} = \frac{c_3\beta L^2}{c_1} 2\pi\, \delta(x)\, \delta(y), \tag{26}$$

which is solved by:

$$g_{(1)} = -\frac{c_3\beta L^2}{c_1} K_0\left(\frac{r}{l_2}\right), \tag{27}$$

in which K_0 is a modified BESSEL function [**6**], [**7**]. The particular solution of (24) then reads:

$$g = -\frac{c_3\beta L^2}{c_1}\left[K_0\left(\frac{r}{l_2}\right) + \ln r + \frac{1}{2}\right]. \tag{28}$$

With the solutions that we already obtained, the right-hand side of the differential equations for A_i and ρ become zero, which is why we set:

$$A_i = 0, \tag{29}$$

and

$$\rho = 0. \tag{30}$$

With (14), (15), and (13), we obtain:

$$\bar{\sigma}_{ik} = e_{ik\alpha}\left(\frac{1+c_3}{2c_3}\partial_\alpha \Delta g + \frac{1}{2}\Delta T_\alpha\right), \qquad \bar{\mu}_{ik} = -\frac{1}{c_3}\partial_i\,\partial_k g - \delta_{ik}\,\Delta g. \tag{31}$$

The tensor field of force-stresses for a screw dislocation in an infinitely-extended continuum ultimately reads, in Cartesian coordinates:

$$\sigma_{11} = \sigma_{22} = \sigma_{33} = \sigma_{12} = \sigma_{21} = 0,$$

$$\sigma_{13} = \frac{Gb}{2\pi}\frac{y}{r^2}\left[-\overset{*}{1} - \frac{c_1}{2}\left(\frac{r}{l_2}\right)K_1\left(\frac{r}{l_2}\right)\right], \qquad \sigma_{13} = \frac{Gb}{2\pi}\frac{y}{r^2}\left[-\overset{*}{1} + \frac{c_1}{2}\left(\frac{r}{l_2}\right)K_1\left(\frac{r}{l_2}\right)\right], \tag{32}$$

$$\sigma_{23} = \frac{Gb}{2\pi}\frac{x}{r^2}\left[\overset{*}{1} + \frac{c_1}{2}\left(\frac{r}{l_2}\right)K_1\left(\frac{r}{l_2}\right)\right], \qquad \sigma_{32} = \frac{Gb}{2\pi}\frac{x}{r^2}\left[\overset{*}{1} - \frac{c_1}{2}\left(\frac{r}{l_2}\right)K_1\left(\frac{r}{l_2}\right)\right],$$

in which the terms that are marked with a * describe the stress field of a screw dislocation in a classical, isotropically elastic continuum.

Due to the rotational symmetry, it is appropriate to also express the stress field in cylindrical coordinates. An elementary intermediate computation delivers:

$$\sigma_{rr} = \sigma_{\varphi\varphi} = \sigma_{zz} = \sigma_{r\varphi} = \sigma_{\varphi r} = \sigma_{rz} = \sigma_{zr} = 0,$$

$$\sigma_{\varphi z} = \frac{Gb}{2\pi}\frac{1}{r}\left[\overset{*}{1} + \frac{c_1}{2}\left(\frac{r}{l_2}\right)K_1\left(\frac{r}{l_2}\right)\right], \qquad \sigma_{z\varphi} = \frac{Gb}{2\pi}\frac{1}{r}\left[\overset{*}{1} - \frac{c_1}{2}\left(\frac{r}{l_2}\right)K_1\left(\frac{r}{l_2}\right)\right], \tag{33}$$

with

$$K_1(\xi) = -\frac{d}{d\xi}K_0(\xi) = -K_0'. \tag{34}$$

For the moment stresses, we obtain, in Cartesian coordinates:

$$\mu_{13} = \mu_{31} = \mu_{23} = \mu_{32} = 0,$$

$$\mu_{11} = \frac{Gb}{2\pi} L^2 \left[\frac{y^2 - x^2}{r^4} + \frac{y^2 + c_3 x^2}{r^4} \left(\frac{r}{l_2} \right)^2 K_0 \left(\frac{r}{l_2} \right) + \frac{x^2 - y^2}{r^4} \left(\frac{r}{l_2} \right)^2 K_1 \left(\frac{r}{l_2} \right) \right],$$

$$\mu_{22} = \frac{Gb}{2\pi} L^2 \left[\frac{x^2 - y^2}{r^4} + \frac{y^2 + c_3 x^2}{r^4} \left(\frac{r}{l_2} \right)^2 K_0 \left(\frac{r}{l_2} \right) + \frac{y^2 - x^2}{r^4} \left(\frac{r}{l_2} \right)^2 K_1 \left(\frac{r}{l_2} \right) \right], \qquad (35)$$

$$\mu_{33} = \frac{Gb}{2\pi} L^2 \left[\frac{c_3}{r^4} \left(\frac{r}{l_2} \right)^2 K_0 \left(\frac{r}{l_2} \right) \right],$$

$$\mu_{12} = \frac{Gb}{2\pi} L^2 \left[-2\frac{xy}{r^4} + \frac{xy}{r^4} \left(\frac{r}{l_2} \right)^2 K_0 \left(\frac{r}{l_2} \right) + \frac{xy}{r^4} \left(\frac{r}{l_2} \right)^2 K_1 \left(\frac{r}{l_2} \right) \right],$$

$$\mu_{21} = \mu_{12},$$

and in cylindrical coordinates:

$$\mu_{r\varphi} = \mu_{\varphi r} = \mu_{z\varphi} = \mu_{\varphi z} = \mu_{rz} = \mu_{zr} = 0,$$

$$\mu_{rr} = \frac{Gb}{2\pi} L^2 \left[\frac{1}{r^2} + \frac{1 + c_3}{r^4} \left(\frac{r}{l_2} \right)^2 K_0 \left(\frac{r}{l_2} \right) + \frac{1}{r^2} \left(\frac{r}{l_2} \right)^2 K_1 \left(\frac{r}{l_2} \right) \right],$$

$$\mu_{\varphi\varphi} = \frac{Gb}{2\pi} L^2 \left[\frac{1}{r^2} + \frac{c_3}{r^4} \left(\frac{r}{l_2} \right)^2 K_0 \left(\frac{r}{l_2} \right) - \frac{1}{r^2} \left(\frac{r}{l_2} \right)^2 K_1 \left(\frac{r}{l_2} \right) \right], \qquad (36)$$

$$\mu_{zz} = \frac{Gb}{2\pi} L^2 \left[\frac{c_3}{r^2} \left(\frac{r}{l_2} \right)^2 K_0 \left(\frac{r}{l_2} \right) \right].$$

The force stresses and the moment stresses of the screw dislocation will be singular along the dislocation axis (z-axis), and decay quite rapidly as $r \to \infty$. In fact, the terms with the modified BESSEL functions vanish exponentially with increasing r.

The stress field of an edge dislocation

The BURGERS vector has the following components in a Cartesian coordinate system:

$$b_1 = b, \qquad b_2 = 0, \qquad b_3 = 0.$$

From (8) and (9), we thus obtain:

$$\tilde{\mu}_{ik} = 0; \qquad \tilde{\sigma}_{13} = \tilde{\sigma}_{31} = \tilde{\sigma}_{23} = \tilde{\sigma}_{32} = 0,$$

$$\tilde{\sigma}_{11} = -\frac{Gb}{\pi}\frac{1-\nu}{1-2\nu}\partial_y \ln r\,, \qquad \tilde{\sigma}_{22} = \tilde{\sigma}_{33} = -\frac{Gb}{\pi}\frac{\nu}{1-2\nu}\partial_y \ln r\,, \tag{37}$$

$$\tilde{\sigma}_{12} = \frac{Gb}{\pi}\left(\frac{1}{2}-\frac{c_1}{4}\right)\partial_x \ln r\,, \qquad \tilde{\sigma}_{21} = \frac{Gb}{\pi}\left(\frac{1}{2}+\frac{c_1}{4}\right)\partial_x \ln r\,,$$

and from (11a, b):

$$\tilde{X}_1 = B_1\, \partial_x\, \partial_y \ln r, \qquad \tilde{X}_2 = -\, B_1\, \partial_x \partial_x \ln r + B_1\, 2\pi\, \delta(x)\delta(y), \qquad \tilde{X}_3 = 0, \tag{38}$$

$$\tilde{Y}_1 = \tilde{Y}_2 = 0, \qquad \tilde{Y}_3 = \beta\, \partial_x \ln r,$$

with the abbreviations:

$$B_1 = \frac{Gb}{2\pi}\left(\frac{1}{1-2\nu}-\frac{c_1}{2}\right), \qquad B_2 = \frac{Gb}{\pi}\frac{\nu}{1-2\nu}, \qquad \beta = \frac{Gbc_1}{2\pi}.$$

The equations (16) then have the following solutions:

$$S_1 = \frac{1}{4} B_1\, (x\, \partial_y \ln r + y\, \partial_x \ln r),$$

$$S_2 = -\,\frac{1}{4} B_1\, (x\, \partial_x \ln r - y\, \partial_y \ln r + 2 \ln r), \qquad S_3 = 0, \tag{39}$$

$$T_1 = T_2 = 0, \qquad T_3 = \frac{1}{2}\beta\, x \ln r.$$

Due to:

$$\partial_\alpha S_\alpha = B_1 \partial_y \ln r, \qquad \partial_\alpha T_\alpha = 0, \tag{40}$$

the solution of (17) reads:

$$f = -\,\frac{\nu}{2(1-\nu)} B_2\, y \ln r, \tag{41}$$

and from $(17)_2$:

$$g = 0. \tag{42}$$

The right-hand side of equation (18) will be zero for $i = 3$, so we set:

$$A_3 = 0. \tag{43}$$

We consider these results in (13), and find, after some conversions, that in order calculate the compatible stresses $\bar{\sigma}_{ik}$ and $\bar{\mu}_{ik} = \mu_{ik}$ one needs only to determine $\partial_\alpha A_\alpha$, $\partial_1 A_2 - \partial_2 A_1$, ΔA_1, and ΔA_2 from (18).

The differential equation for $\partial_\alpha A_\alpha$:

$$\Delta\partial_\alpha A_\alpha - l_1^2 \Delta\Delta\, \partial_\alpha A_\alpha = \partial_\alpha S_\alpha - \frac{2-c_1}{2+c_1} l_1^2 \Delta\partial_\alpha S_\alpha \tag{44}$$

may, on account of the fact that:

$$\partial_\alpha S_\alpha = B_2\, \partial_y \ln r = \Delta\left(\frac{1}{2} B_2 y \ln r\right), \tag{45}$$

be brought into the form:

$$\Delta\partial_\alpha A_\alpha - \frac{1}{l_1^2}\partial_\alpha A_\alpha = -\frac{1}{2l_1^2}\; B_2\, y \ln r + \frac{2-c_1}{2+c_1} B_2\, \partial_y \ln r.$$

The equation:

$$\Delta\Phi - \frac{1}{l^2}\Phi = y \ln r,$$

will, with the solution Ansatz:

$$\Phi = -\, l^2\, y \ln r + \Phi_{(1)}\,,$$

be converted into:

$$\Delta\Phi_{(1)} - \frac{1}{l^2}\Phi_{(1)} = 2\, l^2\, \partial_y \ln r.$$

If we set:

$$\Phi_{(1)} = 2\, l^2\, \partial_y \Phi_{(2)}\,,$$

with

$$\Delta\Phi_{(2)} - \frac{1}{l^2}\Phi_{(2)} = \ln r,$$

then we obtain:

$$\Phi_{(2)} = -\, l^2 \left(\ln r + K_0\left(\frac{r}{l}\right)\right),$$

so:

$$\Phi = -\, l^2 y \ln r - 2l^4 \partial_y \left(\ln r + K_0\left(\frac{r}{l}\right)\right).$$

The solution of (46) then reads:

$$\partial_\alpha A_\alpha = \frac{1}{2} B_2 y \ln r + \frac{2c_1}{2+c_1} B_2 l_1^2 \partial_y \left(\ln r + K_0\left(\frac{r}{l_1}\right)\right). \tag{47}$$

Since:

$$\partial_2 A_1 - \partial_1 A_2 = (B_1 - B_2)\, \partial_x \ln r = \Delta \frac{1}{2} (B_1 - B_2)\, x \ln r, \tag{48}$$

the differential equation for $Q_3 = \partial_2 A_1 - \partial_1 A_2$ that is derived from (18) will be:

$$\Delta Q_3 - \frac{1}{l^2} Q_3 = -\frac{1}{2l_1^2}\left(B_1 - B_2 + \frac{\beta}{2}\right) x \ln r + \left[\frac{2-c_1}{2+c_1}(B_1 - B_2) - \frac{\beta}{2}\right] \partial_x \ln r\,. \tag{49}$$

The solution process for it corresponds to the one for equation (46), and we obtain:

$$Q_3 = \frac{1}{2}\left(B_1 - B_2 + \frac{\beta}{2}\right) x \ln r + \left[\frac{2c_1}{2+c_1}(B_1 - B_2) + \beta\right] l_1^2 \partial_x \left(\ln r + K_0\left(\frac{r}{l_1}\right)\right). \tag{50}$$

The differential equations for ΔA_1 and ΔA_2 will be solved similarly to the equations (46) and (49). We obtain the following results:

$$\Delta A_1 = \frac{1}{4}\left(B_1 + \frac{\beta}{2}\right)(x \partial_y \ln r + y \partial_x \ln r) + \left(\frac{2c_1}{2+c_1} B_1 + \beta\right) l_1^2 \partial_x \partial_y \left(\ln r + K_0\left(\frac{r}{l_1}\right)\right), \tag{51}$$

$$\begin{aligned} \Delta A_1 = \\ = -\frac{1}{4}\left(B_1 + \frac{\beta}{2}\right)(x \partial_y \ln r - y \partial_x \ln r) - \left(\frac{2c_1}{2+c_1} B_1 + \frac{\beta}{2}\right) l_1^2 (\partial_x \partial_x - \partial_y \partial_y) \left(\ln r + K_0\left(\frac{r}{l_1}\right)\right) \\ - \left(\frac{B_1}{2} + \frac{\beta}{4} - B_2\right) \ln r - \left(\frac{\beta}{2} + \frac{c_1}{2+c_1} B_1 - \frac{2c_1}{2+c_1} B_2\right) K_0\left(\frac{r}{l_1}\right) - \frac{\beta}{8}. \end{aligned} \tag{52}$$

We thus know all of the stress functions that are necessary for the representation (13). With (12) and (37), we find that the force stress field of an edge dislocation is:

$$\sigma_{13} = \sigma_{31} = \sigma_{23} = \sigma_{32} = 0,$$

$$\sigma_{13} = \frac{Gb}{2\pi}\left[-\frac{1}{1-\nu}\left(\frac{y}{r^2}+2\frac{x^2y}{r^4}\right)^* + L^2\left(1+\frac{c_2}{2}\right)\partial_x\partial_x\partial_y\left(\ln r + K_0\left(\frac{r}{l_1}\right)\right)\right],$$

$$\sigma_{22} = \frac{Gb}{2\pi}\left[-\frac{1}{1-\nu}\left(\frac{y}{r^2}-2\frac{x^2y}{r^4}\right)^* + L^2\left(1+\frac{c_2}{2}\right)\partial_x\partial_x\partial_y\left(\ln r + K_0\left(\frac{r}{l_1}\right)\right)\right],$$

$$\sigma_{33} = -\frac{Gb}{2\pi}\frac{2\nu}{1-\nu}\frac{y}{r^2}^*, \tag{53}$$

$$\sigma_{12} = \frac{Gb}{2\pi}\left[\frac{1}{1-\nu}\left(\frac{x}{r^2}-2\frac{xy^2}{r^4}\right)^* + L^2\left(1+\frac{c_2}{2}\right)\partial_y\partial_y\partial_x\left(\ln r + K_0\left(\frac{r}{l_1}\right)\right)\right],$$

$$\sigma_{21} = \frac{Gb}{2\pi}\left[\frac{1}{1-\nu}\left(\frac{x}{r^2}-2\frac{xy^2}{r^4}\right)^* - L^2\left(1+\frac{c_2}{2}\right)\partial_x\partial_x\partial_x\left(\ln r + K_0\left(\frac{r}{l_1}\right)\right)\right].$$

The terms that are marked with * are the stresses of an edge dislocation in a classical, isotropically elastic continuum.

Due to the recursion formula:

$$K_1'(\xi) = -K_0(\xi) - \frac{1}{\xi}K_1(\xi),$$

one has:

$$\partial_x\,\partial_x\,\partial_x\,(\ln r + K_0) = \left(3\frac{x}{r^4}-4\frac{x^3}{r^6}\right)\mathcal{K} - \frac{x^3}{r^6}\frac{r^3}{l_1^3}K_1,$$

$$\partial_x\,\partial_y\,\partial_x\,(\ln r + K_0) = \left(\frac{y}{r^4}-4\frac{x^2y}{r^6}\right)\mathcal{K} - \frac{x^2y}{r^6}\frac{r^3}{l_1^3}K_1,$$

$$\partial_y\,\partial_x\,\partial_y\,(\ln r + K_0) = \left(\frac{x}{r^4}-4\frac{y^2x}{r^6}\right)\mathcal{K} - \frac{xy^2}{r^6}\frac{r^3}{l_1^3}K_1,$$

$$\mathcal{K} = -2 + \frac{r^2}{l_1^2}K_0\left(\frac{r}{l_1}\right) + 2\frac{r}{l_1}K_1\left(\frac{r}{l_1}\right).$$

The results become somewhat clearer when we convert to cylindrical coordinates:

$$\sigma_{rz} = \sigma_{zr} = \sigma_{z\varphi} = \sigma_{\varphi z} = 0,$$

$$\sigma_{rr} = \frac{Gb}{2\pi}\left[-\frac{1}{1-\nu}\frac{\sin\varphi^*}{r} + L^2\left(1+\frac{c_2}{2}\right)\left[-\frac{\sin\varphi}{r^3}\mathcal{K}\right]\right],$$

$$\sigma_{\varphi\varphi} = \frac{Gb}{2\pi}\left[-\frac{1}{1-\nu}\frac{\sin\varphi^*}{r} + L^2\left(1+\frac{c_2}{2}\right)\frac{\sin\varphi}{r^3}\mathcal{K}\right],$$

$$\sigma_{zz} = -\frac{Gb}{2\pi}\frac{1}{1-2\nu}\frac{\sin\varphi^*}{r}, \tag{54}$$

$$\sigma_{r\varphi} = \frac{Gb}{2\pi}\left[\frac{1}{1-\nu}\frac{\cos\varphi^*}{r} + L^2\left(1+\frac{c_2}{2}\right)\frac{\cos\varphi}{r^3}\mathcal{K}\right],$$

$$\sigma_{\varphi r} = \frac{Gb}{2\pi}\left[\frac{1}{1-\nu}\frac{\cos\varphi^*}{r} + L^2\left(1+\frac{c_2}{2}\right)\frac{\cos\varphi}{r^3}\left[\mathcal{K}+\frac{r^3}{l_1^3}K_1\right]\right].$$

For the moment stresses of an edge dislocation, we obtain, in Cartesian coordinates:

$$\mu_{11} = \mu_{22} = \mu_{33} = \mu_{12} = \mu_{21} = 0,$$

$$\mu_{13} = \frac{Gb}{2\pi}\left[L^2\left(1+\frac{c_2}{2}\right)\partial_y\partial_y(\ln r + K_0) - \frac{4c_1}{2+c_1}K_0\right],$$

$$\mu_{23} = \frac{Gb}{2\pi}L^2\left(1+\frac{c_2}{2}\right)\partial_x\partial_y(\ln r + K_0), \tag{55}$$

$$\mu_{31} = \frac{2-c_2}{2+c_2}\mu_{13}, \qquad \mu_{32} = \frac{2-c_2}{2+c_2}\mu_{23},$$

and in cylindrical coordinates:

$$\mu_{rr} = \mu_{\varphi\varphi} = \mu_{zz} = \mu_{r\varphi} = \mu_{\varphi r} = 0,$$

$$\mu_{\varphi z} = \frac{Gb}{2\pi}L^2\left(1+\frac{c_2}{2}\right)\frac{\sin\varphi}{r^2}\left[1-\frac{r}{l_1}K_1\left(\frac{r}{l_1}\right)\right], \tag{55}$$

$$\mu_{z\varphi} = \frac{Gb}{2\pi} L^2 \left(1 - \frac{c_2}{2}\right) \frac{\sin\varphi}{r^2} \left[1 - \frac{r}{l_1} K_1\left(\frac{r}{l_1}\right)\right],$$

$$\mu_{rz} = \frac{Gb}{2\pi} L^2 \left(1 + \frac{c_2}{2}\right) \frac{\cos\varphi}{r^2} \left[1 - \frac{r}{l_1} K_1\left(\frac{r}{l_1}\right) - \frac{r^2}{l_1^2} K_0\left(\frac{r}{l_1}\right)\right],$$

$$\mu_{zr} = \frac{Gb}{2\pi} L^2 \left(1 - \frac{c_2}{2}\right) \frac{\cos\varphi}{r^2} \left[1 - \frac{r}{l_1} K_1\left(\frac{r}{l_1}\right) - \frac{r^2}{l_1^2} K_0\left(\frac{r}{l_1}\right)\right].$$

Some of the same things that were already stated for the stress fields of a screw dislocation are true here, as well. The stresses become singular at the dislocation and die off with increasing distance from the dislocation line. The terms that include the modified BESSEL functions go to zero as $r \to \infty$ exponentially.

Bibliography

1. S. KESSEL, "Über die Verträglichkeitsbedingungen in einem COSSERATschen Kontinuum," Abhandlung d. Braunschweig. Wiss. Ges. **XVII** (1965).
2. H. SCHAEFER, private correspondence.
3. S. KESSEL, "Lineare Elastizitätstheorie des anisotropen Cosserat-Kontinuums," Abhandlung d. Braunschweig. Wiss. Ges. **XVI** (1964).
4. S. KESSEL, "Die Spannungsfunktionen des COSSERAT-Kontinuums," ZAMM **47** (1967), 329-336.
5. S. KESSEL, "Verschiebungs- und Drehvectorfeld einer Einzerlkraft und eines Momentes im Cosseratschen Kontinuum," Abhandlung d. Braunschweig. Wiss. Ges. **XIX** (1967).
6. A. N. TYCHONOFF, A. A. SAMARSKI, "Differentialgleichungen der Mathematischen Physik," 1959, VEB Deutscher Verlag der Wissenschaften.
7. JAHNKE-EMDE-LÖSCH, "Tafel höherer Funktionen," Stuttgart 1960, Teubner Verlagsgesellschaft.

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