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The stress functions of the Cosserat continuum

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The equilibrium conditions of the Cosserat continuum are satisfied identically by a complete stress function representation of the stresses. For a kinematically compatible stress state in an isotropic elastic continuum, the stress functions can be expressed in terms of potential functions and solutions of the Helmholtz equations.

Introduction

One can mathematically formulate an elastostatic problem in classical linear elasticity theory as either a boundary-value problem for the displacement vector field or a boundary-value problem for the stress function tensor field. In both cases, the solutions of the field equations can be expressed with the help of certain Ansätze regarding potential functions; the Ansatz for the displacement vector field was found by NEUBER [**1**] and PAPKOWICH [**2**], while SCHAEFER [**3**] found the stress function Ansatz.

Analogously, the elastostatic problem for the linear isotropically-elastic COSSERAT continuum can be formulated as either a boundary-value problem for the displacement and rotation vector fields or as a boundary-value problem for the two tensor fields of the stress functions. NEUBER [**4**] has shown that one can convert the solutions of the six coupled differential equations for the kinematic fields by certain Ansätze on the solutions of potential equations and Helmholtz equations. In the present paper, the corresponding Ansätze for the stress functions will be derived and compared to the NEUBER Ansätze.

Preface

We compute in Cartesian coordinates and employ the summation convention that Greek indices are to be summed over from 1 to 3. It will be assumed that all scalar, vector, and tensor fields are defined in a simply-connected, but possibly multiply-bounded, region G with an outer surface ∂G and that they are continuously-differentiable as many times as is required.

The equations of kinematics and statics of COSSERAT continuum can be written quite simply with the use certain well-defined differential operators. Certain relations between these differential operators characterize the analogies that exist with the differential operators of vector analysis; their interpretation in the calculus of alternating differential forms was recognized by SCHAEFER [**5**].

We combine the tensor fields $\partial_i V_k$ and $(\partial_i W_k - e_{ik\alpha} V_\alpha)$, which are defined by the two vector fields V_i and W_i, and define them as the result of applying a differential operator – "Grad" – to the vector fields **V** and **W**:

(1) $$\mathrm{Grad}\begin{bmatrix}\mathbf{V}\\ \mathbf{W}\end{bmatrix} \equiv \begin{cases}\partial_i V_k\\ \partial_i W_k - e_{ik\alpha}V_\alpha.\end{cases}$$

Similarly, we define a "Div" operator that acts on two tensor fields of rank two **Q** and **R**:

(2) $$\mathrm{Div}\begin{bmatrix}\mathbf{Q}\\ \mathbf{R}\end{bmatrix} \equiv \begin{cases}\partial_\alpha Q_{\alpha k}\\ \partial_\alpha R_{\alpha k} + e_{k\alpha\beta}Q_{\alpha\beta}\end{cases}$$

and a "Rot" operator that likewise acts on two tensor fields:

(3) $$\mathrm{Rot}\begin{bmatrix}\mathbf{Q}\\ \mathbf{R}\end{bmatrix} \equiv \begin{cases}e_{i\alpha\beta}\partial_\alpha Q_{\beta k}\\ e_{i\alpha\beta}(\partial_\alpha R_{\beta k} + e_{k\alpha\gamma}Q_{\beta\gamma}).\end{cases}$$

In this, e_{ikl} is the permutation symbol that is skew-symmetric in all indices.

In addition to the operators defined in (1) to (3), we also define the operators Grad*, Div*, and Rot*:

(1)*, $$\mathrm{Grad}^*\begin{bmatrix}\mathbf{V}\\ \mathbf{W}\end{bmatrix} \equiv \begin{cases}\partial_i V_k\\ \partial_i W_k + e_{ik\alpha}V_\alpha,\end{cases}$$

(2)* $$\mathrm{Div}^*\begin{bmatrix}\mathbf{Q}\\ \mathbf{R}\end{bmatrix} \equiv \begin{cases}\partial_\alpha Q_{\alpha k}\\ \partial_\alpha R_{\alpha k} - e_{k\alpha\beta}Q_{\alpha\beta},\end{cases}$$

(3)* $$\mathrm{Rot}^*\begin{bmatrix}\mathbf{Q}\\ \mathbf{R}\end{bmatrix} \equiv \begin{cases}e_{i\alpha\beta}\partial_\alpha Q_{\beta k}\\ e_{i\alpha\beta}(\partial_\alpha R_{\beta k} - e_{k\alpha\gamma}Q_{\beta\gamma}).\end{cases}$$

One may easily check that the following identities exist:

(4), (5)	Div Grad* $= \Delta$,	Div* Grad $= \Delta$,
(6), (7)	Div Rot $= 0$,	Div* Rot* $= 0$,
(8), (9)	Rot Grad $= 0$,	Rot Rot* $=$ Grad* Div $- \Delta$,

Kinematics, statics, and the material law of the linear, isotropically-elastic Cosserat continuum

We assume: Any "point" of the COSSERAT continuum is orientable and has the possible motions of a rigid body. We describe the six functional degrees of freedom of the continuum by a displacement vector field $\mathbf{u}(x_i)$ and – for small rotations – by a rotation vector field $\boldsymbol{\varphi}(x_i)$. The deformation state of the COSSERAT continuum is described by the two deformation tensors $\boldsymbol{\chi}$ and $\boldsymbol{\varepsilon}$:

(10) $$\begin{bmatrix} \boldsymbol{\chi} \\ \boldsymbol{\varepsilon} \end{bmatrix} = \text{Grad} \begin{bmatrix} \boldsymbol{\varphi} \\ \mathbf{u} \end{bmatrix}.$$

The asymmetric tensor $\boldsymbol{\chi}$ is the tensor of curvature deformation, the symmetric part of $\boldsymbol{\varepsilon}$ is the deformation tensor of classical linear elasticity theory, and the skew-symmetric part of $\boldsymbol{\varepsilon}$ measures the difference between the local rigid rotation that is determined by the displacement vector field and the absolute rotation of the "points" of continuum. If the deformation tensor fields $\boldsymbol{\chi}$ and $\boldsymbol{\varepsilon}$ are given in a simply-connected region G then we can calculate a rotation vector field and a translation vector field uniquely from them – up to a rigid rotation – when the compatibility conditions:

(11) $$\text{Rot} \begin{bmatrix} \boldsymbol{\chi} \\ \boldsymbol{\varepsilon} \end{bmatrix} = 0$$

are fulfilled in G.

The deformation tensor $\boldsymbol{\varepsilon}$ can be associated with a force stress tensor $\boldsymbol{\sigma}$ and the curvature tensor $\boldsymbol{\chi}$, with a moment-stress tensor $\boldsymbol{\mu}$ under the principle of virtual displacements and the LAGRANGE liberation principle [**6**]. For a surface element dF with the external unit normal vector $\mathbf{n}$ that is loaded with a force $\mathbf{p}\,dF$ and a moment $\mathbf{m}\,dF$, one has:

(12) $$\mathbf{n} \cdot \boldsymbol{\sigma} = \mathbf{p}, \qquad \mathbf{n} \cdot \boldsymbol{\mu} = \mathbf{m}.$$

The differential equilibrium conditions for a volume element of the continuum that is loaded with the volume force $\mathbf{X}$ and the volume moment $\mathbf{Y}$ read:

(13) $$\text{Div} \begin{bmatrix} \boldsymbol{\sigma} \\ \boldsymbol{\mu} \end{bmatrix} = - \begin{bmatrix} \mathbf{X} \\ \mathbf{Y} \end{bmatrix}.$$

The material law for the linear isotropically-elastic body is [**7**]:

(14) $$\begin{cases} \sigma_{ik} = 2G\left[\left(\dfrac{1}{2}+\dfrac{c_1}{4}\right)\varepsilon_{ik} + \left(\dfrac{1}{2}-\dfrac{c_1}{4}\right)\varepsilon_{ki} + \dfrac{\nu}{1-2\nu}\delta_{ik}\varepsilon_{\alpha\alpha}\right], \\ \mu_{ik} = 2GL^2\left[\left(\dfrac{1}{2}+\dfrac{c_2}{4}\right)\chi_{ik} + \left(\dfrac{1}{2}-\dfrac{c_2}{4}\right)\chi_{ki} + c_3\delta_{ik}\chi_{\alpha\alpha}\right]; \end{cases}$$

(15) $$\begin{cases} \chi_{ik} = \dfrac{1}{2GL^2}\left[\left(\dfrac{1}{2}+\dfrac{1}{c_2}\right)\chi_{ik} + \left(\dfrac{1}{2}-\dfrac{1}{c_2}\right)\mu_{ki} - \dfrac{c_3}{1+3c_3}\delta_{ik}\mu_{\alpha\alpha}\right], \\ \varepsilon_{ik} = \dfrac{1}{2G}\left[\left(\dfrac{1}{2}+\dfrac{1}{c_1}\right)\sigma_{ik} + \left(\dfrac{1}{2}-\dfrac{1}{c_1}\right)\sigma_{ki} - \dfrac{\nu}{1-2\nu}\delta_{ik}\sigma_{\alpha\alpha}\right] \end{cases}$$

includes six material constants: the shear modulus G, the transverse contraction number ν, a material constant L with the dimension of a length, and the three-dimensional material constants c_1, c_2, c_3 .

For brevity, we write:

$$\begin{bmatrix}\boldsymbol{\sigma}\\ \boldsymbol{\mu}\end{bmatrix} = \begin{bmatrix}0 & \underline{\mathbf{M}}_{(1)}\\ \underline{\mathbf{M}}_{(2)} & 0\end{bmatrix}\begin{bmatrix}\boldsymbol{\chi}\\ \boldsymbol{\varepsilon}\end{bmatrix}, \qquad \begin{bmatrix}\boldsymbol{\chi}\\ \boldsymbol{\varepsilon}\end{bmatrix} = \begin{bmatrix}0 & \underline{\mathbf{M}}_{(2)}^{-1}\\ \underline{\mathbf{M}}_{(1)}^{-1} & 0\end{bmatrix}\begin{bmatrix}\boldsymbol{\sigma}\\ \boldsymbol{\mu}\end{bmatrix}, \tag{16}$$

in which $\underline{\mathbf{M}}_{(i)}$ and $\underline{\mathbf{M}}_{(i)}^{-1}$ are isotropic tensors of rank four.

The basic elastic equations and the solution Ansatz of Neuber

For the determination of the displacement and rotation vector field in a body G, for which the kinematic degrees of freedom are restricted on the outer surface ∂G or that outer surface ∂G is loaded with force and moment stresses, we have the basic elastic equations:

$$\operatorname{Div}\begin{bmatrix}0 & \underline{\mathbf{M}}_{(1)}\\ \underline{\mathbf{M}}_{(2)} & 0\end{bmatrix}\operatorname{Grad}\begin{bmatrix}\boldsymbol{\varphi}\\ \mathbf{u}\end{bmatrix} = 0, \tag{17}$$

with consideration to the corresponding boundary conditions to be solved. More specifically, these equations read:

$$\begin{cases}\left(1+\dfrac{c_1}{2}\right)\Delta\mathbf{u}+\left(\dfrac{1}{1-2\nu}-\dfrac{c_1}{2}\right)\operatorname{grad}\operatorname{div}\mathbf{u}+c_1\operatorname{rot}\boldsymbol{\varphi}=0,\\ \left(1+\dfrac{c_1}{2}\right)\Delta\boldsymbol{\varphi}+\left(1-\dfrac{c_2}{2}+2c_3\right)\operatorname{grad}\operatorname{div}\boldsymbol{\varphi}+\dfrac{2c_1}{L^3}\left(\dfrac{1}{2}\operatorname{rot}\mathbf{u}-\boldsymbol{\varphi}\right)=0;\end{cases} \tag{18}$$

it will be solved by an Ansatz [**4**] that corresponds to the NEUBER-PAPKOWICH Ansatz of classical elasticity theory:

$$\begin{cases}\mathbf{u}=-\operatorname{grad}\left[\Phi_0+\dfrac{1}{4(1-\nu)}\mathbf{r}\cdot\boldsymbol{\Phi}+l_1^2\operatorname{div}\boldsymbol{\psi}\right]+\boldsymbol{\Phi}+\boldsymbol{\psi},\\ \boldsymbol{\varphi}=\dfrac{1}{2}\operatorname{rot}\left(\boldsymbol{\Phi}+\dfrac{2+c_1}{c_1}\boldsymbol{\psi}\right)+\operatorname{grad}\chi,\end{cases} \tag{19}$$

$$\Delta\Phi_0 = 0, \qquad \Delta\boldsymbol{\Phi} = 0, \tag{20, 21}$$

$$\left(\Delta-\frac{1}{l_1^2}\right)\boldsymbol{\psi} = 0, \qquad l_1^2 = L^2\frac{(2+c_1)(2+c_2)}{8c_1}, \tag{22}$$

(23) $$\left(\Delta - \frac{1}{l_2^2}\right)\chi = 0, \qquad l_2^2 = L^2\,\frac{1+c_3}{c_1}\,.$$

The stress function solution

The integration of the elastostatic problem with the help of stress functions comes from an Ansatz for the stresses that the equilibrium conditions satisfy identically.

THEOREM: *Any equilibrium system of force and moment stresses may be represented in the form:*

(24) $$\begin{bmatrix}\boldsymbol{\sigma}\\ \boldsymbol{\mu}\end{bmatrix} = \mathrm{Rot}\begin{bmatrix}\mathfrak{F}\\ \mathfrak{G}\end{bmatrix} + \mathrm{Grad}^*\begin{bmatrix}\mathbf{S}\\ \mathbf{T}\end{bmatrix},$$

in which [**8**]:

(25), (26) $$\mathrm{Div}^*\begin{bmatrix}\mathfrak{F}\\ \mathfrak{G}\end{bmatrix} = 0, \qquad \Delta\begin{bmatrix}\mathbf{S}\\ \mathbf{T}\end{bmatrix} = -\begin{bmatrix}\mathbf{X}\\ \mathbf{Y}\end{bmatrix}.$$

Proof: Let

(27), (28) $$\begin{bmatrix}\mathbf{H}\\ \mathbf{K}\end{bmatrix} = -\frac{1}{4\pi}\int_G \frac{1}{r}\begin{bmatrix}\boldsymbol{\sigma}\\ \boldsymbol{\mu}\end{bmatrix} dV\,, \qquad \mathrm{Div}\begin{bmatrix}\boldsymbol{\sigma}\\ \boldsymbol{\mu}\end{bmatrix} = -\begin{bmatrix}\mathbf{X}\\ \mathbf{Y}\end{bmatrix}.$$

Then, from (27), one has:

(29) $$\Delta\begin{bmatrix}\mathbf{H}\\ \mathbf{K}\end{bmatrix} = \begin{bmatrix}\boldsymbol{\sigma}\\ \boldsymbol{\mu}\end{bmatrix}$$

and with (9):

(30) $$\mathrm{Rot}\,\mathrm{Rot}^*\begin{bmatrix}\mathbf{H}\\ \mathbf{K}\end{bmatrix} + \mathrm{Grad}^*\mathrm{Div}\begin{bmatrix}\mathbf{H}\\ \mathbf{K}\end{bmatrix} = \begin{bmatrix}\boldsymbol{\sigma}\\ \boldsymbol{\mu}\end{bmatrix}$$

We now set:

(31) $$\mathrm{Div}^*\begin{bmatrix}\mathfrak{F}\\ \mathfrak{G}\end{bmatrix} = -\,\mathrm{Div}^*\,\mathrm{Rot}^*\begin{bmatrix}\mathbf{H}\\ \mathbf{K}\end{bmatrix} = 0,$$

with:

(34) $$\Delta\begin{bmatrix}\mathbf{S}\\ \mathbf{T}\end{bmatrix} = \Delta\,\mathrm{Div}\begin{bmatrix}\mathbf{H}\\ \mathbf{K}\end{bmatrix} - \mathrm{Div}\begin{bmatrix}\Delta\mathbf{H}\\ \Delta\mathbf{K}\end{bmatrix} = \mathrm{Div}\begin{bmatrix}\boldsymbol{\sigma}\\ \boldsymbol{\mu}\end{bmatrix} = -\begin{bmatrix}\mathbf{X}\\ \mathbf{Y}\end{bmatrix}.$$

With that, we have shown that we can find a representation of the form (24) for any equilibrium stress state.

For the following, we assume that the volume force **X** and the volume element **Y** are zero. The vectors **S** and **T** in the stress function Ansatz (24) will then be harmonic vectors:

(35) $$\Delta\mathbf{S} = 0, \qquad \Delta\mathbf{T} = 0.$$

We consider the auxiliary condition (25), in which the stress function tensors of first order $\mathfrak{F}$ and $\mathfrak{G}$ are expressed in terms of stress function tensors of second order **F** and **G**:

$$\begin{bmatrix} \mathfrak{F} \\ \mathfrak{G} \end{bmatrix} = \mathrm{Rot}^* \begin{bmatrix} \mathbf{F} \\ \mathbf{G} \end{bmatrix}. \tag{36}$$

The stress function Ansatz (24) then reads:

$$\begin{bmatrix} \boldsymbol{\sigma} \\ \boldsymbol{\mu} \end{bmatrix} = \mathrm{Rot}\ \mathrm{Rot}^* \begin{bmatrix} \mathbf{F} \\ \mathbf{G} \end{bmatrix} + \mathrm{Grad}^* \begin{bmatrix} \mathbf{S} \\ \mathbf{T} \end{bmatrix} \tag{37}$$

or

$$\begin{bmatrix} \boldsymbol{\sigma} \\ \boldsymbol{\mu} \end{bmatrix} = \mathrm{Grad}^* \left\{ \mathrm{Div} \begin{bmatrix} \mathbf{F} \\ \mathbf{G} \end{bmatrix} + \begin{bmatrix} \mathbf{S} \\ \mathbf{T} \end{bmatrix} \right\} - \Delta \begin{bmatrix} \mathbf{F} \\ \mathbf{G} \end{bmatrix}, \tag{38}$$

$$\begin{cases} \sigma_{ik} = \partial_i \partial_\alpha F_{\alpha k} + \partial_i S_k - \Delta F_{ik}, \\ \mu_{ik} = \partial_i \partial_\alpha G_{\alpha k} + \partial_i e_{k\alpha\beta} F_{\alpha\beta} + \partial_i T_k + e_{k\alpha\beta} S_\alpha - \Delta G_{ik}. \end{cases} \tag{38′}$$

Since the body G is simply connected, we can exclude the existence of proper stresses; the stress functions must then be determined in such a way that the equilibrium stress state (38) is compatible, so, from (11), it must satisfy the condition:

$$\mathrm{Rot} \begin{bmatrix} 0 & \underline{\mathbf{M}}_{(2)}^{-1} \\ \underline{\mathbf{M}}_{(1)}^{-1} & 0 \end{bmatrix} \left\{ \mathrm{Grad}^* \left[\mathrm{Div} \begin{bmatrix} \mathbf{F} \\ \mathbf{G} \end{bmatrix} + \begin{bmatrix} \mathbf{S} \\ \mathbf{T} \end{bmatrix} \right] - \Delta \begin{bmatrix} \mathbf{F} \\ \mathbf{G} \end{bmatrix} \right\} = 0. \tag{39}$$

This means that:

$$\begin{bmatrix} 0 & \underline{\mathbf{M}}_{(2)}^{-1} \\ \underline{\mathbf{M}}_{(1)}^{-1} & 0 \end{bmatrix} \left\{ \mathrm{Grad}^* \left[\mathrm{Div} \begin{bmatrix} \mathbf{F} \\ \mathbf{G} \end{bmatrix} + \begin{bmatrix} \mathbf{S} \\ \mathbf{T} \end{bmatrix} \right] - \Delta \begin{bmatrix} \mathbf{F} \\ \mathbf{G} \end{bmatrix} \right\} = \mathrm{Grad} \begin{bmatrix} \boldsymbol{\varphi} \\ \mathbf{u} \end{bmatrix}.$$

On the left-hand side of this equation, after a series of conversions, one may split off a “gradient”:

$$\mathrm{Grad} \begin{bmatrix} \mathbf{N}^{(1)} \\ \mathbf{N}^{(2)} \end{bmatrix} + \begin{bmatrix} \mathfrak{L}^{(1)}(\mathbf{F}, \mathbf{G}) \\ \mathfrak{L}^{(2)}(\mathbf{F}, \mathbf{G}) \end{bmatrix} = \mathrm{Grad} \begin{bmatrix} \boldsymbol{\varphi} \\ \mathbf{u} \end{bmatrix}. \tag{40}$$

Thus, one has:

$$N_i^{(1)} = \frac{1}{2GL^2} (\partial_\alpha G_{\alpha i} + 2F_i + T_i), \qquad N_i^{(2)} = \frac{1}{2G} (\partial_\alpha F_{\alpha i} + S_i), \tag{41}$$

$\mathfrak{L}^{(1)}(\mathbf{F}, \mathbf{G})$ and $\mathfrak{L}^{(2)}(\mathbf{F}, \mathbf{G})$ are tensorial differential expressions in the stress functions F_{ik} and G_{ik} .

We set:

$$\boldsymbol{\varphi} = \mathbf{N}^{(1)}, \qquad \mathbf{u} = \mathbf{N}^{(2)} \tag{42}$$

and

$$\mathfrak{L}^{(1)}(\mathbf{F}, \mathbf{G}) = 0, \qquad \mathfrak{L}^{(2)}(\mathbf{F}, \mathbf{G}) = 0. \tag{43}$$

Equations (43) are 18 coupled partial differential equations of second order for the 18 stress functions F_{ik} and G_{ik} .

When we split them into their symmetric and anti-symmetric parts:

$$\begin{aligned} &\mathfrak{L}^{(m)}_{ik} = \mathfrak{L}^{(m)}_{(ik)} + e_{ik\alpha}\mathfrak{L}^{(m)}_{\alpha}, \qquad m = 1, 2, \\ &\mathfrak{L}^{(m)}_{(ik)}(\mathbf{F},\mathbf{G}) = 0, \qquad \mathfrak{L}^{(m)}_{\alpha}(\mathbf{F},\mathbf{G}) = 0, \end{aligned} \tag{44}$$

they read:

$$\mathfrak{L}^{(1)}_{(ik)} = \frac{1}{2GL^2}\left[-\Delta G_{(ik)} - \frac{c_3}{1+3c_3}\delta_{ik}(-\Delta G_{\alpha\alpha} + \partial_\alpha\partial_\beta G_{(\alpha\beta)} + 2\partial_\alpha F_\alpha + \partial_\alpha T_\alpha)\right] = 0, \tag{45}$$

$$\begin{aligned} \mathfrak{L}^{(1)}_{i} = \frac{1}{2G}\Bigg[&-\left(\frac{1}{2}-\frac{1}{c_1}\right)\delta_{ik}(e_{i\alpha\beta}\partial_\alpha\partial_\gamma G_{(\alpha\beta)} - 2\partial_i\partial_\alpha G_\alpha + e_{i\alpha\beta}\partial_\alpha T_\alpha) \\ &-\left(\frac{1}{2}-\frac{1}{c_2}\right)\Delta G_i + \frac{2}{c_2}(\partial_\alpha F_{(\alpha i)} + S_i) - e_{i\alpha\beta}\partial_\alpha F_\beta\Bigg] = 0, \end{aligned} \tag{46}$$

$$\mathfrak{L}^{(2)}_{(ik)} = \frac{1}{2G}\left[-\Delta F_{(ik)} - \frac{\nu}{1+\nu}\delta_{ik}(-\Delta G_{\alpha\alpha} + \partial_\alpha\partial_\beta F_{(\alpha\beta)} + \partial_\alpha S_\alpha)\right] = 0, \tag{47}$$

$$\begin{aligned} \mathfrak{L}^{(2)}_{i} = \frac{1}{2G}\Bigg[&-\left(\frac{1}{2}-\frac{1}{c_1}\right)\delta_{ik}(e_{i\alpha\beta}\partial_\alpha\partial_\gamma F_{(\alpha\beta)} - 2\partial_i\partial_\alpha F_\alpha + e_{i\alpha\beta}\partial_\alpha S_\alpha) \\ &-\left(\frac{1}{2}-\frac{1}{c_1}\right)\Delta F_i + \frac{1}{L^2}(\partial_\alpha G_{(\alpha i)} + 2F_i + T_i + e_{i\alpha\beta}\partial_\alpha G_\beta)\Bigg] = 0, \end{aligned} \tag{48}$$

in which have introduced:

$$F_{ik} = F_{(ik)} + e_{ik\alpha} F_\alpha, \qquad G_{ik} = G_{(ik)} + e_{ik\alpha} G_\alpha. \tag{49}$$

The differential equations for the stress functions may now be decoupled by means of certain Ansätze, and turn into potential and POISSON equations, as well as homogeneous and inhomogeneous HELMHOTLZ differential equations.

We next solve the equation (47) by the Ansatz:

$$F_{(ik)} = f_{(ik)} + \delta_{ik} f \tag{50}$$

with

$$\Delta f_{(ik)} = 0. \tag{51}$$

By substituting this in (47), we obtain the differential equation for f:

(52) $$\Delta f = -\frac{\nu}{1-\nu}(\partial_\alpha \partial_\beta f_{\alpha\beta} + \partial_\alpha S_\alpha),$$

whose solution, from (35) and (51), reads:

(53) $$f = \overset{0}{f_0} - \frac{\nu}{2(1-\nu)}(x_\alpha \partial_\beta f_{(\alpha\beta)} + x_\alpha S_\alpha),$$

with:

(54) $$\Delta \overset{0}{f_0} = 0.$$

Equation (47) is thus fulfilled.

We next take the trace of (45):

(55) $$\frac{1}{3c_3}\Delta G_{(\alpha\alpha)} + \partial_\alpha \partial_\beta G_{(\alpha\beta)} + 2\partial_\alpha F_\alpha + \partial_\alpha T_\alpha = 0,$$

substitute in (45), and obtain the differential equation for $G_{(ik)}$:

(56) $$\Delta\left(G_{(ik)} - \frac{1}{3}\delta_{ik} G_{(\alpha\alpha)} \right) = 0,$$

which we solve with the Ansatz:

(57), (58) $$G_{(ik)} = g_{(ik)} + \delta_{ik}\, g, \qquad \Delta g_{(ik)} = 0;$$

for the moment, the function g is still arbitrary.

With (57), it follows from (55) that:

(59) $$F_i = -\frac{1+c_3}{2c_3}\partial_i g - \frac{1}{2}\partial_\alpha g_{(\alpha i)} - \frac{1}{2}T_i + \varepsilon_{i\alpha\beta}\partial_\alpha A_\beta,$$

with a still-undetermined vector field A_i .

One deduces the following differential equation for g by taking the divergence of the vector equation (48):

(60) $$\Delta\left(\Delta g - \frac{1}{l_2^2} g + \frac{c_3}{1+c_3}\partial_\alpha T_\alpha \right) = 0, \qquad l_2^2 = L^2 \frac{1+c_3}{c_1}.$$

Up to a potential function, which can be assumed, along with $\delta_{ik}\, g$ in $g_{(ik)}$, g must satisfy the differential equation:

(61) $$\Delta g - \frac{1}{l_2^2} g = -\frac{c_3}{1+c_3}\partial_\alpha T_\alpha.$$

Due to (35), the solution reads:

(62), (63)
$$g = \overset{0}{g} + L^2 \frac{c_3}{c_1} \partial_\alpha T_\alpha , \qquad \left(\Delta - \frac{1}{l_2^2} \right) \overset{0}{g} = 0.$$

We substitute all of the results that we have obtained up to now in (48) and obtain the equation:

(64)
$$e_{i\alpha\beta} \partial_\alpha G_\beta = $$
$$= e_{i\alpha\beta} \partial_\alpha \left[2A_\beta - L^2 \frac{2+c_1}{2c_1} \left(\Delta A_\beta + \frac{1}{2} e_{\beta\lambda\mu} \partial_\lambda T_\mu \right) + L^2 \frac{2-c_1}{2c_1} \left(h_\beta - \frac{1}{2} e_{\beta\lambda\mu} \partial_\lambda T_\mu \right) \right].$$

in which:

(65)
$$h_i = \partial_\alpha f_{(\alpha i)} + \frac{1}{2} \; e_{i\alpha\beta} \partial_\alpha \partial_\gamma g_{(\gamma\beta)} + S_i + \frac{1}{2} \; e_{i\alpha\beta} \partial_\alpha T_\beta$$

is a potential vector.

From (64), it follows that:

(66)
$$G_i = 2A_i - L^2 \frac{2+c_1}{2c_1} \left(\Delta A_i + \frac{1}{2} e_{i\alpha\beta} \partial_\alpha T_\beta \right) + L^2 \frac{2-c_1}{2c_1} \left(h_i - \frac{1}{2} e_{i\alpha\beta} \partial_\alpha T_\beta \right) + \partial_i \rho$$

with a still-undetermined scalar function ρ.

Finally, it follows from equation (46) for the determination of A_i and ρ:

(67)
$$\Delta(A_i - l_1^2 \Delta A_i) + h_i + \partial_i \left[\Delta\rho - f - \partial_\alpha A_\alpha - \frac{2-c_2}{2+c_2} l_1^2 \Delta \partial_\alpha A_\alpha + \frac{(2-c_1)}{(2+c_1)} \frac{(2-c_2)}{(2+c_2)} l_1^2 \partial_\alpha A_\alpha \right] = 0,$$

and we set:

(68)
$$A_i = B_i + C_i + D_i,$$

(69)
$$\left(\Delta - \frac{1}{l_1^2} \right) B_i = 0, \qquad l_1^2 = L^2 \frac{(2+c_1)(2+c_2)}{8c_1},$$

(70), (71)
$$\Delta C_i = 0, \qquad \Delta D_i = h_i ,$$

(72)
$$\Delta\rho = f + \partial_\alpha A_\alpha + \frac{2-c_2}{2+c_2} l_1^2 \Delta \partial_\alpha A_\alpha - \frac{(2-c_1)(2-c_2)}{(2-c_1)(2+c_2)} l_1^2 \partial_\alpha h_\alpha ,$$

in which not only ρ, but also $\partial_i \Delta\rho$, will be required for the computation of $\mathbf{u}$, $\boldsymbol{\varphi}$, $\boldsymbol{\sigma}$, and $\boldsymbol{\mu}$.

The compatible stress functions read thus:

$$F_{(ik)} = f_{(ik)} + \delta_{ik} f, \qquad G_{(ik)} = g_{(ik)} + \delta_{ik} g,$$

$$F_i = -\frac{1+c_3}{2c_3}\partial_i g - \frac{1}{2}\partial_\alpha g_{(\alpha i)} - \frac{1}{2}T_i + e_{i\alpha\beta}\partial_\alpha A_\beta ,$$

$$G_i = 2A_i - L^2\frac{2+c_1}{2c_1}\Delta A_i + L^2\frac{2-c_1}{2c_1}h_i - \frac{L^2}{c_1}e_{i\alpha\beta}\partial_\alpha T_\beta + \partial_i\rho ,$$

with:

$$\Delta S_i = 0, \qquad \Delta T_i = 0, \qquad \Delta \overset{0}{f} = 0, \qquad \Delta C_i = 0, \qquad \Delta D_i = h_i ,$$

$$\Delta f_{(ik)} = 0, \qquad \Delta g_{(ik)} = 0, \qquad \left(\Delta - \frac{1}{l_1^2}\right)B_i = 0, \qquad \left(\Delta - \frac{1}{l_2^2}\right)\overset{0}{g} = 0,$$

and the abbreviations:

$$f = \overset{0}{f} - \frac{\nu}{2(1-\nu)}(x_\alpha\partial_\beta f_{(\alpha\beta)} + x_\alpha S_\alpha) , \qquad g = \overset{0}{g} + L^2\frac{c_3}{c_1}\partial_\alpha T_\alpha ,$$

$$A_i = B_i + C_i + D_i, \qquad h_i = \partial_\alpha f_{(\alpha i)} + \frac{1}{2}e_{i\alpha\beta}\partial_\alpha\partial_\gamma g_{(\gamma\beta)} + S_i + \frac{1}{2}e_{i\alpha\beta}\partial_\alpha T_\beta .$$

Equation (72) is true for ρ.

Comparison of the stress function solution with the Neuber solution

We substitute the compatible stress functions into the equations (42) and obtain:

(73)
$$u_i = \frac{h_i}{G} + \frac{1}{2Gl_1^2}B_i - \partial_i\left(\frac{\eta}{2G} + \frac{1}{4(1-\nu)G}x_\alpha h_\alpha + \frac{1}{2G}\partial_\alpha B_\alpha\right),$$

(74)
$$\varphi_i = \frac{1}{2GL^2c_3}\partial_i\overset{0}{g} + \frac{1}{2}e_{i\alpha\beta}\partial_\alpha\left(\frac{h_\beta}{G} + \frac{2+c_1}{2c_1l_1^2G}B_\beta\right),$$

with:

(75)
$$\eta = -\overset{0}{f} - \frac{1}{2}x_\alpha h_\alpha + \partial_\alpha C_\alpha + \partial_\alpha D_\alpha - \frac{\nu}{4(1-\nu)}x_\alpha e_{\alpha\beta\gamma}\partial_\beta(\partial_\lambda g_{(\lambda\gamma)} + T_\gamma) ,$$

and one has $\Delta\eta = 0$.

A comparison with the NEUBER Ansatz:

(19′)
$$\begin{cases} u_i = \Phi_i + \psi_i - \partial_i\left(\Phi_0 + \dfrac{1}{4(1-\nu)} x_\alpha \Phi_\alpha + l_1^2 \partial_\alpha \psi_\alpha\right), \\ \varphi_i = \partial_i \chi + \dfrac{1}{2} e_{i\alpha\beta} \partial_\alpha \left(\Phi_\beta + \dfrac{2+c_1}{c_1} \psi_\beta\right) \end{cases}$$

yields:

(76)
$$\varphi_0 = \frac{\eta}{2G}, \qquad \chi = -\frac{1}{2GL^2 c_3} \overset{0}{g}, \qquad \Phi_i = \frac{h_i}{G}, \qquad \psi_i = \frac{1}{2Gl_1^2} B_i .$$

With that, we have shown that the particular solution to the equation (40) that was constructed in the previous section is complete if the completeness of the NEUBER Ansatz (19) is known.

Example

We determine the stress state that belongs to the special Ansatz:

(77)
$$\overset{0}{g} = \overset{0}{g}(z), \qquad \overset{0}{f} = 0, \qquad S_i = T_i = C_i = B_i = 0; \quad f_{(ik)} = g_{(ik)} = 0.$$

Since $\overset{0}{g}(z)$ must be a solution to the differential equation:

(78)
$$\overset{0}{g}'' - \frac{1}{l_2^2} \overset{0}{g} = 0,$$

one must have:

(79)
$$\overset{0}{g}(z) = A e^{z/l_2} + B e^{-z/l_2},$$

with the two integration constants A and B. From (38), it then follows that:

(80)
$$\begin{cases} \sigma = \dfrac{c_1}{2L^2 l_3 c_3} (A e^{z/l_2} - B e^{-z/l_2}) \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\ \mu = -\dfrac{c_1}{(1+c_3) L^2} (A e^{z/l_2} + B e^{-z/l_2}) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \dfrac{1+c_3}{c_3} \end{bmatrix}. \end{cases}$$

We thus solve the boundary-value problem for an infinitely extended lamina of thickness $2d$ (Figure 1):

$$z = d: \qquad \mathbf{n} = \mathbf{e}_z, \qquad \mathbf{n} \cdot \boldsymbol{\sigma} = 0, \qquad \mathbf{n} \cdot \boldsymbol{\mu} = m\, \mathbf{e}_z ;$$

$$z = -d: \qquad \mathbf{n} = -\mathbf{e}_z, \qquad \mathbf{n}\cdot\boldsymbol{\sigma} = 0, \qquad \mathbf{n}\cdot\boldsymbol{\mu} = -m\,\mathbf{e}_z;$$

and obtain:

$$\sigma = -\frac{m}{2l_2}\frac{\sinh\zeta}{\cosh\delta}\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\mu = \frac{mc_3}{1+c_3}\frac{\cosh\zeta}{\cosh\delta}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \dfrac{1+c_3}{c_3} \end{bmatrix}, \qquad \zeta = \frac{z}{l_2}, \qquad \delta = \frac{d}{l_2}, \qquad -d \le \zeta \le d.$$

We see from Figure 2 that the stresses in a lamina whose thickness is large compared to the “material constant” l_2: ($\delta \gg 1$) drop off very quickly away from outer surface.

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Literature

1. H. NEUBER, “Ein neuer Ansatz zur Lösung räumlicher Probleme der Elastizitätstheorie,” ZAMM **14** (1934).
2. P. F. PAPKOWICH, “Solution générale des equations différentielles fondamentales d’elasticité, exprimée par trois fonctions harmoniques,” Compt. Rend. Acad. Sci., Paris **195** (1932).
3. H. SCHAEFER, “Die Spannungsfunktionen des dreidimensionalen Kontinuum; statische Deutung und Randwerte,” Ing.-Arch. **28** (1959).
4. H. NEUBER, “On the general solution of linear-elastic problems in isotropic and anisotropic Cosserat continua,” Proc. of the XI. Int. Cong. Appl. Mech., Springer-Verlag (1966).
5. H. SCHAEFER, “Analysis der Motorfelder im Cosserat-Kontinuums,” ZAMM **47**, pp. 319-328 (1967).
6. W. GÜNTHER, “Zur Statik und Kinematik des Cosserat-Kontinuums,” Abhandlung d. Brschw. Wiss. Ges. **X** (1958).
7. S. KESSEL, “Lineare Elastizitätstheorie des anistropen Cosserat-Kontinuum,” Abhandlung d. Brschw. Wiss. Ges. **XVI** (1964).
8. H. SCHAEFER, “Die Spannungsfunktionen eines Kontinuums mit Momentenspannungen I and II,” Bulletin de l’Académie Polonaise de Sciences, Ser. X, XI, **15**/1 (1967).

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