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The composition of continuous, finite, transformation groups

By

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Part One.

In contrast to my previous papers on transformation groups (*), in the present paper, I shall adhere to the notations of Lie exclusively, and refer those readers that are less familiar with those papers to the survey that Engel gave in his paper in volume 27 of Math. Ann (pp. 1, *et seq.*). My newly-introduced notations are also in accord with those of Lie and Engel, which is partially due to oral discussions with them. Thus, I denote those infinitesimal transformations by which an r-term group is determined symbolically by $X_1 f, \ldots, X_r f$ (by $X_1, \ldots, X_r$, where no misunderstanding can arise). An arbitrary infinitesimal transformation $\sum \eta_\iota X_\iota f$ is determined by r coefficients $\eta_1, \ldots, \eta_r$, and can be denoted by merely $(\eta_1, \ldots, \eta_r)$ or (η). Any infinitesimal transformation leads to a simply-infinite family of finite transformations, and conversely, any finite transformation can be obtained by repeating an infinitesimal one infinitely often. However, the finite transformations will be examined in the following investigations only in regard to the question of whether they do or do not commute. As long as one does not consider infinitesimals of higher order, infinitesimal transformations always commute. However, as long as a certain relation exists between two of them, any finite transformation that is derived from one of them will commute with any finite transformation that is obtained from the other one (Programm 1884, pp. 12). We therefore need fear no misunderstanding when we consider all transformations that emerge from the same infinitely-small one by repetition to be identical and refer to that family of finite transformations by means of the associated infinitesimal one. We then speak of a transformation $\sum \eta_\iota X_\iota f$ (perhaps in contrast to Lie).

In order for the r systems of differential equations $X_1 f, \ldots, X_r f$ to lead to a group, one must form the $r(r-1)/2$ expressions $(X_\iota X_\kappa)$ by a known prescription, and one must then have:

(*) "Erweiterung des Raumsbegriffes," Braunsberg, 1884. "Zur Theorie der Lie'schen Transformationsgruppen," Braunsberg, 1886.

Since the first paper was initially included in the directory of lectures for Winter 1884/85, while the other one was included in the directory for Summer 1886, the former shall be cited by Pr. 1884, and the latter by Pr. 1886.

$$(X_\iota X_\kappa) = \sum_\sigma c_{\iota\kappa\sigma} X_\sigma f .$$

Certain relationships must exist between the coefficients $c_{\iota\kappa\sigma}$ that one can derive from either the integration conditions or an equation that is almost self-explanatory for finite groups (*), and which is implied by the Jacobi relations:

$$[X_\iota (X_\kappa X_\lambda)] + [X_\kappa (X_\lambda X_\iota)] + [X_\lambda (X_\iota X_\kappa)] = 0$$

or by:

$$\sum_\rho \{c_{\kappa\lambda\rho} (X_\rho X_\iota) + c_{\lambda\iota\rho} (X_\rho X_\kappa) + c_{\iota\kappa\rho} (X_\rho X_\lambda)\} = 0.$$

Many properties of groups do not depend upon the transformations $X_1 f, \ldots, X_r f$ themselves, but are already derivable from the coefficients $c_{\iota\kappa\lambda}$. Thus, when two groups possess the same coefficients $c_{\iota\kappa\lambda}$, they will be referred to as *equally-composed*. However, since one can alter the $c_{\iota\kappa\lambda}$ in such a way that one replaces the $X_1 f, \ldots, X_r f$ with r mutually-independent, homogeneous linear functions, one will have to refer to two r-term groups as *equally-composed* when they either possess the same coefficients $c_{\iota\kappa\lambda}$ or when one can bring about that equality by a suitable choice of the determining infinitesimal transformations.

As long as the r coefficients $c_{\iota\kappa 1}, \ldots, c_{\iota\kappa r}$ do not all vanish for given values of ι and κ; $\sum_\rho c_{\iota\kappa\rho} X_\rho f$ will once more represent a transformation. It is obvious that at most r of the $r(r-1)/2$ transformations that are obtained in that way will be mutually-independent; in many cases, they can be represented by a smaller number. I have (Programm 1886, pp. 7 and 8) denoted the number of the mutually-independent transformations thus-obtained by p and proved that these p transformations determine an invariant subgroup that exhibits some entirely remarkable properties. One cannot avoid giving these subgroups a special name, and I refer to them as *principal subgroups*. Thus, a group for which $p = r$ is its own principal subgroup; by contrast, a group with nothing but commuting transformations possesses no principal subgroup. It is known that any two-parameter group whose transformations do not commute always has a one-parameter principal subgroup; sometimes that subgroup is referred to as its *principal element*.

One will arrive at second number that is characteristic of the group by the following consideration: The problem of determining the two-parameter subgroups in which a given transformation $(\eta_1, \ldots, \eta_r)$ exists leads to an equation of degree r:

$$\omega^r - \omega^{r-1} \psi_1(\eta) + \omega^{r-2} \psi_2(\eta) - \ldots \pm \omega\, \psi_{r-1}(\eta) = 0,$$

in which the $\psi_\lambda(\eta)$ are homogeneous functions of degree λ in $\eta_1, \ldots, \eta_r$, and in which the absolute term vanishes. The question then arises of how many of the functions $\psi_1, \ldots, \psi_{r-1}$ are mutually-independent, in such a way that all of the other ones could be expressed in terms of them. The number of the mutually-independent ones, which is always smaller than $r - 1$ for $r > 2$, shall be denoted by l and is called the *rank* of the

(*) Engel, "Beiträge zur Gruppentheorie," Leipziger Berichte (1887), pp. 89, *et seq*.

group. If $l > 1$ then not only will the coefficients $\psi_{r-1}, \ldots, \psi_{r-l+1}$ vanish identically, but also all sub-determinants of degree $r - l +$ of a certain determinant that is closely connected with the equation above. Thus, for $l > 1$, an arbitrary finite transformation will commute with not only the ones that can be derived from its infinitely-small transformation, but with some other families, and the associated infinitely-small transformations determine an $(l - 1)$-term subgroup.

The proofs of these theorems, and many other relations in which the number l stands for a property of the group, are based, in one case, on a system of formulas that are derived from the Jacobi relations by simple operations. A certain canonical form in which most groups can be represented leads to a second proof. One arrives at that form in the following way: One starts from an entirely arbitrary infinitesimal transformation (η), poses the equation above for it:

$$\omega^r - \omega^{r-1}\,\psi_1(\eta) + \omega^{r-2}\,\psi_2(\eta) - \ldots \pm \omega\,\psi_{r-1}(\eta) = 0,$$

and determines its roots. In general, it has m vanishing roots, so $\psi_{r-1}, \ldots, \psi_{r-m+1}$ vanish identically, and one can still determine $m - 1$ transformations that are independent of each other, as well as the transformation (η), and which commute with the given one and each other. One chooses the given transformation to be, say, $X_r f$, and the ones that are found to be $X_{r-1} f, \ldots, X_{r-m+1} f$. Now, the $r - m$ non-vanishing roots $\omega_1, \ldots, \omega_{r-m}$ can initially be different from each other. Thus, one can choose $r - m$ transformations $X_1, \ldots, X_{r-m}$ in such a way that:

$$(X_r\,X_1) = \omega_1\,X_1 f, \qquad (X_r\,X_2) = \omega_2\,X_2 f, \qquad \ldots, \qquad (X_r\,X_{r-m}) = \omega_{r-m}\,X_{r-m} f.$$

However, the following equations will likewise be true now:

$$(X_{r-1}\,X_1) = \omega_1'\,X_1 f, \qquad (X_{r-1}\,X_2) = \omega_2'\,X_2 f, \qquad \ldots, \qquad (X_{r-1}\,X_{r-m}) = \omega_{r-m}'\,X_{r-m} f,$$
$$(X_{r-2}\,X_1) = \omega_1''\,X_1 f, \qquad (X_{r-2}\,X_2) = \omega_2''\,X_2 f, \qquad \ldots, \qquad (X_{r-2}\,X_{r-m}) = \omega_{r-m}''\,X_{r-m} f,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

In order to characterize the composition of the group, one must still give the expressions $(X_\iota\,X_\kappa)$ for $\iota, \kappa = 1, \ldots, m$. However, one deduces that immediately from the roots ω, namely, $(X_\iota\,X_\kappa) = c_{\iota\kappa\lambda}\,X_\lambda f$ when $\omega_\iota + \omega_\kappa = \omega_\lambda$, and:

$$(X_\iota\,X_\kappa) = (c_{\iota\kappa r}\,X_r + c_{\iota\kappa r-1}\,X_{r-1} + \ldots + c_{\iota\kappa,\,r-m+1}\,X_{r-m+1})\,f$$

for

$$\omega_\iota + \omega_\kappa = 0.$$

Clearly, the same thing will be true for the roots ω_κ', ω_κ'', … In that way, we will be led to look for certain relations between the roots.

One obtains a similar representation for equal roots ω, such that no essential difference enters for that case. The case in which a transformation of the group commutes with all other transformations, which was excluded from the present sketch, requires special treatment. Meanwhile, the required alteration can also be obtained easily here. By contrast, the present representation of the group will become completely impossible when all of the functions $\psi_1, \ldots, \psi_{r-1}$ vanish identically. I shall also present a certain simple form from which many properties of the group can be easily observed, and which can then be considered to be truly canonical. However, this form does not have the distinguishing properties that are special to the form that was described just now, and which come from the fact that most Jacobi relations are fulfilled by themselves, and among the ones that are not satisfied identically, the independent ones separate from the dependent ones immediately. Those groups belong to a class that Engel considered quite recently [Leipziger Berichte (1887), pp. 95, *et seq.*], namely, the one in which no three-parameter subgroup with the composition of the general projective group contains the simply-extended manifold (conic section groups). He proved that any such group contains an $(r-1)$-parameter invariant subgroup that once again contains an $(r-2)$-parameter invariant subgroup in it, etc. In addition to the groups of rank zero, this class that Engel considered also contains the groups for which all of the functions $\psi_1, \ldots, \psi_{r-1}$ can be represented by linear functions $\eta_1, \ldots, \eta_r$. Engel's theorem follows for the latter from § 4 of our paper, and the statement and proof of it was already known to me. I first found the presentation in § 9 for $l = 0$ after Engel had communicated his theorem to me and sketched out the proof; meanwhile, he had still not communicated Lie's presentation of those groups at the time. On the one hand, I came to my own presentation more easily, while on the other, I brought it into great agreement with that of Lie.

Only the ratios $\eta_1 : \eta_2 : \ldots : \eta_r$ are necessary in order to define the infinitely-small transformation $\sum \eta_\nu X_\nu f$, so it is permissible to multiply all η_ι with the same non-zero number. Along with Lie and Engel (*), we thus associate all of the infinitesimal transformations of the group uniquely and continuously with the points of an $(r-1)$-dimensional projective space and consider $\eta_1, \ldots, \eta_r$ to be the homogeneous coordinates of the associated point that is the image point of the transformation $\sum \eta_\iota X_\iota f$. When the transformations $\sum \eta_\iota X_\iota f$ and $\sum \eta'_\iota X_\iota f$ do not commute, the operation $(\sum \eta_\iota X_\iota, \sum \eta'_\iota X_\iota)$ will again lead to a transformation, and its image point will be referred to as the *product* of the points (η) and (η').

For the time being, I shall present only part of my investigations; I think that the continuation will follow soon.

§ 1.

A remarkable system of equations between the coefficients $c_{\iota\kappa\lambda}$.

As was mentioned already in the introduction, we start with the equations:

$$(X_\iota X_\kappa) = \sum_\sigma c_{\iota\kappa\sigma} X_\sigma f, \tag{1}$$

$$\sum_\rho \{c_{\kappa\lambda\rho}(X_\rho X_\iota) + c_{\lambda\iota\rho}(X_\rho X_\kappa) + c_{\iota\kappa\rho}(X_\rho X_\lambda)\} = 0. \tag{2}$$

(*) Engel, "Zur Theorie der Zusammensetzung der endlichen continuierlichen Transformations-Gruppen," Leipziger Berichte (1886), 83-94.

Our problem will be to draw conclusions about the composition of groups from the system of equations between the $c_{\iota\kappa\rho}$ that are represented by these two equations. A certain system of equations that can be obtained easily from the originally-given one by mere summations will yield one series of conclusions. That system of equations is:

(3) $$\sum_{\kappa,\lambda} c_{\alpha\beta\kappa} c_{\kappa\lambda\lambda} = 0,$$

(4) $$\sum_{\kappa,\lambda,\mu} (c_{\alpha\beta\kappa} c_{\mu\gamma\lambda} + c_{\alpha\gamma\kappa} c_{\mu\beta\lambda})\, c_{\kappa\lambda\mu} = 0,$$

(5) $$\sum_{\kappa,\lambda,\mu,\nu} (c_{\alpha\beta\kappa} c_{\mu\gamma\nu} c_{\gamma\delta\lambda} + c_{\alpha\gamma\kappa} c_{\mu\delta\nu} c_{\nu\beta\lambda} + c_{\alpha\delta\kappa} c_{\mu\beta\nu} c_{\nu\gamma\lambda})\, c_{\kappa\lambda\mu} = 0,$$

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(6) $$\sum_{\iota,\kappa_1,\kappa_2,\cdots,\kappa_s} (c_{\alpha\beta_1\iota} c_{\beta_2\kappa_2\kappa_3} c_{\beta_3\kappa_3\kappa_4} \cdots c_{\beta_s\kappa_s\kappa_1} + c_{\alpha\beta_2\iota} c_{\beta_3\kappa_2\kappa_3} c_{\beta_4\kappa_2\kappa_4} \cdots c_{\beta_1\kappa_s\kappa_1} + \ldots$$
$$\ldots + c_{\alpha\beta_3\iota} c_{\beta_1\kappa_2\kappa_3} c_{\beta_2\kappa_3\kappa_4} \cdots c_{\beta_{s-1}\kappa_s\kappa_1})\, c_{\iota\kappa_1\kappa_2} = 0.$$

In equation (6), the summation extends over the $s + 1$ symbols $\iota, \kappa_1, \kappa_2, \ldots, \kappa_s$, which take the values $1, \ldots, r$; $\alpha, \beta_1, \beta_2, \ldots, \beta_s$ are fixed symbols. There are s products in the brackets, each of which is obtained from the following one when one replaces $\beta_1, \beta_2, \ldots, \beta_s$ with $\beta_2, \beta_3, \ldots, \beta_1$ cyclically.

The first of these equations was already stated and proved by Engel (*); the remaining ones have not been published up to now.

The proof of formula (6) emerges quite clearly from that. We start with the proofs of formulas (4) and (5) and repeat the proof that Engel gave for formula (3).

From equation (2), one has:

$$\sum_{\kappa} (c_{\alpha\beta\kappa} c_{\kappa\lambda\lambda} + c_{\beta\lambda\kappa} c_{\kappa\alpha\lambda} + c_{\lambda\alpha\kappa} c_{\kappa\beta\lambda}) = 0.$$

If we sum over λ then the last two products will drop out, and we will obtain formula (3).

In order to establish the relation (4), we start with the two equations:

$$\sum_{\kappa,\lambda,\mu} (c_{\alpha\beta\kappa} c_{\kappa\lambda\mu} + c_{\beta\lambda\kappa} c_{\kappa\alpha\mu} + c_{\lambda\alpha\kappa} c_{\kappa\beta\mu})\, c_{\gamma\mu\lambda} = 0,$$

$$\sum_{\kappa,\lambda,\mu} (c_{\alpha\gamma\kappa} c_{\kappa\lambda\mu} + c_{\gamma\lambda\kappa} c_{\kappa\alpha\mu} + c_{\lambda\alpha\kappa} c_{\kappa\gamma\mu})\, c_{\beta\mu\lambda} = 0,$$

and add them. Now, one has:

(*) Leipziger Berichte (1886), pp. 89.

$$\sum c_{\beta\lambda\kappa}c_{\kappa\alpha\mu}c_{\gamma\mu\lambda} + \sum c_{\lambda\alpha\kappa}c_{\kappa\gamma\mu}\, c_{\beta\mu\lambda} = 0,$$

as one sees immediately when one replaces the summation symbols κ, λ, μ with μ, κ, λ, resp., in the second sum. One likewise has:

$$\sum c_{\beta\lambda\kappa}c_{\kappa\alpha\mu}c_{\gamma\mu\lambda} + \sum c_{\lambda\alpha\kappa}c_{\kappa\gamma\mu}\, c_{\beta\mu\lambda} = 0,$$

in which κ, λ, μ is replaced with μ, κ, λ, resp., in the second sum. With that, one gets equations (4).

One arrives at equation (5) in the following way: One forms the equation:

$$\sum_{\iota,\kappa,\lambda,\mu}(c_{\alpha\beta\iota}c_{\iota\kappa\lambda} + c_{\beta\kappa\iota}c_{\iota\alpha\lambda} + c_{\kappa\alpha\iota}c_{\iota\beta\lambda})\, c_{\gamma\lambda\mu}c_{\delta\mu\kappa} = 0$$

and then defines the two equations that one obtains when one replaces β, γ, δ with γ, δ, β, resp., and then with δ, β, γ, resp. The left-hand sides of each of these equations contain three summations (over ι, κ, λ, μ). The third sum in the first row will be equal and opposite to the second sum of the second equation. Likewise, the third sum in the second row will be equal and opposite to the second sum in the third equations and the third sum in the third equations will be equal and opposite to the second sum in the first one. If one then adds the three equations then only the first sums will remain, and that will yield equation (5).

In order to prove (6), one forms the equation:

$$\sum_{\iota,\kappa_1,\kappa_2,\cdots,\kappa_s}(c_{\alpha\beta_1\iota}c_{\iota\kappa_1\kappa_2} + c_{\beta_1\kappa_1\iota}c_{\iota\alpha\kappa_2} + c_{\kappa_1\alpha\iota}c_{\iota\beta_1\kappa_2})c_{\beta_1\kappa_2\kappa_3}c_{\beta_1\kappa_3\kappa_4}\cdots c_{\beta_s\kappa_s\kappa_1} = 0.$$

One will define the same equation when one replaces $\beta_1, \beta_2, \ldots, \beta_s$ with one of the cyclic permutations $\beta_2, \beta_3, \beta_4, \ldots, \beta_1$; $\beta_3, \beta_4, \beta_5, \ldots, \beta_2$; …; $\beta_s, \beta_1, \beta_2, \ldots, \beta_{s-1}$. Now:

$$\sum c_{\beta_1\kappa_1\iota}c_{\iota\alpha\kappa_2}c_{\beta_2\kappa_2\kappa_3}c_{\beta_3\kappa_3\kappa_4}\cdots c_{\beta_3\kappa_3\kappa_1}$$

is equal and opposite to:

$$\sum c_{\kappa_1\alpha\iota}c_{\iota\beta_2\kappa_2}c_{\beta_2\kappa_2\kappa_3}c_{\beta_4\kappa_3\kappa_4}\cdots c_{\beta_1\kappa_3\kappa_1}\ ,$$

as one will see immediately when one replaces the summation symbols i, κ_1, κ_2, κ_3, κ_4, … κ_s in the second sum with κ_1, κ_s, ι, κ_2, κ_3, κ_4, … κ_{s-1}, resp. Thus, the second sum in each equation will be equal and opposite to the third sum in the following equations, and the second sum of the s^{th} equation will be equal and opposite to the third sum in the first one. If one then adds the s equations thus-defined then only the first in each of them will remain, such that the validity of (6) will be proved.

§ 2.

The invariants of a certain linear group that is adjoint to the given group.

Every group of transformations is closely related to two groups of linear transformations. For $\kappa = 1, \ldots, r$, the individual infinitesimal transformations of the one group will be represented by the system of equations:

$$dx_\iota = dt \sum_\rho c_{\rho\kappa\iota} x_\rho ,$$

or, with Lie's symbolic notation, by the r equations:

$$\sum_{\iota,\rho} c_{\rho\kappa\iota} x_\rho \frac{\partial f}{\partial x_\iota} .$$

Lie and Engel (*) referred to this group as the *adjoint linear group*.

A second one is:

$$d\eta_\iota = dt \sum_\rho \eta_\rho c_{\iota\kappa\rho} \qquad \text{or} \qquad \sum_{\iota,\rho} \eta_\rho c_{\iota\kappa\rho} \frac{\partial f}{\partial \eta_\iota} .$$

It will be referred to as the *second adjoint linear group*. Both of them are composed in the same way as the given one.

I have addressed the invariants of the second adjoint group previously (Programm, 1886); I will now derive the invariants of the other one.

To that end, I pose the equation:

$$(7) \qquad \begin{vmatrix} \sum_\rho \eta_\rho c_{\rho 11} - \omega & \sum_\rho \eta_\rho c_{\rho 11} & \cdots & \sum_\rho \eta_\rho c_{\rho r1} \\ \sum_\rho \eta_\rho c_{\rho 12} & \sum_\rho \eta_\rho c_{\rho 22} - \omega & \cdots & \sum_\rho \eta_\rho c_{\rho r2} \\ \cdots & \cdots & \cdots & \cdots \\ \sum_\rho \eta_\rho c_{\rho 1r} & \sum_\rho \eta_\rho c_{\rho 2r} & \cdots & \sum_\rho \eta_\rho c_{\rho rr} - \omega \end{vmatrix} = 0.$$

For the sake of brevity, we set:

$$(8) \qquad \sum \eta_\rho c_{\rho\kappa\lambda} = \gamma_{\kappa\lambda}(\eta) = \gamma_{\kappa\lambda} ,$$

and then the foregoing equation can also be written:

(*) In a letter. Confer Engel, Leipziger Berichte (1886), pp. 88.

$$\begin{vmatrix} \gamma_{11}-\omega & \gamma_{21} & \cdots & \gamma_{r1} \\ \gamma_{12} & \gamma_{22}-\omega & \cdots & \gamma_{r2} \\ \cdots & \cdots & \cdots & \cdots \\ \gamma_{1r} & \gamma_{2r} & \cdots & \gamma_{rr}-\omega \end{vmatrix} = 0.$$

Since $c_{\iota\kappa\lambda} + c_{\kappa\iota\lambda} = 0$, one also has:

$$\sum_\sigma \eta_\sigma \gamma_{\sigma\lambda} = \sum_{\rho,\sigma} \eta_\rho \eta_\sigma c_{\rho\sigma\lambda} = 0,$$

and therefore the determinant of the $\gamma_{\iota\kappa}$ will vanish identically; that is:

(9) $$|\gamma_{\iota\kappa}| = 0.$$

If we then develop equation (7) in powers of ω then we will obtain:

(10) $$\omega^r - \psi_1(\eta)\cdot\omega^{r-1} + \psi_2(\eta)\cdot\omega^{r-2} - \psi_3(\eta)\cdot\omega^{r-3} + \ldots \pm \omega\,\psi_{r-1}(\eta) = 0.$$

For $\nu = 1, \ldots, r - 1$, each ψ_ν will be a homogeneous function of degree ν in the quantities $\eta_1, \ldots, \eta_r$ when it does not vanish identically.

One now has the theorem:

The coefficients $\psi_1(\eta), \psi_2(\eta), \ldots, \psi_{r-1}(\eta)$ *in equation* (10) *are invariant functions of the adjoint linear group,*

or in other words:

The determinant:

$$\begin{vmatrix} \sum_\rho \eta_\rho c_{\rho11}-\omega & \sum_\rho \eta_\rho c_{\rho11} & \cdots & \sum_\rho \eta_\rho c_{\rho r1} \\ \sum_\rho \eta_\rho c_{\rho12} & \sum_\rho \eta_\rho c_{\rho22}-\omega & \cdots & \sum_\rho \eta_\rho c_{\rho r2} \\ \cdots & \cdots & \cdots & \cdots \\ \sum_\rho \eta_\rho c_{\rho1r} & \sum_\rho \eta_\rho c_{\rho2r} & \cdots & \sum_\rho \eta_\rho c_{\rho rr}-\omega \end{vmatrix}$$

will not change for an arbitrary of ω *when one subjects it to any transformation of the group that is determined by the r infinitely-small transformations:*

$$\mathrm{H}_\alpha f = \sum_{\rho,\iota} \eta_\rho c_{\rho\alpha\iota} \frac{\partial f}{\partial \eta_\iota}.$$

We first prove this theorem for the lowest-indexed functions ψ_1, ψ_2, ψ_3. One has $\psi_1 = \sum_{\iota,\sigma} \eta_\iota c_{\iota\sigma\rho}$. In order to apply the infinitesimal transformation $\mathrm{H}_\alpha f$, one must replace $d\eta_\iota$ with $dt \sum_\rho \eta_\rho c_{\rho\alpha\iota}$; One will then have:

$$d\psi_1 = -\,dt \sum_{\iota,\rho,\sigma} \eta_\rho c_{\rho\alpha\iota} c_{\iota\sigma\sigma},$$

in which, the coefficient of each η_ρ will vanish as a result of (3). Since the function ψ_1 remains unchanged by each infinitesimal transformation of the group, it will also not change for any transformation that belongs to the group.

The coefficient ψ_2 of ω^{r-2} is:

$$\underset{(\mu\lambda)}{S}\,(\gamma_{\lambda\lambda}\,\gamma_{\mu\mu} - \gamma_{\lambda\mu}\,\gamma_{\mu\lambda}),$$

in which the various combinations $(\lambda\mu)$ are to be taken from the set $1, \ldots, r$. In place of this, we can set:

$$\begin{aligned}\psi_2 &= \tfrac{1}{2} \sum_{\lambda,\mu} (\gamma_{\lambda\lambda}\,\gamma_{\mu\mu} - \gamma_{\lambda\mu}\,\gamma_{\mu\lambda}), \\ &= \tfrac{1}{2} \sum_{\gamma,\kappa,\lambda,\mu} \eta_\gamma\,\eta_\kappa\,(c_{\gamma\lambda\lambda}\,c_{\kappa\mu\mu} - c_{\kappa\lambda\mu}\,c_{\gamma\mu\lambda}).\end{aligned}$$

Thus, when one applies the infinitesimal transformation $\mathrm{H}_\alpha f$:

$$d\psi_2 = -\,\tfrac{1}{2} dt \sum_{\beta,\gamma,\kappa,\lambda,\mu} \eta_\beta\,\eta_\gamma\,c_{\alpha\beta\kappa}\,(c_{\gamma\lambda\lambda}\,c_{\kappa\mu\mu} - c_{\kappa\lambda\mu}\,c_{\gamma\mu\lambda}).$$

Here, the coefficient of $-\,\tfrac{1}{2}\,dt \cdot \eta_\beta\,\eta_\gamma$ is:

$$\sum_{\kappa,\lambda,\mu} \{(c_{\alpha\beta\kappa}\,c_{\gamma\lambda\lambda}\,c_{\kappa\mu\mu} + c_{\alpha\gamma\kappa}\,c_{\gamma\lambda\lambda}\,c_{\kappa\mu\mu}) - c_{\kappa\lambda\mu}\,(c_{\alpha\beta\kappa}\,c_{\gamma\mu\lambda} + c_{\alpha\gamma\kappa}\,c_{\beta\mu\lambda})\}.$$

Each of the first two sums vanishes because of relation (3), while the remaining part vanishes because of (4). One then has, in fact, $d\psi_2(\eta) = 0$.

In a completely corresponding way, one has:

$$\psi_3 = \tfrac{1}{6} \sum_{\gamma,\delta,\kappa,\lambda,\mu,\nu} \eta_\gamma \eta_\delta \eta_\kappa \begin{vmatrix} c_{\kappa\lambda\lambda} & c_{\kappa\lambda\mu} & c_{\kappa\lambda\nu} \\ c_{\gamma\mu\lambda} & c_{\gamma\mu\mu} & c_{\gamma\mu\nu} \\ c_{\delta\nu\lambda} & c_{\delta\nu\mu} & c_{\delta\nu\nu} \end{vmatrix},$$

so, when one applies the same infinitesimal transformation:

$$d\psi_3 = -\tfrac{1}{6}dt \sum_{\substack{\beta,\gamma,\delta,\\ \kappa,\lambda,\mu,\nu}} \eta_\gamma\eta_\delta\eta_\kappa c_{\alpha\beta\kappa} \begin{vmatrix} c_{\kappa\lambda\lambda} & c_{\kappa\lambda\mu} & c_{\kappa\lambda\nu} \\ c_{\gamma\mu\lambda} & c_{\gamma\mu\mu} & c_{\gamma\mu\nu} \\ c_{\delta\nu\lambda} & c_{\delta\nu\mu} & c_{\delta\nu\nu} \end{vmatrix}.$$

Up to the part that vanishes because of the relations (3) and (4), the coefficient of:

$$\tfrac{1}{6}dt\,\eta_\beta\eta_\gamma\eta_\delta$$

is equal to:

$$\sum_{\kappa,\lambda,\mu,\nu} c_{\kappa\lambda\mu}(c_{\alpha\beta\kappa}c_{\gamma\mu\nu}c_{\delta\nu\lambda} + \cdots),$$

in which the additional terms in the bracket are the ones that are obtained from the one that is written by permuting $\beta\gamma\delta$. Those permutations can be decomposed into two sequences of cycles; the sum over each cycle will vanish due to (5); one will then have $d\psi_3 = 0$.

We can proceed in the same way. We have:

$$\psi_s(\eta) = \frac{1}{s!}\sum \eta_\iota\eta_{\beta_2}\eta_{\beta_3}\cdots\eta_{\beta_s} \begin{vmatrix} c_{\iota\kappa_1\kappa_1} & c_{\iota\kappa_1\kappa_2} & \cdots & c_{\iota\kappa_1\kappa_s} \\ c_{\beta_2\kappa_2\kappa_1} & c_{\beta_2\kappa_2\kappa_2} & \cdots & c_{\beta_2\kappa_2\kappa_s} \\ \cdots & \cdots & \cdots & \cdots \\ c_{\beta_s\kappa_s\kappa_1} & c_{\beta_s\kappa_s\kappa_2} & \cdots & c_{\beta_s\kappa_s\kappa_s} \end{vmatrix}.$$

If we then once more apply the infinitesimal transformation:

$$d\eta_\iota = -\,dt \sum_{\beta_1} \eta_{\beta_1} c_{\alpha\beta_1\iota}$$

then, up to terms whose vanishing will follow from eq. (3), (4), (5) and the corresponding ones for 4, 5, …, $s - 1$, the coefficient of:

$$-\frac{dt}{s!}\eta_{\beta_1}\cdots\eta_{\beta_s}$$

will be equal to:

$$c_{\iota\kappa_1\kappa_2}(c_{\alpha\beta_1\iota}c_{\beta_2\kappa_2\kappa_3}c_{\beta_3\kappa_2\kappa_4}\cdots c_{\beta_s\kappa_s\kappa_1} + \cdots),$$

in which the additional terms in the bracket are the ones that are obtained from the product that was written down by permuting $\beta_1\ \beta_2\ \ldots\ \beta_s$. Since all permutations decompose into a sequence of cycles, the theorem will follow from (6). (*)

(*) Engel, to whom I communicated the present theorem, along with the proof that was given here, at the end of July, 1886, has provided another proof, which I will describe here in words:

"I set:

Of the functions $\psi_1, \ldots, \psi_{r-1}$ that are considered here, the function ψ_1 has already been considered by Lie for a long time. Namely, he posed the theorem (*) that when not all expressions $\sum c_{\alpha\lambda\lambda}$ vanish (so when the function $\psi_1(\eta)$ does not vanish identically), the group will possess an $(r-1)$-parameter invariant subgroup, and it will be represented by $\psi_1(\eta) = 0$. It then follows from this that (Pr., 1886, pp. 11) the number p that I introduced is less than r. Engel has provided a very simple proof for the two aforementioned theorems (**). One can give its proof a form that is closely connected with the method that is applied in the following investigations. Hence, I do not consider it to be superfluous to derive another proof from formula (3), upon which Engel also based his own, except that I will prove the theorem in the opposite sequence, and first show that:

If $\psi_1(\eta)$ does not vanish identically then the coordinates $\zeta_1, \ldots, \zeta_r$ of the transformations to which one will arrive by the operation:

$$B_\kappa f = \sum_{i,j}^{1-r} c_{jki} y_j \frac{\partial f}{\partial y_i}, \qquad \sum_{j=1}^{r} c_{jki} y_j - \varepsilon_{ki}\,\omega = u_{ki},$$

in which $\varepsilon_{ik} = 0$ for $i \neq k$, and $\varepsilon_{ii} = 1$. Then let:

$$\Delta = \sum \pm u_{11} \ldots u_{rr};$$

it shall be proved that $B_k\,\Delta \equiv 0$.

I find that:

$$B_s\,\Delta = \sum_{k,i}^{1-r} \frac{\partial \Delta}{\partial u_{\kappa i}} B_s u_{ki},$$

$$B_s\,u_{ki} = \sum_{\nu,j}^{1-r} c_{jki} c_{\nu sj} y_\nu,$$

$$\sum_{j=1}^{r} (c_{\nu sj} c_{jki} + c_{ski} c_{j\nu i} + c_{k\nu i} c_{jsi}) = 0,$$

so:

$$B_s\,u_{ki} = \sum_{j=1}^{r} c_{skj} \sum_{\nu=1}^{r} c_{\nu ji} y_\nu + \sum_{j=1}^{r} c_{jsi} \sum_{\nu=1}^{r} c_{\nu kj} y_\nu,$$

$$= \sum_{j=1}^{r} c_{skj} (u_{ji} + \varepsilon_{ji}\omega) + \sum_{j=1}^{r} c_{jsi} (u_{kj} + \varepsilon_{kj}\omega);$$

since $(c_{ski} + c_{ksi})\,\omega$ vanishes:

$$B_s\,u_{ki} = \sum_{j=1}^{r} (c_{skj} u_{ji} + c_{jsi} u_{kj}),$$

$$B_s\,\Delta = \sum_{k,j=1}^{r} c_{skj} \sum_{i=1}^{r} u_{ji} \frac{\partial \Delta}{\partial u_{ki}} + \sum_{i,j=1}^{r} c_{jsi} \sum_{k=1}^{r} u_{kj} \frac{\partial \Delta}{\partial u_{ki}},$$

$$= \Delta \sum_{k,j=1}^{r} c_{skj}\,\varepsilon_{jk} + \Delta \sum_{i,j=1}^{r} c_{jsi}\,\varepsilon_{ji}$$

$$= \Delta \left\{ \sum_{k=1}^{r} c_{skk} + \sum_{i=1}^{r} c_{isi} \right\} \equiv 0.\text{”}$$

(*) Archiv for Math. og Naturw. X, pp. 88.
(**) Leipziger Berichte, 1886, pp. 89.

$$\left(\sum \eta_\iota X_\iota, \sum \eta'_\iota X_\iota\right) = \sum \zeta_\rho X_\rho f$$

for entirely arbitrary values of η and η' will satisfy the equation:

$$\psi_1(\zeta) = 0.$$

In fact, let $\sum \eta_\iota X_\iota f$ and $\sum \eta'_\iota X_\iota f$ be two arbitrary infinitesimal transformations, and let:

$$\left(\sum \eta_\iota X_\iota, \sum \eta'_\iota X_\iota\right) = \sum \zeta_\rho X_\rho f\,,$$

so

$$\zeta_\rho = \sum_{\iota,\kappa} \eta_\iota \eta'_\kappa c_{\iota\kappa\rho}\,,$$

and:

$$\psi_1(\zeta) = \sum_{\rho,\sigma} \zeta_\rho c_{\rho\sigma\sigma} = \sum_{\iota,\kappa,\rho,\sigma} \eta_\iota \eta'_\kappa c_{\iota\kappa\rho} c_{\rho\sigma\sigma} = 0,$$

as a result of relation (3).

The image point of each infinitesimal transformation to which one will arrive by the operation $\left(\sum \eta_\iota X_\iota, \sum \eta'_\iota X_\iota\right)$ or by multiplying the points η and η' will always belong to the plane $\psi_1(\eta) = 0$ then. Thus, the image point will also belong to that plane when one of the given points is assumed to be on it; i.e., that plane represents an invariant subgroup. However, that is the theorem that was first proved by Lie.

§ 3.

Derivation of an equation that is characteristic of a group.

The previous two paragraphs had the goal of making it possible for us to be able to properly explain the relevance of the following developments. For that reason, I have also presented equation (7) without mentioning the problem that led to that equation in a natural way. We shall now do that, and it will then be possible for us to deduce some further consequences of the theorems that were proved in the previous paragraphs.

Any two-parameter group whose transformations do not commute with each other contains a one-parameter principal subgroup, and when $X_1 f$ represents it for the group that is determined by $X_1 f$ and $X_2 f$ then one will have:

$$(X_1 X_2) = \omega X_1 f,$$

in which ω is non-zero.

We now look for the two-parameter subgroups of the given group $(X_1, X_2, \ldots, X_r)$ in which a given infinitesimal transformation $(\eta_1, \ldots, \eta_r)$ is included. Initially, the transformations of the desired two-parameter groups should not commute, and $(\zeta_1, \ldots, \zeta_r)$ should represent the one-parameter subgroup. The problem then arises of looking for all two-parameter subgroups to which the given transformations $\eta_1, \dots, \eta_r$ belong, and determining $r + 1$ quantities $\zeta_1, \dots, \zeta_r$, ω such that the following equation is valid:

$$(11) \qquad \left(\sum \eta_\iota X_\iota, \sum \zeta_\iota X_\iota\right) = \omega \sum \zeta_\iota X_\iota f .$$

This equation also yields the two-parameter subgroups with commuting transformations, since one would then have $\omega = 0$. One must start with another equation only when the given transformation $(\eta_1, \dots, \eta_r)$ represents the principal subgroup for a two-parameter subgroup. However, we will show that this is possible only for distinguished positions of the point $(\eta_1, \dots, \eta_r)$, so we can omit that case.

When we replace the values in equation (11) with the ones in (1), we will obtain an equation that decomposes into the following r equations:

$$(12) \qquad \sum_{\iota,\kappa} \eta_\iota\, \zeta_\kappa\, c_{\iota\kappa\rho} = \omega\, \zeta_\rho .$$

In order to solve this equation, it is known that one must first calculate ω by means of the equation:

$$(7) \qquad \begin{vmatrix} \sum \eta_\rho c_{\rho 11} - \omega & \sum \eta_\rho c_{\rho 21} & \cdots & \sum \eta_\rho c_{\rho r1} \\ \sum \eta_\rho c_{\rho 12} & \sum \eta_\rho c_{\rho 22} - \omega & \cdots & \sum \eta_\rho c_{\rho r2} \\ \cdots & \cdots & \cdots & \cdots \\ \sum \eta_\rho c_{\rho 1r} & \sum \eta_\rho c_{\rho 2r} & \cdots & \sum \eta_\rho c_{\rho rr} - \omega \end{vmatrix} = 0.$$

This is the same equation that we presented before, as well as writing it in the form:

$$(10) \qquad \omega^r - \omega^{r-1}\psi_1(\eta) + \omega^{r-2}\psi_2(\eta) - \dots \pm \omega\, \psi_{r-1}(\eta) = 0.$$

Each non-zero root ω determines a system of quantities ζ that satisfy eq. (12), and therefore also eq. (11). However, when ω is an α-fold non-zero root, and all sub-determinants of degree $r - \alpha + 1$ on the left-hand side of (7) vanish for that value of ω, α can determine α systems $\zeta, \zeta', \zeta'', \dots, \zeta^{(\alpha-1)}$ in such a way that for any arbitrary coefficients $\lambda, \lambda_1, \lambda_2, \dots, \lambda_{\alpha-1}$, the following equation will be true:

$$\left(\sum \eta_\iota X_\iota, \sum\left(\lambda\zeta_\iota + \lambda_1\zeta'_\iota + \lambda_2\zeta'_\iota + \cdots + \lambda_{\alpha-1}\zeta_1^{(\alpha-1)}\right) X_\iota\right)$$
$$= \omega \sum_\iota \left(\lambda\zeta_\iota + \lambda_1\zeta'_\iota + \lambda_2\zeta'_\iota + \cdots + \lambda_{\alpha-1}\zeta_1^{(\alpha-1)}\right) X_\iota f .$$

In this case, an α-dimensional plane will go through the image point of the given transformation $(\eta_1, \dots, \eta_r)$, and every line in that plane that is drawn through that point will represent a two-parameter subgroup.

Furthermore, if ω is a non-zero multiple root for which at least all sub-determinants of degree $r - 1$ do not vanish then that will yield only a single two-parameter subgroup.

However, one can determine α systems $\zeta, \zeta', \ldots, \zeta^{(\alpha-1)}$ in this case such that for arbitrary values of $\lambda, \lambda_1, \lambda_2, \ldots, \lambda_{\alpha-1}$ the following equation can be fulfilled:

$$\left(\sum \eta_\iota X_\iota, \sum\left(\lambda\zeta_\iota + \lambda_1\zeta'_\iota + \lambda_2\zeta'_\iota + \cdots + \lambda_{\alpha-1}\zeta_1^{(\alpha-1)}\right) X_\iota\right)$$
$$= \sum_\iota \left(\mu\zeta_\iota + \mu_1\zeta'_\iota + \mu_2\zeta'_\iota + \cdots + \mu_{\alpha-1}\zeta_1^{(\alpha-1)}\right) X_\iota f \,,$$

as long as the $\mu, \mu_1, \mu_2, \ldots, \mu_{\alpha-1}$ are determined as linear functions of $\lambda, \lambda_1, \lambda_2, \ldots, \lambda_{\alpha-1}$ in a suitable way.

The fact that one of the r roots of eq. (7) always vanishes was stated above already. However, a simple vanishing root will lead to no two-parameter subgroups, but only to the obvious relation:

$$\left(\sum \eta_\iota X_\iota, \sum \eta_\iota X_\iota\right) = 0.$$

However, when $\omega = 0$ is an α-fold root of eq. (7), and likewise all sub-determinants of degree $r - \alpha + 1$ vanish, we will arrive at an α-parameter subgroup whose transformations can all commute with the given ones. Finally, as far as the case is concerned for which $\omega = 0$ is an α-fold root without all sub-determinants of degree $r - \alpha + 1$ vanishing, we will initially pass over it, since we will address that case more rigorously in § 10. Here, it shall suffice to remark that in that case, either all functions $\psi_1(\eta), \ldots, \psi_{r-1}(\eta)$ vanish for such a value or all sub-determinants of $|\gamma_{\iota\kappa}|$ of degree $r - 1$ are equal to zero.

§ 4.

Classification of groups according to rank.

If one again writes equation (7) in the form (10):

$$\omega^r - \omega^{r-1}\psi_1(\eta) + \omega^{r-2}\psi_2(\eta) - \ldots \pm \omega\,\psi_{r-1}(\eta) = 0$$

then the smallest number of functions in terms of which all functions $\psi_1, \psi_2, \ldots, \psi_{r-1}$ can be expressed shall be denoted by l, and groups shall be classified according to their associated value of l. For $l = 0$, all functions $\psi_1, \psi_2, \ldots, \psi_{r-1}$ vanish identically; for $l = 1$, all functions ψ_ν can be represented rationally in terms of a single function of η as long as they do not vanish identically. Moreover, for any arbitrary l, l functions $P_1, \ldots, P_l$ can be chosen such that all functions $\psi_1, \psi_2, \ldots, \psi_{r-1}$ can be expressed rationally in terms of $P_1, \ldots, P_l$. I refer to this number l as the *rank* of the group.

If one again sets:

$$\gamma_{\iota\kappa} = \sum_\rho \eta_\rho c_{\rho\kappa\iota} \,,$$

to abbreviate, then, from the developments of the second paragraph, the following equation will be true for any function $\psi_\nu(\eta)$:

(13) $$\sum_\iota \frac{\partial \psi_\nu}{\partial \eta_\iota} \gamma_{\iota\kappa} = 0$$

when one takes $\nu = 1, \ldots, r - 1$.

Since the functions $\psi_\nu(\eta)$ can be expressed in terms of the l functions $P_1, \ldots, P_l$, the following equation must be true for every value of κ:

$$\gamma_{\kappa 1} \frac{\partial P_\alpha}{\partial \eta_1} + \gamma_{\kappa 1} \frac{\partial P_\alpha}{\partial \eta_2} + \ldots + \gamma_{\kappa r} \frac{\partial P_\alpha}{\partial \eta_r} = 0,$$

in which one must set $\alpha = 1, \ldots, l$. However, the $P_1, \ldots, P_l$ are mutually-independent here, and one therefore has the theorem:

If all of the the functions $\psi_1, \psi_2, \ldots, \psi_{r-1}$ can be represented by l mutually-independent functions then all sub-determinants of $|\gamma_{\iota\kappa}|$ of degree $r - l + 1$ will vanish identically. Naturally, $\psi_{r-1}, \ldots, \psi_{r-l+1}$ will likewise be zero identically.

If we couple this theorem with a remark about the vanishing of ω that was made in the previous paragraph then we will come to the following result:

If the rank of a group is greater than one then every transformation will belong to an $(l - 1)$*-dimensional manifold whose transformations all commute with the given ones.*

Here, I shall recall the sense in which the word "commute" is to be taken with the convention that was made in the introduction. If we then start with an arbitrary finite transformation for $l > 1$ then it will not only commute with all of the transformations that are derived from the same infinitesimal transformations as the given transformation, but also with those transformations for which the associated infinitely-small transformations define a certain $(l - 1)$-fold extended manifold.

Conversely, we conclude:

If a $(k - 1)$*-dimensional plane goes through any point of the image space that is defined by only the given commuting transformations, but no k-dimensional plane of that kind, then $\psi_{r-1}, \ldots, \psi_{r-k+1}$ will vanish in a natural way; likewise, all of the functions $\psi_1, \psi_2, \ldots, \psi_{r-k}$ can be represented rationally in terms of at most k functions, or else one has $l \leq k$.*

In particular, this implies the theorem:

If there is a finite transformation in a group that commutes with only the transformations that emerge from the same infinitesimal transformation (as it does) then all $r - 1$ functions $\psi_1, \psi_2, \ldots, \psi_{r-1}$ can be represented rationally in terms of a single function.

Lie's theorem that the group has an $(r - 1)$-parameter invariant subgroup when the function $\psi_1(\eta)$ does not vanish identically can now be generalized in the following way:

If one chooses the l functions P_1, …, P_l in terms of which all ψ_1, …, ψ_{r-1} can be represented rationally in the simplest way, and if one of them – say, $P_1(\eta)$ – is a linear function then $P_1(\eta) = 0$ will represent an invariant subgroup.

The proof is precisely the same as the one that was given for Lie's theorem at the conclusion of § 2. If we set:

$$P_1(\eta) = \sum p_\iota \eta_\iota$$

then, from (13), one will have:

$$\sum_\nu c_{\alpha\beta\nu}\, p_\nu = 0$$

for every combination α, β.

If we then form:

$$\left(\sum \eta_\iota X_\iota, \sum \eta'_\iota X_\iota\right) = \sum \zeta_\rho X_\rho f$$

then we will have:

$$\zeta_\rho = \sum_{\iota,\kappa} \eta_\iota \eta'_\kappa c_{\iota\kappa\rho} ,$$

so

$$P_1(\zeta) = \sum p_\rho \zeta^\rho = \sum_{\iota,\kappa,\rho} \eta_\iota \eta'_\kappa c_{\iota\kappa\rho} p_\rho ,$$

such that the coefficient of each product $\eta_\iota \eta'_\kappa$ will vanish, and the image point of each infinitesimal transformation that is obtained from the operation:

$$\left(\sum \eta_\iota X_\iota, \sum \eta'_\iota X_\iota\right)$$

will lie on the plane $P_1(\eta) = 0$.

As a special case of the foregoing theorem, I mention:

If one of the functions $\psi_1(\eta)$, …, $\psi_{r-1}(\eta)$ is a power of a linear function then its vanishing will represent an invariant subgroup.

However, one can still extend that theorem:

If i of the l functions P_1, …, P_l in terms of which the functions ψ_1, …, ψ_{r-1} can be expressed – say, P_1, …, P_i – are linear then the principal subgroup will have at most r – i parameters, and is either contained in each group that is represented by the equations:

$$P_1(\eta) = P_2(\eta) = \ldots = P_i(\eta) = 0$$

or identical with it. For these principal subgroups, the functions ψ^ν can be represented in terms of the functions P_{i+1}, …, P_i.

The theorem requires no proof. I point out only that when all of the functions P_1, …, P_l are linear, the principal subgroups will have at most $r - 1$ parameters, and its rank will be equal to zero, in such a way that all two-parameter subgroups of the latter will contain commuting transformations.

The foregoing theorems are also quite closely linked to the following ones:

If a group is not its own principal subgroup then rank of the latter will be less than that of the given group.

If a group possesses an $(r - 1)$*-parameter invariant subgroup then the rank of the latter will be one less than that of the given group.*

In this, we have assumed only that we do not already have $l = 0$ for the given group itself. In particular, it follows that:

If each transformation of an r-parameter group does not commute with any other one, and at the same time, either $\psi_1(\eta)$ *does not vanish identically or one of the functions* $\psi_2(\eta), \ldots, \psi_{r-1}(\eta)$ *is a power of a linear function then it will have an* $(r - 1)$*-parameter subgroup for which one has* $l = 0$.

§ 5.

The principal transformations of the two-parameter subgroups that are contained in a group.

If the infinitesimal transformation $\sum \eta'_\iota X_\iota f$ is the one-parameter principal subgroup and $\sum \eta_\iota X_\iota f$ is an arbitrary transformation of a two-parameter subgroup then one must have equations (12) for non-vanishing ω, which we now give in the form:

$$\sum_{\iota,\kappa} \eta_\iota \eta'_\kappa c_{\iota\kappa\rho} = -\sum_\iota \eta_\iota \gamma'_{\iota\rho} = \omega \eta'_\rho, \tag{12a}$$

if $\gamma_{\iota\rho}(\eta')$ is written briefly as $\gamma'_{\iota\rho}$.

If any of the functions $\psi_1, \ldots, \psi_{r-1}$ is once more denoted briefly by ψ_ν, and $\psi_\nu(\eta')$ is written briefly as ψ'_ν then, from (13), one will have:

$$\sum_\rho \frac{\partial \psi'_\nu}{\partial \eta'_\rho} \gamma'_{\iota\rho} = 0.$$

If we then multiply both sides of eq. (12a) by $\partial \psi'_\nu / \partial \eta'_\rho$ and sum over r then the left-hand side will vanish, and we will have:

$$\psi_1(\eta') = 0, \quad \psi_2(\eta') = 0, \quad \ldots, \quad \psi_{r-1}(\eta') = 0 \tag{14}$$

for non-vanishing values of ω.

That implies the following theorem:

The coefficients $\eta'_1, \ldots, \eta'_r$ *of any transformation* $\sum \eta'_\iota X_\iota f$ *that is the principal transformation of any one of the two-parameter subgroups that belong to the group satisfy equations* (14), *and equation* (7) *has only vanishing roots for them.*

This theorem also admits the following statement:

Any point that the principal transformation of a two-parameter subgroup that is contained in the group maps lies on the structure that is defined by the vanishing of the function $\psi_1, \ldots, \psi_{r-1}$. *The totality of these points then defines a manifold that is at least* $(r - l - 1)$*-fold extended.*

It by no means follows from the foregoing developments that any transformation for which the equations:

$$\psi_1(\eta) = \psi_2(\eta) = \ldots = \psi_{r-1}(\eta) = 0$$

are fulfilled is the principal transformation of a two-parameter subgroup; in fact, in many groups, the coordinates of the transformations that were referred to will satisfy even more equations, and their totality will then define a manifold of dimension less than $r - l - 1$.

As a corollary to the foregoing theorem, the following might be mentioned:

If a transformation that is the principal element of a two-parameter subgroup belongs to even more two-parameter subgroups then either their transformations will commute with each other or the given transformation will also be the principal element for the other subgroups.

This theorem can then lead us to the proof of the assertion that was made at the conclusion of § 3; however, we will encounter the theorem in the context of another consideration, and it will then link up with the further consequences immediately.

With a slight change in notations, the same development that led from formula (12a) to (14) will imply the following theorem:

All of the two-parameter subgroups whose transformation do not commute with each other maps as tangents to an $(r - l - 1)$*-dimensional structure, and it will be determined by the vanishing of all the functions* $\psi_1, \psi_1, \ldots, \psi_{r-2}$.

However, I emphasize that even in the case where all points of the stated structure define principal transformations of two-parameter subgroups, not all of the tangents correspond to two-parameter subgroups. With the exception of the conic section group (cf., *infra*, the end of § 8), no further groups can exist in which all tangents to the structure:

$$\psi_1 = \ldots = \psi_{r-1} = 0$$

define two-parameter subgroups.

§ 6.

Examples of the present developments.

As an example of the results that the foregoing paragraphs yielded, and to prepare for the investigations that will be carried out in the second part, we would like to consider two well-known classes of simple groups somewhat more closely. With Lie, we refer to a group as *simple* when it possesses no invariant subgroups.

We next examine the group that produces the most general projective reassignment of an l-dimensional space. By a suitable choice of the coordinates $x_1, \dots, x_l$, one will arrive at the most general infinitely-small reassignment of the point (x_κ) to $(x_\kappa + \delta x_\kappa)$, provided that the completely-arbitrary, infinitely-small quantities $\alpha_{0\kappa}$, $\alpha_{\kappa 0}$, $\alpha_{\kappa\lambda}$ (κ; $\lambda = 1, \dots, l$) satisfy the equation:

$$x_\kappa + \delta x_\kappa = \frac{x_\kappa + \sum_\lambda \alpha_{\kappa\lambda} x_\lambda + \alpha_{\kappa 0}}{1 + \sum_\lambda \alpha_{0\lambda} x_\lambda}.$$

Thus:

$$\delta x_\kappa = \alpha_{\kappa 0} + \sum_\lambda \alpha_{\kappa\lambda} x_\lambda - \sum_\lambda \alpha_{0\lambda} x_\kappa x_\lambda .$$

When one uses those transformations for which all $\alpha_{\rho\sigma}$ vanish, except for one, as a basis and sets the non-vanishing one equal to δt, one will obtain $l\,(l + 2)$ infinitesimal transformations $X_{\kappa 0}$, $X_{0\kappa}$, $X_{\iota\kappa}$, which can be represented in the following way:

$$(a) \qquad X_{\kappa 0} = \frac{\partial f}{\partial x_\kappa}, \qquad X_{0\kappa} = -\sum_\nu x_\kappa x_\nu \frac{\partial f}{\partial x_\nu}, \quad X_{\iota\kappa} = \frac{\partial f}{\partial x_\kappa} x_\kappa .$$

When either $\iota = \mu$ or $\kappa = \lambda$, one will have $(X_{\iota\kappa}\, X_{\lambda\mu}) = 0$. By contrast, one has, moreover:

$$(b) \qquad \begin{aligned} &(X_{\iota\kappa} X_{\lambda\mu}) = - X_{\iota\lambda} + \varepsilon_{\iota\lambda}\, X_{\kappa\kappa}, \\ &(X_{\kappa 0}\, X_{0\lambda}) = - X_{\kappa\lambda} + \varepsilon_{\kappa\lambda}\, X_{\mu\mu}, \\ &(X_{0\kappa} X_{\kappa\lambda}) = - X_{0\lambda}, \qquad (X_{\iota\kappa} X_{\kappa 0}) = - X_{\iota 0}, \end{aligned}$$

in which $\varepsilon_{\kappa\lambda} = 1$ or 0 according to whether κ and λ are equal or unequal, resp.

In order to show that this group is simple, we start with an arbitrary infinitesimal transformation $\sum \eta_{\rho\sigma} X_{\rho\sigma} f$. If not all $\eta_{\rho 1}$ vanish here then we couple them to all $X_{1\nu}$ for $\nu = 0, 1, \dots, l$ by way of $\left(\sum \eta_{\rho\sigma} X_{\rho\sigma}, X_{1\nu}\right)$; one will then get at least some of them by the transformations:

$$X_{10}, \dots, X_{l0}, \quad 2\, X_{11} + X_{22} + \dots + X_{ll}, \qquad X_{12}, \dots, X_{1l}.$$

Any of these $l\,(l + 2)$ basic infinitesimal transformations will necessarily lead to others, and repeating the operation for these will lead to all infinitesimal transformations. The group will then have no invariant subgroups.

We would like to cite some properties of this group, and first give the functions that remain unchanged under all transformations:

$$\mathsf{H}_{\lambda\mu} f = \sum_{\iota,\kappa,\rho,\sigma} c_{\iota\kappa,\lambda\mu,\rho\sigma} \eta_{\rho\sigma} \frac{\partial f}{\partial \eta_{\iota\kappa}},$$

which then belongs to this transformation group, according to something that Lie said. It can be represented in the following way when one introduces an undetermined quantity z:

$$(c) \qquad D(z) = \begin{vmatrix} -z+\eta_{11}+\cdots+\eta_{ll} & \eta_{10} & \eta_{20} & \cdots & \eta_{l0} \\ \eta_{01} & z+\eta_{11} & \eta_{21} & \cdots & \eta_{l1} \\ \eta_{02} & \eta_{12} & z+\eta_{22} & \cdots & \eta_{l2} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \eta_{0l} & \eta_{1l} & \eta_{2l} & \cdots & z+\eta_{ll} \end{vmatrix}.$$

In this determinant, the coefficient of z^{l+1} is equal to – 1, that of z^l is equal to 0, and the coefficients of the powers of z^{l-1}, z^{l-2}, …, z, z^0 will be the desired functions.

Since the transformations are essentially of the same type, it will suffice to show that the function $D(z)$ is not changed by one of them – say, $\mathsf{H}_{12} f$. However, as long as ρ is not equal to 1, for this transformation, one must replace $d\eta_{\rho 2}$ with $dt \cdot h_{\rho 1}$, and when σ is not equal to 2, one must replace $d\eta_{1\sigma}$ with – $dt \cdot \eta_{2\sigma}$. However, one must make both changes for $d\eta_{12}$ that one gets under the assumptions that $\rho = 1$ and $\sigma = 2$. Thus, the change in $D(z)$ that is provoked by the transformation $\mathsf{H}_{12} f$, as long as it is divided by dt, will consist of the sum:

$$\begin{vmatrix} -z+\eta_{11}+\cdots+\eta_{ll} & \eta_{10} & -\eta_{10} & \cdots & \eta_{10} \\ \eta_{01} & \eta_{11}+z & -\eta_{11} & \cdots & \eta_{l1} \\ \eta_{02} & \eta_{12} & -\eta_{12} & \cdots & \eta_{l2} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \eta_{0l} & \eta_{1l} & -\eta_{1l} & \cdots & z+\eta_{ll} \end{vmatrix}$$

$$+ \begin{vmatrix} -z+\eta_{11}+\cdots+\eta_{ll} & \eta_{10} & \eta_{20} & \cdots & \eta_{l0} \\ \eta_{02} & \eta_{12} & \eta_{22} & \cdots & \eta_{l2} \\ \eta_{02} & \eta_{12} & z+\eta_{22} & \cdots & \eta_{l2} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \eta_{0l} & \eta_{1l} & \eta_{2l} & \cdots & z+\eta_{ll} \end{vmatrix},$$

the first term of which will be obtained by replacing the third column of $D(z)$ with – η_{10}, …, $-\eta_{l0}$, and the second by replacing the second row with η_{02}, …, η_{l2}. However, one will see directly that this sum vanishes.

The functions that remain unchanged under the l (l + 2) infinitesimal transformations:

$$\sum_{\rho,\sigma,\iota,\kappa} \eta_{\rho\sigma} c_{\rho\sigma,\alpha\beta,\iota\kappa} \frac{\partial f}{\partial \eta_{\iota\kappa}}$$

can be represented in an entirely similar way. When one sets $\eta_{11} + \eta_{22} + \ldots + \eta_{ll} = \sigma$, it will be given for arbitrary z by the determinant:

$$(d) \qquad E(z) = \begin{vmatrix} -z+\dfrac{\sigma}{l+1} & \eta_{01} & \eta_{02} & \cdots & \eta_{0l} \\ \eta_{10} & z+\eta_{11}-\dfrac{\sigma}{l+1} & \eta_{12} & \cdots & \eta_{1l} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \eta_{l0} & \eta_{l1} & \cdots & \cdots & z+\eta_{ll}-\dfrac{\sigma}{l+1} \end{vmatrix}$$

$$= - z^{l+1} + z^{l-1} P_1 + P_2 \cdot z^{l-1} + \ldots + P_{l-1} \cdot z + P_l .$$

It will not be necessary to carry out the proof. Likewise, it might be permissible to state the following theorems without proof.

If the system (η_{01}, η_{02}, …, η_{ll}) satisfies none of the l equations:

$$P_1 = 0, \qquad P_2 = 0, \qquad \ldots, \qquad P_l = 0$$

then the transformation $\sum \eta_{\rho\sigma} X_{\rho\sigma} f$ will belong to precisely l (l + 1) two-parameter subgroups with no commuting elements. The principal elements of all of the two-parameter subgroups that are obtained in that way will define a manifold that is only (2l – 1)-extended. It will be represented analytically by the vanishing of all determinants of degree two in the determinant $E(z)$ for z = 0. That manifold also belongs to all l structures $P_1 = P_2 = \ldots = P_l = 0$.

If a point is a principal point of a two-parameter subgroup, and the latter does not belong to the structure $P_1 = \ldots = P_l = 0$, then the point will also be a principal point for an (l^2 – 1)-parameter family of two-parameter subgroups, and all of them will define an (l^2 + 1)-parameter subgroup.

The group belongs to the general projective groups of rank 1, 2, …, l – 1. The general projective groups of rank l – 1 all belong to the structure $P_l = 0$; those of rank l – 2 lie on the structure $P_l = P_{l-1}$ =, etc. Finally, the conic section groups (viz., the general projective groups for l = 1) belong to the structure $P_l = P_{l-1} = \ldots = P_2 = 0$.

As one easily sees, and as will be emphasized in Part Two especially, for every general projective group of rank l', one can define a composite group of the same rank l' whose number of parameters will amount to $l'^2 + 2l' + 1$. At the same time, for $l' < l$, each such group will be contained as a subgroup of the given group, and indeed the groups of rank l – 1 in question will fill up the structure $P_l = 0$, while those of rank l – 2 will fill up the structure $P_l = P_{l-1} = 0$, etc. A planar manifold of dimension (l – 1)(l + 2) will then go through every point of $P_l = 0$. The intersection of the λ structures $P_l = 0$, $P_{l-1} = 0$, …, $P_{l-\lambda+1} = 0$ will likewise be $(l - \lambda)(l - \lambda + 3)$-dimensional.

An $(l - 1)$-dimensional planar structure whose points are mapped by mutually-commuting transformations goes through each point of the image space. Conic section groups that are subgroups of the given group go through each of the $\frac{l(l+1)}{2}$ points of such a plane.

A second class of simple groups will be determined by those continuous projective transformations that translate a proper structure of order two in a space of more than three dimensions into itself, in which, a "proper" structure, in contrast to a conic structure, will be understood to mean one whose determinant does not vanish. That group is defined in the same way as the group of an $(m + 1)$-dimensional space when all variables are changed by entire, linear transformations and remain unchanged under a proper form of degree two. If one establishes the relationship:

$$x_1^2 + \ldots + x_{m+1}^2 = 1 \qquad (e)$$

between the $m + 1$ variables $x_1, \ldots, x_{m+1}$ then the group shall be based upon the $\frac{m(m+1)}{2}$ inf. transformations:

$$X_{\iota\kappa} f = x_\iota \frac{\partial f}{\partial x_\kappa} - x_\kappa \frac{\partial f}{\partial x_\iota}. \qquad (e')$$

It follows from this that:

$$(X_{\iota\kappa} X_{\iota\lambda}) = X_{\kappa\lambda}, \qquad (X_{\iota\kappa} X_{\lambda\mu}) = 0 \qquad (f)$$

when $\iota, \kappa; \lambda, \mu$ are all unequal. The groups that are defined in that way will be simple for $m > 3$, as Lie proved in the tenth volume of his Archiv on pp. 412.

The adjoint linear groups that were presented at the beginning of § 2 will be identical for those groups. The invariants are implied by the following determinant, in which one takes $\eta_{\iota\kappa} + \eta_{\kappa\iota} = 0$:

$$\begin{vmatrix} z & \eta_{12} & \cdots & \eta_{1,m+1} \\ \eta_{21} & z & \cdots & \eta_{2,m+1} \\ \cdots & \cdots & \cdots & \cdots \\ \eta_{m+1,1} & \eta_{m+1,2} & \cdots & z \end{vmatrix} = z^{m+1} + P_1 z^{m-1} + P_3 z^{m-2} + \ldots \qquad (g)$$

Therefore, l will be equal to $\frac{m+1}{2}$ or $\frac{m}{2}$ according to whether m is odd or even. If m is even then the $2l^2$ roots of the characteristic equation (7) will be $\pm\,\omega_\kappa$, $\pm\,\omega_\kappa \pm\,\omega_\lambda$. For even m, they will be associated with a $(4l - 3)$-fold extended manifold that is determined by the vanishing of all expressions $(\iota\kappa\lambda\mu)$ and by $P_1 = \sum \eta_{\iota\kappa}^2$ (*). That $(4l - 3)$-extended structure has a $(4l - 5)$-dimensional double structure that is given by the equations:

$$\sum_\rho \eta_{\iota\rho}\eta_{\kappa\rho} = 0$$

for arbitrary values of ι and κ. That double structure will imply the principal points that belong to the roots $\pm\,\omega_\iota \pm \omega_\kappa$.

For even m, the given group will contain those subgroups that are defined in corresponding ways for $m' = 1, \ldots, m - 1$. If one sets $m' = m - 1$ then the corresponding subgroups will have the same rank $l = m / 2$ and will fill up the same manifold as the given group such that every transformation of the given group will belong to such a subgroup. The correspondingly-defined subgroups of rank $l - 1$ whose number of parameters amounts to $(l - 1)(2l - 1)$ or $(l - 1)(2l - 3)$ belong to a $(2l^2 + l - 2)$-fold extended manifold, etc. The only points that lie in any $(l - 1)$-dimensional plane of the image space that goes through an arbitrary point and maps by nothing but mutually-commuting transformations are the ones for which the corresponding infinitesimal transformation belongs to a conic section group. General projective groups whose rank l' is less than l also belong to the given group.

These results change very little for an odd m. Every general transformation belongs to $2l\,(l - 1)$ two-parameter subgroups with no commuting elements, and their principal points fill up a $(4l - 7)$-dimensional manifold for which all equations $(\iota\kappa\lambda\mu) = 0$ and $\sum_\rho \eta_{\iota\rho}\eta_{\iota\rho} = 0$ must be true. In order for a transformation $\sum \eta_{\iota\kappa} X_{\iota\kappa} f$ to belong to an r-parameter subgroup for which $p = r$, the coefficients $\eta_{\iota\kappa}$ must satisfy certain conditions, but this is not the place to go into them in detail.

§ 7.

The characteristic equation has nothing but unequal roots.

We shall make an assumption that might, on first glance, seem purely general, but which has an entirely specialized character due to the conditions that exist between the c, such as would follow from the developments in § 4. Namely, we assume that equation (7) has no two of its roots equal for a certain system of values $\eta_1, \ldots, \eta_r$. As was remarked already, one root will be equal to zero; the other $\omega_1, \ldots, \omega_{r-1}$ shall then be different from each other and from zero. A single system of coefficients $\zeta_1^{(\nu)}, \ldots, \zeta_r^{(\nu)}$ belongs to each root ω_ν, and the $r - 1$ infinitesimal transformations $\sum_\iota \zeta_\iota^{(\nu)} X_\iota f$ that are thus determined will be independent of each other and of the given transformation $\sum \eta_i X_i f$. We can then employ those r infinitesimal transformations for the

(*) Should all $(\iota\kappa\lambda\mu) = \eta_{\iota\lambda}\,\eta_{\lambda\mu} + \eta_{\iota\lambda}\,\eta_{\mu\kappa} + \eta_{\iota\mu}\,\eta_{\kappa\lambda}$ vanish, then, as is known, it would be necessary and sufficient that all expressions $(12\lambda\mu)$ should vanish for $\eta_{12} \neq 0$, which would yield $(l - 1)(2l - 1)$ equations.

determination of the group. In order to obtain directly the form that that is most suitable for further investigation, we denote the given transformation by $X_{r-1}f$, and the other ones by $X_1, \ldots, X_{r-2}$, such that the following equations will be true:

(15) $$(X_{r-1}\,X_1) = \omega_1\,X_1 f,\ \ldots,\ (X_{r-1}\,X_{r-2}) = \omega_{r-2}\,X_{r-2} f, \qquad (X_{r-1}\,X_r) = \omega_r\,X_r f.$$

When we form the Jacobi relation for $r - 1$, k, l, it will follow that:

(16) $$(\omega_\kappa + \omega_\lambda - \omega_\mu)\,c_{\kappa\lambda\mu} = 0,$$

in which the equation for $\nu = r - 1$ is also contained:

(16a) $$(\omega_\kappa + \omega_\lambda)\,c_{\kappa\lambda\,r-1} = 0.$$

This equation allows us to recognize that when well-defined relations do not exist between the roots $\omega_\kappa, \ldots, \omega_{r-2}$, ω_r of equation (7), the $r - 1$ infinitesimal transformations $X_1f, \ldots, X_{r-2}f, X_r f$ will determine a group in which two arbitrary transformations commute with each other.

We now add a further assumption to the one that we made above, namely, that r of the $r\,(r - 1) / 2$ transformations $\sum_\rho c_{\kappa\lambda\rho} X_\rho f$ are independent of each other, so $p = r$. In that case, due to (16a), at least one root must be equal and opposite to another. Therefore, let $\omega_{r-2} + \omega_r = 0$, and let $c_{r,\,r-2,\,r-1}$ be non-zero, such one can set:

$$(X_{r-2}\,X_r) = X_{r-1} f.$$

Now, apply the Jacobi relation to the numbers r, $r - 2$, κ; where κ is one of the numbers $2, \ldots, r - 3$, and obtain:

(17) $$\omega_\kappa X_\kappa f = \sum_\rho \left\{ c_{r-2,\kappa\rho}(X_\rho X_r) - c_{r\kappa\rho}(X_\rho X_{r-2}) \right\}.$$

Since the left-hand side of this equation cannot vanish, it will follow from this equation, in conjunction with (16), that when ω_κ is a root, either $\omega_\kappa + \omega_r$ or $\omega_\kappa - \omega_r$ will be a new root, or that one can obtain a new root from ω_κ by either adding or subtracting ω_r.

With no loss of generality, we can set $\omega_r = 2$. If ω_κ is an arbitrary root then at least one of the two quantities $\omega_\kappa + 2$ and $\omega_\kappa - 2$ will be contained among the roots. We start from an arbitrary root ω_κ and look for all of the roots that one will arrive at by repeated addition and subtraction of 2. Let the number of them be m, and for the sake of simplicity, they might be denoted by $\omega_1, \ldots, \omega_m$. Since they differ only by multiples of 2, one can order them according to the magnitudes of their real parts and set:

$$\omega_2 = \omega_1 - 2, \quad \omega_3 = \omega_1 - 4, \quad \omega_4 = \omega_1 - 6,\ \ldots,\ \omega_m = \omega_1 - (m - 1)\,2.$$

Equation (17) then yields the result:

(17a)
$$\begin{cases} \omega_1 = & c_{r-2,1,2}c_{2r1}, \\ \omega_2 = & c_{r-2,2,3}c_{3r2} - c_{r21}c_{1,r-2,2}, \\ \omega_3 = & c_{r-2,3,4}c_{4r3} - c_{r32}c_{2,r-2,3}, \\ & \dots\dots\dots\dots\dots\dots \\ \omega_m = & -c_{rmm-1}c_{m-1,r-2,m}, \end{cases}$$

which implies, by addition, that:

$$\omega_1 + \omega_2 + \omega_3 + \ldots + \omega_m = 0,$$

such that $\omega_1 = m - 1$. If m is an odd number here then the middle root must be equal to zero. However, when m is an even number, the middle roots must be equal to ± 1. Our assumption that all roots should be unequal would then imply that $m = r - 3$, and it would be an even number. At the same time, one would get:

$$\omega_1 = r - 4, \qquad \omega_2 = r - 6, \qquad \omega_3 = r - 8, \ldots, \omega_{r-3} = -r + 4.$$

One concludes directly from (17) that $(X_r\, X_{r-2-\nu}) = 0$ for $\nu = 1, \ldots, r - 3$, and it will then follow easily that $(X_\alpha\, X_\beta) = 0$ when α and β are any two of the numbers $1, \ldots, r - 3$. The further coefficients $c_{\iota\kappa\lambda}$ can be simplified by multiplying the $X_1 f, \ldots, X_{r-3} f$ by suitable constants. In that way, one will find from (17a) that:

$$(X_r\, X_\beta) = (r - \alpha - 3)\, X_{\alpha+1} f \qquad \text{for} \qquad \alpha = 1, \ldots, r - 4,$$
$$(X_r\, X_{r-3}) = 0, \quad (X_{r-2}\, X_\alpha) - (\alpha - 1)\, X_{\alpha-1} f\,.$$

In that way, one will arrive at the result that the assumptions that were made determine the composition of the group completely, as long as r is given, and that this must be an odd number. In order to be able to express this result most conveniently, one considers that equal roots of equation (7) will lead to either just a single two-parameter subgroup or to a family of them. One then expresses the result in the following way now:

In an r-parameter group that is its own principal subgroup, a given one-parameter subgroup should adhere to precisely $r - 1$ two-parameter subgroups, and none with no commuting transformations shall occur among them. r must then be an odd number. The group contains one (and indeed only one) invariant subgroup that is defined by $r - 3$ parameters. All transformations that belong to that subgroup will commute. The composition of the group is determined completely by the assumptions that were made. If one denotes the given one-parameter subgroup by $X_{r-1} f$ then one can choose the remaining $X_1, X_2, \ldots, X_{r-2}$ such that:

$$(X_{r-2}\, X_r) = X_{r-1} f\,, \; (X_{r-1}\, X_r) = 2\, X_r f\,, \qquad (X_{r-1}\, X_{r-2}) = -\,2X_{r-2} f\,,$$
$$(X_{r-1}\, X_\alpha) = (r - 2(\alpha + 1))\, X_\alpha f,$$
$$(X_r\, X_\alpha) = (r - \alpha + 3)\, X_{\alpha+1} f, \qquad (X_{r-2}\, X_\alpha) = -\,(\alpha - 1)\, X_{\alpha-1} f,$$
$$(X_\alpha\, X_\beta) = 0 \qquad \text{for} \qquad \alpha, \beta = 1, \ldots, r - 3.$$

If we define a group for an even r by the same prescription then $X_{r-1} f$ and $X_{(r-2)/2} f$ will determine a two-parameter group with commuting transformations. However, in that case, one will have $l = 1$, and all functions $\psi_\nu(\eta)$ will vanish for an odd ν and will define powers of $\eta_{r-1}^2 - \eta_r \, \eta_{r-2}$, up to a constant factor, for even ν.

A group that is constructed in that way will exist for two variables and arbitrary r. It was listed in the penultimate place by Lie in his enumeration (*) under B), and has the form:

$$q, xq, \ldots, x^{r-4} q, p, 2xp + (r-4)\, yq, x^2 p + (r-4)\, xyq,$$

with his notation.

§ 8.

The characteristic equation has equal non-vanishing roots.

The determination of groups that was given in the previous paragraphs is based essentially upon the assumption that all roots of equation (7) were unequal. We would now like to examine whether the simplicity of the result does not also remain true for multiple roots.

Let Xf (with no subscript) denote an arbitrary transformation. We exhibit equation (7) for it and determine its roots. If that equation has equal roots, and if ω_α is one of them then ω_α shall be counted as a $(\lambda + 1)$-fold root if and only if one can determine $\lambda + 1$ infinitesimal transformations $X_{\alpha_0}, X_{\alpha_1}, \ldots, X_{\alpha_\lambda}$ that are independent of each other, as well as Xf, and for which, one will have the equations:

$$(18)\qquad \begin{aligned} (X X_{\alpha_0}) &= \omega_\alpha X_{\alpha_0}, \\ (X X_{\alpha_1}) &= \omega_\alpha X_{\alpha_1} + e_{\alpha_1\alpha_0} X_{\alpha_0}, \\ (X X_{\alpha_2}) &= \omega_\alpha X_{\alpha_2} + e_{\alpha_2\alpha_1} X_{\alpha_1} + e_{\alpha_2\alpha_0} X_{\alpha_0}, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ (X X_{\alpha_\lambda}) &= \omega_\alpha X_{\alpha_\lambda} + e_{\alpha_\lambda\alpha_{\lambda-1}} X_{\alpha_{\lambda-1}} + \ldots + e_{\alpha_\lambda\alpha_1} X_{\alpha_1} + e_{\alpha_\lambda\alpha_0} X_{\alpha_0}, \end{aligned}$$

and that system cannot be decomposed into several systems with a smaller number λ and the same composition. The arbitrariness in the choice of $X_{\alpha_0}, \ldots, X_{\alpha_\kappa}$ can be employed to make all coefficients $e_{\alpha_\kappa\alpha_0}$ vanish for $\kappa > \iota + 1$. One can then use the following form as a basis:

$$(18a)\qquad \begin{aligned} (X X_{\alpha_0}) &= \omega_\alpha X_{\alpha_0}, \\ (X X_{\alpha_1}) &= \omega_\alpha X_{\alpha_1} + e_{\alpha_1\alpha_0} X_{\alpha_0}, \\ (X X_{\alpha_2}) &= \omega_\alpha X_{\alpha_2} + e_{\alpha_2\alpha_1} X_{\alpha_1}, \\ &\ldots\ldots\ldots\ldots\ldots\ldots, \end{aligned}$$

(*) Math. Ann. XVI, pp. 524.

$$(X\,X_{\alpha_\lambda}) = \omega_\alpha\; X_{\alpha_\lambda} + e_{\alpha_\lambda\,\alpha_{\lambda-1}} X_{\alpha_{\lambda-1}}\,,$$

in which none of the coefficients can vanish. When we use the infinitesimal transformation Xf as a basis, we would like to say that the infinitesimal transformations X_{α_0}, …, X_{α_λ} *belong* to the roots ω_α, resp.

Now, let ω_α, ω_β, ω_γ, … be the roots of the characteristic equation, and let each root lead to a sequence of transformations X_α (a single such transformation, resp.). It was by no means assumed that ω_α and ω_β are unequal in this.

We now form the Jacobi relation for X, X_{α_0}, X_{β_0}, and it will follow that:

$$\sum_\rho c_{\alpha_0\beta_0\rho}\,[(\omega_\alpha + \omega_\beta)\,X_\rho f - (X\,X_\rho)] = 0.$$

We develop this equation completely in those X_ρ that belong to an arbitrary root ω_γ. We then obtain the equation:

$$\text{(19)}\qquad (\omega_\alpha + \omega_\beta)\,(c_{\alpha_0\,\beta_0\,\gamma_0} X_{\gamma_0} + c_{\alpha_0\,\beta_0\,\gamma_1} X_{\gamma_1} + \cdots + c_{\alpha_0\,\beta_0\,\gamma_\nu} X_{\gamma_\nu})$$
$$= c_{\alpha_0\,\beta_0\,\gamma_0}\,\omega_\gamma X_{\gamma_0} + c_{\alpha_0\,\beta_0\,\gamma_1}(\omega_\gamma X_{\gamma_1} + e_{\gamma_1\gamma_0} X_{\gamma_\nu}) + \ldots + c_{\alpha_0\,\beta_0\,\gamma_\nu}(\omega_\gamma X_{\gamma_\nu} + e_{\gamma_\nu\,\gamma_{\nu-1}} X_{\gamma_{\nu-1}})\,.$$

If one does not have $\omega_\alpha + \omega_\beta = \omega_\gamma$ in this then one must have $c_{\alpha_0\,\beta_0\,\gamma_\nu} = 0$ in order for $X_{\gamma_\nu} f$ to have the same coefficients on both sides; now, one must have $c_{\alpha_0\,\beta_0\,\gamma_{\nu-1}} = 0$ in order for $X_{\gamma_{\nu-1}} f$ to have the same coefficients on both sides. In the same way, one recognizes that all of the coefficients $c_{\alpha_0\,\beta_0\,\gamma_\nu}$, …, $c_{\alpha_0\,\beta_0\,\gamma_1}$, $c_{\alpha_0\,\beta_0\,\gamma_0}$ will be equal if one does not have $\omega_\alpha + \omega_\beta = \omega_\gamma$. However, when that condition is fulfilled, one can compare the coefficients of $X_{\gamma_{\nu-1}} f$, , X_{γ_1}, X_{γ_0} on both sides and obtain:

$$c_{\alpha_0\,\beta_0\,\gamma_\nu} e_{\gamma_\nu\gamma_{\nu-1}} = 0, \qquad c_{\alpha_0\,\beta_0\,\gamma_{\nu-1}} e_{\gamma_{\nu-1}\gamma_{\nu-2}} = 0, \qquad \ldots, \qquad c_{\alpha_0\,\beta_0\,\gamma_1} e_{\gamma_1\gamma_0} = 0,$$

from which, it will follow that:

$$c_{\alpha_0\,\beta_0\,\gamma_\nu} = c_{\alpha_0\,\beta_0\,\gamma_{\nu-1}} = \ldots = c_{\alpha_0\,\beta_0\,\gamma_1} = 0$$

or

$$\text{(20)}\qquad (X_{\alpha_0} X_{\beta_0}) = \sum c_{\alpha_0\,\beta_0\,\gamma_0} X_{\gamma_0} f \qquad (\omega_\alpha + \omega_\beta = \omega_\gamma),$$

in which the summation refers to the first transformation X_{γ_0} that belongs to one of the roots $\omega_\alpha + \omega_\beta = \omega_\gamma$.

The Jacobi relations X, X_{α_0}, X_{β_1} lead to an equation that differs from equation (19) only by the fact that one adds:

$$e_{\beta_1\,\beta_0} c_{\alpha_0\,\beta_0\,\gamma_0} X_{\gamma_0}$$

to the right-hand side.

As before, it follows from this that only those coefficients:

$$c_{\alpha_0 \beta_0 \gamma_0}, \ldots, c_{\alpha_0 \beta_1 \gamma_\nu}$$

for which $\omega_\alpha + \omega_\beta = \omega_\gamma$ can be non-zero, and with that condition, one will have:

(20a) $$(X_{\alpha_0} X_{\beta_1}) = \sum (c_{\alpha_0 \beta_1 \gamma_1} X_{\gamma_1} + c_{\alpha_0 \beta_1 \gamma_0} X_{\gamma_0}) \qquad (\omega_\alpha + \omega_\beta = \omega_\gamma).$$

One can proceed in the same way. With the condition $\omega_\alpha + \omega_\beta = \omega_\gamma$, $(X_{\alpha_0} X_{\beta_1})$ can be expressed in terms of $X_{\gamma_0}, X_{\gamma_1}, \ldots, X_{\gamma_\iota}$, and correspondingly, $(X_{\alpha_\iota} X_{\beta_\kappa})$ can be expressed in terms of only those X_{γ_ρ} for which ρ goes up to at most $\iota + \kappa$.

One can derive the equations:

$$(X_{\alpha_0} X_{\alpha_1}) = \sum c_{\alpha_0 \beta_1 \gamma_0} X_{\gamma_0} f, \qquad (X_{\alpha_0} X_{\alpha_2}) = \sum (c_{\alpha_0 \alpha_2 \gamma_0} X_{\gamma_0} + c_{\alpha_0 \alpha_2 \gamma_1} X_{\gamma_1}) f,$$
$$(2\omega_\alpha = \omega_\gamma)$$

in an entirely corresponding way.

These theorems suffice to find all groups that satisfy the two conditions:

a) $p = r$.
b) Not every transformation of the group shall commute with another one.

When we start with the transformation $X_{r-1} f$, equation (7) shall have only one simple vanishing root. We set:

$$(X_{r-1} X_{r-2}) = -2 X_{r-2} f, \qquad (X_{r-1} X_r) = 2 X_r f, \qquad (X_{r-2} X_r) = X_{r-1} f,$$

and assume in this that $X_{r-1} f$ enters into an expression for one of the $(X_\iota X_\kappa)$, in which X_ι and X_κ belong to the roots ± 2 as the *first* transformations. The justification for this assumption will be proved in the conclusion. Meanwhile, it is not excluded that even more transformations belong to the roots + 2 and – 2; neither is it assumed that $X_{r-1} f$ does not enter into any other expressions $(X_\iota X_\kappa)$. We group the $X_\lambda f$ for $\lambda = 1, \ldots, r - 3$ according to the associated roots ω_α, ω_β, ...

The relation $(r, r - 2, \alpha_0)$ yields:

(21) $$\omega_\alpha X_{\alpha_0} f = \sum_\rho [c_{r-2,\alpha_0,\rho} (X_\rho X_r) - c_{r,\alpha_0,\rho} (X_\rho X_{r-2})].$$

If $c_{r-2,\alpha_0,\rho}$ is to not vanish then one must have $\rho = \beta_0$ and $\omega_\beta = \omega_\alpha - 2$, and likewise, $c_{r,\alpha_0,\rho}$ can be non-zero only when $\rho = \gamma_0$ and $\omega_\gamma = \omega_\alpha + 2$. Therefore, if ω_α is any root then at least one of the two quantities $\omega_\alpha + 2$ or $\omega_\alpha - 2$ must be a root. As in the previous paragraphs, one can determine the roots when one forms the foregoing equation for all X_{α_0} that belong to the roots that are obtained from one of them by repeated addition and subtraction of 2 as the *first* transformation. Since the root zero is excluded from them, only the distinct groups:

$$-2m-1,\ -2m+1,\ \ldots,\ -1,\ +1,\ \ldots,\ 2m-1,\ 2m+1$$

are possible.

It follows from this that no further transformations belong to the roots ± 2, which can also be seen in many other ways.

One now forms the Jacobi relation for r, $r-2$, α_λ, where $X_{\alpha_\lambda} f$ is the *last* transformation that belongs to the root ω_α. One then obtains:

$$\omega_\alpha = \sum_\rho (c_{r-2,\alpha_\lambda\rho}\, c_{\rho r \alpha_\lambda} - c_{r\alpha_\lambda\rho}\, c_{\rho,r-2,\alpha_\lambda}) .$$

Here, one observes that X_{α_λ} can enter into the expression for $(X_{r-2}\ X_{\beta_\lambda})$ only when $\omega_\beta - 2 = \omega_\alpha$ and into the expression for $(X_r\ X_{\gamma_\lambda})$ only when $\omega_\gamma + 2 = \omega_\alpha$. Therefore, ρ can be equal to one β_λ or one γ_λ in the foregoing equation. That implies that the same number of transformations X_{α_0}, X_{α_1}, …, X_{α_λ} belong to all of the same sequence of associated roots ω_κ.

That implies the theorem:

In order to obtain all of the roots that can appear for the transformation groups that satisfy the requirements that were posed, along with the roots + 2 *and* – 2, *one chooses two numbers* λ *and m in such a way that* $2(m+1)(\lambda+1) \le r-3$, *defines the roots* ± 1, ± 3, ±, …, ± (2m + 1), *and lets each of these* 2m + 2 *numbers be a* $\lambda+1$*-fold root. One then chooses two new numbers* λ' *and* m' *such that* $2(m'+1)(\lambda'+1) \le r-3-2(m+1)(\lambda+1)$ *and lets each of the numbers* ± 1, ± 3, ±, …, ± (2m' + 1) *be a* $\lambda'+1$*-fold root. One proceeds in that way until all* $r-3$ *numbers are exhausted.*

Meanwhile, that will imply the theorem:

If r is an even number then any transformation of an r-parameter group that is its own principal subgroup must belong to a two-parameter group with commuting transformations.

or

For an even r > 2, *an r-parameter group in which not every transformation commutes with one of the other ones will necessarily be an* $(r-1)$*-parameter invariant subgroup.*

As in the previous paragraphs, one also easily sees here that for $\iota, \kappa = 1, 2, \ldots, r-3$, one must necessarily have $(X_\iota\, X_\kappa) = 0$. It is likewise immediately clear that for $\iota = 1, \ldots, r-3$, $(X_{r-2}\, X_\iota)$ and $(X_r\, X_\iota)$ can be represented in terms of the first $r-3$ transformations X_1, …, X_{r-3}. It will not be necessary to write down all of the coefficients. I remark only that all these functions $\psi_\nu(\eta)$ can be expressed rationally in terms of $\eta_{r-1}^2 - \eta_r \eta_{r-2}$, and that all sub-determinants of degree $r - 1$ of $|\gamma_{\iota\kappa}|$ will vanish for:

$$\eta_r \eta_{r-2} = \eta_{r-1}^2 .$$

That implies the theorem:

If an r-parameter group satisfies the two conditions that p = r and that no line that represents commuting transformations goes through every point of the image space then r must be an odd number. The group has an $(r - 3)$*-term invariant subgroup, and the transformations of that subgroup all commute with each other. If the coefficients of a transformation* $\sum \eta_\iota X_\iota f$ *satisfy the condition* $\eta_r \eta_{r-2} = \eta_{r-1}^2$ *then it will commute with other transformations.*

The fact that the invariant subgroup can be decomposed into several groups here, each of which is an invariant subgroup, is not excluded by this version of the theorem.

I have previously (Programm 1884) presented the theorem that any group whose order is larger than three contains commuting transformations without publishing its proof. Engel then gave a proof of it [Leipziger Berichte (1886), pp. 91], and thus determined $r - 3$ to be the number of dimensions of that structure whose points belong to a two-parameter subgroup of commuting transformations. Here, we recognize that this number is $r - 2$ for $p = r$, and that only the groups that are given in this and the foregoing paragraphs will satisfy that condition. Moreover, one sees easily that for $p < r$, a straight line of commuting transformations will go through either each point of the $(r - 2)$-dimensional image space or through each point of an $(r - 1)$-dimensional plane. Namely, should such a line not go through each point then one must have $l = 1$. Hence, all $\psi_\nu(\eta)$ must be expressible in terms of a function $P(\eta)$, and since the group possesses an invariant subgroup, $P(\eta) = 0$ must represent it, so it must be a line; however, $l = 0$ for the subgroup. It will then follow that:

If the order r of a group is larger than three then a straight line that maps to mutually-commuting transformations will go through either every point of the $(r - 1)$-*dimensional image space or through every point of an* $(r - 2)$-*dimensional second-order cone or through every point of an* $(r - 2)$-*dimensional plane.*

As I have remarked before, there are only three groups in which no commuting transformations occur, namely, a one-parameter one, a two-parameter one, and a three-parameter one. The last one, for which all two-parameter subgroups map to tangents to a conic section, was referred to as the *conic section group* by Lie and Engel.

An assumption that must be justified was made in the foregoing. Namely, we have assumed that the two transformations X_λ and X_μ for which the transformation X_{r-1} enters into the expression for $(X_\lambda X_\mu)$ are, in both cases, the first transformations that belong to the roots in question. Since equation (7) has only one vanishing root, all $(X_{\alpha_i} X_{\beta_\kappa})$ for which $\omega_\alpha + \omega_\beta = 0$ must be expressed in terms of $X_{r-1} f$, as long as they do not vanish.

Up to now, we have assumed that $(X_{\alpha_0}\ X_{\beta_0}) = \chi \cdot X_{r-1}$; we now assume that $(X_{\alpha_0}\ X_{\beta_0})$ = 0.

It then follows from the Jacobi relation ($\alpha_0\ \beta_0\ \alpha_1$) that:

$$c_{\beta_0 \alpha_1, r-1} \omega_\alpha X_{r-1} + c_{\alpha_0 \alpha_1 \gamma_0} (X_{\gamma_0} X_{\beta_0}) = 0,$$

and since $\omega_\alpha + \omega_\beta = 0$, where $\omega_\gamma = 2\,\omega_\alpha$, that $(X_{\gamma_0} X_{\beta_0})$ does not include $X_{r-1} f$, and thus that one also has $(X_{\alpha_0} X_{\beta_1}) = 0$; the same thing will be true for $(X_{\alpha_1} X_{\beta_0})$, etc. Therefore, if one assumes that $(X_{\alpha_0} X_{\beta_0}) = 0$ then one can never have $(X_{\alpha_\iota} X_{\beta_\kappa}) = X_{r-1} f$.

With that, the theorems that were presented are proved in all of their respects.

Addendum: The requirement that not every arbitrary transformation should commute with one of the others subsumes the two conditions that, first of all, the number r of parameters must be *odd*, and that secondly, every $\lambda + 1$ must be associated with an *even* number $2m + 2$ of $\lambda + 1$-fold roots. However, neither condition is necessary in the slightest when one ignores that requirement. We shall now construct groups by the following prescription:

For any arbitrary r, one initially chooses transformations X_r, X_{r-1}, X_{r-2} such that:

$$(X_{r-1}\ X_r) = 2\, X_r f, \qquad (X_{r-1}\ X_{r-2}) = -\,2\, X_{r-2} f, \qquad (X_{r-2}\ X_r) = X_{r-1} f.$$

One then chooses two numbers σ and s such that $\sigma s \leq r - 3$, forms the roots:

$$\pm (s-1),\ \pm (s-3),\ \pm (s-5),\ \ldots,$$

and lets each of these s numbers be a σ-fold root of the characteristic equation for $X_{r-1} f$. One then chooses two new numbers σ' and s' such that $\sigma' s' \leq r - 3 - s\sigma$ and lets each of the s' numbers $\pm (s-1)$, $\pm (s-3)$, … be a σ'-fold root. One then proceeds in that way until all $r - 3$ numbers have been exhausted. One assigns each of those σ-fold roots with σ transformations according to equations (18), *and then each of the σ'-fold roots with σ' transformations, etc.*

As for the groups that were considered before, one also sees immediately here that for $\iota, \kappa = 1, 2, \ldots, r - 3$, one will necessarily have $(X_\iota\ X_\kappa) = 0$, and that $(X_{r-2}\ X_\iota)$ and $(X_r\ X_\iota)$ can be represented in terms of the first $r - 3$ transformations. The entire character of the groups then remains essentially unchanged; an arbitrary transformation will commute with another one only when r is even or one of the numbers $s, s', \ldots$ is odd. However, all of the theorems that were derived in the last paragraphs will also be true for the groups that are defined by the general prescription. Namely, $l = 1$ and $p = r$ for the groups that are defined in that way. The question of whether that is the only group for which $l = 1$ and $p = r$ will be first answered in a later part of our paper.

§ 9.

Some properties of the groups of rank zero.

The results of the two foregoing paragraphs allow us to write down in explicit form all groups that satisfy the conditions that were given there as soon as the number of parameters is chosen. The method that was applied there, which will also prove to be important under other assumptions, cannot be used for groups of rank zero. However, it is not possible for me to give theorems that allow one to represent such groups explicitly. Nonetheless, since I will need a theorem from the theory of those groups in what follows, please permit me to summarize here the most important theorems that are true for $l = 0$, along with their proofs.

If follows immediately from the concluding remark in § 3 that when $\psi_1, \ldots, \psi_{r-1}$ vanish identically, any arbitrary transformation of the group will commute with at least one second transformation of it. Therefore, at least one straight line that maps to transformations that commute with each other will go through every point of the image space. Hence, all sub-determinants of degree $r - 1$ of $| \gamma_{\iota\kappa} |$ will also vanish identically. For many of these groups, all sub-determinants of lower degree will also vanish identically, and then every transformation will commute with a multiply-extended manifold of transformations. One can see the same thing without referring to the prior theorems when one establishes a certain form for the groups and considers that each of the two-parameter subgroups that belongs to the group will contain only commuting elements.

One arrives at that form in the following way: One chooses $X_0 f$ and $X_1 f$ to be two completely general (infinitesimal) transformations. $(X_0\ X_1)$ will then yield a transformation that is independent of both of them, and which will be denoted by $X_2 f$. Likewise, $(X_0\ X_2)$ might not be representable in terms of X_0, $X_1\ X_2$, and will be denoted by $X_3 f$. One proceeds in that way such that one will have:

$$(X_0\ X_1) = X_2 f, \quad (X_0\ X_2) = X_3 f, \quad (X_0\ X_3) = X_4 f, \quad \ldots, \quad (X_0\ X_{m-1}) = X_m f .$$

However, let the transformation $X_m f$ be the first one to which one arrives for which $(X_0\ X_m)$ can be represented in terms of $X_0, X_1, \ldots, X_m$. Then let:

$$(X_0\ X_m) = \sum_{\nu=0}^{m} e_\nu X_\nu f .$$

However, if the $e_0, e_1, \ldots, e_m$ do not all vanish here, and one takes ω to be a non-vanishing root of the equation:

$$\omega^m = e_1 + e_2\ \omega + e_3\ \omega^2 + \ldots + e_m\ \omega^{m-1},$$

then one can determine coefficients $p_0, p_1, \ldots, p_m$ such that one has:

$$\left(X_0, \sum p_\iota X_\iota\right) = \omega \sum p_\iota X_\iota ,$$

so the group will contain a two-parameter subgroup with no commuting elements. Therefore, $e_0, e_1, \ldots, e_m$ must all vanish, and since $(X_0\, X_m)$ must be expressible in terms of $X_0, X_1, \ldots, X_m$ at the very latest for $m = r - 1$, one will see that any transformation will commute with one of the others.

However, if one arrives at the equation $(X_0\, X_m) = 0$ along the path that was described for $m < r - 1$ then one can assume that m is the largest number that is possible in that regard. One then again chooses a transformation $X_{m+1}f$ that is completely arbitrary, but independent of $X_0, X_1, \ldots, X_m$, forms $(X_0\, X_{m+1})$, and if the expression for it is independent of $X_0, X_1, \ldots, X_m, X_{m+1}$ then one will set:

$$(X_0\, X_{m+1}) = X_{m+2}\,;$$

one defines:

$$(X_0\, X_{m+2}) = X_{m+3}\, f,\ \ldots,\ (X_0\, X_{m+m'-1}) = X_{m+m'}\, f,\ \ (X_0\, X_{m+m'}) = \sum_{\nu=0}^{m+m'} e_\nu X_\nu\, f\,,$$

from which the case of $m' = 1$ is excluded. We next seek to find a two-parameter subgroup of $X_0, X_1, \ldots, X_{m+m'}$. Due to the equation:

$$\omega^{m'} = e_{m+1} + \omega\, e_{m+2} + \ldots + \omega^{m'}\, e_{m+m'}\,,$$

it will always have a principal element if the coefficients $e_{m+1}, e_{m+2}, \ldots, e_{m+m'}$ do not vanish. However, if all or some of the coefficients $e_0, e_1, \ldots, e_m$ are non-zero then one will observe that from our assumption, the m-fold repetition of the operation $(X_0\, Y_1) = Y_2$, … must lead to $(X_0\, Y_m) = 0$. If we apply this to X_{m+1} then we will see that m' can be equal to at most m, and that for $m' = m$, one must necessarily have $(X_0\, X_{2m}) = 0$. On the other hand, performing that operation for $m' < m$ will show immediately that one needs only to replace X_{m+1} with a linear function of $X_{m+1}, X_1, \ldots, X_{m-1}$ in order for the coefficients $e_1, \ldots, e_m$ to vanish, and therefore e_0, as well.

One can proceed in the same way. If one again forms:

$$(X_0\, X_{m+m'+1}) = X_{m+m'+2}\,,\ \ldots,\ (X_0\, X_{m+m'+m''}) = \sum_{\nu=0}^{m+m'+m''} e_\nu X_\nu\, f$$

then one must have $m'' \leq m'$, and since at the very most an m-fold repetition of the given operation applied to $X_{m+m'+1}$ will lead to $(X_0\, X_\rho) = 0$, one can again assume that $(X_0\, X_{m+m'+m''}) = 0$.

That implies the proposition:

One can choose r mutually-independent infinitesimal transformations $X_0, X_1, \ldots X_{r-1}$ *in any group of rank zero such that one has the equations:*

(A) $\quad (X_0\, X_1) = a_1\, X_2\, f,\quad (X_0\, X_2) = a_2\, X_3\, f,\quad \ldots,\quad (X_0\, X_r) = a_r\, X_{r+1}\, f,\ \ldots,\ (X_0\, X_{r-1}) = 0,$

in which all coefficients $a_1, \ldots, a_{r-2}$ *are equal to* 1 *or* 0. If one already has $a_m = 0$ *for a general choice of* $X_0 f$ *and* $X_1 f$ *for* $m < r - 1$ *then at least one vanishing coefficient must appear in the* $a_{m+1}, \ldots, a_{2m}$; *if* $a_{m+m'}$ *is the first one then at least one of the coefficients* $a_{m+m'+1}, \ldots, a_{m+2m'}$ *will also vanish; etc.*

The Jacobi relation for $(0, m, m + m')$, $(0, m, m + m' + m'')$, …, as well as for $(0, \alpha, m)$, $(0, \alpha, m+m')$, …, where α means a symbol that is different from m, $m + m'$, …, yields:

(*a*) $(X_0\,(X_m\, X_{m+m'})) = 0, \ldots$

(*b*) $(X_0\,(X_\alpha\, X_m)) = (X_{\alpha+1}\, X_m)$, $(X_0\,(X_\alpha\, X_{m+1})) = (X_{\alpha+1}\, X_{m+m'})$, …

It follows immediately from (*a*), in conjunction with (*b*), for $a = m - 1$, $m + m' - 1$, …, that $(X_m\, X_{m+m'})$ and the corresponding expression can be expressed in terms of X_m, $X_{m+m'}$, $X_{m+m'+m''}$, … However, when the equations (*A*) are also valid, if one considers that X_{m+1}, $X_{m+m'+1}$, … can still be chosen in such a way that $X_{m+m'}$ can be added to an arbitrary linear function of X_m and $X_{m+m'}$, and if one further considers the character of the two-parameter subgroups that are even possible then it will follow that:

$$(X_m\, X_{m+m'}) = 0, \qquad (X_m\, X_{m+m'+m''}) = 0, \ldots$$

That consideration will also now show that one also has:

$$(X_{m-1}\, X_m) = (X_{m-1}\, X_{m+m'}) = \ldots = 0.$$

Assuming that, one will find in that way that:

$$(X_{\alpha+1}\, X_m) = (X_{\alpha+1}\, X_{m+m'}) = \ldots = 0.$$

It will then follow immediately from equations (*b*) that $(X_\alpha\, X_m)$, $(X_\alpha\, X_{m+m'})$ can contain at most X_m, $X_{m+m'}$, …, and the argument that was thus sketched out will teach us that $(X_\alpha\, X_m)$, $(X_\alpha\, X_{m+m'})$, … will also vanish. Therefore, all transformations that can be exhibited in the given way as commuting with X_0 will commute with all transformations of the group. We then see:

Any group of rank zero has a subgroup whose transformations commute with all transformations of the group. If all sub-determinants of rank $r - k$ *in the determinant* $|\gamma_{\iota\kappa}|$ *vanish then any manifold of transformations that commute with an arbitrarily-chosen one will be a* $(k - 1)$*-dimensional manifold of ones that commute with all of them.*

Lie called a subgroup of the given type a *distinguished subgroup* (Ann. Bd. 25, pp. 77, note). We will always find distinguished subgroups in the manner that was considered here then.

Similarly, he gave a simple method for finding that subgroup.

If α and β are different from m, $m + m'$, … then the Jacobi relation for $(0, \alpha, \beta)$ will yield the equation:

$$(c) \qquad (X_0\,(X_\alpha\,X_\beta)) = (X_{\alpha+1}\,X_\beta) + (X_\alpha\,X_{\beta+1}).$$

When we employ this equation, for the sake of convenience, we restrict ourselves to groups for which only all of the sub-determinants of degree $r - 1$ of $|\,\gamma_{\iota\kappa}\,|$ vanish, and thus, for which one must set $a_1 = a_2 = \ldots = a_{r-2} = 1$ in (A). When one sets $\beta = \alpha + 1$, $\beta = \alpha + 3$, …, in sequence, and further considers the equation $(X_{r-2}\,X_{r-1}) = 0$, one will see immediately that the $X_0 f$ and $X_1 f$ do not occur in the expression for $(X_\alpha\,X_\beta)$.

If not all sub-determinants of degree $r - 2$ *in the characteristic determinant vanish identically for* $l = 0$ *then the transformations* $X_2 f$, …, $X_{r-1} f$ *that are determined by the prescription above will define the principal subgroup.*

Similarly, (c) immediately implies that $(X_{r-3}\,X_{r-2})$, and in general $(X_\alpha\,X_{r-2})$, can be expressed in terms of only $X_{r-1} f$. Had we proved in general that $(X_{\alpha+1}\,X_{\beta+1})$ and $(X_\alpha\,X_{\beta+1})$ could be expressed in terms of only $X_{\beta+1}$, $X_{\beta+2}$, …, X_r for $\alpha < \beta$, then equations (c) would show that only X_β, $X_{\beta+1}$, …, X_r could occur in $(X_\alpha\,X_\beta)$, although the coefficient of X_β would again have to vanish in order for only two-parameter subgroups with commuting elements to occur in the group. The changes that would be necessary for that when the form (A) is assumed in full generality do not need to be given. We then arrive at the following theorem that Engel first presented and proved under a somewhat more general assumption:

Any r-parameter group G_r *for which* $l = 0$ *has an* $(r - 1)$*-parameter invariant subgroup* G_{r-1}, *which will also have an* $(r - 2)$*-parameter subgroup* G_{r-2} *that is invariant with respect to* G_r, *as well as* G_{r-1}; *it, in turn, will have an* $(r - 3)$*-parameter one that is invariant under* G_{r-2}, G_{r-1}, G_r, *etc.*

Since any invariant $(r - 1)$-parameter subgroup of the principal subgroup closes on itself, it will then follow that:

If $l = 0$, *and only the sub-determinants of degree* $r - 1$ *of* $|\,\gamma_{\iota\kappa}\,|$ *vanish identically then the group will have a simply-infinite family of* $(r - 1)$*-parameter invariants subgroups; any transformation that does not belong to the principal subgroup will belong to an* $(r - 1)$*-parameter invariant subgroup.*

We now infer some further consequences of equations (c) when we let α, β be the smallest pair of values in turn. It was proved already that for $\alpha = 1$, any $(X_\alpha\,X_\beta)$ could be represented in terms of those $X_\iota f$ for which $\iota \geq \alpha + \beta - 1$. Assume that this was proved for α, β and α, $\beta + 1$; from (c), that will then imply the same property for $(X_{\alpha+1}\,X_\beta)$. Therefore, the property will be true in full generality [naturally, only for $a_1 = \ldots, a_{r-2} = 1$ in (A)]. Hence, a well-defined $(r - 3)$-dimensional plane E_{r-3} lies in the $(r - 1)$-dimensional image space, and it exhibits the principal subgroup; a well-defined $(r - 4)$-dimensional plane E_{r-4} lies in it, etc. The product of an arbitrary point of the image space with a point of $E_{r-\alpha}$ will lead to a point in $E_{r-\alpha-1}$, and the product of a point in $E_{r-\alpha}$ with a point in $E_{r-\lambda}$ will lead to a point in $E_{r-\kappa-\lambda}$, and will vanish when that index become negative. In particular, one has:

If one constructs a k+1-*parameter group from the* k *transformations* $X_{r-\kappa}$, $X_{r-\kappa+1}$, …, X_{r-1}, *and a completely general transformation then it will be an invariant subgroup that has the group that is defined by the transformations* $X_{r-\kappa+1}$, …, X_{r-1} *as its principal subgroup.*

Now, in addition to equations (A), one can also assume the following equations:

$$(X_1\, X_2) = \alpha_2\, X_2 + \ldots + \alpha_{r-1}\, X_{r-1}\,,$$
$$(X_2\, X_3) = \beta_4\, X_4 + \ldots + \beta_{r-1}\, X_{r-1}\,,$$
$$(X_3\, X_1) = \gamma_6\, X_6 + \ldots + \gamma_{r-1}\, X_{r-1}\,, \ldots,$$

and derive all $(X_\alpha\, X_\beta)$ from this with the help of (c). However, for $r > 6$, further conditions must be added, and therefore it has not been possible for me to represent those groups explicitly, up to now.

Braunsberg, beginning of November 1887.