"Ueber das Gleichgewicht und die Bewegung eines unendlich dünnen elastischen Stabes," J. f. d. reine u. angew. Math. **56** (1858), 285-313.

On the equilibrium and the motion of an infinitely-thin elastic rod.

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Poisson, in his treatise on mechanics, developed a theory of the finite changes of form that an infinitely-thin, originally straight or curved, elastic rod would experience as a result of forces that acted, in part, on its interior, and in part, at its ends. *De Saint-Venant* then proved that the assumptions that *Poisson* started with were partially incorrect, and for the first time rigorously investigated the torsion and bending of an infinitely-thin rod of arbitrary cross-section while starting with the basic equations of the theory of elasticity. However, *de Saint-Venant* treated only the case for which the rod was originally cylindrical, the changes in form were infinitely small, and the axis of the rod was an axis of elasticity. In the present treatise, I will examine the changes of form of an infinitely-thin rod of everywhere equal cross-section, while starting with the equations of the theory of elasticity, but without making those restricting assumptions.

In the first paragraph, I will present some considerations regarding the basic equations of the theory of elasticity that precede their application to the case of an infinitely-thin rod. In § 2, that application will be made in full generality in the context of equilibrium and motion. In § 3, I will treat the equilibrium of an originally cylindrical rod that suffers a finite change of form as a result of forces that act upon the ends. There, we will find that the problem of determining the shape of the rod will lead to the same differential equations as the problem of the rotation of a massive body around a fixed point. Finally, in § 4, I will develop an example of the equilibrium of an originally-curved rod that is under the influence of forces that act upon the ends by examining the change of form that a helix that is defined by a wire surface of circular cross-section will experience as a result of a force that acts on a point on its axis in the direction of that axis when one end of the helix is fixed.

§ 1.

Let x, y, z be the rectangular coordinates of a point of a homogeneous, elastic body in its natural state; I will use that term to refer to the state in which no dilatations (or contractions) are present anywhere. The body might suffer a change in form as a result of infinitely-small forces that partially act in the interior and partially on the outer surface, and which I would like to call *external*, in order to distinguish them from *elastic* forces. After that deformation, let $x + u$, $y + v$, $z + w$ be the coordinates of the previouslyconsidered point that I would like to refer to as the point (x, y, z). One thinks of a plane that is perpendicular to the x-axis as being laid through that point after the change of form. It will divide the body into two pieces: Let the components along the coordinate axes of the elastic force per unit area that the part that corresponds to the larger values of x exerts upon the other one at the point (x, y, z) be:

$$X_x, Y_x, Z_x,$$

and the symbols:

$$X_y, Y_y, Z_y,$$
$$X_z, Y_z, Z_z$$

shall have analogous meanings. One will then have:

$$Y_x = X_y, \qquad Z_y = Y_x, \qquad X_z = Z_x .$$

Moreover, I set:

$$x_x = \frac{\partial u}{\partial x}, \qquad y_z = z_y = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z},$$
$$y_y = \frac{\partial v}{\partial y}, \qquad z_x = x_z = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x},$$
$$z_z = \frac{\partial w}{\partial z}, \qquad x_y = y_x = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}.$$

I let:

$$iX, iY, iZ$$

denote the components of the external force per unit volume that acts upon the point (x, y, z) inside of the body after the change of form. Furthermore, let:

$$i(X), i(Y), i(Z)$$

be the components per unit area that is exerted upon the outer surface at the point (x, y, z), in which, I understand i to mean an infinitely-small constant and X, Y, Z, (X), (Y), (Z) to mean finite quantities. Finally, I write the equation of the outer surface of the body in its natural state as:

$$g = 0.$$

Under the assumption that the nine differential quotients of u, v, w with respect to x, y, z are infinitely small, the six quantities X_x, X_y, … will be linear, homogeneous functions of the six quantities x_x, x_y, …, whose coefficients will be the constants of elasticity of the body, and for the case of equilibrium, one will have:

(1)
$$\begin{cases} \dfrac{\partial X_x}{\partial x}+\dfrac{\partial X_y}{\partial y}+\dfrac{\partial X_z}{\partial z}=-iX, \\ \dfrac{\partial Y_x}{\partial x}+\dfrac{\partial Y_y}{\partial y}+\dfrac{\partial Y_z}{\partial z}=-iY, \\ \dfrac{\partial Z_x}{\partial x}+\dfrac{\partial Z_y}{\partial y}+\dfrac{\partial Z_z}{\partial z}=-iZ \end{cases}$$

for every point inside of the body, and:

(2)
$$\begin{cases} X_x\dfrac{\partial g}{\partial x}+X_y\dfrac{\partial g}{\partial y}+X_z\dfrac{\partial g}{\partial z}=i(X)\sqrt{\left(\dfrac{\partial g}{\partial x}\right)^2+\left(\dfrac{\partial g}{\partial y}\right)^2+\left(\dfrac{\partial g}{\partial z}\right)^2}, \\ Y_x\dfrac{\partial g}{\partial x}+Y_y\dfrac{\partial g}{\partial y}+Y_z\dfrac{\partial g}{\partial z}=i(Y)\sqrt{\left(\dfrac{\partial g}{\partial x}\right)^2+\left(\dfrac{\partial g}{\partial y}\right)^2+\left(\dfrac{\partial g}{\partial z}\right)^2}, \\ Z_x\dfrac{\partial g}{\partial x}+Z_y\dfrac{\partial g}{\partial y}+Z_z\dfrac{\partial g}{\partial z}=i(Z)\sqrt{\left(\dfrac{\partial g}{\partial x}\right)^2+\left(\dfrac{\partial g}{\partial y}\right)^2+\left(\dfrac{\partial g}{\partial z}\right)^2}, \end{cases}$$

where the positive square roots should be taken for the points inside of the body for which g is negative.

Six arbitrary constants enter into the general solutions of these differential equations for u, v, w. Namely, the expressions for u, v, w contain the additive terms:

$$a_0 + cy - bz, \quad b_0 + za - cx, \quad c_0 + bx - ay,$$

in which a_0, b_0, c_0, a, b, c are arbitrary constants. It follows from this that the quantities u, v, w enter into equations (1) and (2) only to the extent that they imply the values of x_x, x_y, …, and those values will remain unchanged when one adds the given terms to the u, v, w. The six constants a_0, b_0, c_0, a, b, c shall be obtained by requiring that:

(3)
$$u = 0,\ v = 0,\ w = 0, \qquad \frac{\partial v}{\partial x} = 0,\ \frac{\partial w}{\partial x} = 0,\ \frac{\partial w}{\partial y} = 0$$

for the point ($x = 0$, $y = 0$, $z = 0$), which might lie inside the body. Of the nine quantities:

$$\begin{matrix} 1, & \dfrac{\partial v}{\partial x}, & \dfrac{\partial w}{\partial x}, \\ \dfrac{\partial u}{\partial y}, & 1, & \dfrac{\partial w}{\partial y}, \\ \dfrac{\partial u}{\partial z}, & \dfrac{\partial v}{\partial z}, & 1, \end{matrix}$$

the three in the top row are the cosines of the angle that a line element that is originally parallel to the x-axis defines with that axis after the deformation, and the other ones have corresponding meanings, so the last three equations in (3) express the idea that the direction of a line element that is originally laid through the point ($x = 0$, $y = 0$, $z = 0$) parallel to the x-axis will not change, and a line element that is originally laid through the same point and parallel to the y-axis will remain perpendicular to the z-axis.

Equations (1), (2), and (3) determine the functions u, v, w uniquely, as we shall verify later on [after equation (9)]. The expressions for u, v, w and their differential quotients with respect to x, y, z must therefore contain the factor i. They will then be of order i when all dimensions of the body are finite, or if I am to be more precise, when only finite constants are present in the function g. The assumption that the nine differential quotients of u, v, w with respect to x, y, z are infinitely-small, which is the only case in which equations (1) and (2) will be correct, will be fulfilled in that case. That assumption will not generally be fulfilled if g contains an infinitely-small constant; however, it will also be satisfied in the case that I shall now consider.

Let all dimensions of the body be infinitely thin, and of the same order. Rather, to speak more definitely, let the equation $g = 0$ be such that when one sets:

$$x = \mathfrak{i}\mathfrak{x}, \quad y = \mathfrak{i}\mathfrak{y}, \quad z = \mathfrak{i}\mathfrak{z} \tag{4}$$

in it, where $\mathfrak{i}$ means an infinitely-small constant, it will go to an equation:

$$\mathfrak{g} = 0,$$

whose left-hand side is a function of $\mathfrak{x}$, $\mathfrak{y}$, $\mathfrak{z}$ that contains only finite constants. The quantities X, Y, Z, (X), (Y), (Z) shall be functions of x, y, z, and $\mathfrak{i}$, but in such a way that they will stay finite for all values of x, y¸ z that are inside the body.

One thinks of the substitutions (4) as having been carried out in equations (1), (2), and (3). If one makes:

$$\begin{aligned}
\mathfrak{x}_x &= \frac{\partial u}{\partial \mathfrak{x}}, &\qquad \mathfrak{y}_z &= \frac{\partial w}{\partial \mathfrak{y}} + \frac{\partial v}{\partial \mathfrak{z}}, \\
\mathfrak{y}_y &= \frac{\partial v}{\partial \mathfrak{y}}, &\qquad \mathfrak{z}_x &= \frac{\partial u}{\partial \mathfrak{z}} + \frac{\partial w}{\partial \mathfrak{x}}, \\
\mathfrak{z}_x &= \frac{\partial w}{\partial \mathfrak{z}}, &\qquad \mathfrak{x}_y &= \frac{\partial v}{\partial \mathfrak{x}} + \frac{\partial u}{\partial \mathfrak{y}},
\end{aligned}$$

and lets $\mathfrak{X}_x$, $\mathfrak{Y}_x$, $\mathfrak{Z}_x$, … denote the expressions that one obtains when one replaces x_x , y_x , z_x , … with $\mathfrak{x}_x$, $\mathfrak{y}_x$, $\mathfrak{z}_x$, …, respectively, in the expressions that represent X_x, Y_x, Z_x, … as functions of x_x, y_x, z_x, …, respectively, then one will get:

(5)
$$\left\{\begin{aligned} \frac{\partial \mathfrak{X}_x}{\partial \mathfrak{x}} + \frac{\partial \mathfrak{X}_y}{\partial \mathfrak{y}} + \frac{\partial \mathfrak{X}_z}{\partial \mathfrak{z}} &= -i\mathfrak{i}^2 X, \\ \frac{\partial \mathfrak{Y}_x}{\partial \mathfrak{x}} + \frac{\partial \mathfrak{Y}_y}{\partial \mathfrak{y}} + \frac{\partial \mathfrak{Y}_z}{\partial \mathfrak{z}} &= -i\mathfrak{i}^2 Y, \\ \frac{\partial \mathfrak{Z}_x}{\partial \mathfrak{x}} + \frac{\partial \mathfrak{Z}_y}{\partial \mathfrak{y}} + \frac{\partial \mathfrak{Z}_z}{\partial \mathfrak{z}} &= -i\mathfrak{i}^2 Z, \end{aligned}\right.$$

so for $\mathfrak{g} = 0$:

(6)
$$\left\{\begin{aligned} \mathfrak{X}_x \frac{\partial \mathfrak{g}}{\partial \mathfrak{x}} + \mathfrak{X}_y \frac{\partial \mathfrak{g}}{\partial \mathfrak{y}} + \mathfrak{X}_z \frac{\partial \mathfrak{g}}{\partial \mathfrak{z}} &= i\mathfrak{i}(X)\sqrt{\left(\frac{\partial \mathfrak{g}}{\partial \mathfrak{x}}\right)^2 + \left(\frac{\partial \mathfrak{g}}{\partial \mathfrak{y}}\right)^2 + \left(\frac{\partial \mathfrak{g}}{\partial \mathfrak{z}}\right)^2}, \\ \mathfrak{Y}_x \frac{\partial \mathfrak{g}}{\partial \mathfrak{x}} + \mathfrak{Y}_y \frac{\partial \mathfrak{g}}{\partial \mathfrak{y}} + \mathfrak{Y}_z \frac{\partial \mathfrak{g}}{\partial \mathfrak{z}} &= i\mathfrak{i}(Y)\sqrt{\left(\frac{\partial \mathfrak{g}}{\partial \mathfrak{x}}\right)^2 + \left(\frac{\partial \mathfrak{g}}{\partial \mathfrak{y}}\right)^2 + \left(\frac{\partial \mathfrak{g}}{\partial \mathfrak{z}}\right)^2}, \\ \mathfrak{Z}_x \frac{\partial \mathfrak{g}}{\partial \mathfrak{x}} + \mathfrak{Z}_y \frac{\partial \mathfrak{g}}{\partial \mathfrak{y}} + \mathfrak{Z}_z \frac{\partial \mathfrak{g}}{\partial \mathfrak{z}} &= i\mathfrak{i}(Z)\sqrt{\left(\frac{\partial \mathfrak{g}}{\partial \mathfrak{x}}\right)^2 + \left(\frac{\partial \mathfrak{g}}{\partial \mathfrak{y}}\right)^2 + \left(\frac{\partial \mathfrak{g}}{\partial \mathfrak{z}}\right)^2}, \end{aligned}\right.$$

and for $\mathfrak{x} = 0$, $\mathfrak{y} = 0$, $\mathfrak{z} = 0$:

(7)
$$u = 0,\ v = 0,\ w = 0, \qquad \frac{\partial v}{\partial \mathfrak{x}} = 0,\ \frac{\partial w}{\partial \mathfrak{x}} = 0,\ \frac{\partial w}{\partial \mathfrak{y}} = 0.$$

If the quantity $\mathfrak{i}$ does not vanish by carrying out the substitutions (4) in the expressions for X, Y, Z, (X), (Y), (Z) then one can set those six quantities equal to the finite boundary values that they must approach when $\mathfrak{i}$ approaches zero as a result of the assumption that was made; i.e., one can consider X, Y, Z, (X), (Y), (Z) to be finite and functions of $\mathfrak{x}$, $\mathfrak{y}$, $\mathfrak{z}$ that are independent of $\mathfrak{i}$. On that basis, the values that u, v, w get from equations (5), (6), (7) must have the same order as the products $i\mathfrak{i}$. The differential quotients of u, v, w with respect to $\mathfrak{x}$, $\mathfrak{y}$, $\mathfrak{z}$ are of the same order, and thus the differential quotients of u, v, w with respect to x, y, z are of order i.

For that reason, equations (1) and (2) will also be true for the case that is now considered; however, they can be simplified essentially. Namely, if one ponders the fact that the right-hand sides of equations (5) are infinitely small with respect to the right-hand sides of equations (6) then one will see that the quantities X, Y, Z exert only a vanishingly small influence on the values of u, v, w that one can therefore neglect, and equations (1) can be replaced with the following ones:

$$(8)\qquad \begin{cases} \dfrac{\partial X_x}{\partial x}+\dfrac{\partial X_y}{\partial y}+\dfrac{\partial X_z}{\partial z}=0, \\ \dfrac{\partial Y_x}{\partial x}+\dfrac{\partial Y_y}{\partial y}+\dfrac{\partial Y_z}{\partial z}=0, \\ \dfrac{\partial Z_x}{\partial x}+\dfrac{\partial Z_y}{\partial y}+\dfrac{\partial Z_z}{\partial z}=0. \end{cases}$$

These equations were derived under the assumption that iX, iY, iZ had the same order as $i(X)$, $i(Y)$, $i(Z)$. Obviously, they will also be true in the case for which each of the three forces are infinitely small with respect to them, but not in the converse case. The conclusion that should be inferred below from equations (8) will thus also be true only under the assumption that the latter case is not present.

Equations (1) and (2) can be combined into *one* equation. Of the 36 constants that are contained in the equations that represent X_x, Y_x, … as functions of x_x, y_x, …, fifteen of them must be equal to fifteen of the other ones, such that the expression:

$$X_x\, dx_x + Y_x\, dy_y + Z_x\, dz_y + Y_z\, dy_z + Z_x\, dz_x + X_y\, dx_y$$

is the complete differential of a homogeneous function of degree two in the six variables x_x, y_x, z_z, y_z, z_x, x_y (*). If F is that function then equations (1) and (2) will mean the same thing as the one equation:

$$(9)\qquad \delta\Omega - \delta \int F\, dx\, dy\, dz = 0,$$

in which, the first term on the left-hand side means the moment of the external forces for infinitely-small variations of u, v, w, and the second one means the corresponding variation of the integral that it enters into over the volume of the body.

The aforementioned assertion that equations (1), (2), (3) determine the functions u, v, w uniquely can be proved from a certain property of the function F.

If that were not the case then there would have to be non-zero values of u, v, w that would satisfy equations (1), (2), (3) when the right-hand sides of equations (1) and (2) are set equal to zero. It shall be shown that such values do not exist. If one multiplies equation (1) by $u\, dx\, dy\, dz$, $v\, dx\, dy\, dz$, $w\, dx\, dy\, dz$, and integrates them over the volume of the body, divides equations (2) by:

(*) The validity of this assertion follows easily from the principles of the mechanical theory of heat. Namely, if the expression were not a complete differential then one could get work with the help of the elastic body when one let pressure forces act upon its outer surface that one varied in such a way that one would take the body back to its original state again. From those principles, that could not be the case without a corresponding loss of heat. If such a thing took place then the agreement with the aforementioned principles would not be produced, since one would convert heat into work with the help of the elastic body without it being necessary for the body to have a different temperature. I believe that this argument was already presented by *W. Thomson* in the Quarterly Mathematical Journal (April, 1855); I have not been able to find the cited reference, though.

$$\sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + \left(\frac{\partial g}{\partial z}\right)^2},$$

multiplies that by $u\,dO$, $v\,dO$, $w\,dO$, where dO means an element of the outer surface of the body, and integrates that over that outer surface then one will get:

$$0 = \int F\,dx\,dy\,dz$$

by addition for the case in which X, Y, Z, (X), (Y), $(Z) = 0$. The function F is presented below [in equation (29)] for a body whose elasticity is the same in all directions. Here, F has the property that it never becomes negative and vanishes only when the six arguments x_x, x_y, … are equal to zero. Since the differences between the elasticities are typically small in bodies that possess different elasticities in different directions, one can assume that F will have that property for all bodies that exist in nature. It will then follow from the equation that was derived that the six quantities x_x, x_y, … must be equal to zero in the entire body. In order to prove that $u = 0$, $v = 0$, $w = 0$ for the entire body from this and equations (3), I shall show how one can find u when x_x, x_y, … are given. One has:

$$u = (u)_0 + \int \left(\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz\right),$$

in which $(u)_0$ means the value of u for $x = 0$, $y = 0$, $z = 0$, and the integration is extended from an arbitrary point of the path of the point ($x = 0$, $y = 0$, $z = 0$) to the point (x, y, z). With a similar notation, one will have:

$$\frac{\partial u}{\partial y} = \left(\frac{\partial u}{\partial y}\right)_0 + \int \left(\frac{\partial^2 u}{\partial x\,\partial y}dx + \frac{\partial^2 u}{\partial y^2}dy + \frac{\partial^2 u}{\partial y\,\partial z}dz\right),$$

and one will have:

$$\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial x_x}{\partial y},$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial y_x}{\partial y} - \frac{\partial y_y}{\partial x},$$

$$\frac{\partial^2 u}{\partial y\,\partial z} = \frac{1}{2}\left(-\frac{\partial y_z}{\partial x} + \frac{\partial z_x}{\partial y} + \frac{\partial x_y}{\partial z}\right).$$

One thinks of substituting these values into the equation for $\partial u / \partial y$. One can derive an expression for $\partial u / \partial z$ that is similar to what one gets for $\partial u / \partial y$, while $\partial u / \partial x$ has the simpler expression x_x. Now, if the six quantities x_x, x_x, … are equal to zero and equations (3) are valid then it will follow from this that $\partial u / \partial x$, $\partial u / \partial y$, $\partial u / \partial z$ are equal to zero, and then furthermore that $u = 0$. One can obviously derive the facts that v and w also vanish in that way.

Equation (9) – like equations (1) and (2) – is true only when all of the dimensions of the body have the same order. However, one can also exhibit an equation of a similar form for the case in which that condition is not fulfilled. In that case, one thinks of the body as having been divided into parts that each have one dimension of the same order. One puts one of those parts into its natural state and then brings into a position that shall be characterized immediately: Let x, y, z then be the coordinates of a point of the part relative to a rectangular coordinate system whose origin might lie in the part itself. Let $x + u$, $y + v$, $z + w$ be the coordinates of that point relative to the same coordinate system when the part is brought back to its changed form and position. That position of the part that is still left undetermined in its natural state shall be chosen in such a way that equations (3) are still valid for $x = 0$, $y = 0$, $z = 0$. Equation (9) is then true for the part considered when one considers the elastic force that is exerted on its outer surface by the neighboring parts in the construction of $\delta\Omega$. If one exhibits equation (9) for all parts into which the body is thought to be decomposed and takes the sum then one will get:

$$\delta\Omega - \int F\,dx\,dy\,dz = 0, \tag{10}$$

in which $\delta\Omega$ means the moment of the external forces that act, in part on the interior of the body and in part on the outer surface, since the moment of the elastic forces that act upon the boundary surfaces of the individual components will vanish.

Equation (10), which includes equation (9) as a special case, admits a further useful generalization. Namely, it can be made independent of the assumption that x, y, z, u, v, w refer to the *natural* state of the corresponding component of the body. If the x, y, z, u, v, w refer to a state in which arbitrary, but infinitely small, dilatations exist, and if u', v', w' are the values that u, v, w assume when one lets the component considered go from its natural state to an arbitrary state then equations (1), (2), (3) will be valid when one sets u, v, w equal to $u - u'$, $v - v'$, $w - w'$ in them. The same substitution must then also make equations (9) and (10) valid for that case.

One can easily go from equation (10), which relates to the equilibrium of elastic body, to the case of its motion by a known principle of mechanics: If t is time and T is one-half of the *vis viva* then the equation:

$$\int dt\,\{\delta T + \delta\Omega - \delta\Sigma \int F\,dx\,dy\,dz\} = 0 \tag{11}$$

will be true for the motion.

§ 2.

Equations (10) and (11) shall now be applied to an infinitely-thin rod of everywhere equal cross-section upon whose lateral surface no external forces act.

First, one might assume that the rod is cylindrical in its natural state. In that state, one imagines a rectangular system of axes in the rod. The first axis shall be the line in which the center of mass of the cross-section lies, while the other two shall be parallel to the principal axes of a cross-section that goes through the center of mass itself. One chooses a point P on the first axis and directs one's attention to three line elements that are drawn from P outward in the directions of the three axes; I call them 0, 1, 2. Thus, 0 shall be the one that has the direction of the length of the cylinder. When the rod has suffered a change in form, these three line elements will no longer be perpendicular to each other, in general, but will define angles that deviate from right angles by quantities that have the order of the dilatations that are present. The position of the points of the rod that are in the vicinity of *P* shall be referred to a rectangular coordinate system whose origin is *P*, whose x-axis has the direction of the line element 0, and whose z-axis is perpendicular to the line element 1. Let the coordinates of a point of the rod relative to that coordinate system be:

$x + u, y + v, z + w,$	when the rod has its altered form and position,
$x, y, z,$	when the rod is in its original state and is in the position for which the line elements 0, 1, 2, fall along the x, y, z axes, respectively.

If one then establishes that x shall take on only values that have the order of the cross-sectional dimensions of the rod then the symbols x, y, z, u, v, w will have the same meaning that they had in the derivation of equation (10). Moreover, let ξ, η, ζ be the coordinates of a point *P* after the deformation of the rod relative to another rectangular coordinate system that is chosen in space arbitrarily. I let:

$$\begin{matrix} \alpha_0, & \beta_0, & \gamma_0, \\ \alpha_1, & \beta_1, & \gamma_1, \\ \alpha_2, & \beta_2, & \gamma_2 \end{matrix}$$

be the cosines of the angles that the x, y, z axes define with the ξ, η, ζ axes, in such a way that the index 0 refers to the x-axis, the index 1, to the y-axis, and the index 2, to the z-axis, respectively. It shall be assumed that the two coordinate systems that we speak of have the property that the x-axis can be made parallel to the ξ-axis, the y-axis, to the η-axis, and the z-axis, to the ζ-axis by rotation.

The coordinates of the point whose coordinates are $x + u, y + v, z + w$ relative to the x, y, z axes can be expressed relative to the ξ, η, ζ axes with the help of the symbols that were introduced. The aforementioned coordinates will have the values:

$$(12) \qquad \left\{ \begin{aligned} &\xi + \alpha_0(x+u) + \alpha_1(y+v) + \alpha_2(z+w), \\ &\eta + \beta_0(x+u) + \beta_1(y+v) + \beta_2(z+w), \\ &\zeta + \gamma_0(x+u) + \gamma_1(y+v) + \gamma_2(z+w). \end{aligned} \right.$$

If s means the distance from the point *P* to the origin of the rod in its original state then these three quantities must be functions of $s + x$; i.e., their partial differential quotients with respect to x must be equal to their partial differential quotients with respect to s. If one ponders the fact that ξ, η, ζ and the quantities α, β, γ do not include x then it will follow from this that:

$$\alpha_0\left(1+\frac{\partial u}{\partial x}\right)+\alpha_1\frac{\partial v}{\partial x}+\alpha_2\frac{\partial w}{\partial x} = \frac{d\xi}{ds}+\frac{d\alpha_0}{ds}(x+u)+\frac{d\alpha_1}{ds}(y+v)+\frac{d\alpha_2}{ds}(z+w)$$
$$+\ \alpha_0\frac{\partial u}{\partial s}+\alpha_1\frac{\partial v}{\partial s}+\alpha_2\frac{\partial w}{\partial s},$$

$$\beta_0\left(1+\frac{\partial u}{\partial x}\right)+\beta_1\frac{\partial v}{\partial x}+\beta_2\frac{\partial w}{\partial x} = \frac{d\eta}{ds}+\frac{d\beta_0}{ds}(x+u)+\frac{d\beta_1}{ds}(y+v)+\frac{d\beta_2}{ds}(z+w)$$
$$+\ \beta_0\frac{\partial u}{\partial s}+\beta_1\frac{\partial v}{\partial s}+\beta_2\frac{\partial w}{\partial s},$$

$$\gamma_0\left(1+\frac{\partial u}{\partial x}\right)+\gamma_1\frac{\partial v}{\partial x}+\gamma_2\frac{\partial w}{\partial x} = \frac{d\zeta}{ds}+\frac{d\gamma_0}{ds}(x+u)+\frac{d\gamma_1}{ds}(y+v)+\frac{d\gamma_2}{ds}(z+w)$$
$$+\ \alpha_0\frac{\partial u}{\partial s}+\alpha_1\frac{\partial v}{\partial s}+\alpha_2\frac{\partial w}{\partial s}.$$

These equations shall be multiplied, first, by α_0, β_0, γ_0, then by α_1, β_1, γ_1, and then finally by α_2, β_2, γ_2, and then added. If one sets:

$$(13)\qquad \left\{\begin{array}{l} \varepsilon = \sqrt{\left(\dfrac{d\xi}{ds}\right)^2+\left(\dfrac{d\eta}{ds}\right)^2+\left(\dfrac{d\zeta}{ds}\right)^2}-1, \\ \text{from which, it will follow that:} \\ \dfrac{d\xi}{ds}=\alpha_0(1+\varepsilon),\ \dfrac{d\eta}{ds}=\beta_0(1+\varepsilon),\ \dfrac{d\zeta}{ds}=\gamma_0(1+\varepsilon), \end{array}\right.$$

and one further sets:

$$p = \alpha_1\frac{d\alpha_2}{ds}+\beta_1\frac{d\beta_2}{ds}+\gamma_1\frac{d\gamma_2}{ds},$$
$$q = \alpha_2\frac{d\alpha_0}{ds}+\beta_2\frac{d\beta_0}{ds}+\gamma_2\frac{d\gamma_0}{ds},$$
$$r = \alpha_0\frac{d\alpha_1}{ds}+\beta_0\frac{d\beta_1}{ds}+\gamma_0\frac{d\gamma_1}{ds}$$

then one will find, in the manner that was described, and upon recalling the known relations between the quantities α, β, γ:

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial s}+r\,(y+v)-q\,(z+w)+\varepsilon,$$
$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial s}+p\,(z+w)-r\,(x+u),$$

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial s} + q\,(x+u) - p\,(y+v)\,.$$

It will then emerge from the considerations that were presented in § 1 that u, v, w are infinitely small compared to $\partial u / \partial x$, $\partial v / \partial x$, $\partial w / \partial x$. Assuming that $\partial u / \partial s$, $\partial v / \partial s$, $\partial w / \partial s$ are not infinitely large in comparison to u, v, w, the differential quotients with respect to s will then be infinitely small in comparison to x. If one neglects infinitely smaller quantities of higher order then one will have:

$$\frac{\partial u}{\partial x} = ry - qz + \varepsilon,$$
$$\frac{\partial v}{\partial x} = pz - rx,$$
$$\frac{\partial w}{\partial x} = qx - py.$$

By integration, one will find from this that:

$$(14)\qquad \left\{\begin{aligned} u &= u_0 + (ry - qz + \varepsilon)x \\ v &= v_0 + pzx - \frac{r}{2}x^2, \\ w &= w_0 + \frac{q}{2}x^2 - pxy, \end{aligned}\right.$$

in which u_0, v_0, w_0 refer to quantities that are independent of x.

If one constructs the values of x_x, y_x, … with the help of these expressions for u, v, w then that will yield:

$$(15)\qquad \left\{\begin{aligned} x_x &= ry - qz + \varepsilon, & y_z &= \frac{\partial v_0}{\partial z} + \frac{\partial w_0}{\partial y}, \\ y_y &= \frac{\partial v_0}{\partial y}, & z_x &= \frac{\partial u_0}{\partial z} - py, \\ z_z &= \frac{\partial w_0}{\partial z}, & x_y &= \frac{\partial u_0}{\partial y} + pz. \end{aligned}\right.$$

All of these values are independent of x. As a result, equations (8), which can substitute for equations (1), will assume the following form:

(16)
$$\begin{cases} \dfrac{\partial X_y}{\partial y} + \dfrac{\partial X_z}{\partial z} = 0, \\ \dfrac{\partial Y_y}{\partial y} + \dfrac{\partial Y_z}{\partial z} = 0, \\ \dfrac{\partial Z_y}{\partial y} + \dfrac{\partial Z_z}{\partial z} = 0. \end{cases}$$

If one understands $g = 0$ to be the equation of the contour of the cross-section of the rod, considers that g is independent of x and that (X), (Y), $(Z) = 0$ for the lateral surface of the rod then for $g = 0$, equations (2) will give:

(17)
$$\begin{cases} X_y \dfrac{\partial g}{\partial y} + X_z \dfrac{\partial g}{\partial z} = 0, \\ Y_y \dfrac{\partial g}{\partial y} + Y_z \dfrac{\partial g}{\partial z} = 0, \\ Z_y \dfrac{\partial g}{\partial y} + Z_z \dfrac{\partial g}{\partial z} = 0. \end{cases}$$

Finally, equations (3) say that for $y = 0$, $z = 0$:

(18)
$$u_0 = 0, \qquad v_0 = 0, \qquad w_0 = 0, \qquad \frac{\partial w_0}{\partial y} = 0.$$

If one substitutes the expressions for X_y, X_z, … in terms of x_x, y_x in equations (16) and (17) then equations (16), (17), (18) will determine the quantities u_0, v_0, w_0 uniquely, and indeed, as linear, homogeneous functions of p, q, r, ε. The fact that u_0, v_0, w_0 are determined follows from the fact that when one sets $p = 0$, $q = 0$, $r = 0$, $\varepsilon = 0$, equations (16), (17), (18) cannot be fulfilled except when $u_0 = 0$, $v_0 = 0$, $w_0 = 0$. The validity of that assertion is deduced from considerations that are similar to the ones that were presented above on pp. 6. If one substitutes the values of u_0, v_0, w_0 that (16), (17), (18) yield into equations (14) then one will also find u, v, w as linear, homogeneous functions of p, q, r, ε; their coefficients will be independent of s. Therefore, if dp / ds, dq / ds, dr / ds, $d\varepsilon / ds$ are not infinitely large in comparison to p, q, r, s, respectively, then equations (13) will fulfill the assumption that was made about $\partial u / \partial s$, $\partial v / \partial s$, $\partial w / \partial s$. If one substitutes the values of u_0, v_0, w_0 in equations (15) then that will also yield x_x, x_y, … as linear, homogeneous functions of p, q, r, ε, and if one introduces them into the expression for F then one will obtain a homogeneous function of degree two of the same four quantities for F. That function will be independent of x, since x_x, x_y, … are independent of x. If one sets:

$$\int F \, dy \, dz = f,$$

in which the integration is extended over the cross-section of the rod, then f will be a homogeneous function of degree two in p, q, r, ε whose coefficients depend upon only the constants of the cross-section and the elasticity of the rod. By introducing that quantity f, equations (10) and (11) will become:

(19) $$\delta\Omega - \delta\int f\,ds = 0$$

and

(20) $$\int dt\,\{\delta T + \delta\Omega - \delta\int f\,ds\} = 0.$$

From the considerations that were discussed, the determination of the coefficients of the function f demands the solution of three simultaneous partial differential equations, in general. That determination will be lightened substantially when one assumes that the axis of the cylindrical rod is parallel to an elastic axis. In that case, the expressions for X_x, X_y, … in terms of x_x, x_y, … will be the following ones (*):

$$\begin{aligned}
X_x &= A_{00}\,x_x + A_{01}\,y_y + A_{02}\,z_z + A_{03}\,y_x\,,\\
Y_y &= A_{10}\,x_x + A_{11}\,y_y + A_{12}\,z_z + A_{13}\,y_x\,,\\
Z_z &= A_{20}\,x_x + A_{21}\,y_y + A_{22}\,z_z + A_{23}\,y_x\,,\\
Y_z &= A_{30}\,x_x + A_{31}\,y_y + A_{32}\,z_z + A_{33}\,y_x\,,\\
Z_x &= A_{44}\,z_x + A_{45}\,x_y\,,\\
X_y &= A_{54}\,z_x + A_{55}\,x_y\,,
\end{aligned}$$

in which:

$$A_{01} = A_{10}\,, \qquad A_{01} = A_{10}\,, \ldots$$

If one recalls equations (15) then the first of equations (16) will become:

(21) $$A_{44}\frac{\partial^2 u_0}{\partial z^2} + 2A_{44}\frac{\partial^2 u_0}{\partial y\,\partial z} + A_{55}\frac{\partial^2 u_0}{\partial y^2} = 0,$$

and the first of equations (17):

(22) $$\left[A_{44}\left(\frac{\partial u_0}{\partial z} - py\right) + A_{45}\left(\frac{\partial u_0}{\partial y} + pz\right)\right]\frac{\partial g}{\partial z} + \left[A_{54}\left(\frac{\partial u_0}{\partial z} - py\right) + A_{55}\left(\frac{\partial u_0}{\partial y} + pz\right)\right]\frac{\partial g}{\partial y} = 0.$$

u_0 can be determined from these two equations and the first of equations (18). The remaining equations (16), (17), (18) serve to determine v_0 and w_0 . As *Saint-Venant* first remarked in his investigations into the torsion of prisms, one will satisfy them when one sets:

(23) $$Y_y = 0,\ Z_z = 0,\ Y_z = 0.$$

If fact, if one solves these equations for y_y, z_z, y_z then when one sets x_x equal to its value in (15), one will obtain linear expressions for y and z for these three quantities, and thus expressions that fulfill the equation:

(*) Berl. Ber., "über die Fortschritte der Physik in den Jahren 1850 and 1851," pp. 245.

$$\frac{\partial^2 y_y}{\partial z^2} + \frac{\partial^2 z_z}{\partial y^2} = \frac{\partial^2 y_z}{\partial y \, \partial z},$$

in which the condition for that to be true is that the two quantities u_0, w_0 can be determined from the three equations:

$$y_y = \frac{\partial v_0}{\partial y}, \qquad z_z = \frac{\partial w_0}{\partial z}, \qquad y_z = \frac{\partial w_0}{\partial y} + \frac{\partial v_0}{\partial z}.$$

The integration of these equations will introduce three arbitrary constants, and the last three of equations (18) can be satisfied by a suitable choice of them.

The value of T in equation (20) shall now be developed. To that end, one must differentiate the expression (12) with respect to t and take the sum of the squares of the differential quotients. One has:

$$u = a_0\, p + a_1\, q + a_2\, r + a_3\, \varepsilon,$$

in which a_0, a_1, a_2, a_3 are independent of t, and in which, if the dimensions of the cross-section of the rod are denoted by infinitely-small first-order quantities then a_3 will be of order one, while a_0, a_1, a_2 are of order two; similar expressions will be true for v and w. One sees that the orders of the coefficients a are given correctly most easily from equations (14). It follows from the expression for u that:

$$\frac{du}{dt} = a_0 \frac{dp}{dt} + a_1 \frac{dq}{dt} + a_2 \frac{dr}{dt} + a_3 \frac{d\varepsilon}{dt};$$

similarly equations will be true for dv / dt and dw / dt. One can conclude from the value of ε that is given in (13) that $d\varepsilon / dt$ will not be infinitely large in comparison to the quantities $d\xi / dt$, $d\eta / dt$, $d\zeta / dt$ if one assumes that the differential quotients of these quantities with respect to s are not themselves infinitely large in comparison to the latter quantities. It follows from the values of p, q, r, by a similar assumption, that none of the quantities dp / dt, dq / dt, dr / dt are infinitely large in comparison to the nine quantities:

$$\frac{d\alpha_0}{dt}, \quad \frac{d\beta_0}{dt}, \quad \frac{d\gamma_0}{dt}, \quad \frac{d\alpha_1}{dt}, \quad \frac{d\beta_1}{dt}, \quad \frac{d\gamma_1}{dt}, \quad \frac{d\alpha_2}{dt}, \quad \frac{d\beta_2}{dt}, \quad \frac{d\gamma_2}{dt}.$$

That will yield the fact that by neglecting infinitely-small quantities of higher degree, one can next write the differential quotients of the expressions (12) as:

$$\frac{d\xi}{dt} + x\frac{d\alpha_0}{dt} + y\frac{d\alpha_1}{dt} + z\frac{d\alpha_2}{dt},$$

$$\frac{d\eta}{dt} + x\frac{d\beta_0}{dt} + y\frac{d\beta_1}{dt} + z\frac{d\beta_2}{dt},$$

$$\frac{d\psi}{dt}+x\frac{d\gamma_0}{dt}+y\frac{d\gamma_1}{dt}+z\frac{d\gamma_2}{dt}.$$

It now follows further from equations (13) then none of the quantities $d\alpha_0 / dt$, $d\beta_0 / dt$, $d\gamma_0 / dt$ will be infinitely large in comparison to the three quantities $d\xi / dt$, $d\eta / dt$, $d\zeta / dt$. On that basis, the terms in the expressions that were just given that were endowed with x can be neglected. By comparison, the terms that contained y and z cannot be omitted, in general, since their coefficients can be infinitely large in comparison to $d\xi / dt$, $d\eta / dt$, $d\zeta / dt$.

If one defines the sum of the squares of the given expressions in light of that, multiplies that by $dy\,dz$, and integrates over the cross-section of the rod then since, from the assumptions that were made at the start of this paragraph, one has:

$$\int y\,dy\,dz = 0, \quad \int y\,dy\,dz = 0, \quad \int y\,dy\,dz = 0,$$

one will get:

$$\left[\left(\frac{d\xi}{dt}\right)^2+\left(\frac{d\eta}{dt}\right)^2+\left(\frac{d\zeta}{dt}\right)^2\right]\int dy\,dz$$
$$+\left[\left(\frac{d\alpha_1}{dt}\right)^2+\left(\frac{d\beta_1}{dt}\right)^2+\left(\frac{d\gamma_1}{dt}\right)^2\right]\int y^2 dy\,dz$$
$$+\left[\left(\frac{d\alpha_2}{dt}\right)^2+\left(\frac{d\beta_2}{dt}\right)^2+\left(\frac{d\gamma_2}{dt}\right)^2\right]\int z^2 dy\,dz\,.$$

One now sets:

$$\begin{aligned}
\alpha_1\frac{d\alpha_2}{dt}+\beta_1\frac{d\beta_2}{dt}+\gamma_1\frac{d\gamma_2}{dt} &= P,\\
\alpha_2\frac{d\alpha_0}{dt}+\beta_2\frac{d\beta_0}{dt}+\gamma_2\frac{d\gamma_0}{dt} &= Q,\\
\alpha_1\frac{d\alpha_0}{dt}+\beta_1\frac{d\beta_0}{dt}+\gamma_1\frac{d\gamma_0}{dt} &= -R;
\end{aligned}$$

one then has:

$$\begin{aligned}
\left(\frac{d\alpha_1}{dt}\right)^2+\left(\frac{d\beta_1}{dt}\right)^2+\left(\frac{d\gamma_1}{dt}\right)^2 &= P^2+R^2,\\
\left(\frac{d\alpha_2}{dt}\right)^2+\left(\frac{d\beta_2}{dt}\right)^2+\left(\frac{d\gamma_2}{dt}\right)^2 &= P^2+Q^2.
\end{aligned}$$

If one substitutes these values in the last expression and considers that of the quantities P, Q, R, only the first one can be infinitely large in comparison to $d\xi / dt$, $d\eta / dt$, $d\zeta / dt$, then when one sets, to abbreviate:

$$\int dy\,dz = \lambda, \qquad \int (y^2+z^2)\,dy\,dz = \mu,$$

it will become:

$$\lambda\left[\left(\frac{d\xi}{dt}\right)^2+\left(\frac{d\eta}{dt}\right)^2+\left(\frac{d\zeta}{dt}\right)^2\right]+\mu P^2.$$

If one multiplies this expression by $\frac{1}{2}\rho\,ds$ (when one understands ρ to mean the density of the rod) and integrates over the length of the rod then one will get the value of T that is to be substituted in equation (20). If no external forces act upon the rod then that equation will become:

(24) $$0=\delta\iint dt\,ds\left\{\tfrac{1}{2}\rho\left[\lambda\left(\left(\frac{d\xi}{dt}\right)^2+\left(\frac{d\eta}{dt}\right)^2+\left(\frac{d\zeta}{dt}\right)^2\right)+\mu P^2\right]-f\right\}.$$

Equations (19), (20), and (24) were derived under the assumption that the rod was cylindrical in its natural state. However, they are also true with a certain modification when the rod is not cylindrical in its natural state and is curved arbitrarily, as long as the cross-section is still the same everywhere. Under that assumption, the rod can be made cylindrical by suitable forces that inside of it. Hence, its parts will suffer infinitely-small dilatations. If one refers the quantities x, y, z and u, v, w to the state in which the rod is then found, instead of to its natural state, and lets u', v', w' denote the values that u, v, w assume when one lets the rod go from its natural state into an arbitrary position then equations (10) and (11) will be correct when one sets $u-u'$, $v-v'$, $w-w'$ in F in place of u, v, w. Equations (19), (20), and (24) will then also be true when one sets $p-p'$, $q-q'$, $r-r'$, $\varepsilon-\varepsilon'$ in f in place of p, q, r, ε, in which p', q', r', ε' mean the values that p, q, r, ε assume when one lets the rod go from its natural state to an arbitrary position. In fact, in that case, $u-u'$, $v-v'$, $w-w'$ will be the same linear functions of $p-p'$, $q-q'$, $r-r'$, $\varepsilon-\varepsilon'$ as the previous ones u, v, w were of p, q, r, ε. In order to see the truth of that assertion, one must only ponder the facts that equations (14) are also true here, and that those equations can be derived in the same was as the one that will arise from (14) when one applies a prime to the symbols u, v, w, u_0, v_0, w_0, p, q, r (so that u_0', v_0', w_0' mean the values of u', v', w' for $x = 0$), and equations (16), (17), (18) will be correct when one replaces u, v, w with $u-u'$, $v-v'$, $w-w'$ in them.

§ 3.

Equation (19) shall now be developed further under the assumption that no other external forces act upon the rod than ones that have their point of application at its ends.

Only four desired functions of s enter into the expression for f, namely, p, q, r, ε. However, they are defined by the differential quotients of ξ, η, ζ, α_0, β_0, γ_0, α_1, β_1, γ_1, α_2, β_2, γ_2, between which, certain condition equations exist. For the moment, I would like to denote the sixteen stated unknown functions of s by y_1, y_2, …, and the condition equations that exist between them by $\varphi_1 = 0$, $\varphi_2 = 0$, … If one sets:

$$U = f + \lambda_1\varphi_1 + \lambda_2\varphi_2 + \ldots,$$

in which λ_1, λ_2, … mean new unknown functions of s, then the equation (19) will be equivalent to the following one:

(25) $$0 = \delta\Omega - \delta \int U \, ds \, .$$

Since, by assumption, $\delta\Omega$ should depend upon only variations of the ends of the rod, one must then have:

(26) $$\frac{\partial U}{\partial y} = \frac{d}{ds} \frac{\partial U}{\partial \frac{dy}{ds}}$$

for every y.

I shall now combine the condition equations $\varphi_1 = 0$, $\varphi_2 = 0$, … and add the notations for the factors λ_1, λ_2, … that I would like to introduce.

Condition equations	**Factors**
$\frac{d\xi}{ds} - \alpha_0 (1 + \varepsilon) = 0,$	$A,$
$\frac{d\eta}{ds} - \beta_0 (1 + \varepsilon) = 0,$	$B,$
$\frac{d\zeta}{ds} - \gamma_0 (1 + \varepsilon) = 0,$	$C,$
$\alpha_1 \frac{d\alpha_2}{ds} + \beta_1 \frac{d\beta_2}{ds} + \gamma_1 \frac{d\gamma_2}{ds} - p = 0,$	M_0 ,
$\alpha_2 \frac{d\alpha_0}{ds} + \beta_2 \frac{d\beta_0}{ds} + \gamma_2 \frac{d\gamma_0}{ds} - q = 0,$	M_1 ,
$\alpha_0 \frac{d\alpha_1}{ds} + \beta_0 \frac{d\beta_1}{ds} + \gamma_0 \frac{d\gamma_1}{ds} - r = 0,$	M_2 ,
$\alpha_0^2 + \beta_0^2 + \gamma_0^2 - 1 = 0,$	$\frac{1}{2}\lambda_{00}$,
$\alpha_1^2 + \beta_1^2 + \gamma_1^2 - 1 = 0,$	$\frac{1}{2}\lambda_{11}$,
$\alpha_2^2 + \beta_2^2 + \gamma_2^2 - 1 = 0,$	$\frac{1}{2}\lambda_{22}$,
$\alpha_1\alpha_2 + \beta_1\beta_2 + \gamma_1\gamma_2 = 0,$	λ_{12} ,
$\alpha_2\alpha_0 + \beta_2\beta_0 + \gamma_2\gamma_0 = 0,$	λ_{20} ,
$\alpha_0\alpha_1 + \beta_0\beta_1 + \gamma_0\gamma_1 = 0,$	λ_{01} .

If one sets y equal to p, q, r, ε, ξ, η, ζ, α_0, β_0, γ_0, α_1, β_1, γ_1, α_2, β_2, γ_2 in turn in equation (26) then one will get:

$$\frac{\partial f}{\partial p} = M_0 \, , \qquad \frac{\partial f}{\partial q} = M_1 \, , \qquad \frac{\partial f}{\partial r} = M_2 \, ,$$

$$\frac{\partial f}{\partial \varepsilon} = A\,\alpha_0 + B\,\beta_0 + C\,\gamma_0\,,$$

$$\frac{dA}{ds} = 0, \qquad \frac{dB}{ds} = 0, \qquad \frac{dC}{ds} = 0,$$

$$\frac{d(\alpha_2 M_1)}{ds} = M_2\,\frac{d\alpha_2}{ds} + \lambda_{00}\,\alpha_0 + \lambda_{01}\,\alpha_1 + \lambda_{02}\,\alpha_2 - A\,(1+\varepsilon),$$
$$\frac{d(\beta_2 M_1)}{ds} = M_2\,\frac{d\beta_2}{ds} + \lambda_{00}\,\beta_0 + \lambda_{01}\,\beta_1 + \lambda_{02}\,\beta_2 - B\,(1+\varepsilon),$$
$$\frac{d(\gamma_2 M_1)}{ds} = M_2\,\frac{d\gamma_2}{ds} + \lambda_{00}\,\gamma_0 + \lambda_{01}\,\gamma_1 + \lambda_{02}\,\gamma_2 - C\,(1+\varepsilon),$$

$$\frac{d(\alpha_0 M_2)}{ds} = M_0\,\frac{d\alpha_2}{ds} + \lambda_{10}\,\alpha_0 + \lambda_{11}\,\alpha_1 + \lambda_{12}\,\alpha_2\,,$$
$$\frac{d(\beta_0 M_2)}{ds} = M_0\,\frac{d\beta_2}{ds} + \lambda_{10}\,\beta_0 + \lambda_{11}\,\beta_1 + \lambda_{12}\,\beta_2\,,$$
$$\frac{d(\gamma_0 M_2)}{ds} = M_0\,\frac{d\gamma_2}{ds} + \lambda_{10}\,\gamma_0 + \lambda_{11}\,\gamma_1 + \lambda_{12}\,\gamma_2\,,$$
$$\frac{d(\alpha_1 M_0)}{ds} = M_1\,\frac{d\alpha_0}{ds} + \lambda_{20}\,\alpha_0 + \lambda_{21}\,\alpha_1 + \lambda_{22}\,\alpha_2\,,$$
$$\frac{d(\beta_1 M_0)}{ds} = M_1\,\frac{d\beta_0}{ds} + \lambda_{20}\,\beta_0 + \lambda_{21}\,\beta_1 + \lambda_{22}\,\beta_2\,,$$
$$\frac{d(\gamma_1 M_0)}{ds} = M_1\,\frac{d\gamma_2}{ds} + \lambda_{20}\,\gamma_0 + \lambda_{21}\,\gamma_1 + \lambda_{22}\,\gamma_2\,,$$

in which:

$$\lambda_{21} = \lambda_{12}\,, \quad \lambda_{02} = \lambda_{20}\,, \quad \lambda_{10} = \lambda_{01}\,.$$

If one takes three equations from the last three groups of these equations, multiplies them by α_0, β_0, γ_0 , then α_0, β_0, γ_0 , and then α_0, β_0, γ_0 , and adds them then one will find the following:

$$\begin{aligned}
-M_1\,q &= \lambda_{00} + M_2\,r - (1+\varepsilon)\,(A\alpha_0 + B\beta_0 + C\gamma_0),\\
M_1\,p &= \lambda_{01} \qquad\quad - (1+\varepsilon)\,(A\alpha_1 + B\beta_1 + C\gamma_1),\\
\frac{dM_1}{ds} &= \lambda_{02} - M_2\,p - (1+\varepsilon)\,(A\alpha_2 + B\beta_2 + C\gamma_2),\\
\frac{dM_2}{ds} &= \lambda_{10} - M_0\,q,\\
-M_2\,r &= \lambda_{11} + M_2\,p,\\
M_2\,q &= \lambda_{12}\,,\\
M_0\,r &= \lambda_{20}\,,\\
\frac{dM_0}{ds} &= \lambda_{21} - M_1\,r,
\end{aligned}$$

$$-M_0\, p \;\; = \lambda_{22} + M_1\, q.$$

Now, since $\lambda_{21} = \lambda_{12}$, $\lambda_{02} = \lambda_{20}$, $\lambda_{10} = \lambda_{01}$, when one employs the fact that e is infinitely small that will yield:

$$(27) \qquad \left\{ \begin{aligned} \frac{dM_0}{ds} &= M_2 q - M_1 r, \\ \frac{dM_1}{ds} &= M_0 r - M_2 p - (A\alpha_2 + B\beta_2 + C\gamma_2), \\ \frac{dM_2}{ds} &= M_1 p - M_0 q + A\alpha_1 + B\beta_1 + C\gamma_1. \end{aligned} \right.$$

The quantities M_0, M_1, M_2, A, B, C that enter into these equations have a simple meaning that I would like to derive.

The beginning of the rod shall be assumed to be fixed, and equation (25) will apply to the part of the rod from the beginning up to a cross-section that is determined by a certain value of s. One can then understand $\delta\Omega$ to mean the moment of the elastic forces that are exerted upon that cross-section by the other parts of the rod that correspond to larger values of s. Equation (25) will then become:

$$0 = \delta\Omega - \sum \frac{\partial U}{\partial \frac{dy}{ds}} \delta y \,,$$

or, when developed:

$$\begin{aligned} \delta\Omega = A\ \delta\xi + B\ \delta\eta + C\ \delta\zeta & \\ + M_0\, (\alpha_1\ \delta\alpha_2 + \beta_1\ \delta\beta_2 + \gamma_1\ \delta\gamma_2) & \\ + M_1\, (\alpha_2\ \delta\alpha_0 + \beta_2\ \delta\beta_0 + \gamma_2\ \delta\gamma_0) & \\ + M_2\, (\alpha_0\ \delta\alpha_1 + \beta_0\ \delta\beta_1 + \gamma_0\ \delta\gamma_1) & \,. \end{aligned}$$

It follows from this that A, B, C are the sums of the components along the ξ, η, ζ-axes, respectively, of the elastic forces that are exerted upon the cross-section that is determined by the assumed value of s by those ports of the rods that correspond to larger values of s, and the M_0, M_1, M_2 are the rotational moments of those forces relative to the x, y, z-axes. The sense in which that rotational moment is regarded as positive can be given as follows:

If one establishes *the* sequence of axes that makes the y-axis follow the x-axis, and then the z-axis follow the y-axis, and finally, the x-axis follow the z-axis then the rotational moment relative to one of the axes will be positive when (assuming that axis is the first one) the points of the third axis make it rotate towards the direction of the second one.

It might be remarked that the equations that express the idea that A, B, C are independent of s and equations (27) can be derived from this interpretation for A, B, C, M_0, M_1, M_2 when one applies the six equilibrium conditions for a rigid body to a piece of the rod that is bounded by two arbitrary cross-sections. Namely, if one sets:

$$M_\xi = M_0\,\alpha_0 + M_1\,\alpha_1 + M_2\,\alpha_2\,,$$
$$M_\eta = M_0\,\beta_0 + M_1\,\beta_1 + M_2\,\beta_2\,,$$
$$M_\zeta = M_0\,\gamma_0 + M_1\,\gamma_1 + M_2\,\gamma_2\,,$$

then these conditions will give:

$$A = \text{const.,} \qquad B = \text{const.,} \qquad C = \text{const.,}$$

$$M_\xi + (B\zeta - C\eta) = \text{const.,}$$
$$M_\eta + (C\xi - A\zeta) = \text{const.,}$$
$$M_\zeta + (A\eta - B\xi) = \text{const.}$$

If one differentiates the last three equations, multiplies them first by α_0, β_0, γ_0, then by α_1, β_1, γ_1, and finally, by α_2, β_2, γ_2, and adds them each time then one will get equations (27).

One now must set:

$$M_0 = \frac{\partial f}{\partial p} = a_{00}\,(p - p') + a_{01}\,(q - q') + a_{02}\,(r - r') + a_{03}\,(\varepsilon - \varepsilon'),$$
$$M_1 = \frac{\partial f}{\partial q} = a_{10}\,(p - p') + a_{11}\,(q - q') + a_{12}\,(r - r') + a_{13}\,(\varepsilon - \varepsilon'),$$
$$M_2 = \frac{\partial f}{\partial r} = a_{20}\,(p - p') + a_{21}\,(q - q') + a_{22}\,(r - r') + a_{23}\,(\varepsilon - \varepsilon')$$

in those equations; if one sets:

$$A\alpha_0 + B\beta_0 + C\gamma_0 = S,$$

to abbreviate, then one will get:

$$S = \frac{\partial f}{\partial \varepsilon} = a_{30}\,(p - p') + a_{31}\,(q - q') + a_{32}\,(r - r') + a_{33}\,(\varepsilon - \varepsilon')\,.$$

Here, the quantities a depend upon the constants of the cross-section and the elasticity of the rod, and the relations $a_{01} = a_{10}$, $a_{02} = a_{20}$, … exist between them. The quantities a are not all of the same order. Since $\varepsilon - \varepsilon'$ is a number and $p - p'$, $q - q'$, $r - r'$ are reciprocal lengths, the quantities a that have the index 3 once must have one dimension less than the quantities a for which the index the index 3 does not occur, and one dimension more than a_{33}. However, the lengths that enter into the expressions for the quantities a are of order the dimensions of the cross-section of the rod, and thus, infinitely small. Hence, a_{03}, a_{13}, a_{23} must be infinitely small compared to a_{33} and infinitely large compared to the other quantities a. On that basis, the terms in the equations that were just given that are endowed with $\varepsilon - \varepsilon'$ will not be neglected when $p - p'$, $q - q'$, $r - r'$ are finite. It follows from the last of these equations:

$$\varepsilon - \varepsilon' = -\frac{a_{30}\,p + a_{31}\,q + a_{32}\,r - S}{a_{33}}\,;$$

one thinks of these values as being substituted in the expressions for M_0, M_0, M_2. Due to the relationships between the quantities a that were just given, the terms in them that contain S will then be infinitely small compared to M_0, M_0, M_2, in the event that S is not infinitely large compared to M_0, M_0, M_2 . If that case were excluded then one would have:

$$(28) \qquad \begin{cases} M_0 = b_{00}(p-p') + b_{01}(q-q') + b_{02}(r-r'), \\ M_1 = b_{10}(p-p') + b_{01}(q-q') + b_{02}(r-r'), \\ M_2 = b_{20}(p-p') + b_{21}(q-q') + b_{22}(r-r'), \end{cases}$$

in which the quantities b can be expressed in a simple way in terms of the quantities a, and where $b_{01} = b_{10}$, $b_{12} = b_{21}$, $b_{20} = b_{02}$.

The condition under which these equations are true – namely, the condition that S is not infinitely large compared to M_0, M_1, M_2 – will be fulfilled when the axis of the rod is finitely curved for the equilibrium figure considered, regardless of whether that axis is or is not curved in the natural state. In fact, it follows from equations (27) that the expressions $A\alpha_1 + B\beta_1 + C\gamma_1$ and $A\alpha_2 + B\beta_2 + C\gamma_2$ are of the same order as M_0, M_1, M_2 . The same expressions must then be infinitely small compared to S when S is infinitely large compared to M_0, M_1, M_2 . However, if that were the case then the ratios $A : B : C$ would deviate infinitely little from the ratios $\alpha_0 : \beta_0 : \gamma_0$; i.e., the direction of the tangent to the axis of the rod must deviate infinitely little everywhere from the direction of the resultant of the constant forces A, B, C.

It shall be assumed that the rod is cylindrical in its natural state – i.e., that $p' = 0$, $q' = 0$, $r' = 0$. If one substitutes the values that M_0, M_1, M_2 take on as a result of equations (28) into equations (27) then one will arrive at differential equations that are identical with the ones that the investigation of the rotation of a massive body about a fixed point will lead to when one gives the following meanings to the symbols that are used here in the context of rotating bodies:

The ξ, η, ζ axes are the axes of a coordinate system that is fixed in space. The x, y, z axes are the axes of a coordinate system that is fixed in the body at time s. The origin of the latter is the point of rotation, and its x-axis goes through the center of mass. A, B, C are the negative components of the weight of the body along the ξ, η, ζ axes, when it is multiplied by the x-coordinate of the center of mass. Finally, if m is the mass of a spatial element of the body that has the coordinates x, y, z then one will have:

$$\begin{aligned} b_{00} &= \sum m\,(y^2 + z^2), & b_{12} &= -\sum m\,y\,z, \\ b_{11} &= \sum m\,(z^2 + x^2), & b_{20} &= -\sum m\,z\,x, \\ b_{22} &= \sum m\,(x^2 + y^2), & b_{01} &= -\sum m\,x\,y. \end{aligned}$$

If the corresponding problem of the rotation has been solved then the determination of the forms of elastic rods will require that one perform three quadratures. Namely, one obtains the running coordinates of a point on the axis of the rod from the equations:

$$\xi = \int \alpha_0\, ds, \qquad \eta = \int \beta_0\, ds, \qquad \zeta = \int \gamma_0\, ds .$$

§ 4.

Finally, I would like to apply the theory that was developed to a simple case in which the rod is not straight in its natural state. Let the rod be a wire with a circular cross-section and equal elasticities in all directions whose axis defines a helix in the natural state.

For a body whose elasticity is the same in all directions, one has the equations (*):

$$X_x = 2K\{(1+\theta)\,x_x + \theta\,y_y + \theta\,z_z\},$$
$$Y_y = 2K\{\theta\,x_x + (1+\theta)\,y_y + \theta\,z_z\},$$
$$Z_z = 2K\{\theta\,x_x + \theta\,y_y + (1+\theta)\,z_z\},$$

$$Y_z = K\,y_z\,, \qquad Z_x = K\,z_x\,, \qquad X_y = K\,x_y\,.$$

For the case in which x, y, z, u, v, w are referred to the natural state of the body, it will then follow that:

$$F = K\,\{x_x^2 + y_y^2 + z_z^2 + \tfrac{1}{2}y_x^2 + \tfrac{1}{2}z_x^2 + \tfrac{1}{2}x_y^2 + \theta(x_x + y_y + z_z)^2\,. \tag{29}$$

Equations (23) give:

$$y_y = z_z = -\frac{\theta}{1+2\theta}\,x_x\,, \qquad y_z = 0.$$

Equations (21) and (22) become:

$$\frac{\partial^2 u_0}{\partial y^2} + \frac{\partial^2 u_0}{\partial z^2} = 0$$

and for $g = 0$:

$$\left(\frac{\partial u_0}{\partial z} - py\right)\frac{\partial g}{\partial z} + \left(\frac{\partial u_0}{\partial y} + pz\right)\frac{\partial g}{\partial y} = 0.$$

Since the cross-section of the wire is supposed to be a circle, one must have $g = y^2 + z^2$ – const. It then follows from these two equations, in conjunction with the first of equations (18), that $u_0 = 0$. Equations (15) then give:

$$x_x = ry - qz + \varepsilon, \qquad z_x = -\,py, \qquad x_y = pz.$$

One will then have:

$$F = K\left\{\frac{1+3\theta}{1+2\theta}(ry - qz + \varepsilon)^2 + \tfrac{1}{2}p^2(y^2 + z^2)\right\}.$$

(*) In these equations, the quantities K and θ have the same meaning as they did in my treatise "über das Gleichgewicht und die Bewegung einer elastische Scheibe," this journal, Band 40. I take this occasion to remark that the theory of equilibrium and motion of an infinitely-thin, elastic plate can be developed more rigorously than it was there in a manner that is similar to the one that have pursued in this treatise for a rod, and that the case in which the plate has different elasticities in different directions can be treated in that way.

If one now defines:

$$f = \int F\, dy\, dz,$$

and then employs the fact that:

$$\int y\, dy\, dz = 0, \quad \int z\, dy\, dz = 0, \quad \int yz\, dy\, dz = 0,$$

and sets:

$$\int dy\, dz = \lambda, \quad \int y^2\, dy\, dz = \int z^2\, dy\, dz = \frac{\mu}{2},$$

according to the notation that was used above, then one will find:

$$f = K \left\{ \frac{\mu}{2} p^2 + \frac{1+3\theta}{1+2\theta} \frac{\mu}{2} (q^2 + r^2) + \frac{1+3\theta}{1+2\theta} \lambda \varepsilon^2 \right\}.$$

Therefore, all of the quantities a_{00}, a_{01}, … that have unequal indices are equal to zero, and one will have:

$$a_{00} = K\mu, \qquad a_{11} = a_{22} = \frac{1+3\theta}{1+2\theta} K\mu, \qquad a_{33} = 2\, \frac{1+3\theta}{1+2\theta} K\lambda .$$

Of the quantities b_{00}, b_{01}, … that are introduced by means of (28), the ones that have unequal indices are likewise equal to zero, and one will have:

$$b_{00} = K\mu, \qquad b_{11} = b_{22} = \frac{1+3\theta}{1+2\theta} K\mu .$$

For the sake of brevity, we shall set:

$$b_{00} = L, \qquad b_{11} = b_{22} = N.$$

I shall let ξ', η', ζ', α_0', β_0', γ_0', α_1', β_1', γ_1', α_2', β_2', γ_2' denote the values that ξ, η, ζ, α_0, β_0, γ_0, α_1, β_1, γ_1, α_2, β_2, γ_2 assume when the line is brought from its natural state into a certain position. That position can be chosen such that:

$$\begin{aligned} \xi' &= s \cdot \cos \vartheta', \\ \eta' &= \frac{1}{n'} \sin \vartheta' \cdot \sin n's, \\ \zeta' &= -\frac{1}{n'} \sin \vartheta' \cdot \cos n's, \end{aligned}$$

in which ϑ' and n' are constants. In this, ϑ' means the angle that a tangent to the helix defines with its axis, and $\dfrac{\sin \vartheta'}{n'}$ is the radius of the cylinder surface on which the helix lies. These values of ξ, η, ζ yield the following values of α_0', β_0', γ_0':

$$\alpha_0' = \cos\vartheta',$$
$$\beta_0' = \sin\vartheta' \cdot \sin n's,$$
$$\gamma_0' = \sin\vartheta' \cdot \cos n's.$$

If the cross-section of the wire were not a circle then its natural form would also determine the values of α_1', β_1', γ_1', α_2', β_2', γ_2', up to a sign. However, since the cross-section is assumed to be circular, one of these quantities will remain arbitrary and can be assumed to be equal to an arbitrary function of s; I set:

$$\alpha_1' = \sin\vartheta' \cdot \cos l's,$$

in which I understand l' to mean an arbitrary constant (*). When one arbitrarily assigns the sign of α_2', which remains undetermined, the relations between the quantities α', β', γ' will yield:

$$\beta_1' = -\cos\vartheta' \cdot \cos n's \cdot \cos l's - \sin n's \cdot \sin l's,$$
$$\gamma_1' = -\cos\vartheta' \cdot \sin n's \cdot \cos l's + \cos n's \cdot \sin l's,$$
$$\alpha_2' = \sin\vartheta' \cdot \sin l's,$$
$$\beta_2' = -\cos\vartheta' \cdot \cos n's \cdot \sin l's + \sin n's \cdot \cos l's,$$
$$\gamma_2' = -\cos\vartheta' \cdot \sin n's \cdot \sin l's - \cos n's \cdot \cos l's.$$

One further finds from this that:

$$p' = l' - n'\cos\vartheta',$$
$$q' = -n'\sin\vartheta' \cdot \cos l's,$$
$$r' = -n'\sin\vartheta' \cdot \sin l's.$$

One now sets ξ, η, ζ, α_0, β_0, γ_0, α_1, β_1, γ_1, α_2, β_2, γ_2 equal to the expressions that arise from the expressions for ξ, η, ζ, … when one replaces the constants ϑ', n', l' in them with the new constants ϑ, n, l. All of the differential equations of the problem, with the exception of equations (27), will then be satisfied, which might also be the values of ϑ, n, l. Due to the fact that one established suitable relations between the constants ϑ, n, l, ϑ', n', l', A, B, C, one can also fulfill them. In fact, when one applies equations (28) and the values of the quantities b that were developed above, equations (27) will become:

$$L\,\frac{d(p-p')}{ds} = N\,(rq' - qr'),$$
$$N\,\frac{d(q-q')}{ds} = L\,(p-p')\,r - N\,(r-r')\,p - (A\,\alpha_2 + B\,\beta_2 + C\,\gamma_2),$$
$$N\,\frac{d(r-r')}{ds} = N\,(q-q')\,p - L\,(p-p')\,q + A\,\alpha_1 + B\,\beta_1 + C\,\gamma_1.$$

(*) The calculation can be shortened somewhat by setting $l' = 0$; still, I prefer to leave that constant undetermined.

These will be satisfied when one sets:

$$l = l',$$
$$A = \frac{n}{\sin\vartheta}\{L\,(n\cos\vartheta - n\cos\vartheta)\sin\vartheta - N\,(n\sin\vartheta - n\sin\vartheta)\cos\vartheta\},$$
$$B = 0,\quad C = 0.$$

The last two of these equations express one of the conditions under which the expressions that are assumed for $\xi, \eta, \zeta, \ldots$ will be valid. They say that the force that acts upon the end of the wire must have the same direction as the axis of the helix. Another condition gets appended to that one that refers to the rotational moment that is exerted upon the end of the wire. Equations (28) give:

$$M_0 = -L\,(n\cos\vartheta - n'\cos\vartheta'),$$
$$M_1 = -N\,(n\sin\vartheta - n'\sin\vartheta')\cos l's,$$
$$M_2 = -N\,(n\sin\vartheta - n'\sin\vartheta')\sin l's.$$

In these equations, one might understand (?) to mean the value that relates to the end of the wire. If one forms:

$$M_\xi = M_0\,\alpha_0 + M_1\,\alpha_1 + M_2\,\alpha_2,$$
$$M_\eta = M_0\,\beta_0 + M_1\,\beta_1 + M_2\,\beta_2,$$
$$M_\zeta = M_0\,\gamma_0 + M_1\,\gamma_1 + M_2\,\gamma_2$$

then M_ξ, M_η, M_ζ will be the rotational moments that act externally upon the end of the wire relative to the three axes that are laid parallel to the end of the axes of the $\xi, \eta, \zeta, \ldots$ One finds:

$$M_\xi = -\{L\,(n\cos\vartheta - n'\cos\vartheta')\cos\vartheta + N\,(n\sin\vartheta - n'\sin\vartheta')\sin\vartheta\},$$
$$M_\eta = -\{L\,(n\cos\vartheta - n'\cos\vartheta')\sin\vartheta - N\,(n\sin\vartheta - n'\sin\vartheta')\cos\vartheta\}\cos ns,$$
$$M_\zeta = -\{L\,(n\cos\vartheta - n'\cos\vartheta')\sin\vartheta - N\,(n\sin\vartheta - n'\sin\vartheta')\cos\vartheta\}\sin ns.$$

The last two of these equations can be written:

$$M_\eta = A\,\zeta,\quad M_\zeta = -A\,\eta,$$

where η and ζ and refer to the end of the wire. It follows from this that M_η and M_ζ are equal to precisely the rotational moments that the force A would produce if it had its point of application at a point in the axis of the helix that was fixed on the end of the wire.

Thus, the expressions that were assumed for $\xi, \eta, \zeta, \ldots$ will be valid when the deformation of the line is produced by a force A that acts upon a point of the axis of the helix that is fixed on the end of the wire and in the direction of that axis, and a rotational moment M_ξ around that axis. If A and M_ξ are given then the equations for A and M_ξ will determine the two unknown constants n and ϑ that enter into the expressions for ξ, η, ζ. If they are found then one will have the expression for the elongation of the helix:

$$s\,(\cos\vartheta - \cos\vartheta'),$$

and the expression:

$$s\,(n - n')$$

for the rotation of its end around its axis, in which s again refers to the end.

One must set $M = 0$ in order to arrive at the case that was referred to in the introduction as subject of this paragraph. *J. Thomson* (*) has already treated that case; however, the considerations that he applied to it are not rigorous, and the result to which he arrived is not precise.

Heidelberg, 1858.

(*) Mech. Mag., L, pp. 160 and 207.