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Lecture twenty-eight

(Finite deformations of an infinitely-thin, originally-cylindrical rod. Dilatations of a small part of it. Simplifications that arise when the cross-section is an ellipse or its plane is a symmetry plane. Potential of the forces that produce the dilatations. *Vis viva* of the rod. Equilibrium of the rod under the influence of thrusts that act upon the ends. Agreement of the problem that is treated here with the problem of the rotation of a ponderable body around a fixed point. The rod can define a helix. Equilibrium of a curved rod that originally defines a helix.)

§ 1.

We shall now address the equilibrium and the motion of bodies, some of whose dimensions are infinitely small; thin rods and plates can be regarded in that way approximately. The bodies that we would now like to consider can suffer *finite* deformations while the dilatations continue to be infinitely small. We can also apply our theory to such cases when we think of the body as being divided into pieces whose dimensions all have the same order and then think of the equations that are imposed as relating to *one* of those pieces.

We imagine a body (or part of a body) whose dimensions are all of the same order as the infinitely-small quantity i and summarize the conditions for its equilibrium. Equations (9) of the previous lecture belong to that case, and thus the equations:

$$\begin{aligned}\mu X &= \frac{\partial X_x}{\partial x}+\frac{\partial X_y}{\partial y}+\frac{\partial X_z}{\partial z},\\ \mu Y &= \frac{\partial X_x}{\partial x}+\frac{\partial X_y}{\partial y}+\frac{\partial X_z}{\partial z},\\ \mu Z &= \frac{\partial X_x}{\partial x}+\frac{\partial X_y}{\partial y}+\frac{\partial X_z}{\partial z}.\end{aligned}\tag{1}$$

Let g be a function of x, y, z, so:

$$g = 0$$

is the equation of the outer surface of the body, and g is positive in its interior; n, in turn, is the interior normal of an element of that surface. One will then have:

$$\cos(n\,x) : \cos(n\,y) : \cos(n\,z) = \frac{\partial g}{\partial x} : \frac{\partial g}{\partial y} : \frac{\partial g}{\partial z},$$

and those cosines will have *the same* sign as the differential quotients, since $\frac{\partial g}{\partial n}$ is positive. One will then have:

$$X_x\frac{\partial g}{\partial x}+X_y\frac{\partial g}{\partial y}+X_z\frac{\partial g}{\partial z} = X_n\sqrt{\left(\frac{\partial g}{\partial x}\right)^2+\left(\frac{\partial g}{\partial y}\right)^2+\left(\frac{\partial g}{\partial z}\right)^2},$$

$$Y_x \frac{\partial g}{\partial x} + Y_y \frac{\partial g}{\partial y} + Y_z \frac{\partial g}{\partial z} = Y_n \sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + \left(\frac{\partial g}{\partial z}\right)^2}, \tag{2}$$

$$X_x \frac{\partial g}{\partial x} + X_y \frac{\partial g}{\partial y} + X_z \frac{\partial g}{\partial z} = Z_n \sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + \left(\frac{\partial g}{\partial z}\right)^2}$$

on the outer surface, in which the roots are taken to be positive, and X_n, Y_n, Z_n shall be considered to have been given.

In order for u, v, w to be determined completely, we establish that the position of the body in its natural state (in which u, v, w are calculated) is chosen in such a way that one has:

$$\begin{aligned} &u = 0, && v = 0, && w = 0, \\ &\frac{\partial u}{\partial z} = 0, && \frac{\partial v}{\partial z} = 0, && \frac{\partial v}{\partial x} = 0 \end{aligned} \tag{3}$$

at the origin of the coordinates (which shall be found inside of the body), and thus for $x = 0$, $y = 0$, $z = 0$.

We now set:

$$x = i x', \qquad y = i y', \qquad z = i z'. \tag{4}$$

As a result of the assumptions that were made, x', y', z' will then be finite in the body, and:

$$g' = 0$$

will be the equation in x', y', z' that corresponds to the outer surface, such that g' contains only finite constants.

One also imagines that the substitutions (4) have been carried out in equations (1), (2), and (3). If one makes:

$$\begin{aligned} x'_x &= \frac{\partial u}{\partial x'}, & y'_z &= \frac{\partial u}{\partial z'} + \frac{\partial w}{\partial y'}, \\ y'_y &= \frac{\partial v}{\partial y'}, & z'_x &= \frac{\partial w}{\partial x'} + \frac{\partial u}{\partial z'}, \\ z'_z &= \frac{\partial w}{\partial z'}, & x'_y &= \frac{\partial u}{\partial y'} + \frac{\partial v}{\partial x'}, \end{aligned}$$

and if one lets X'_x, Y'_y, … denote the expressions that one obtains when one replaces x_x, x_y, … with x'_x, y'_y, … in the expressions that represent X_x, X_y, … as functions of x_x, x_y, … then one will get:

$$\frac{\partial X'_x}{\partial x'} + \frac{\partial X'_y}{\partial y'} + \frac{\partial X'_z}{\partial z'} = i^2\, X\, \mu,$$

$$\frac{\partial Y'_x}{\partial x'} + \frac{\partial Y'_y}{\partial y'} + \frac{\partial Y'_z}{\partial z'} = i^2\, Y\, \mu, \tag{5}$$

$$\frac{\partial Z'_x}{\partial x'} + \frac{\partial Z'_y}{\partial y'} + \frac{\partial Z'_z}{\partial z'} = i^2\, X\, \mu\,.$$

For $g' = 0$:

$$X'_x \frac{\partial g'}{\partial x'} + X'_y \frac{\partial g'}{\partial y'} + X'_z \frac{\partial g'}{\partial z'} = i\, X_n \sqrt{\left(\frac{\partial g'}{\partial x'}\right)^2 + \left(\frac{\partial g'}{\partial y'}\right)^2 + \left(\frac{\partial g'}{\partial z'}\right)^2}\,,$$

$$Y'_x \frac{\partial g'}{\partial x'} + Y'_y \frac{\partial g'}{\partial y'} + Y'_z \frac{\partial g'}{\partial z'} = i\, Y_n \sqrt{\left(\frac{\partial g'}{\partial x'}\right)^2 + \left(\frac{\partial g'}{\partial y'}\right)^2 + \left(\frac{\partial g'}{\partial z'}\right)^2}\,, \tag{5}$$

$$Z'_x \frac{\partial g'}{\partial x'} + Z'_y \frac{\partial g'}{\partial y'} + Z'_z \frac{\partial g'}{\partial z'} = i\, Z_n \sqrt{\left(\frac{\partial g'}{\partial x'}\right)^2 + \left(\frac{\partial g'}{\partial y'}\right)^2 + \left(\frac{\partial g'}{\partial z'}\right)^2}\,,$$

and:

$$u = 0, \qquad v = 0, \qquad w = 0,$$

$$\frac{\partial u}{\partial z'} = 0, \quad \frac{\partial v}{\partial z'} = 0, \quad \frac{\partial v}{\partial x'} = 0 \tag{7}$$

for $x' = 0$, $y' = 0$, $z' = 0$. The values of u, v, w that (5), (6), and (7) yield can be represented as the sums of terms, one of which fulfills equations (6) and (7), but instead of fulfilling equation (5), it will fulfill the one that arises from (5) when one replaces the right-hand side with zero, and the other of which will fulfill equations (5) and (7), but instead of fulfilling equation (6), it will fulfill the one that arises from (6) when one replaces the right-hand side with zero. The former terms have the same order as iX_n, iY_n, iZ_n, while the latter have the same order as $i^2\, X\, \mu$, $i^2\, Y\, \mu$, $i^2\, Z\, \mu$. They are then infinitely-small in comparison to the ones that we would get if we assumed that the forces X, Y, Z, were not infinitely large in comparison to the thrusts X_n, Y_n, Z_n; i.e., that the relative displacements that would be induced for a body whose dimensions are all finite would not be infinitely large in comparison to the ones that they would generate in the body itself. With those assumptions, for our infinitely-small body, one must replace equations (5) with the ones that arise when one sets X, Y, Z equal to zero, so equations (1) will get replaced with:

$$\frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z} = 0,$$
$$\frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z} = 0, \qquad (8)$$
$$\frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z} = 0.$$

The consideration that one carries out likewise shows that u, v, w have the same order as iX_n, iY_n, iZ_n ; the differential quotients of u, v, w with respect to x', y', z' have that same order, so the differential quotients of u, v, w with respect to x, y, z will have the same order as X_n, Y_n, Z_n .

These results are also true for the case of motion, and equations (8) appear in place of equations (3) ([1]) from the previous lecture, assuming that the accelerations $\frac{\partial^2 u}{\partial t^2}$, $\frac{\partial^2 v}{\partial t^2}$, $\frac{\partial^2 w}{\partial t^2}$ do not exceed the limits that we have assumed for the forces X, Y, Z. It will then follow from this that in order to go from equilibrium to motion, we would have to replace X, Y, Z with $X - \frac{\partial^2 u}{\partial t^2}$, $Y - \frac{\partial^2 v}{\partial t^2}$, $Z - \frac{\partial^2 w}{\partial t^2}$.

§ 2.

We would now like to assume that the body that we are dealing with is an infinitely-thin, cylindrical rod in its natural state. In that state, one imagines that there is a right-angled system of axes in the rod. One axis should be the line in which the center of mass of the cross-section lies, while the other two shall be parallel to the principal axes of a cross-section that goes through the center of mass itself. One chooses a point P in the first axis, calls the distance from it to the beginning of the rod s, and turns one's attention to three line elements that are drawn from P in the directions of the three axes. They might be called 3, 1, 2, respectively, and 3 shall be the one that has the direction of the length of the cylinder. When the state of the rod changes, those three line elements will not generally remain perpendicular to each other, but define angles that deviate from right angles by quantities that have the order of the dilatations that have come about. The points of the rod in the neighborhood of P shall be referred to a right-angled coordinate system whose origin is P, whose z-axis has the direction of the line-element 3, and whose zx-plane goes through the line elements 3 and 1. Let $x + u$, $y + v$, $z + w$ be the coordinates of a point of the rod after the change, and let x, y, z be the coordinates of that point when the rod is in its natural state, and in that position, one will find that the line elements 1, 2, 3 fall upon the x, y, z axes, resp. With those assumptions, equations (3) of this lecture and equations (11) of the previous lecture will be true, which would emerge from the

([1]) In the previous editions, (1) was incorrectly printed as [1]), (2) was printed as [2]), and (3), as [3]).

remark that was made for them. An equation between x and y will exist for the outer surface of the rod; it is:

$$\int x\,dx\,dy = 0, \qquad \int y\,dx\,dy = 0, \qquad \int xy\,dx\,dy = 0, \tag{9}$$

when the integrations are extended over the cross-section. Finally, every material point of the rod is characterized by certain values of x, y, $s + z$.

Furthermore, let ξ, η, ζ be the coordinates of the point P after the deformation of the rod relative to an arbitrarily-chosen coordinate system that might have the property that the x, y, z axes can be made parallel to the ξ, η, ζ axes by a rotation. Let:

$$\begin{matrix} \alpha_1, & \beta_1, & \gamma_1 \\ \alpha_2, & \beta_2, & \gamma_2 \\ \alpha_3, & \beta_3, & \gamma_3 \end{matrix}$$

be the cosines of the angles that the ξ, η, ζ axes make with the x, y, z axes, such that the indices 1, 2, 3 refer to the x, y, z axes, resp. These nine quantities, as well as ξ, η, ζ, are functions of the one variable s in the case of equilibrium and functions of s and t in the case of motion.

With those notations:

$$\begin{aligned} &\xi + \alpha_1 (x + u) + \alpha_2 (y + v) + \alpha_3 (z + w) \\ &\eta + \beta_1 (x + u) + \beta_2 (y + v) + \beta_3 (z + w) \\ &\zeta + \gamma_1 (x + u) + \gamma_2 (y + v) + \gamma_3 (z + w) \end{aligned} \tag{10}$$

will be the coordinates, relative to the ξ, η, ζ axes, of the point whose coordinates are $x + u$, $y + v$, $z + w$ relative to the x, y, z axes. The expressions (10) must be functions of $s + z$, since the values of $s + x$¸ x, and y determine a material point of the rod. The partial differential quotients of these expressions with respect to z and s must then be equal to each other. One will then have:

$$\alpha_1 \frac{\partial u}{\partial z} + \alpha_2 \frac{\partial v}{\partial z} + \alpha_3 \left(1 + \frac{\partial w}{\partial z}\right) = \alpha_1 \frac{\partial u}{\partial s} + \alpha_2 \frac{\partial v}{\partial s} + \alpha_3 \frac{\partial w}{\partial s}$$
$$+ \frac{d\xi}{ds} + \frac{d\alpha_1}{ds}(x+u) + \frac{d\alpha_2}{ds}(y+v) + \frac{d\alpha_3}{ds}(z+w),$$

$$\beta_1 \frac{\partial u}{\partial z} + \beta_2 \frac{\partial v}{\partial z} + \beta_3 \left(1 + \frac{\partial w}{\partial z}\right) = \beta_1 \frac{\partial u}{\partial s} + \beta_2 \frac{\partial v}{\partial s} + \beta_3 \frac{\partial w}{\partial s}$$
$$+ \frac{d\eta}{ds} + \frac{d\beta_1}{ds}(x+u) + \frac{d\beta_2}{ds}(y+v) + \frac{d\beta_3}{ds}(z+w),$$

$$\gamma_1 \frac{\partial u}{\partial z} + \gamma_2 \frac{\partial v}{\partial z} + \gamma_3 \left(1 + \frac{\partial w}{\partial z}\right) = \gamma_1 \frac{\partial u}{\partial s} + \gamma_2 \frac{\partial v}{\partial s} + \gamma_3 \frac{\partial w}{\partial s}$$

$$+\frac{d\zeta}{ds}+\frac{d\gamma_1}{ds}(x+u)+\frac{d\gamma_2}{ds}(y+v)+\frac{d\gamma_3}{ds}(z+w).$$

One multiplies these equations successively by α_1, β_1, γ_1, and α_2, β_2, γ_2, and α_3, β_3, γ_3, and then adds them each time. One then sets:

$$\sigma = \sqrt{\left(\frac{d\xi}{ds}\right)^2+\left(\frac{d\eta}{ds}\right)^2+\left(\frac{d\zeta}{ds}\right)^2} - 1. \tag{11}$$

Since, from the assumptions that were made, one has:

$$\frac{d\xi}{ds} : \frac{d\eta}{ds} : \frac{d\zeta}{ds} = \alpha_3, \beta_3, \gamma_3,$$

and it will then follow that:

$$\frac{d\xi}{ds} = \alpha_3 (1 + \sigma), \quad \frac{d\eta}{ds} = \beta_3 (1 + \sigma), \quad \frac{d\zeta}{ds} = \gamma_3 (1 + \sigma), \tag{12}$$

and σ will be the dilatation that the element *ds* has experienced. One further sets:

$$\begin{aligned} p &= \alpha_3\frac{d\alpha_2}{ds}+\beta_3\frac{d\beta_2}{ds}+\gamma_3\frac{d\gamma_2}{ds}, \\ q &= \alpha_1\frac{d\alpha_3}{ds}+\beta_1\frac{d\beta_3}{ds}+\gamma_1\frac{d\gamma_3}{ds}, \\ r &= \alpha_2\frac{d\alpha_1}{ds}+\beta_2\frac{d\beta_1}{ds}+\gamma_2\frac{d\gamma_1}{ds}. \end{aligned} \tag{13}$$

If we compare these expressions with equations (19) that we gave in the fifth lecture, which were set equal to p', q', r', and recall the interpretation that we gave for p', q', r' there, then we will see that $p\,ds$, $q\,ds$, $r\,ds$ are the angles around which the *x*, *y*, *z* system of axes rotates around the *x*, *y*, *z* when its origin runs through the element *ds*. $r\,ds$ is called the *torsion* of the part of the rod that corresponds to the element *ds*, and *p*, *q* are the reciprocal radii of curvature of the projections of the element *ds* onto the *yz* and *xz* planes.

With the help of the six relations that exist between the cosines α, β, γ, and the ones that result from differentiating them with respect to *s*, one will then obtain:

$$\begin{aligned} \frac{\partial u}{\partial z} &= \frac{\partial u}{\partial s} + q(z + w) - r(y + v), \\ \frac{\partial v}{\partial z} &= \frac{\partial v}{\partial s} + r(x + u) - p(z + w), \\ \frac{\partial w}{\partial z} &= \frac{\partial w}{\partial s} + p(y + v) - q(x + u) + \sigma. \end{aligned}$$

Based upon the remark that was made at the end of the previous §, we assume that $\frac{\partial u}{\partial z}$, $\frac{\partial v}{\partial z}$, $\frac{\partial w}{\partial z}$ will be infinitely large in comparison to *u*, *v*, *w* when we give *z* only values that have the same order of dimensions as the cross-section of the rod. We further assume that $\frac{\partial u}{\partial z}$, $\frac{\partial v}{\partial z}$, $\frac{\partial w}{\partial z}$ have the same order of magnitude as *u*, *v*, *w*. If we employ the fact that *u*, *v*, *w* are infinitely small in comparison to *x*, *y*, *z*, in addition, then the derived equations will become:

$$\frac{\partial u}{\partial z} = qz - ry,$$
$$\frac{\partial v}{\partial z} = rx - pz,$$
$$\frac{\partial w}{\partial z} = py - qx + \sigma.$$

In then follows from this by integration that:

$$\begin{aligned} u &= u_0 + \frac{q}{2} z^2 - ryz, \\ v &= v_0 + rxz - \frac{p}{2} z^2, \\ w &= w_0 + (py - qx + \sigma)\, z, \end{aligned} \tag{14}$$

in which u_0, v_0, w_0 mean functions of *x* and *y*, namely, the values that *u*, *v*, *w* take on for *z* = 0. Those functions are determined by equations (8), (2), and (3).

The expressions that are found for *u*, *v*, *w* yield:

$$\begin{aligned} x_x &= \frac{\partial u_0}{\partial x}, & y_z &= \frac{\partial w_0}{\partial y} + rx, \\ y_y &= \frac{\partial v_0}{\partial y}, & z_x &= \frac{\partial w_0}{\partial x} - ry, \\ z_z &= py - qx + \sigma, & x_y &= \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}. \end{aligned} \tag{15}$$

All of these values are independent of *z*. As a result, equations (8) simplify into:

$$\begin{aligned} \frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} &= 0, \\ \frac{\partial Y_x}{\partial x} + \frac{\partial Y_y}{\partial y} &= 0, \end{aligned} \tag{16}$$

$$\frac{\partial Z_x}{\partial x} + \frac{\partial Z_y}{\partial y} = 0.$$

We would like to assume that the original cylindrical outer surface of the rod is not acted upon by tension, and understand that g is the function of x and y that will define the equation for the contour of the cross-section when it is set equal to zero. When $g = 0$, equations (2) will then give:

$$\begin{aligned} X_x \frac{\partial g}{\partial x} + X_y \frac{\partial g}{\partial y} &= 0, \\ Y_x \frac{\partial g}{\partial x} + Y_y \frac{\partial g}{\partial y} &= 0, \\ Z_x \frac{\partial g}{\partial x} + Z_y \frac{\partial g}{\partial y} &= 0. \end{aligned} \tag{17}$$

Finally, two of equations (3) will be satisfied identically, while the other one will demand that one must have:

$$u_0 = 0, \qquad v_0 = 0, \qquad w_0 = 0, \qquad \frac{\partial v_0}{\partial x} = 0 \tag{18}$$

for $x = 0$ and $y = 0$.

We derived equations (17) under the assumption that the tensions that acted upon the outer surface of the rod were equal to zero. However, we will also preserve those equations when the tensions have any values that do not exceed certain limits. They must have values such that tensions with their order of magnitude would provoke only dilatations that are infinitely small in comparison to the dilatations that are determined from (15) for a body whose dimensions all have the same order. When one neglects the quantities that define the right-hand sides of equations (17), one will then neglect only quantities that are infinitely small in comparison to the individual terms that comprise the left-hand sides.

If one sets X_x, X_y, … equal to their expressions in terms of x_x, x_y, … in equations (16) and (17), and sets the latter quantities equal to the values that were given in (15) then equations (16), (17), and (18) will determine the quantities u_0, v_0, w_0 uniquely as linear homogeneous functions of p, q, r, σ. In order to prove that assertion, one has to show that the aforementioned equations can be fulfilled only by $u_0 = 0$, $v_0 = 0$, $w_0 = 0$ when p, q, r, σ vanish, and one will arrive at that by considerations that are entirely similar to the ones by which a similar theorem was proved in § 2 of the previous lecture. If u_0, v_0, w_0 are expressed in the stated way then equations (15) will yield x_x, x_y, … as linear homogeneous functions of p, q, r, σ; the components X_x, X_y, … of the tension will be such functions, and f will be a second-degree homogeneous function of the same four elements.

Here, we would like to add a remark that will extend the applicability of our considerations essentially. We imagine that the rod, in its natural, cylindrical state, is acted upon by forces that act upon its interior and thrusts that act upon its end surfaces, which take it to one state in one case and another state in another case. The symbols x_x, x_y, …, p, q, r, σ might refer to the second of those states, while the symbols x'_x, x'_y, …,

p', q', r', σ' might refer to the first one. If the rod goes from the first state to the second one then the differences $x_x - x'_x$, $x_y - x'_y$, … will determine the dilatations that then come about in precisely the same way that x_x, x_y, … themselves determine the dilatations that come about during the transition from the rod in its cylindrical state to the one that we have called the second one. That will also be true when the natural state is not the cylindrical one, but the one that was referred to as the first one, so when the rod is bent and twisted in its natural state in the way that would correspond to the values of p', q', r'. In that case, then, the tension components X_x, X_y, …, and the quantity f will be the same functions of $x_x - x'_x$, $x_y - x'_y$, … that they previously were in terms of x_x, x_y, …, and (since $x_x - x'_x$, $x_y - x'_y$, … are the same linear functions of $p - p'$, $q - q'$, $r - r'$, $\sigma - \sigma'$ that x_x, x_y, … are of p, q, r, σ) the same functions of $p - p'$, $q - q'$, $r - r'$, $\sigma - \sigma'$ that they previously were of p, q, r, σ. That remark has particular importance when the material that rod is composed of is isotropic. With its help, one can then always exhibit the equations of equilibrium and motion for an infinitely-thin rod whose cross-section has the same form everywhere when it is arbitrarily bent and twisted in its natural state. The quantity that we have denoted by σ' can then be set equal to zero.

§ 3.

Carrying out the determination of u_0, v_0, w_0 is relatively easy when the cross-section of the rod is an ellipse, which might also give the constants of elasticity. Corresponding to that assumption, we set:

$$g = 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}.$$

Equations (16) and (17) (the latter, not just for $g = 0$, but in general) will then be fulfilled by:

$$X_x = 0, \qquad Y_y = 0, \qquad X_y = 0,$$
$$Z_x = c\frac{y}{b^2}, \qquad Z_y = -c\frac{x}{a^2},$$

in which c means an arbitrary constant. Those five equations, in conjunction with the equation:

$$z_z = py - qx + \sigma$$

that entered into (15), along with the help of the relations that exist between the six quantities x_x, x_y, …, and the six tension components X_x, X_y, …, allow one to express x_x, y_y, x_y, and z_x, z_y as linear functions of x and y. When one recalls equations (15), the first three of them will lead to the determination of u_0, v_0, …, and the last two, to the determination of w_0. In order for these determinations to be possible, one must have:

$$\frac{\partial^2 x_x}{\partial y^2} + \frac{\partial^2 y_y}{\partial x^2} = \frac{\partial^2 x_y}{\partial x \partial y}$$

and

$$\frac{\partial y_z}{\partial x} - \frac{\partial x_z}{\partial y} = 2r.$$

The first of these equations [which follows from considerations that are entirely similar to the ones by which we derived equations (13) and (14) in the previous lecture] is fulfilled as a result of the fact that x_x, y_y, x_y are linear in x and y; the second one determines the quantity c. The integrations that must be performed in order to calculate u_0 and v_0 then bring three arbitrary constants with them, and the integration that gives w_0 will introduce *one* of them. Those constants are precisely sufficient for equations (18) to be fulfilled. One then gets u_0, v_0, w_0 as second-degree functions of x and y.

A simplification in the determination of u_0, v_0, w_0 for a cross-section of arbitrary form will come about when its plane is a symmetry plane. In that case, from equations (5) of the previous lecture, one will have:

$$\begin{aligned}
\tfrac{1}{2}X_x &= a_{11}\,x_x + a_{12}\,y_y + a_{13}\,z_z + a_{16}\,x_y\,, \\
\tfrac{1}{2}Y_y &= a_{21}\,x_x + a_{22}\,y_y + a_{23}\,z_z + a_{26}\,x_y\,, \\
\tfrac{1}{2}Z_z &= a_{31}\,x_x + a_{32}\,y_y + a_{33}\,z_z + a_{36}\,x_y\,, \\
\tfrac{1}{2}X_y &= a_{61}\,x_x + a_{62}\,y_y + a_{63}\,z_z + a_{66}\,x_y\,, \\
\tfrac{1}{2}Z_y &= a_{44}\,z_y + a_{45}\,z_x\,, \\
\tfrac{1}{2}Z_x &= a_{54}\,z_y + a_{55}\,z_x\,,
\end{aligned}$$

in which:

$$a_{12} = a_{21}\,, \qquad a_{13} = a_{31}\,, \qquad \ldots$$

With hindsight of equations (15), the last of equations (16) will now be:

$$a_{55}\frac{\partial^2 w_0}{\partial x^2} + 2a_{45}\frac{\partial^2 w_0}{\partial x\,\partial y} + a_{44}\frac{\partial^2 w_0}{\partial y^2} = 0, \tag{19}$$

and the last of equations (17) will become:

$$\left[a_{54}\left(\frac{\partial w_0}{\partial y} + rx\right) + a_{55}\left(\frac{\partial w_0}{\partial x} - ry\right)\right]\frac{\partial g}{\partial x} + \left[a_{44}\left(\frac{\partial w_0}{\partial y} + rx\right) + a_{45}\left(\frac{\partial w_0}{\partial x} - ry\right)\right]\frac{\partial g}{\partial y} = 0. \tag{20}$$

w_0 is determined from these two equations and the third of equations (18). The rest of equations (16), (17), and (18) serve to determine u_0 and v_0 .One will satisfy them when one sets:

$$X_x = 0, \qquad Y_y = 0, \qquad X_y = 0. \tag{20a}$$

In fact, when one solves those equations for x_x, y_y, x_y, one will obtain linear expressions in x and y for those quantities when one sets z equal to its value in (15). As a result of that, it will be possible to determine u_0 and v_0 from the equations:

$$x_x = \frac{\partial u_0}{\partial x}, \qquad y_y = \frac{\partial v_0}{\partial y}, \qquad x_y = \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}.$$

Integrating them will introduce three arbitrary constants by which one can fulfill equations (18), which one still has to consider.

§ 4.

If u_0, v_0, w_0 have been found then one will be dealing with the determinations of p, q, r, s as functions of s in the equilibrium case and as functions of s and t in the case of motion. To that end, one can appeal to the principal of virtual displacement in the first case and Hamilton's principle in the second. In both cases, it is requisite that one must have an expression for the potential of the force that produces the dilatations. If f denotes the same second-degree homogeneous function of x_x, x_y, … as before then that potential will be:

$$= \int f\, dx\, dy\, ds,$$

in which the integration over x and y is extended over the cross-section, while the integration over s extends along the length of the rod. One sets x_x, x_y, … equal to their values in (15) here. Since those values are linear, homogeneous functions of p, q¸ r, σ, f will be a second-degree homogeneous function of p, q, r, σ; the coefficients depend upon only x and y. If one now makes:

$$F = \int f\, dx\, dy \tag{21}$$

then F will be a second-degree homogeneous function of p, q, r, σ with constant coefficients, and that potential will be:

$$= \int F\, ds.$$

If U' denotes the work that is done by the force that acts in the interior and the tensions that act upon the outer surface and the end surfaces of the rod for certain variations of p, q, r, σ, and T denotes the *vis viva* then the condition for equilibrium will be:

$$U' + \delta \int F\, ds = 0, \tag{22}$$

and for motion, one will have the equation:

$$\int dt \left(U' + \delta T + \delta \int F\, ds \right) = 0. \tag{23}$$

In order to define the value of T, we must differentiate the expressions (10) with respect to t, multiply the sum of the squares of the differential quotients by one-half the element of mass of the rod, and integrate over it. In that, we neglect $\frac{\partial u}{\partial t}$, $\frac{\partial v}{\partial t}$, $\frac{\partial w}{\partial t}$ as being infinitely small in comparison to terms that appear additively-linked, and set $z = 0$, which is allowed, since the expressions (10) are functions of $s + z$, and we consider s to be variable. The differential quotients of those expressions are then:

$$\begin{gathered}
\frac{\partial \xi}{\partial t} + x\frac{\partial \alpha_1}{\partial t} + y\frac{\partial \alpha_2}{\partial t}, \\
\frac{\partial \eta}{\partial t} + x\frac{\partial \beta_1}{\partial t} + y\frac{\partial \beta_2}{\partial t}, \\
\frac{\partial \zeta}{\partial t} + x\frac{\partial \gamma_1}{\partial t} + y\frac{\partial \gamma_2}{\partial t}.
\end{gathered}$$

As a result of equations (9), the sum of the squares of these expressions, when multiplied by $dx\,dy$ and integrated over the cross-section of the rod, will be:

$$\begin{aligned}
&\left[\left(\frac{\partial \xi}{\partial t}\right)^2 + \left(\frac{\partial \eta}{\partial t}\right)^2 + \left(\frac{\partial \zeta}{\partial t}\right)^2\right]\int dx\,dy \\
&+ \left[\left(\frac{\partial \alpha_1}{\partial t}\right)^2 + \left(\frac{\partial \beta_1}{\partial t}\right)^2 + \left(\frac{\partial \gamma_1}{\partial t}\right)^2\right]\int x^2 dx\,dy \qquad (24) \\
&+ \left[\left(\frac{\partial \alpha_2}{\partial t}\right)^2 + \left(\frac{\partial \beta_2}{\partial t}\right)^2 + \left(\frac{\partial \gamma_2}{\partial t}\right)^2\right]\int y^2 dx\,dy .
\end{aligned}$$

One now sets:

$$\begin{aligned}
-P &= \alpha_2\frac{\partial \alpha_3}{\partial t} + \beta_2\frac{\partial \beta_3}{\partial t} + \gamma_2\frac{\partial \gamma_3}{\partial t} \\
Q &= \alpha_1\frac{\partial \alpha_3}{\partial t} + \beta_1\frac{\partial \beta_3}{\partial t} + \gamma_1\frac{\partial \gamma_3}{\partial t}, \qquad (25) \\
R &= \alpha_2\frac{\partial \alpha_1}{\partial t} + \beta_2\frac{\partial \beta_1}{\partial t} + \gamma_2\frac{\partial \gamma_1}{\partial t} .
\end{aligned}$$

From the equations that one can define using the model of equations (20) in the fifth lecture, one will get:

$$\begin{aligned}
\left(\frac{\partial \alpha_1}{\partial t}\right)^2 + \left(\frac{\partial \beta_1}{\partial t}\right)^2 + \left(\frac{\partial \gamma_1}{\partial t}\right)^2 &= Q^2 + R^2, \\
\left(\frac{\partial \alpha_2}{\partial t}\right)^2 + \left(\frac{\partial \beta_2}{\partial t}\right)^2 + \left(\frac{\partial \gamma_2}{\partial t}\right)^2 &= P^2 + R^2.
\end{aligned}$$

One now ponders the fact that as a result of equations (12), $\frac{\partial \alpha_3}{\partial t}$, $\frac{\partial \beta_3}{\partial t}$, $\frac{\partial \gamma_3}{\partial t}$ cannot be infinitely large in comparison to $\frac{\partial \xi}{\partial t}$, $\frac{\partial \eta}{\partial t}$, $\frac{\partial \zeta}{\partial t}$, assuming that the differential quotients of those quantities with respect to s are infinitely large in comparison to them. It will then follows that P and Q cannot be infinitely large in comparison to $\frac{\partial\xi}{\partial t}, \frac{\partial\eta}{\partial t}, \frac{\partial\zeta}{\partial t}$, while the corresponding statement in regard to R cannot be asserted. Finally, if one imagines that of the three integrals that enter into the expression (24), the last two are infinitely small in comparison to the first one, then one will see that this expression is:

$$= \left[\left(\frac{\partial\xi}{\partial t}\right)^2 + \left(\frac{\partial\eta}{\partial t}\right)^2 + \left(\frac{\partial\zeta}{\partial t}\right)^2\right]\int dx\,dy + R^2\int (x^2+y^2)\,dx\,dy\,.$$

If one makes:

$$\int dx\,dy = \lambda, \qquad \int (x^2+y^2)\,dx\,dy = \kappa; \tag{26}$$

and once more denotes the density by μ then one will have:

$$T = \frac{\mu}{2}\int ds\left\{\lambda\left[\left(\frac{\partial\xi}{\partial t}\right)^2 + \left(\frac{\partial\eta}{\partial t}\right)^2 + \left(\frac{\partial\zeta}{\partial t}\right)^2\right]\right\} + \kappa R^2\,. \tag{27}$$

§ 5.

We would now like to examine the equilibrium of rods more closely under the assumption that no forces act upon its parts, and thrusts act upon only its end surfaces. However, instead of making use of the principle of virtual displacements, we would like to immediately appeal to the definition of tension that was given by equations (1) and (2) of the eleventh lecture. We apply it to the part of the rod that is between two arbitrary cross-sections. If we let Α, Β, Γ denote the sums of the components of the tension along the ξ, η, ζ axes, resp., that will be exerted in the element of the cross-section that is determined by an arbitrary value of s by *the* part of the rod in which s has a smaller value upon the one in which s possesses a greater value, and let M_α, M_β, M_γ denote the rotational moments of that tension relative to the same axes, resp., then as a result of the assumption that equilibrium exists and no forces act upon the interior of the rod:

$$\mathrm{A} = \text{const.}, \qquad \mathrm{B} = \text{const.}, \qquad \Gamma = \text{const.},$$
$$M_\alpha = \text{const.}, \qquad M_\beta = \text{const.}, \qquad M_\gamma = \text{const.}$$

If $s = 0$ for one end of the rod and $s = l$ for the other one, and l is positive, then Α, Β, Γ, M_α, M_β, M_γ will be equal to the component sums and rotational moments of the tensions that act upon the element of the cross-section $s = 0$ from the outside; – Α, – Β, – Γ, – M_α, – M_β, – M_γ have the same interpretations for the other end.

We would now like to introduce the rotational moments of the same tensions from which M_α, M_β, M_γ originate relative to the x, y, z axes that correspond to the chosen value of s and denote them by M_x, M_y, M_z, resp. We likewise choose the ζ-axis such that Α = 0, Β = 0, and Γ is negative or equal to 0 (which is always possible). As a result of the relations in § 4 of the fifth lecture, one will then have:

$$\begin{aligned} M_\alpha &= \alpha_1 M_x + \alpha_2 M_y + \alpha_3 M_z + \eta\,\Gamma = \text{const.} \\ M_\beta &= \beta_1 M_x + \beta_2 M_y + \beta_3 M_z - \xi\,\Gamma = \text{const.} \\ M_\gamma &= \gamma_1 M_x + \gamma_2 M_y + \gamma_3 M_z = \text{const.} \end{aligned} \tag{28}$$

One differentiates these equations with respect to s, multiplies then by α_1, β_1, γ_1 or α_2, β_2, γ_2 or α_3, β_3, γ_3 , resp., and adds them. Recalling the relations that exist between these nine cosines, along with equations (12) and (13), will then yield:

$$\begin{aligned} \frac{dM_x}{ds} &= r\,M_y - q\,M_z + \gamma_2\,\Gamma, \\ \frac{dM_y}{ds} &= p\,M_z - r\,M_z + \gamma_1\,\Gamma, \\ \frac{dM_z}{ds} &= q\,M_x - p\,M_y\,. \end{aligned} \tag{29}$$

We now derive the relationship that exists between the rotational moments M_x, M_y , M_z and the function F in the previous §. To that end, we consider the increase δf that f experiences when the state of the rod in the neighborhood of a cross-section that corresponds to a constant value of s changes in such a way that p, q, r, σ increase by δp, δq, δr, $\delta\sigma$. One will then have:

$$\delta f = X_x\,\delta x_x + Y_y\,\delta y_y + Z_z\,\delta z_z + Y_z\,\delta y_z + Z_x\,\delta z_x + X_y\,\delta x_y\,,$$

since X_x, Y_y, … are the partial differential quotients of f with respect to x_x, y_y, … With the help of equations (15), one will then obtain:

$$\begin{aligned} \delta f = {} & X_x \delta\frac{\partial u_0}{\partial x} + Y_y \delta\frac{\partial v_0}{\partial y} + Z_z (y\,\delta p - x\,\delta q + \delta\sigma) \\ & + Y_z\left(\delta\frac{\partial w_0}{\partial y} + x\,\delta r\right) + Z_x\left(\delta\frac{\partial w_0}{\partial x} - y\,\delta r\right) + X_y\left(\delta\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}\right). \end{aligned}$$

One multiplies these equations by $dx\,dy$ and integrates them over the cross-section of the rod. From (21), the left-hand side of it is then δF; one transforms the right-hand side with the help of the equation:

$$0 = \int dx\,dy \left\{ X_x \delta\frac{\partial u_0}{\partial x} + Y_y \delta\frac{\partial v_0}{\partial y} + Y_z \delta\frac{\partial w_0}{\partial y} + Z_x \delta\frac{\partial w_0}{\partial x} + X_y \delta\left(\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}\right)\right\},$$

which one obtains by partial integrations, while taking into account equations (17), in which cos (nx) and cos (ny) can be written $\frac{\partial g}{\partial x}$ and $\frac{\partial g}{\partial y}$, when one multiplies equations (16) by $dx\,dy\,\delta u_0$, $dx\,dy\,\delta v_0$, $dx\,dy\,\delta w_0$, adds them, and integrates over the cross-section. If one sets:

$$\begin{aligned} Z &= \int dx\ dy\, Z_z\,, \\ M_x &= \int dx\ dy\, y\, Z_z\,, \\ M_y &= -\int dx\ dy\, x\, Z_z\,, \\ M_z &= \int dx\ dy\, (x\, Y_z - y\, X_z), \end{aligned}$$

in which Z denotes the component of the force Γ along the z-axis, and M_x , M_y , M_z have the meanings that they had in equations (28), then one will obtain:

$$dF = M_x\ \delta p + M_y\ \delta q + M_z\ \delta r + Z\ \delta s,$$

from which, it will follow that:

$$\frac{\partial F}{\partial p} = M_x\,, \qquad \frac{\partial F}{\partial q} = M_y\,, \qquad \frac{\partial F}{\partial r} = M_z\,, \qquad \frac{\partial F}{\partial \sigma} = Z\,. \tag{30}$$

$2F$ is a homogeneous function of degree two of p, q, r¸ σ whose coefficients depend upon the constants of elasticity and the constants of the cross-section of the rod; one then has:

$$\begin{aligned} \frac{\partial F}{\partial \sigma} &= \gamma_3\,\Gamma = A_{00}\sigma + A_{01}\,p + A_{02}\,q + A_{02}\,q\,, \\ \frac{\partial F}{\partial p} &= M_{\mathrm{x}} \ = A_{10}\sigma + A_{11}\,p + A_{12}\,q + A_{13}\,q\,, \\ \frac{\partial F}{\partial q} &= M_y \ = A_{20}\sigma + A_{21}\,p + A_{22}\,q + A_{23}\,q\,, \\ \frac{\partial F}{\partial r} &= M_z \ = A_{30}\sigma + A_{31}\,p + A_{32}\,q + A_{33}\,q\,, \end{aligned} \tag{31}$$

in which A_{00} , $A_{01} = A_{10}$, A_{11} , … are the aforementioned coefficients. They do not all have the same order of magnitude. Since σ is a pure number, but p, q¸ r are reciprocal lengths, the A's that contain the index 0 must have a dimension that is one less than the ones in which the index 0 does not appear, and one greater than A_{00} . The lengths that enter into the quantities A, however, have the same order of dimensions as the cross-section of the rod, and are thus infinitely small. A_{01}, A_{02}, A_{03} must then be infinitely small in comparison to A_{00}, and infinitely large in comparison to the other A's. On that basis, the terms in (31) that are endowed with σ cannot be neglected, even when σ is infinitely small, but p, q, r should be regard as being finite. It follows from the first of equations (31) that:

$$\sigma = -\frac{A_{01}p + A_{02}q + A_{03}r - \gamma_3\Gamma}{A_{00}}\,. \tag{32}$$

If one substitutes this value for σ in the expressions that were given for M_x , M_y , M_z in (31) and assumes that Γ is not infinitely large in comparison to M_x , M_y , M_z then it will follow from the relationships between the *A*'s that were cited above that the terms that then appear that are independent of Γ cannot be neglected as infinitely-small in comparison to M_x, M_y, M_z . One will then get these rotational moments as linear homogeneous functions of *p*, *q*, *r*. They can then be represented as follows: Let *G* be the function of *p*, *q*, *r* that *F* goes to when one expresses σ in terms of *p*, *q*, *r* with the help of equation $\frac{\partial F}{\partial \sigma} = 0$. One will then have:

$$M_x = \frac{\partial G}{\partial p}, \qquad M_y = \frac{\partial G}{\partial q}, \qquad M_z = \frac{\partial G}{\partial r}. \tag{33}$$

In fact, when σ is expressed in terms of *p*, *q*, *r* using $\frac{\partial F}{\partial \sigma} = 0$:

$$\frac{\partial F}{\partial p} = \frac{\partial G}{\partial p},$$

since, when *G* is derived from *F* in such a way that one sets σ equal to an arbitrary function of *p*, *q*, *r*, one will have:

$$\frac{\partial G}{\partial p} = \frac{\partial F}{\partial p} + \frac{\partial F}{\partial \sigma}\frac{\partial \sigma}{\partial p},$$

and the corresponding differential quotients of *q* and *r* will behave similarly. Equations (29) will then be:

$$\begin{aligned}
\frac{d}{ds}\frac{\partial G}{\partial p} &= r\,\frac{\partial G}{\partial p} - q\frac{\partial G}{\partial r} + \Gamma\gamma_2 , \\
\frac{d}{ds}\frac{\partial G}{\partial q} &= p\frac{\partial G}{\partial r} - r\frac{\partial G}{\partial p} - \Gamma\gamma_1 , \\
\frac{d}{ds}\frac{\partial G}{\partial r} &= q\frac{\partial G}{\partial p} - p\frac{\partial G}{\partial q} .
\end{aligned} \tag{34}$$

These equations, in which *G* means a second-degree homogeneous function of *p*, *q*, *r* with constant coefficients, have the same form as equations (17) of lecture seven, which relates to the rotation of a ponderable rigid body around a fixed point. They will agree with them completely when one sets $s = t$, $G = T$, and $-\Gamma$ equal to the product of the weight of the body with the distance from its center of mass to the fixed point. The meanings of the nine cosines α, β, γ, and the quantities *p*, *q*, *r* will then be the same here and there. Since the line that is drawn from the fixed point through the center of mass was chosen to be the *z*-axis there, there will always be a ponderable, rigid body that rotates around a fixed point that corresponds to the rod in such a way that the line that goes through the fixed point and the center of mass will always be parallel to the tangent to the rod when one assumes that $s = t$. When the rotational problem is solved, one will have to define the equations:

$$\xi = \int \alpha_3\, ds, \quad \eta = \int \beta_3\, ds, \quad \zeta = \int \gamma_3\, ds, \tag{34a}$$

if one is to discern the form of the rod.

§ 6.

The problem of the rotation of a ponderable body around a fixed point is not generally soluble as it was posed in the seventh lecture. One case in which it can be solved is the one in which gravity does not act upon it. Here, that case would correspond to the one in which $\Gamma = 0$; i.e., the one in which the sum of the components along any direction of the thrust that is exerted upon the element at one end of the rod vanishes. Another case in which the rotational problem can be solved is the one in which gravity does act, but the body is a rotating body and the fixed point is a point of the rotational axis. Here, that case corresponds to the one in which certain relations exist between the constants of elasticity of the rod and the constants of its cross-section. As we would now like to show, those relations exist when the material of the rod is isotropic and its cross-section is a circle.

From § 1 of the previous lecture, for an isotropic body, one will have:

$$f = -K\left\{x_x^2 + y_y^2 + z_z^2 + \tfrac{1}{2}y_z^2 + \tfrac{1}{2}z_x^2 + \tfrac{1}{2}x_y^2 + \theta(x_x + y_y + z_z)^2\right\}.$$

It will then follow from equations (20a) that:

$$x_x = y_y = -\frac{\theta}{1+2\theta}z_z, \quad x_y = 0.$$

Equations (19) and (20) will then become:

$$\frac{\partial^2 w_0}{\partial x^2} + \frac{\partial^2 w_0}{\partial y^2} = 0, \tag{35}$$

and for $g = 0$, one will have:

$$\left(\frac{\partial w_0}{\partial x} - ry\right)\frac{\partial g}{\partial x} + \left(\frac{\partial w_0}{\partial y} + rx\right)\frac{\partial g}{\partial y} = 0. \tag{36}$$

The cross-section of the rod shall be a circle. We then have to set:

$$g = x^2 + y^2 - \text{const.}$$

For that value of g, it will then follow from (35), (36), and (18) that:

$$w_0 = 0.$$

Equations (15) will then give:

$$z_z = py - qx + \sigma, \qquad y_z = rx, \qquad x_z = -ry.$$

One will then have:

$$f = -K\left[\frac{1+3\theta}{1+2\theta}(py - qx + \sigma)^2 + \tfrac{1}{2}r^2(x^2+y^2)\right],$$

and from (21), when one employs the symbols κ, λ that are defined by (26):

$$F = -K\left[\frac{1+3\theta}{1+2\theta}\frac{\kappa}{2}(p^2+q^2) + \frac{\kappa}{2}r^2 + \frac{1+3\theta}{1+2\theta}\lambda\sigma^2\right].$$

With that, one will finally obtain the function G that is defined in (33) as:

$$G = -K\,\frac{\kappa}{2}\left[\frac{1+3\theta}{1+2\theta}(p^2+q^2) + r^2\right]. \tag{37}$$

With that, the statement that was made above is proved, namely, that for an isotropic rod of circular cross-section, G is the same function of p, q, r as the *vis viva* is for a rotating body that rotates around a point on its symmetry axis, and that will then show that the general solution of equations (34) can be found for a rod of the stated kind in the same way that was given in § 4 of the seventh lecture for the corresponding rotational problem.

We would like to restrict ourselves to actually constructing the solution for a special case. We set:

$$A_{11} = -K\kappa\,\frac{1+3\theta}{1+2\theta}, \qquad A_{23} = -K\kappa; \tag{38}$$

and introduce the angles ϑ, φ, f that were defined by equations (8) in the fifth lecture, whereby the symbol f takes on a different meaning than the one that we used up to now in our present investigations. Equations (34) will then become:

$$\begin{aligned} A_{11}\frac{dp}{ds} &= rq\,(A_{11} - A_{33}) + \Gamma \sin f \cos \vartheta, \\ A_{11}\frac{dq}{ds} &= rp\,(A_{31} - A_{11}) - \Gamma \cos f \sin \vartheta, \\ \frac{dr}{ds} &= 0. \end{aligned} \tag{39}$$

To these, we add the equations:

$$\frac{d\vartheta}{ds} = p \sin f - q \cos f,$$

$$\sin \vartheta \frac{d\varphi}{ds} = p \cos f + q \sin f, \qquad (40)$$
$$\frac{df}{ds} = \cos \vartheta \frac{d\varphi}{ds} - r,$$

which are obtained from equations (21), (13), and (15) of the seventh lecture, while recalling equations (8) of the fifth one, when one writes *s* instead of *t*. We will then see that equations (39) and (40) can be satisfied by the assumption that:

$$\vartheta = \text{const.}$$

The solution that that one obtains under this assumption is just the one that we would like to construct. It corresponds to *the* motion of a ponderable rotating body about a point on the symmetry axis in such a way that the axis describes a right cone about a vertical line. If ϑ is constant then the first of equations (40) will become:

$$0 = p \sin f - q \cos f,$$

so that we can write:

$$p = \sqrt{p^2 + q^2} \cos f, \quad q = \sqrt{p^2 + q^2} \sin f, \qquad (41)$$

in which the sign of $\sqrt{p^2 + q^2}$ remains unchanged. Thus, when one multiplies the first two of equations (39) by p and q, resp., and adds them, that will give:

$$p^2 + q^2 = \text{const.},$$

whereas it will always follow from the third one that:

$$r = \text{const.}$$

When one understands that φ_0 and f_0 are two arbitrary constants, the last two of equations (40) will further yield:

$$\varphi - \varphi_0 = \frac{\sqrt{p^2 + q^2}}{\sin \vartheta} s, \qquad f - f_0 = \left(\frac{\sqrt{p^2 + q^2}}{\tan \vartheta} - r \right) s. \qquad (42)$$

We still have to fulfill one of the first two of equations (39). If one replaces p and q in it with their values in (41) then that will convert the equation into an equation between constants, namely, the equation:

$$0 = A_{11} \frac{\sqrt{p^2 + q^2}}{\tan \vartheta} - A_{33}\, r + \Gamma \frac{\sin \vartheta}{\sqrt{p^2 + q^2}}. \qquad (43)$$

In order to find the form that the rod will have when the equations that were posed are valid, one must still develop equations (34a). If one sets:

$$\alpha_3 = \cos\varphi \sin\vartheta, \qquad \beta_3 = \sin\varphi \sin\vartheta, \qquad \gamma_3 = \cos\vartheta$$

in them, according to equations of the fifth lecture, makes:

$$ds = \frac{\sin\vartheta}{\sqrt{p^2+q^2}}\, d\varphi$$

in the calculation of ξ and η from (42), and has the origin of the ξ, η, ζ at one's disposal in a certain way then one will obtain:

$$\xi = \frac{\sin^2\vartheta}{\sqrt{p^2+q^2}}\sin\varphi, \quad \eta = -\frac{\sin^2\vartheta}{\sqrt{p^2+q^2}}\cos\varphi, \qquad \zeta = s\cos\vartheta. \tag{44}$$

With that, the rod will define a *helix* whose axis is the ζ-axis. The radius of the cylinder that lies along it is:

$$= \frac{\sin^2\vartheta}{\sqrt{p^2+q^2}}, \tag{45}$$

and the height of a screw step is:

$$= \frac{2\pi\sin\vartheta\cos\vartheta}{\sqrt{p^2+q^2}}. \tag{46}$$

As far as the thrust is concerned that must be exerted externally upon the end $s = 0$ of the rod in order for it to be in equilibrium in the calculated form for arbitrary values of the constants ϑ, $\sqrt{p^2+q^2}$, and r, the force Γ must be determined by (43). In order to complete the analogy between the problem of equilibrium of an elastic rod and the problem of the rotation of a ponderable body, the ζ-axis must be chosen in such a way that Γ is negative when it does not vanish. However, that condition drops out when we (as we would like to do) renounce the completeness of the analogy and allow Γ to take on positive and negative values. It still remains to ascertain the rotational moments M_α, M_β, M_γ. One then finds from (33), (37), and (38) that:

$$M_x = A_{11}\, p, \qquad M_y = A_{11}\, q, \qquad M_z = A_{33}\, r,$$

with which, (41) can be written:

$$M_x = A_{11}\frac{\sqrt{p^2+q^2}}{\sin\vartheta}\gamma_1, \qquad M_y = A_{11}\frac{\sqrt{p^2+q^2}}{\sin\vartheta}\gamma_2,$$

$$M_z = A_{11}\frac{\sqrt{p^2+q^2}}{\sin\vartheta}\gamma_3 + A_{33}\,r - A_{11}\frac{\sqrt{p^2+q^2}}{\sin\vartheta}.$$

If one substitutes these values in equations (28) then one will find, when one takes into account the relations that exist between the nine cosines α_1, α_2, …, along with equations (43) and (44), that:

$$M_\alpha = 0, \qquad M_\beta = 0, \qquad M_\gamma = A_{11}\sqrt{p^2+q^2}\,\sin\vartheta + A_{33}\,r\cos\vartheta.$$

A special case that belongs here might also be mentioned. If the relation:

$$\tan\vartheta = \frac{\sqrt{p^2+q^2}}{\sin\vartheta} \tag{47}$$

exists between the constants ϑ, $\sqrt{p^2+q^2}$, r then, as would follow from (42), f will be equal to a constant – namely, f_0 . From (41), p and q, like r, will also be constant. One can then assign arbitrary constant values to the three quantities p, q, r, as long as one has suitable values of $\sqrt{p^2+q^2}$, f_0, r available. The case in which p, q, r are constant is always subsumed then by the one that was treated above. In that case, the rod will also define a helix. The radius of the cylinder upon which it lies will be:

$$= \frac{\sqrt{p^2+q^2}}{p^2+q^2+r^2},$$

and the height of a screw step will be:

$$= \frac{2\pi r}{p^2+q^2+r^2},$$

as would follow from the expressions (45) and (46) when one ponders the fact that (47) imply that:

$$\cos\vartheta = \frac{r}{\sqrt{p^2+q^2+r^2}}, \qquad \sin\vartheta = \frac{\sqrt{p^2+q^2}}{\sqrt{p^2+q^2+r^2}}, \tag{48}$$

in which one must determine the sign of the root $\sqrt{p^2+q^2+r^2}$ in a suitable way.

§ 7.

We shall now treat an example of equilibrium in an isotropic rod that is *curved* in its natural state. From the summary that was made at the end of § 2, in order to go from the case of an originally *straight* isotropic rod to one that is originally *curved*, we must replace p, q, r with $p - p'$, $q - q'$, $r - r'$ in the expression for the function f, in which p', q', r' denote the values that the p, q, r take one when the rod goes to its natural state from one in which it is *straight*. If one carries out the same substitution for F and G then the conclusion that related to the function f in §§ 4 and 5 will also be valid, and equations (34) will preserve their validity.

If the cross-section of the rod is a circle then the following equations will appear in place of equations (39):

$$\begin{aligned} A_{11}\frac{d(p-p')}{ds} &= A_{11}\, r\,(q-q') - A_{33}\, q\,(r-r') + \Gamma \sin f \sin \vartheta, \\ A_{11}\frac{d(q-q')}{ds} &= A_{33}\, p\,(r-r') - A_{11}\, r\,(p-p') - \Gamma \cos f \sin \vartheta, \\ A_{33}\frac{d(r-r')}{ds} &= A_{11}\,[q\,(p-p') - p\,(q-q')]\,. \end{aligned} \tag{49}$$

In that way, equations (40) will be unchanged.

In general, p', q', r' will be functions of s that are required by the original form of the rod. We would like to assume that they are constant; i.e., from the remark that was made at the end of the previous §, that the rod will originally be a helix. We would like to show that equations (49) and (40) can then be satisfied by the assumption that p, q, r are also constant; i.e., by the assumption that the rod remains a helix. With that assumption, the last of equations (49) will give:

$$\frac{p'}{p} = \frac{q'}{q},$$

and with consideration given to that, the other two will reduce to the one:

$$0 = A_{11}\, r\left(1 - \frac{p'}{p}\right) - A_{33}\,(r - r') + \frac{\Gamma}{\sqrt{p^2+q^2+r^2}},$$

when one employs the fact that, from (41) and (48), one will have:

$$\sin f \sin \vartheta = \frac{r}{\sqrt{p^2+q^2+r^2}}, \qquad \cos f \sin \vartheta = \frac{p}{\sqrt{p^2+q^2+r^2}}.$$

However, equations (40) will fulfilled when one sets:

$$\begin{aligned} \cos \vartheta &= \frac{r}{\sqrt{p^2+q^2+r^2}}, \\ \varphi &= \varphi_0 + s\,\sqrt{p^2+q^2+r^2}\,, \end{aligned}$$

$$\tan f = \frac{q}{p},$$

which are equations that were derived in the previous § under the assumption that ϑ and f are constant.

That further implies that:

$$\xi = \frac{\sqrt{p^2+q^2}}{p^2+q^2+r^2}\sin\varphi, \qquad \eta = -\frac{\sqrt{p^2+q^2}}{p^2+q^2+r^2}\cos\varphi, \qquad \xi = \frac{r}{\sqrt{p^2+q^2+r^2}}\,s,$$

and when one employs the fact that:

$$M_x = A_{11}\,(p-p'), \qquad M_y = A_{11}\,(q-q'), \qquad M_z = A_{33}\,(r-r'),$$

one will get:

$$M_\alpha = 0, \qquad M_\beta = 0, \qquad M_\gamma = A_{11}\frac{p^2+q^2}{\sqrt{p^2+q^2+r^2}}\left(1-\frac{p'}{p}\right) + A_{33}\frac{r\,(r-r')}{\sqrt{p^2+q^2+r^2}}.$$

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