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Twenty-ninth lecture

(Infinitely-small deformations of an infinitely-thin, originally-cylindrical rod. Bending and torsion for the case in which the rod is isotropic and unstressed. Work done by the forces that produce dilatations for an isotropic stressed rod. Bending of a stressed rod. s'Gravesande's method for determining the elasticity coefficients of wires. Bending of a horizontally-stressed wire by its weight. Longitudinal and torsional oscillations of a rod. Transversal oscillations of a weakly-stressed and a strongly-stressed string.)

§ 1.

We shall now further investigate the equilibrium and motion of a cylindrical, infinitely-thin rod under the assumption that the displacements of its parts are infinitely small, so p, q, and r will be infinitely small. We first focus our attention upon the case in which the rod is in equilibrium and no forces act upon its parts. Equations (34) of the previous lecture are then true. Since the changes that the nine cosines α_1, β_1, … experience along the entire length are infinitely small, γ_1 and γ_2 can be assumed to be constant in them, assuming that they themselves are finite, so the direction of the parts of the rods do not coincide with the direction of the force Γ, up to infinitely-small differences. For the time being, we shall exclude that case. We can then set:

$$\gamma_1 \Gamma = A, \qquad \gamma_2 \Gamma = B,$$

in which we understand A and B to mean constants. By neglecting infinitely-small quantities of higher order, the aforementioned equations will become:

$$\frac{d}{ds}\frac{\partial G}{\partial p} = B, \qquad \frac{d}{ds}\frac{\partial G}{\partial q} = -A, \qquad \frac{d}{ds}\frac{\partial G}{\partial r} = 0, \tag{1}$$

and these equations, which also might be true for the ξ, η, ζ coordinate system, might be true regardless of whether a special direction has been assumed for the ζ-axis. One will see that immediately when one ponders the fact that the meanings of the quantities p, q, r are completely independent of the ξ, η, ζ coordinate system, just like the coefficients that enter into the function G. By integrating those equations, one will get expressions for p, q, r in terms of linear functions of s that contain arbitrary constants. They can be determined by the values that $\frac{\partial G}{\partial p}$, $\frac{\partial G}{\partial q}$, $\frac{\partial G}{\partial r}$ – i.e., the rotational moments M_x, M_y, M_z – possess at the end of the rod. The ξ, η, ζ axes can be oriented (and we would like to do this) in such a way that the directions of the x, y, z axes deviate from the latter directions infinitely-little everywhere. α_1, β_2, γ_3 then differ from 1 infinitely little, and α_2, α_3, β_3, β_1, γ_1, γ_2 are infinitely small. It then follows from:

$$-p = \alpha_2 \frac{d\alpha_3}{ds} + \beta_2 \frac{d\beta_3}{ds} + \gamma_2 \frac{d\gamma_3}{ds},$$

$$p = \alpha_1 \frac{d\alpha_3}{ds} + \beta_1 \frac{d\beta_3}{ds} + \gamma_1 \frac{d\gamma_3}{ds},$$
$$r = \alpha_2 \frac{d\alpha_1}{ds} + \beta_2 \frac{d\beta_1}{ds} + \gamma_2 \frac{d\gamma_1}{ds},$$

that:

$$p = -\frac{d\beta_2}{ds}, \qquad q = \frac{d\alpha_2}{ds}, \qquad r = \frac{d\beta_1}{ds}.$$

If we consider equations (12) of the preceding lecture and write ψ for β_1 then we will get:

$$p = -\frac{d^2\eta}{ds^2}, \qquad q = \frac{d^2\xi}{ds^2}, \qquad r = \frac{d\psi}{ds}. \tag{2}$$

One will get ξ and η as third-degree functions of s and ψ, as a second-degree function of s by integrating these equations. ξ and η will then determine the bending, and ψ will determine the torsion of the rod.

We now specialize the case considered further by the assumption that the material of the rod is isotropic; however, we shall leave its cross-section indeterminate. We denote the elastic coefficient that were defined at the end of § 3 of the twenty-seventh lecture – i.e.:

$$2K\,\frac{1+3\theta}{1+2\theta},$$

by E, set:

$$\int x^2\,dx\,dy = \kappa_1, \qquad \int y^2\,dx\,dy = \kappa_2, \qquad \int x^2\,dx\,dy = \lambda, \tag{3}$$

and employ the fact that the x and y axes are chosen such that

$$\int x\,dx\,dy = 0, \qquad \int y\,dx\,dy = 0, \qquad \int xy\,dx\,dy = 0.$$

An argument that is similar to the one that was made in the beginning of § 6 of the previous lecture will exhibit F and G. The quantity that was denoted by w_0 there must contain the factor r; by the use of that fact, one will find that:

$$F = -\frac{E}{2}(\kappa_1 q^2 + \kappa_2 p^2 + \rho r^2 + \lambda\sigma^2), \tag{4}$$

in which ρ means a constant that is:

$$= \frac{1+2\theta}{1+3\theta}\,\frac{\kappa_1+\kappa_2}{2}$$

for the case in which the cross-section of the rod is a circle, while for any other form for the cross-section, ρ will be equal to that expression, multiplied by a numerical factor that can be given easily for an elliptical form by the calculation that was performed in § 3 of the previous lecture. It further follows from (4) that:

$$G = -\frac{E}{2}(\kappa_1 q^2 + \kappa_2 p^2 + \rho r^2).$$

Equations (1) will then give:

$$E\,\kappa_2 \frac{dp}{ds} = -B, \qquad E\,\kappa_1 \frac{dq}{ds} = A, \qquad \frac{dr}{ds} = 0.$$

Let s equal 0 and l for the two ends of the rod, respectively; let l be positive in that. A and B can then be defined to be the sums of the components along the x and y axes of the thrusts that are exerted externally on the end of the rod with s equal to 0. In place of A and B, we would prefer to introduce the corresponding component sums of the thrusts that act externally upon the other end. If we call then X' and Y' then:

$$A = -X', \qquad B = -Y',$$

and then

$$E\,\kappa_2 \frac{dp}{ds} = Y', \qquad E\,\kappa_1 \frac{dq}{ds} = -X', \qquad \frac{dr}{ds} = 0.$$

One integrates these equations and determines the integration constants from the rotational moment of the thrust that acts externally upon the end with s equal to l relative to the x, y, z axes that correspond to that end. If one calls these rotational moments M'_x, M'_y, M'_z then, as a result of equations (33) of the previous lecture, one will have:

$$E\,\kappa_2\, p = M'_x, \qquad E\,\kappa_1\, q = M'_y, \qquad E\,\rho\, r = M'_z$$

for s equal to l. It will then follow that for other values of s:

$$E\,\kappa_2\, p = M'_x - Y'(l-s), \qquad E\,\kappa_1\, q = M'_y + X'(l-s), \qquad E\,\rho\, r = M'_z.$$

With the help of equations (2), one will get:

$$E\,\kappa_1 \xi = \frac{s^2}{2}\left[X'\left(l - \frac{s}{3}\right) + M'_y\right], \qquad E\,\kappa_2 \xi = \frac{s^2}{2}\left[Y'\left(l - \frac{s}{3}\right) - M'_x\right], \qquad E\,\rho\,\psi = M'_z$$

for a suitable choice of the ξ, η, ζ coordinate system.

A method for determining the elastic coefficient E from measurements of the bending of a rod is based upon the first two of these equations. If one knows the elastic coefficient then the third equation will serve as a means of calculating the constant θ that enters into the expression for ρ from measurements of the torsion. Poisson asserted that θ equals 1 / 2 for all of the bodies that we consider here. One can neither prove nor contradict this assertion with any certainty, since one can assume with certainty that no body is homogeneous and isotropic.

§ 2.

In the previous §, we excluded the case in which the directions of the parts of the rod agreed with the direction of the force (which was denoted by Γ in the previous lecture) up to infinitely-small deviations. We would now like to direct our attention to that case. We shall employ the principle of virtual displacements in it and start from equation (4). We must set p, q, r equal to their values in (2). In order to construct an expression for σ, we make:

$$\zeta = s + \omega,$$

in which ω then means an infinitely-small quantity. From the definition of σ that was given in equation (11) of the previous lecture, one will then have:

$$(1 + \sigma)^2 = \left(\frac{d\xi}{ds}\right)^2 + \left(\frac{d\eta}{ds}\right)^2 + \left(1 + \frac{d\omega}{ds}\right)^2,$$

and when we leave undetermined the relations between the orders of magnitude, from which, ξ, η, ζ will be infinitely small, it will follow from this that:

$$\sigma = \frac{d\omega}{ds} + \frac{1}{2}\left[\left(\frac{d\xi}{ds}\right)^2 + \left(\frac{d\eta}{ds}\right)^2\right]. \tag{5}$$

The expression for the work that is done by the force that produces a displacement for which ξ, η, ω, ψ, are shifted by $\delta\xi$, $\delta\eta$, $\delta\omega$, $\delta\psi$, resp., and thus, the expression for:

$$\delta \int_0^l F\, ds$$

(in which 0 and l are assumed to be the values of s that correspond to the ends of the rod) is, as a result of equation (4):

$$-E \int_0^l ds \left[\kappa_1 \frac{d^2\xi}{ds^2}\frac{d^2\delta\xi}{ds^2} + \kappa_2 \frac{d^2\eta}{ds^2}\frac{d^2\delta\eta}{ds^2} + \rho \frac{d\psi}{ds}\frac{d\,\delta\psi}{ds} + \lambda\sigma\left(\frac{d\,\delta\omega}{ds} + \frac{d\xi}{ds}\frac{d\,\delta\xi}{ds} + \frac{d\eta}{ds}\frac{d\,\delta\eta}{ds}\right)\right].$$

It can be brought into the following form by partial integration:

$$-E \int_0^l ds \left[\kappa_1 \frac{d^4\xi}{ds^4} - \lambda \frac{d}{ds}\left(\sigma \frac{d\xi}{ds}\right)\right]\delta\xi$$

$$
\begin{aligned}
&- E \left[\frac{d^2\xi}{ds^2}\frac{d\,\delta\xi}{ds}\right]_0^l + E\left[\left(\kappa_1 \frac{d^3\xi}{ds^3} - \lambda\sigma\frac{d\xi}{ds}\right)\delta\xi\right]_0^l \\
&- E \int_0^l ds\left[\kappa_2 \frac{d^4\eta}{ds^4} - \lambda\frac{d}{ds}\left(\sigma\frac{d\eta}{ds}\right)\right]\delta\eta \\
&- E\,\kappa_2\left[\frac{d^2\eta}{ds^2}\frac{d\,\delta\eta}{ds}\right]_0^l + E\left[\left(\kappa_2 \frac{d^3\eta}{ds^3} - \lambda\sigma\frac{d\eta}{ds}\right)\delta\eta\right]_0^l \\
&+ E\,\lambda \int_0^l ds \frac{d\sigma}{ds}\delta\omega \\
&- E\,\lambda\,[\sigma\,\delta\omega]_0^l \\
&+ E\,\rho \int_0^l ds \frac{d^2\psi}{ds^2}\delta\psi \\
&- E\,\rho\left[\frac{d\psi}{ds}\delta\psi\right]_0^l .
\end{aligned}
\tag{6}
$$

We would now like to impose the restriction on the variations $\delta\xi$, $\delta\eta$, $\delta\frac{d\xi}{ds}$, $\delta\frac{d\eta}{ds}$, $\delta\omega$, $\delta\psi$ that they must vanish for s equal to 0, and construct an expression for the work that is done by the thrusts that act externally upon the end of the rod for which s is equal to l. With the help of the expression (24) and equations (18) and (19) of the fifth lecture, as well as equations (12) in the previous one, we find that this work is:

$$= X'\,\delta\xi + Y'\,\delta\eta + Z'\,\delta\omega - M'_x\,\delta\frac{d\eta}{ds} + M'_y\,\delta\frac{d\xi}{ds} + M'_z\,\delta\psi\ , \tag{7}$$

in which the variations are taken for s equal to l, the symbols X', Y', M'_x, M'_y, M'_z have the same meaning as in the previous §, and Z' means the sum of the components of the thrusts that relates to that symbol along the z-axis.

The condition for equilibrium is that the sum of the expressions (6) and (7) must vanish for any arbitrary values that one might give to the variations in them. The equations that follow from that include the results that were derived in the previous § for an isotropic rod, but are more general than them insofar as they also encompass the case that was excluded there.

One will obtain the same expression for the torsion ψ that was found there. It follows further that σ is constant, and in fact, it is determined from the equation:

$$E\,\lambda\,\sigma = Z'. \tag{8}$$

Each of the quantities ξ, η, ζ, which determine the bending, are calculated with the help of this value of s from the differential equation that is true for them, along with the boundary conditions that are associated with them. If that happens then equation (5) will tell one what $d\omega / ds$ is, and when one further establishes that ω vanishes with s, one will know ω itself.

The differential equation for ξ is:

$$E\,\kappa_1 \frac{d^4\xi}{ds^4} - Z' \frac{d^2\xi}{ds^2} = 0 \,. \tag{9}$$

This is given the boundary conditions that:

$$\xi = 0, \qquad \frac{d\xi}{ds} = 0 \tag{10}$$

for s equal to 0 and:

$$E\,\kappa_1 \frac{d^2\xi}{ds^2} = M_y', \qquad E\,\kappa_1 \frac{d^3\xi}{ds^3} - Z' \frac{d\xi}{ds} = -X' \tag{11}$$

for s equal to l.

When Z' is not infinitely large in comparison to X', the second term on the left-hand side of the last of these equations will be infinitely small in comparison to its right-hand side. The stated equation can then be written:

$$E\,\kappa_1 \frac{d^3\xi}{ds^3} = -X'.$$

Assuming that $\frac{d^3\xi}{ds^3}$ and $\frac{d\xi}{ds}$ have the same order of magnitude, it will likewise follow that Z' will be infinitely small in comparison to $E\kappa_1$, and it will follow further from this that equation (9) can be written:

$$\frac{d^4\xi}{ds^4} = 0.$$

This will yield the same value for ξ that was derived in the previous §.

Considerations that are similar to the ones that were applied to ξ can also be applied to η.

§ 3.

In order to apply the formulas that were exhibited in the previous § to an example, we address a method for determining the elasticity coefficient that is very convenient for thin wires and goes back to s'Gravesande. The method is this: The wire is stretched horizontally between two clamps, a weight hangs from its midpoint, and one observes the drop that it experiences by that means. We regard one half of the wire as a rod to which our formulas refer with the point that carries the weight as the end with s equal to 0; we assume that the ξ-axis points vertically upwards. The rod is then found in the $\xi\zeta$-plane, η equals 0, l is one-half the length of the wire, ξ is the observed drop for s equal to l, and X' is the magnitude of the hanging weight. M_y' and Z' are not given directly, here. In order to determine those quantities, one must impose the conditions that:

$$\frac{d\xi}{ds} = 0 \qquad \text{and} \qquad \omega = \omega'$$

for s equal to l, when ω' means the elongation that the half of the rod would experience if it were stretched between the clamps.

One sets:

$$h^2 = \frac{Z'}{E\,\kappa_1},$$

or, what, from (8), is the same thing:

$$h^2 = \frac{\lambda}{\kappa_1}\,\sigma, \tag{12}$$

so equation (9) will then become:

$$\frac{d^4\xi}{ds^4} = h^2\,\frac{d^2\xi}{ds^2}.$$

The integral of that equation that satisfies the conditions (10) that are to be fulfilled for s equal to 0 is:

$$\xi = A\,(e^{hs} - hs - 1) + B\,(e^{-hs} + hs - 1),$$

in which A and B are arbitrary constants. Conditions (11) give:

$$E\,\kappa_1\,h^3\,A\,(e^{hl} + e^{-hl}) = h\,M_y' - e^{-hl}\,X',$$
$$E\,\kappa_1\,h^3\,B\,(e^{hl} + e^{-hl}) = h\,M_y' + e^{hl}\,X',$$

while it follows from the fact that $d\xi / ds$ vanishes for s equal to l that:

$$A\,(e^{hl} - 1) + B\,(-\,e^{-hl} + 1) = 0\,.$$

Those three equations then yield:

$$h\,M_y'\,(e^{hl/2} + e^{-hl/2}) = -\,(e^{hl/2} - e^{-hl/2})\,X',$$
$$E\,\kappa_1\,h^3\,A\,(e^{hl/2} + e^{-hl/2}) = -\,e^{-hl/2}\,X',$$
$$E\,\kappa_1\,h^3\,B\,(e^{hl/2} + e^{-hl/2}) = e^{hl/2}\,X'.$$

If one denotes the value of ξ for s equal to l by ξ' and sets:

$$\frac{hl}{2} = p,$$

to abbreviate, then one will find that:

$$\xi' = \frac{X' l^3}{4E\,\kappa_1}\frac{1}{p^2}\left(1 - \frac{1}{p}\frac{e^p - e^{-p}}{e^p + e^{-p}}\right). \tag{13}$$

In order to be able to calculate the elasticity coefficient E from this equation, one must still ascertain p. It follows from (5) and (12) that:

$$4p^2\frac{\kappa_1}{\lambda} = \omega' l + \frac{l}{2}\int_0^l \left(\frac{d\xi}{ds}\right)^2 ds\,.$$

However, one has:

$$\frac{d\xi}{ds} = \frac{X' l^2}{4E\kappa_1}\frac{1}{p^2}\left(1 - \frac{e^{p\left(\frac{2s}{l}-1\right)} + e^{-p\left(\frac{2s}{l}-1\right)}}{e^p + e^{-p}}\right);$$

when one introduces ξ', as determined by (13), into this, the previous equation will become:

$$4p^2\frac{\kappa_1}{\lambda} = \omega' l + \frac{\xi'^2}{2}\frac{e^{2p} + e^{-2p} + 4 - \frac{5}{2}\frac{1}{p}(e^{2p} - e^{-2p})}{\left[e^p + e^{-p} - \frac{1}{p}(e^p - e^{-p})\right]^2}. \tag{14}$$

The factor of ξ'^2 is always positive, so we assume that ω' is positive. It will then follow that p must be infinitely large when one of the quantities $\omega' l$ and ξ'^2 is infinitely large in comparison to $\kappa_1 / 2$, or when both of them are. That case is realized approximately by the method that was discussed. For it, one has, from (13) and (14), that:

$$\xi' = \frac{X' l^3}{4E\,\kappa_1}\frac{1}{p^2}, \qquad 4p^2\,\frac{\kappa_1}{\lambda} = \omega' l + \frac{\xi'^2}{2},$$

in the first approximation, so:

$$E\,\lambda\,\xi'\left(\omega' l + \frac{\xi'^2}{2}\right) = X' l^3.$$

If one would like to consider the terms of next-highest order then one would have to employ the equations:

$$\xi' = \frac{X' l^3}{4E\,\kappa_1}\frac{1}{p^2}\left(1 - \frac{1}{p}\right), \qquad 4p^2\,\frac{\kappa_1}{\lambda} = \omega' l + \frac{\xi'^2}{2}\left(1 + \frac{1}{2p}\right),$$

the second of which will tell one what p is when one sets the p in its right-hand side equal to its first approximation.

We make the following remark: The factor of ξ'^2 in equation (14) will be infinite for no finite value of p. It then follows that p must be infinitely small when $\omega' l$ and ξ'^2 are infinitely small in comparison to κ_1 / λ. In that case, equation (13) will then give:

$$\xi' = \frac{X' l^3}{12E\,\kappa_1}.$$

§ 4.

We would now like to treat an example of equilibrium in a rod on whose parts *forces* act. We think of a wire as stretched horizontally between two clamps and look for the bending that it suffers when gravity acts upon its parts.

Let the ξ-axis point vertically upwards, let g be gravity, and let μ be the density of the wire. It will then follow from the expression (6) that:

$$\kappa_1 \frac{d^4\xi}{ds^4} - \lambda\sigma \frac{d^2\xi}{ds^2} = \frac{\mu\lambda g}{E}, \qquad \frac{d\sigma}{ds} = 0.$$

If s equals l and $s = -l$ for the ends of the wire then one shall have:

$$\xi = 0 \quad \text{and} \quad \frac{d\xi}{ds} = 0$$

for that value of s. If ω' means the elongation that one-half of it suffers by the stretching then one will finally have:

$$\sigma = \frac{\omega'}{l} + \frac{1}{2l}\int_0^l ds \left(\frac{d\xi}{ds}\right)^2.$$

These equations can be treated in a manner that is entirely similar to the equations that we developed in the previous §. Here, however, we would like to restrict ourselves to the consideration of the limiting case to the cases in which κ_1 is infinitely large or infinitely small in comparison to the $\lambda\sigma$ (or $\lambda\, l^2\sigma$, which is the same thing, since we regard l as finite).

If κ_1 is infinitely large in comparison to $\lambda\sigma$ then the differential equation for ξ will become:

$$\frac{d^4\xi}{ds^4} = \frac{\mu\lambda g}{E\kappa_1},$$

assuming that $\frac{d^2\xi}{ds^2}$ is not infinitely large in comparison to $\frac{d^4\xi}{ds^4}$. It and the four boundary conditions will be satisfied by:

$$\xi = \frac{\mu\lambda g}{24E\kappa_1}(l^2 - s^2)^2.$$

κ_1 will be infinitely large in comparison to $\lambda\sigma$ when $\sqrt{\omega'}$ and ξ are infinitely small in comparison to the dimensions of the cross-section of the wire.

By contrast, if one of the quantities $\sqrt{\omega'}$ and ξ is infinitely large in comparison to the dimensions of the cross-section, or if both of them are, then κ_1 will be infinitely small in comparison to $\lambda\sigma$, and the differential equation for ξ will become:

$$\frac{d^2\xi}{ds^2} = -\frac{\mu g}{E\sigma},$$

assuming that $\frac{d^4\xi}{ds^4}$ is not infinitely large in comparison to $\frac{d^2\xi}{ds^2}$. The integral of this equation that satisfies the condition that ξ vanishes for $s = \pm\, l$ is:

$$\xi = \frac{\mu g}{2E\sigma}(l^2 - s^2).$$

The condition that $d\xi / ds$ also vanishes for the ends of the wire cannot be imposed in it. $d\xi / ds$ varies infinitely fast infinitely close to the ends, so $\frac{d^4\xi}{ds^4}$ will be infinitely large in comparison to $\frac{d^2\xi}{ds^2}$ there, and the simplified differential equation will not be valid. That will yield the equation:

$$\sigma = \frac{\omega'}{l} + \frac{l^2}{6}\left(\frac{\mu g}{E\sigma}\right)^2$$

for the determination of σ.

§ 5.

The following considerations shall relate to the *oscillations* of an infinitely-thin rod. We restrict them to the case in which the oscillations are infinitely small and the rod is originally straight and isotropic. One easily finds the differential equations of motion from the expression (6) and equation (27) of the previous lecture with the help of Hamilton's principle. In the latter, one must observe that with our present assumptions, from (25) of the previous lecture:

$$R = \frac{\partial \beta_1}{\partial t},$$

or, when we again write ψ for β_1 :

$$R = \frac{\partial \psi}{\partial t};$$

if we again set:

$$\zeta = s + \omega$$

and introduce the constants κ_1 and κ_2 that were defined by (3) then the stated equation will become:

$$T = \frac{\mu}{2}\int ds\left\{\lambda\left[\left(\frac{\partial\xi}{\partial t}\right)^2 + \left(\frac{\partial\eta}{\partial t}\right)^2 + \left(\frac{\partial\omega}{\partial t}\right)^2\right] + (\kappa_1 + \kappa_2)\left(\frac{\partial\psi}{\partial t}\right)^2\right\}.$$

When one sets $\delta\xi$, $\delta\eta$, $\delta\omega$, $\delta\psi$ equal to zero at the limits of the time interval, it will follow that:

$$\delta\int T\,dt$$

has the expression:

$$-\mu\lambda\iint ds\,dt\left(\frac{\partial^2\xi}{\partial t^2}\delta\xi + \frac{\partial^2\eta}{\partial t^2}\delta\eta + \frac{\partial^2\omega}{\partial t^2}\delta\omega\right) - \mu(\kappa_1+\kappa_2)\iint ds\,dt\,\frac{\partial^2\psi}{\partial t^2}\delta\psi. \qquad (15)$$

We examine some special cases. We first assume that the rod remains straight in its motion; i.e., we set:

$$\xi = 0 \quad \text{and} \qquad \eta = 0.$$

Since one has, from (5), that:

$$\sigma = \frac{\partial\omega}{\partial s},$$

Hamilton's principle will yield the differential equations:

$$\frac{\partial^2\omega}{\partial t^2} = \frac{E}{\mu}\frac{\partial^2\omega}{\partial s^2}$$

and

$$\frac{\partial^2\psi}{\partial t^2} = \frac{E\rho}{\mu(\kappa_1+\kappa_2)}\frac{\partial^2\psi}{\partial s^2}.$$

The first of these determines the *longitudinal oscillations* of the rod, while the second one determines the *torsional oscillations*. Both of them have the same form, which is a form that we have already had to deal with in lecture twenty-three. It represents waves that propagate with constant velocity, partly in the direction of increasing s, and partly in the opposite direction. The speed of propagation for the longitudinal waves is:

$$\sqrt{\frac{E}{\mu}},$$

while for torsional waves, it is:

$$\sqrt{\frac{E\rho}{\mu(\kappa_1+\kappa_2)}}.$$

The longitudinal, as well as the torsional, oscillations can give simple tones; it is easy to calculate their oscillation numbers and the positions of the *nodes* that correspond to them. It will suffice to show that for the longitudinal oscillations, since the torsional oscillations differ from then only by a different value of the speed of propagation. We write the differential equation for motion as:

$$\frac{\partial^2\omega}{\partial t^2} = a^2\frac{\partial^2\omega}{\partial s^2},$$

in which we let a denote the speed of propagation of a longitudinal wave, and set:

$$\omega = u \sin 2\pi nt,$$

in which u should be a function of one variable s; n is then oscillation number for the tone. That will then yield the ordinary differential equation:

$$\frac{d^2u}{ds^2} = -\left(\frac{2\pi n}{a}\right)^2 u\,;$$

the general integral of it is:

$$u = A \sin \frac{2\pi n}{a}\, s + B \cos \frac{2\pi n}{a}\, s,$$

in which A and B mean arbitrary constants. Now, there are three cases to distinguish: The case in which both ends are fixed, the one in which both ends are free, and the one in which one end is fixed, while the other one is free. For a fixed end, one will always have:

$$\omega = 0, \qquad \text{so} \qquad u = 0,$$

while for a free one, it will emerge from the expression (6) that:

$$\frac{\partial\omega}{\partial t} = 0, \qquad \text{so} \qquad \frac{du}{ds} = 0.$$

Let:

$$s = 0 \quad \text{and} \quad s = l$$

for the end of the rod. If both ends as fixed then one will satisfy the conditions that are true for u when one sets:

$$u = A \sin \frac{2\pi n}{a} s,$$
$$n = h \frac{a}{2l},$$

in which h means a whole number. If both ends are free then one will have:

$$u = A \sin \frac{2\pi n}{a} s,$$
$$n = (2h - 1) \frac{a}{4l}.$$

For each of the types of oscillation, there is a point for which u is equal to 0, which will then remain at rest; these are the *nodes*. For them, when k means a whole number, one will have:

$$s = l\,\frac{k}{h},$$
$$s = l\,\frac{2k-1}{2h},$$

and

$$s = l \frac{2k}{2h-1}$$

in the three different cases, respectively.

§ 6.

We now drop the assumption that the rod remains straight, but make the assumption that ψ is equal to 0 and η is equal to 0. The cases in which ψ equals 0 and ξ equals 0 will be treated likewise.

It follows from the expressions (15), (6), and (5), with the help of Hamilton's principle, that:

$$\begin{aligned}
&\mu \frac{\partial^2 \xi}{\partial t^2} + E \frac{\kappa_1}{\lambda} \frac{\partial^4 \xi}{\partial s^4} - E \frac{\partial}{\partial s}\left(\sigma \frac{\partial \xi}{\partial s}\right) = 0, \\
&\mu \frac{\partial^2 \omega}{\partial t^2} - E \frac{\partial \sigma}{\partial s} = 0, \\
&s = \frac{\partial \omega}{\partial s} + \frac{1}{2}\left(\frac{\partial \xi}{\partial s}\right)^2 .
\end{aligned} \tag{16}$$

These are associated with certain conditions that must be true for the ends of the rod (s equals 0 and s equal l), and which can be read off from the expression (6).

One will get a particular solution of the problem that was just posed when one sets:

$$\omega = 0 \quad \text{and} \quad \sigma = 0.$$

(16) will then yield the partial differential equation for ξ:

$$\mu\lambda \frac{\partial^2 \xi}{\partial t^2} = - E\ \kappa_1 \frac{\partial^4 \xi}{\partial s^4};$$

from (6), one must have:

$$\frac{\partial^2 \xi}{\partial s^2} = 0 \qquad \text{and} \qquad \frac{\partial^3 \xi}{\partial s^3} = 0,$$

for a free end, while for an end that can neither displace nor rotate:

$$\xi = 0 \quad \text{and} \quad \frac{\partial \xi}{\partial s} = 0.$$

We assume that the rod gives a simple tone of oscillation number n and set:

$$\xi = u \sin 2\pi nt, \tag{17}$$

in which u means a function of s that satisfies the differential equation:

$$\frac{d^4 u}{ds^4} = \frac{\mu\lambda}{E\kappa_1}(2\pi n)^2\, u.$$

If one introduces a constant p by the equation:

$$\frac{\mu\lambda}{E\kappa_1}(2\pi n)^2 = \left(\frac{p}{l}\right)^4 \tag{18}$$

then its general integral will be:

$$u = A \cos\frac{ps}{l} + B \sin\frac{ps}{l} + C\frac{e^{ps/l} + e^{-ps/l}}{2} + D\frac{e^{ps/l} - e^{-ps/l}}{2},$$

in which A, B, C, D mean arbitrary constants. The four boundary conditions determine three of them, and give a transcendental equation for p whose roots will tell one the values that n can have, when one recalls (18).

Let the end where s ([1]) equals 0 be free; the two conditions that have to be fulfilled will then give:

$$C = A, \qquad D = B$$

here, so:

([1]) In the previous edition, l appeared in place of s.

$$u = A\left(\cos\frac{ps}{l} + \frac{e^{ps/l} + e^{-ps/l}}{2}\right) + B\left(\sin\frac{ps}{l} + \frac{e^{ps/l} - e^{-ps/l}}{2}\right). \tag{19}$$

If the end with s equal to l is also free then the equations:

$$A\left(\frac{e^p + e^{-p}}{2} - \cos p\right) + B\left(\frac{e^p - e^{-p}}{2} - \sin p\right) = 0,$$
$$A\left(\frac{e^p - e^{-p}}{2} + \sin p\right) + B\left(\frac{e^p + e^{-p}}{2} - \cos p\right) = 0$$

must be true. They determine the ratio $A : B$ and give the equation:

$$\left(\frac{e^p + e^{-p}}{2} - \cos p\right)^2 + \left(\frac{e^p - e^{-p}}{2}\right)^2 + \sin^2 p = 0$$

for p; i.e., the equation:

$$\cos p\,\frac{e^p + e^{-p}}{2} = 1.$$

Its roots are the values of x that correspond to the intersection points of the curves whose equations are:

$$y = \cos x \qquad \text{and} \qquad y = \frac{2}{e^x + e^{-x}}.$$

A discussion of these equations would show that p equal to 0 is a root, and in fact, a multiple one, and that the next-higher root would be somewhat greater than $3\pi/2$, the following one would be somewhat smaller than $5\pi/2$, etc., and that the roots would approach an odd multiple of $\pi/2$ as their order numbers increase. p equal to 0 would correspond to an infinite period of oscillation, so there would be no tone; for the deepest tone of the rod – viz., its *basic tone* – one will have, approximately, $p = 3\pi/2$; i.e., 4.712. One will get a more precise approximation when one calculates p from the equation:

$$\cos p = \frac{2}{e^{3\pi/2} + e^{-3\pi/2}},$$

which implies that p is equal to 4.730. One can find all roots of the equation in question with arbitrary precision by a similar process.

The *nodes* are determined by the equation:

$$u = 0;$$

if one sets:

$$\frac{s}{l} = x$$

then that will be:

$$\left(\frac{e^p - e^{-p}}{2} - \sin p\right)\left(\frac{e^p + e^{-p}}{2} + \cos px\right) = \left(\frac{e^p + e^{-p}}{2} - \cos p\right)\left(\frac{e^p - e^{-p}}{2} + \sin px\right)$$

From Strehlke's calculation (*), the values of x for the first tones are:

Tone 1	**Tone 2**	**Tone 3**
0.2242	0.1321	0.0944
0.7758	0.5000	0.3585
	0.8679	0.6415
		0.9056

If the s equal to l is fixed, while the s equal to 0 end is free then equation (19) will still be true, but one will have:

$$A\left(\frac{e^p + e^{-p}}{2} + \cos p\right) + B\left(\frac{e^p - e^{-p}}{2} + \sin p\right) = 0,$$
$$A\left(\frac{e^p - e^{-p}}{2} - \sin p\right) + B\left(\frac{e^p + e^{-p}}{2} + \cos p\right) = 0$$

for the determination of $A : B$ and p, which will imply that:

$$\cos p\ \frac{e^p + e^{-p}}{2} = -1.$$

The smallest positive root of this equation is somewhat larger than $\pi / 2$ (precisely, 1.875), the following one somewhat smaller than $3\pi / 2$, the next one, somewhat larger than $5\pi / 2$, etc.

When one again sets s / l equal to x, one will have:

$$\left(\frac{e^p - e^{-p}}{2} + \sin p\right)\left(\frac{e^p + e^{-p}}{2} + \cos px\right) = \left(\frac{e^p + e^{-p}}{2} + \cos p\right)\left(\frac{e^p - e^{-p}}{2} + \sin px\right)$$

for the nodes.

We would still like to consider the case in which, whereas the s equal to 0 end is free, the s equal to l end maintains a certain periodic motion. Let:

(*) Dove's Repertorium der Physik III, 110.

$$\xi = \alpha \sin 2\pi n\, t, \qquad \frac{\partial \xi}{\partial s} = \beta \sin 2\pi n\, t \tag{20}$$

for s equal to l, in which α, β, and n are given constants. One also satisfies the partial differential equation that is true for ξ and the boundary conditions for s equal to 0 by equations (17) and (19), when one calculates p from (18); the conditions that are imposed for s equal to l give:

$$\alpha = A\left(\frac{e^p + e^{-p}}{2} + \cos p\right) + B\left(\frac{e^p - e^{-p}}{2} + \sin p\right),$$

$$\frac{l}{p} \cdot \beta = A\left(\frac{e^p - e^{-p}}{2} - \sin p\right) + B\left(\frac{e^p + e^{-p}}{2} + \cos p\right),$$

which are two equations that determine A and B completely, in general. Only when the determinant of the coefficients of A and B vanishes – i.e., when p and n correspond to the tones that the rod can give with one free end and one fixed one – will A and B be undetermined, in the event that the ratio $\alpha : \beta$ has a certain value that is infinite for other values of that ratio.

In an entirely similar way, one can treat the case in which one has the equations:

$$\xi = \alpha' \cos 2\pi n\, t, \qquad \frac{\partial \xi}{\partial s} = \beta' \cos 2\pi n\, t$$

for s equal to l, instead of equations (20). If one sets ξ equal to the sum of the expressions that are true for ξ in both cases then one will know that in the case for which s equals l the motion of the rod is:

$$\xi = \alpha \sin 2\pi n\, t + \alpha' \cos 2\pi n\, t,$$

$$\frac{\partial \xi}{\partial s} = \beta \sin 2\pi n\, t + \beta' \cos 2\pi n\, t.$$

§ 7.

We would now like to look for particular solutions of the equations (16) for which ω and σ do not vanish, and which relate to the transversal oscillations of *strings*. One calls a stretched rod a *string* when its lateral dimensions are also sufficiently small in comparison to the displacement of its parts. The factor κ_1 / λ enters into the second term of the first of equations (16); that factor has the order of the cross-section. We will assume that the cross-section is small enough in comparison to the displacements that are present that the stated term is infinitely small in comparison to the third term in that equation. Equations (16) will then become:

$$\frac{\mu}{E}\frac{\partial^2\xi}{\partial t^2} = \frac{\partial\sigma}{\partial s}\frac{\partial\xi}{\partial s}+\sigma\frac{\partial^2\xi}{\partial s^2},$$
$$\frac{\mu}{E}\frac{\partial^2\omega}{\partial t^2} = \frac{\partial\sigma}{\partial s}, \tag{21}$$
$$\sigma = \frac{\partial\omega}{\partial s}+\frac{1}{2}\left(\frac{\partial\xi}{\partial s}\right)^2.$$

We add the conditions that:

for $s = 0$, $\xi = 0$, $\omega = 0$,
for $s = l$, $\xi = 0$, $\omega = \omega'$,

in which ω' means a given constant; that expresses the idea that both ends of the string are fixed. The value of ω' determines the *tension* that is given for it.

We would now like to look for those motions for which $\frac{\partial^2\omega}{\partial t^2}$ is infinitely small compared to $\frac{\partial^2\xi}{\partial t^2}$. With that assumption, it will follow from the first two of equations (21) that $\frac{\partial\sigma}{\partial s}$ is infinitely small compared to $\frac{\partial\sigma}{\partial s}\frac{\partial\xi}{\partial s}+\sigma\frac{\partial^2\xi}{\partial s^2}$. However, if $\frac{\partial\sigma}{\partial s}\frac{\partial\xi}{\partial s}$ is infinitely small compared to $\frac{\partial\sigma}{\partial s}$ then $\sigma\frac{\partial^2\xi}{\partial s^2}$ must be infinitely large compared to $\frac{\partial\sigma}{\partial s}$, and even more infinitely large compared to $\frac{\partial\sigma}{\partial s}\frac{\partial\xi}{\partial s}$. Thus, the first of equations (21) will become:

$$\frac{\mu}{E}\frac{\partial^2\xi}{\partial t^2} = \sigma\frac{\partial^2\xi}{\partial s^2}. \tag{22}$$

It follows from this that $\frac{\partial\sigma}{\partial s}$ is infinitely small compared to $\sigma\frac{\partial^2\xi}{\partial s^2}$, from which, it will follow that $\frac{\partial\sigma}{\partial s}$ is even more infinitely small compared to s, so σ is independent of s. From the third of equations (21), one will then have:

$$\sigma = \frac{\omega'}{l}+\frac{1}{2l}\int_0^l\left(\frac{\partial\xi}{\partial s}\right)^2 ds,$$

and thus, from (22):

$$\frac{\mu}{E}\frac{\partial^2\xi}{\partial t^2} = \left[\frac{\omega'}{l}+\frac{1}{2l}\int_0^l\left(\frac{\partial\xi}{\partial s}\right)^2\right]\frac{\partial^2\xi}{\partial s^2}. \tag{23}$$

That equation simplifies essentially when the tension in the string is large enough, namely, when ω is large enough compared to ξ that the second term in the factor of $\frac{\partial^2 \xi}{\partial s^2}$ can be neglected in comparison to the first. Before we go into a consideration of that case more closely, we would like to derive certain particular solutions of equation (23) that will be true no matter how small the tension might be.

We set:

$$\xi = u \sin \frac{ms}{l} \pi,$$

in which m means a whole number, and u is a function of t that is to be determined. The conditions that ξ has to fulfill for s equal to 0 and s equal l will be satisfied by that. Equation (23) will also be fulfilled when one determines u from the differential equation:

$$\frac{d^2 u}{dt^2} = -\left(\frac{m\pi}{l}\right)^2 \frac{E}{\mu} u \left[\frac{\omega'}{l} + \left(\frac{m\pi}{2l}\right)^2 u^2\right]. \tag{24}$$

Its general integral is:

$$u = a \cos \operatorname{am} h\,(t - t_0) \qquad (\operatorname{mod} k),$$

in which a and t_0 are two arbitrary constants, while h and k are two constants that depend upon a in a certain way. With that assumption for u, one will, in fact, get:

$$\frac{d^2 u}{dt^2} = -h^2\, u \left(1 - 2\kappa^2 + \frac{2\kappa^2}{a^2} u^2\right),$$

and that equation will become identical with (24) when one sets:

$$2\kappa^2 = \frac{m^2 \pi^2 a^2}{m^2 \pi^2 a^2 + 4l\omega'},$$

$$h^2 = \frac{m^2 \pi^2}{4l^4} \frac{E}{\mu} (m^2 \pi^2 a^2 + 4l\, \omega').$$

§ 8.

We now turn to a discussion of the case that was mentioned before, in which the tension in the string is large enough that the factor of $\frac{\partial^2 \xi}{\partial s^2}$ in the second term of equation (23) can be neglected in comparison to the first one. The stated equation will then be:

$$\frac{\partial^2 \xi}{\partial t^2} = \frac{E}{\mu}\frac{\omega'}{l}\frac{\partial^2 \xi}{\partial s^2}.$$

That must be combined with the conditions that ξ vanishes for s equal to 0 and s equal to l.

We have already had to treat that same differential equation several times, and the last time was in the investigation of the longitudinal and torsion oscillations of an elastic rod. Among the cases that were considered there, one also finds the one that fulfills the same boundary conditions as the ones here. The particular solutions that are true for this case are also valid here, and what was said there about the possible simple tones and the corresponding nodes is also true here. We would now like to summarize the particular solutions that we referred to for the transversally-oscillating string, in general. In order to shorten the formulas somewhat, we then introduce units of length and time such that l will equal π and the period of a simple oscillation will be π for the basic tone. A particular solution will then be:

$$\xi = \sin mt \sin ms,$$

and another one will be:

$$\xi = \cos mt \sin ms,$$

in which m means any positive whole number. Another solution will then be:

$$\xi = \sum (A_m \sin mt + B_m \cos mt) \sin ms,$$

in which A_m, B_m are arbitrary constants, and the sum is taken over all m from 1 to ∞. If one sets:

$$\xi = U, \qquad \frac{\partial \xi}{\partial t} = U'$$

for t equal to 0, in which U and U' mean functions of s that are given arbitrarily for s equal to 0 up to s equal to π, then that will require that one must have:

$$U = \sum B_m \sin ms, \qquad U' = \sum m\, A_m \sin ms \tag{25}$$

for that interval. Assuming that the functions U and U' can be represented in that way, the values that must be given to the constants A_m and B_m can be found easily with the help of the theorem that when m and m' are two different whole numbers:

$$\int_0^\pi \sin ms \sin m's\, ds = 0,$$

and when m is an arbitrary whole number, one will have:

$$\int_0^\pi \sin^2 ms\,ds = \frac{\pi}{2}.$$

One will prove that theorem easily when one employs the fact that:

$$2 \sin ms \sin m's = \cos (m - m')\, s - \cos (m + m')\, s,$$
$$2 \sin^2 ms = 1 - \cos 2ms.$$

With its help, one will find from (25) that:

$$B_m = \frac{2}{\pi}\int_0^\pi U \sin ms\,ds,$$

$$m\,A_m = \frac{2}{\pi}\int_0^\pi U' \sin ms\,ds.$$

Dirichlet (*) was the first to prove rigorously that U and U' can always be represented in the way that we imagine when he showed that the infinite series (viz., a so-called Fourier series):

$$\sum C_m \sin ms,$$

in which the coefficients are determined from the equation:

$$C_m = \frac{2}{\pi}\int_0^\pi f(s) \sin ms\,ds,$$

in which $f(s)$ means an arbitrary, everywhere single-valued, finite, continuous function of s, converges to $f(s)$ for all values of s between 0 and π.

We mention another form for a solution of the problem of string oscillations that is being treated. We preserve the units of length and time that we just used – i.e., we again set the length of the string and the period of a simple oscillation of the basic tone equal to π – then the differential equation for the displacement ξ will become:

$$\frac{\partial^2 \xi}{\partial t^2} = \frac{\partial^2 \xi}{\partial s^2},$$

and its general integral will be:

$$\xi = \varphi(t + s) + \psi(t - s),$$

in which φ and ψ mean two arbitrary functions of the relevant arguments. It follows from the condition that ξ always vanishes for s equal to 0 that:

$$0 = \varphi(t) + \psi(t),$$

(*) Dove's Repertorium der Physik I, 152; Crelle's Journal, Bd. 4, pp. 157.

so

$$\xi = \varphi(t + s) + \psi(t - s),$$

and it follows from that condition that ξ is always equal to 0 for s equal to π that:

$$\varphi(t + \pi) = \varphi(t - \pi)$$

or

$$\varphi(x + 2\pi) = \varphi(x) ;$$

i.e., φ is a periodic function of period 2π. φ, and therefore ξ, will be determined completely when $\varphi(x)$ is known for the interval from $\xi = -\pi$ to $\xi = +\pi$. Knowing the initial state of the string will lead to that. Once more, let:

$$\xi = U, \qquad \frac{\partial \xi}{\partial t} = U',$$

for t equal 0, in which U and U' mean functions of s that are given for s equal to 0 up to s equal to π. One must then have:

$$\begin{aligned} U &= \varphi(s) - \varphi(-s), \\ U' &= \varphi'(s) - \varphi'(-s) \end{aligned}$$

for that interval, in which φ' means the differential quotient of the function φ with respect to the argument. If one multiplies the last equation by ds and integrates it then one will obtain:

$$\int U' ds = \varphi(s) + \varphi(-s),$$

in which the lower limit of the integral is an arbitrary constant, and then furthermore:

$$\begin{aligned} \varphi(s) &= \tfrac{1}{2} U + \tfrac{1}{2} \int U' ds, \\ \varphi(-s) &= -\tfrac{1}{2} U + \tfrac{1}{2} \int U' ds . \end{aligned}$$

$\varphi(s)$ is determined (up to an additive constant) by these equations for the interval from $s = -\pi$ to $s = +\pi$, and will then be determined in general (up to an additive constant). However, its value has no influence on the value of ξ, since the latter is equal to the difference of two values of φ.

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