Excerpted from G. Kirchhoff, *Vorlesungen über mathematische Physik*, v. I: Mechanik, Teubner, Leipzig, 1897. Translated by D. H. Delphenich.

Lecture twenty-six

(Friction in an incompressible fluid. Presentation of the differential equations and boundary conditions. Flow of the fluid through a long, cylindrical tube. Introduction of the assumptions that the fluid adheres to a solid body with which it is in contact and that the velocities are infinitely small. Uniform rotation of a ball in the fluid around a diameter or an ellipsoid of rotation around its symmetry axis in the case in which the fluid is externally unbounded or bounded by a concentric spherical surface (confocal ellipsoid, resp.). Calculation of the rotational moment of the force that must act upon the ball or the ellipsoid. Resistance on a ball that advances uniformly in the fluid. Rotational oscillations of a ball. Oscillations of a ball in which the center goes back and forth along a line.)

§1.

We would now like to conclude our hydrodynamical investigations by considering certain motions of an incompressible fluid that are influenced by *friction*. We have already presented the differential equations for such motions in the eleventh lecture. We again let u, v, w denote the components of the velocity at the point (x, y, z) at the time t, set:

$$\begin{aligned}
X_x &= p - 2k\,\frac{\partial u}{\partial x}, & Y_z &= Z_y = -k\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right),\\
Y_y &= p - 2k\,\frac{\partial v}{\partial y}, & Z_x &= X_z = -k\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right),\\
Z_z &= p - 2k\,\frac{\partial w}{\partial z}, & X_y &= Y_x = -k\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right),
\end{aligned} \tag{1}$$

in which k is a constant of the fluid that is required by the friction, p means an unknown function of x, y, z, t that expresses the components of the acceleration, as was done in the fifteenth lecture. When one assumes that no forces act upon the parts of the fluid and denotes it density by μ, the differential equations will then be:

$$\begin{aligned}
&\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}+\frac{1}{\mu}\left(\frac{\partial X_x}{\partial x}+\frac{\partial X_y}{\partial y}+\frac{\partial X_z}{\partial z}\right)=0,\\
&\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}+\frac{1}{\mu}\left(\frac{\partial Y_x}{\partial x}+\frac{\partial Y_y}{\partial y}+\frac{\partial Y_z}{\partial z}\right)=0,\\
&\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}+\frac{1}{\mu}\left(\frac{\partial Z_x}{\partial x}+\frac{\partial Z_y}{\partial y}+\frac{\partial Z_z}{\partial z}\right)=0,\\
&\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0.
\end{aligned} \tag{2}$$

Certain conditions must be fulfilled on the outer surface of the fluid, and thus, on the surfaces at which it contacts other bodies, which can be solid or fluid. Some of these can be gathered from § 6 of lecture ten and § 4 of lecture eleven. If we let ds denote an element of the contacting surfaces and let n denote the normal to ds that is directed into the interior of the fluid considered then the components of the velocities of the particles in the direction of n must have equal values on both sides of ds, while X_n, Y_n, Z_n must have equal values for those particles. However, these conditions are not sufficient to determine the solutions of the differential equations (2) or (3); another hypothesis is necessary in order to extend them. A suitable hypothesis that has proved itself in certain cases is that u, v, w themselves possesses the same values on both sides of ds, so the particles of the two bodies always remain in contact once they are first in contact. We cite a more general hypothesis that is established. We let refer u, v, w to the particles of the fluid in question that lie on ds, and let refer u_1, v_1, w_1 refer to the particles on the other side of ds; as mentioned, one then has:

$$(u - u_1) \cos (nx) + (v - v_1) \cos (ny) + (w - w_1) \cos (nz) = 0.$$

We can refer to $u - u_1$, $v - v_1$, $w - w_1$ as the components of the *relative velocity* of the particles in question and this equation will then express the idea that this relative velocity is perpendicular to n, and thus parallel to ds. We think of the pressure that is exerted upon ds – namely, the pressure whose components along the coordinate axes are X_n , Y_n , Z_n – as being divided into two components, one of which is parallel to n, while the other is parallel to ds. From the hypothesis that we spoke of, the latter will have the opposite direction, like the relative velocity, and is proportional to it. One finds the analytical expression for this hypothesis by the following argument: One has:

$$X_n \cos (nx) + Y_n \cos (ny) + Z_n \cos (nz) = 0$$

for the component of the pressure that is exerted upon ds in the direction of n. If one multiplies this expression by cos (nx), cos (ny), cos (nz), resp., then one will obtain the projections of that component along the coordinate axes. If one subtracts these products from X_n , Y_n , Z_n then one will have in these differences, the projections onto the coordinate axes of the component of the pressure that acts upon ds that is parallel to ds. From the hypothesis that was expressed, one will then have:

$$\begin{aligned}
X_n - [X_n \cos (nx) + Y_n \cos (ny) + Z_n \cos (nz)] \cos (nx) &= \lambda (u_1 - u), \\
Y_n - [X_n \cos (nx) + Y_n \cos (ny) + Z_n \cos (nz)] \cos (ny) &= \lambda (v_1 - v), \qquad (4) \\
Z_n - [X_n \cos (nx) + Y_n \cos (ny) + Z_n \cos (nz)] \cos (nz) &= \lambda (w_1 - w),
\end{aligned}$$

in which λ means a constant that depends upon the nature of the fluid and the bodies that it contacts.

If one assumes that λ is infinitely large then equations (4) will lead to the special hypothesis that was mentioned before by which $u = u_1$, $v = v_1$, $w = w_1$. The other boundary condition is that $\lambda = 0$. Equations (4) will give:

$$X_n : Y_n : Z_n = \cos(nx) : \cos(ny) : \cos(nz)$$

for it, as one sees when one divides it by cos (nx), cos (ny), cos (nz) and subtracts any two of them from each other. The pressure whose components are X_n, Y_n, Z_n is then a perpendicular one. That must be the case when the body that is contacted is a fluid in which one can neglect friction.

§ 2.

We would now like to look for particular solutions of the equations that were presented in the previous §. We first assume that:

$$u = 0 \quad \text{and} \quad v = 0;$$

i.e., that the motion is everywhere parallel to the z-axis. The first, second, and fourth of equations (3) will then become:

$$\frac{\partial p}{\partial x} = 0, \qquad \frac{\partial p}{\partial y} = 0, \qquad \frac{\partial w}{\partial z} = 0;$$

i.e., p is independent of x and y, and w is independent of z. The third of equations (3) will become:

$$\mu \frac{\partial w}{\partial t} + \frac{\partial p}{\partial z} - k\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) = 0,$$

from which, with the remark that was just made, it will follow that:

$$\frac{\partial p}{\partial z} = c, \qquad k\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) - \mu \frac{\partial w}{\partial t} = c,$$

in which c is independent of x, y, z, and thus a function of the one variable t. We specialize the case in question even further by assuming that the motion is stationary; c will then be a constant, and:

$$\frac{dp}{dz} = c, \qquad k\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) = c. \tag{5}$$

According to these equations, a fluid can move in a fixed and immobile cylindrical tube that is parallel to the z-axis. We now define the boundary conditions that must be fulfilled on the internal surface of such a tube. For our case, (1) will imply that:

$$X_x = p, \qquad Y_z = Z_y = -k \frac{\partial w}{\partial y},$$

$$Y_y = p, \qquad Z_x = X_z = -\,k\,\frac{\partial w}{\partial x},$$
$$Z_z = p, \qquad X_y = Y_x = 0,$$

and since one has:

$$\cos(nz) = 0,$$

it will follow from equations (7) in lecture eleven that:

$$X_n = p\cos(nx), \qquad Y_n = p\cos(ny), \qquad Z_n = -\,k\left(\frac{\partial w}{\partial x}\cos(nx) + \frac{\partial w}{\partial y}\cos(ny)\right),$$

and

$$X_n\cos(nx) + Y_n\cos(ny) + Z_n\cos(nz) = p.$$

One must set $u_1 = v_1 = w_1 = 0$ in equations (4), which we would like to apply as boundary conditions. The first two of them will then be fulfilled immediately, and the third one will give:

$$k\left(\frac{\partial w}{\partial x}\cos(nx) + \frac{\partial w}{\partial y}\cos(ny)\right) = \lambda w,$$

or, what amounts to the same thing:

$$\frac{\partial w}{\partial n} = \frac{\lambda}{k}\,w. \tag{6}$$

We now assume that the cross-section of the tube is a circle of radius R whose center lies in the z-plane, and that the motion is the same at the same distance from that axis. If one sets:

$$\rho = \sqrt{x^2 + y^2}$$

then the second of equations (5) will become:

$$\frac{d^2 w}{d\rho^2} + \frac{1}{\rho}\frac{dw}{d\rho} = \frac{c}{k},$$

from which it will follow that:

$$w = \frac{1}{4}\frac{c}{k}\rho^2 + A\ln\rho + B,$$

in which A and B mean arbitrary constants. The first of these these must vanish, since w cannot become infinite for $\rho = 0$. The second one is implied by (6); i.e., by the condition that one must have:

$$\frac{dw}{d\rho} = -\frac{\lambda}{k}w$$

for $\rho = R$. It will then follow that:

$$B = -\frac{c}{2\lambda}R - \frac{c}{4k}R^2,$$

so

$$w = -\frac{c}{4k}\left(R^2 + \frac{2k}{\lambda}R - \rho^2\right).$$

One finds the constant c from the first of equations (5) when the value of p is known for two values of z. If $p = p_0$ for $z = 0$, and $p = p_l$ for $z = \lambda$ then:

$$c = \frac{p_l - p_0}{l}.$$

If one lets Q denote the volume of the fluid that flows through a cross-section in the direction of the z-axis in a unit time then:

$$Q = 2\pi\int_0^R w\rho\, d\rho,$$

so

$$Q = \pi\frac{p_0 - p_l}{8kl}\left(R^4 + 4\frac{k}{\lambda}R^3\right). \tag{7}$$

This result is very close to valid for the case in which a gravitating fluid flows through a very long, thin, horizontal tube from a voluminous vessel into the atmosphere. One can choose the cross-sections $z = 0$ and $z = l$ to be at distances from the two ends of the tube that are large in comparison to its diameter, but small in comparison to l, and set p_0 equal to the pressure that is present in the tube when the fluid is at rest, and p_l is the pressure of the atmosphere.

Measurements of the discharge volume have been performed by Poiseuille for an arrangement of the type described; he found that:

$$Q = K\frac{p_0 - p_l}{l}R^4,$$

in which K means a quantity that remains unchanged when p_0, l, or R are changed. A comparison of this equation with (7) then leads to the conclusion that λ should be regarded as infinitely large, so one must assume that the fluid particles that contact the wall of the tube will adhere to it. The values of that were found for K then further allow the value of k to be calculated for the fluids that were subjected to the tests.

§ 3.

We would like to simplify the further considerations that we would like to make of the friction in a fluid by the assumption that the fluid particles that contact a solid body adhere to it and the assumption that the velocities are infinitely small. The latter will imply that equations (3) will become:

$$\mu\frac{\partial u}{\partial t}+\frac{\partial p}{\partial x} = k\,\Delta u,$$

$$\mu\frac{\partial v}{\partial t}+\frac{\partial p}{\partial y} = k\,\Delta v,$$

$$\mu\frac{\partial w}{\partial t}+\frac{\partial p}{\partial z} = k\,\Delta w, \tag{8}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z} = 0.$$

If the motion is stationary then they will go to:

$$\frac{\partial p}{\partial x} = k\,\Delta u, \qquad \frac{\partial p}{\partial y} = k\,\Delta v, \qquad \frac{\partial p}{\partial z} = k\,\Delta w,$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z} = 0. \tag{9}$$

One solution of these equations is:

$$\mu = \text{const.}, \qquad u = \frac{\partial W}{\partial y}, \qquad v = -\frac{\partial W}{\partial x}, \qquad w = 0, \tag{10}$$

if W satisfies the equation:

$$\Delta W = \text{const.}$$

We can then set:

$$W = \frac{c}{r}, \qquad r = \sqrt{x^2+y^2+z^2}\,,$$

in which c means a constant, and get a particular solution of our differential equation in:

$$p = \text{const.}, \qquad u = -\frac{c}{r^3}\,y\,, \qquad v = \frac{c}{r^3}\,x\,, \qquad w = 0. \tag{11}$$

The motion that it represents is easy to visualize. Considerations that first carried out in § 5 of lecture four show that the point for which:

$$u = -\,\psi y, \qquad v = \psi x, \qquad w = 0, \tag{12}$$

in which ψ means a constant, do not change their relative positions and thus move as if they belonged to a solid body that rotates around the z-axis with an angular velocity ψ. As a result of equations (11), conditions (12) will be fulfilled for the points of a spherical surface that is described about the coordinate origin with the arbitrary radius r when one sets:

$$\psi = \frac{c}{r^3}.$$

If a solid ball whose outer surface has the equation $r = r_1$ is found in the fluid and which rotates around the z-axis with the constant angular velocity ψ_1 then equations (11) will represent a possible motion of the fluid when one sets:

$$c = \psi_1 r_1^3$$

in it.

If the fluid is bounded by two concentric spherical surfaces whose equations are $r = r_1$ and $r = r_2$, the first of which is the smaller one and rotates around the z-axis with the constant angular velocity ψ_1, while the second one is the larger and remains at rest, then equations (10) will give a possible motion when one sets:

$$W = \frac{c}{r} - \frac{b}{2}(x^2 + y^2)$$

in them, and determines the constants b and c suitably. With that assumption on W, one will have:

$$u = -\left(\frac{c}{r^3} + b\right)y, \qquad v = \left(\frac{c}{r^3} + b\right)x, \qquad w = 0, \tag{13}$$

and the boundary conditions will be fulfilled when one makes:

$$\psi_1 = \frac{c}{r_1^3} + b, \quad 0 = \frac{c}{r_2^3} + b;$$

it will then follow from this that:

$$c\left(\frac{1}{r_1^3} - \frac{1}{r_2^3}\right) = \psi_1 .$$

Should the ball of radius r_1 rotate a velocity that remains the same, then an rotational moment M must act upon it in the sense of the motion that is equal to the rotational moment of the pressure that is exerted upon the fluid. If ds is an element of the outer surface of the ball, and n is the normal to ds, which coincides with the lengthening of the radius, then one will have:

$$M = \int ds\,(xY_n - yX_n). \tag{14}$$

However, one has:

$$Y_n = \frac{1}{r}(xY_x + y\,Y_y + z\,Y_z),$$
$$X_n = \frac{1}{r}(xX_x + y\,X_y + z\,X_z),$$

and from (1) and (13):

$$\begin{aligned} X_x &= p - 6kc\,\frac{xy}{r^5}, & Y_z = Z_y &= 3kc\frac{xz}{r^5}, \\ Y_y &= p + 6kc\,\frac{xy}{r^5}, & Z_x = X_z &= -\,3kc\frac{yz}{r^5}, \\ Z_z &= p, & X_y = Y_x &= 3kc\frac{x^2 - y^2}{r^5}, \end{aligned} \tag{15}$$

in which equations, one sets $r = r_1$ everywhere. That will imply:

$$Y_n = \frac{y}{r}\,p + \frac{3kc}{r^4}\,x, \qquad X_n = \frac{x}{r}\,p - \frac{3kc}{r^4}\,y,$$

so:

$$M = \frac{3kc}{r^4}\int ds\,(x^2 + y^2),$$

or, since:

$$\int x^2\,ds = \int y^2\,ds = \int z^2\,ds = \frac{4\pi}{3}r^4,$$

one will have:

$$M = 8\pi kc.$$

Since r does not enter into that expression, it will suffer no alteration when one sets $r = r_1$.

Equations (10) can also be adapted to the case in which the fluid is bounded by two confocal ellipsoids of rotation whose rotational axis defines the z-axis, where the outer one is at rest and the inner one rotates around the z-axis with constant angular velocity ψ_1. We write the equation for the inner ellipsoid:

$$\frac{x^2 + y^2}{a_1^2} + \frac{z^2}{c_1^2} = 1, \tag{16}$$

and let Ω denote the potential of the mass 1 that fills up the same bounded space with an equal density, relative to the external point (x, y, z). In the case where the fluid is regarded as externally unbounded, which is the case that we shall consider next, one can satisfy the boundary conditions by setting:

$$W = c\Omega,$$

and determine the constant c in a suitable way. As a result of equation (3) of lecture eighteen, one has, in fact:

$$\Omega = \frac{3}{4}\int_\sigma^\infty d\lambda \frac{1 - \frac{x^2+y^2}{a_1^2+\lambda} - \frac{z^2}{c_1^2+\lambda}}{(a_1^2+\lambda)\sqrt{c_1^2+\lambda}},$$

in which σ means the positive root of the equation:

$$\frac{x^2+y^2}{a_1^2+\sigma} + \frac{z^2}{c_1^2+\sigma} = 1;$$

equations (10) will then give:

$$u = -\psi y, \qquad v = \psi x, \qquad w = 0,$$

when one makes:

$$\psi = \frac{3}{2}c\int_\sigma^\infty \frac{d\lambda}{(a_1^2+\lambda)^2\sqrt{c_1^2+\lambda}}.$$

Thus, the points of the fluid that lie on an ellipsoid that is confocal to the ellipsoid (16) and is determined by a value of σ move as if they adhered to a solid body that rotates around the z-axis with the angular velocity ψ; the value of c is then determined from the equation:

$$\psi_1 = \frac{3}{2}c\int_0^\infty \frac{d\lambda}{(a_1^2+\lambda)^2\sqrt{c_1^2+\lambda}}. \tag{17}$$

Equation (14) is also true here for the rotational moment M that must act upon the ellipsoid in order to make it rotate uniformly. Its calculation will be eased by the following remark, which is connected with the definition that was given in lecture eleven of the pressure forces in equations (1) and (2). If we apply the last of these equations to an arbitrary part of the fluid then we will observe that the motion is stationary, the velocity is infinitely small, and forces do not act upon that part of the fluid, then we will obtain:

$$\int ds\,(x\,Y_n - y\,X_n) = 0,$$

in which ds means an element of the outer surface of the chosen part, and n means the normal to ds that is directed to the interior. Now, let that part be bounded by the ellipsoid and an infinitely-large, concentrated spherical surface; the remark that was made above will then show that M is equal to the integral:

$$\int ds\,(x\,Y_n - y\,X_n)$$

when it is taken over the infinite spherical surface, and n is understood to be the normal that coincides with the extension of the radius. However, equations (15) are also true here at infinity; one therefore also has:

$$M = 8\pi kc, \tag{18}$$

in which c must be determined by (17), though.

If the fluid is externally bounded by the ellipsoid at rest:

$$\frac{x^2+y^2}{a_2^2} + \frac{z^2}{c_2^2} = 1,$$

which is confocal to the ellipsoid (16) such that:

$$a_2^2 - a_1^2 = c_2^2 - c_1^2,$$

then one must set:

$$W = c\,\Omega - \frac{b}{2}(x^2+y^2),$$

and determine the constants b and c in such a way that:

$$\psi_1 = \frac{3}{2}c\int_0^\infty \frac{d\lambda}{(a_1^2+\lambda)^2\sqrt{c_1^2+\lambda}} + b,$$

$$0 = \frac{3}{2}c\int_0^\infty \frac{d\lambda}{(a_2^2+\lambda)^2\sqrt{c_2^2+\lambda}} + b,$$

from which, it follows that:

$$\psi_1 = \frac{3}{2}c\int_0^{a_2^2-a_1^2} \frac{d\lambda}{(a_1^2+\lambda)^2\sqrt{c_1^2+\lambda}} + b. \tag{19}$$

If one computes the rotational moment M that must act upon the inner ellipsoid in order to obtain uniform motion with the help of equation (14) then the constant b will not appear, and one will find that it is expressed in terms of the constant c in precisely the same way as when the fluid is bounded externally. Therefore, equation (18) will also be true here, when the value of c in (19) is assumed to be true.

§ 4.

It follows from equations (9) that:

$$\Delta p = 0 .$$

If one has chosen p according to this condition and determined a function V that satisfies the equation:

$$\Delta V = \frac{1}{k} p,$$

then one will fulfill equations (9) when one sets:

$$u = \frac{\partial V}{\partial x} + u', \quad v = \frac{\partial V}{\partial y} + v', \quad w = \frac{\partial V}{\partial z} + w'$$

and chooses u', v', w', such that:

$$\Delta u' = 0, \qquad \Delta v' = 0, \qquad \Delta w' = 0,$$

and:

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = -\frac{1}{k} p.$$

One can then make:

$$\frac{1}{k} p = 2c \frac{\partial \frac{1}{r}}{\partial z}, \qquad u' = 0, \qquad v' = 0, \qquad w' = -\frac{2c}{r},$$

and since one has:

$$\Delta \frac{r}{2} = \frac{1}{r},$$

one will have:

$$V = az + b \frac{\partial \frac{1}{r}}{\partial z} + c \frac{\partial r}{\partial z},$$

and

$$\begin{aligned} u &= 3b \frac{xz}{r^5} - c \frac{xz}{r^3}, \\ v &= -3b \frac{xz}{r^5} - c \frac{xz}{r^3}, \\ w &= a + b \left(\frac{3z^2}{r^5} - \frac{1}{r^3} \right) - c \left(\frac{z^2}{r^3} + \frac{1}{r} \right), \end{aligned} \tag{20}$$

in which a, b, c mean arbitrary constants. They can be determined in such a way that for a value of r that might be called R, one will have:

$$u = 0, \quad v = 0, \quad w = 0.$$

The equations:

$$\frac{3b}{R^2} = c, \qquad a = \frac{b}{R^3} + \frac{c}{R}$$

serve this purpose, from which, it will follow that:

$$b = \frac{R^3 a}{4}, \qquad c = \frac{3Ra}{4}.$$

Equations (20) then represent the motion of a fluid that flows to infinity everywhere with velocity a in the direction of the z-axis, and in which, a ball that is described at the coordinate origin with radius R is at rest.

Let Z be the force that must be exerted on the ball in the direction of the z-axis in order to keep its position; one then has:

$$Z = \int ds\, Z_n = \int \frac{ds}{r}(x\, Z_x + y\, Z_y + z\, Z_z), \tag{21}$$

in which ds means an element of the ball that is described around the coordinate origin of radius r, and one sets $r = R$. However, an arbitrarily larger value can be chosen for r than this one. It then emerges from the third of equations (1) in lecture eleven that:

$$\int ds\, Z_n = 0,$$

in which ds means an arbitrary part of the fluid. There is an advantage to choosing r to be infinitely large in equation (21). Namely, one can neglect the terms that are endowed with the factor b in the calculation of Z_x, Z_y, Z_z from equations (1) with the help of (20) here. For an infinitely-large r, one finds that:

$$Z_x = -6kc\,\frac{xz^2}{r^5}, \qquad Z_y = -6kc\,\frac{yz^2}{r^5}, \qquad Z_z = -6kc\,\frac{z^3}{r^5},$$

and therefore that:

$$\begin{aligned} Z &= -6kc\,\frac{1}{r^4}\int z^2 ds \\ &= -8\pi kc \quad \text{or} = -6\pi kRa. \end{aligned} \tag{22}$$

From a remark that we have already made many times, the equations that were developed here will also be valid in the case where the coordinate system to which they refer advances in some direction with a velocity that remains the same, rather than remaining at rest. If we let it advance in the direction of the z-axis with a velocity of $-a$ then the fluid will be at rest at infinity, and the ball of radius R will move in it in the direction of the z-axis with velocity $-a$. Equation (22) will then tell us the *resistance* that this ball then suffers.

§ 5.

We would now like to make two applications of equations (8), which are true for non-stationary motions, that refer to the oscillation that a ball can experience in an externally-unbounded fluid under the influence of certain forces.

The aforementioned equations will be satisfied when one sets:

$$p = \text{const.}$$

and chooses u, v, w in such a way that one has:

$$\frac{\mu}{k}\frac{\partial u}{\partial t} = \Delta u, \qquad \frac{\mu}{k}\frac{\partial v}{\partial t} = \Delta v, \qquad \frac{\mu}{k}\frac{\partial w}{\partial t} = \Delta w,$$
$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z} = 0.$$

One will fulfill these equations when one makes:

$$u = \frac{\partial W}{\partial y}, \qquad v = \frac{\partial W}{\partial x}, \qquad w = 0,$$

and W is determined by the equation:

$$\frac{\mu}{k}\frac{\partial W}{\partial t} = \Delta W. \tag{23}$$

We now assume that W is a function of the two variables t and r, where r once more means the quantity $\sqrt{x^2+y^2+z^2}$; we will then have:

$$u = \frac{1}{r}\frac{\partial W}{\partial r}y, \quad v = -\frac{1}{r}\frac{\partial W}{\partial r}x, \qquad w = 0.$$

These equations represent a motion for which the points that lie at a distance of r from the coordinate origin move as if they were attached to a solid body that rotates around the z-axis with angular velocity ψ when one sets:

$$\psi = -\frac{1}{r}\frac{\partial W}{\partial r}. \tag{24}$$

We can then assume that one finds a ball in the fluid whose outer surface is $r = R$, and which rotates around the z-axis with an angular velocity that is equal to the value that one obtains for the expression that gives ψ when $r = R$.

If M is the rotational moment of the pressure that ball we imagine exerts upon the fluid then equation (14) will also be true here, and a calculation that is entirely similar to the ones that we connected with this equation will give:

$$M = \frac{8\pi}{3} k r^4 \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial W}{\partial r}\right) \qquad \text{for } r = R.$$

Now, let ϑ be the angle through which the ball rotates from a certain position in the time t such that:

$$\frac{d\vartheta}{dt} = \psi \qquad \text{for} \qquad r = R. \tag{25}$$

Furthermore, let M' be the rotational moment of the forces that act upon the ball, in addition to the pressures that are exerted upon the fluid, and let K its moment of inertia; one will then have:

$$K\frac{d^2\vartheta}{dt^2} = M' - M.$$

When M' is given, these equations define a boundary condition for the function W, which has been defined only by the partial differential equation (23), up to now. Having set it, let:

$$M' = -\alpha^2 \vartheta,$$

in which α should be an arbitrarily-given constant; When one differentiates this condition with respect to t, one will then get:

$$\frac{\alpha^2}{r}\frac{\partial W}{\partial r} - \frac{8\pi}{3} k r^4 \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial^2 W}{\partial r\,\partial t}\right) + \frac{K}{r}\frac{\partial^3 W}{\partial r\,\partial t^2} = 0 \qquad \text{for } r = R. \tag{26}$$

Equation (23), which can be written as:

$$\frac{\mu}{k}\frac{\partial(rW)}{\partial t} = \frac{\partial^2(rW)}{\partial r^2},$$

has the particular solution:

$$W = Ce^{\beta^2 t}\frac{1}{r}e^{\beta\sqrt{\frac{\mu}{k}}r}, \tag{27}$$

where C and β mean arbitrary constants. The second of them can be determined in such a way that equation (26) will be satisfied; for that, it will be necessary for β to be a root of the equation:

$$\left(\sqrt{\frac{k}{\mu}} - R\beta\right)(\alpha^2 + K\beta^4) + \frac{8\pi}{3}R^3\mu\sqrt{\frac{k}{\mu}}\beta^2\left(3\frac{k}{\mu} - 3\sqrt{\frac{k}{\mu}}R\beta + R^2\beta^2\right) = 0. \tag{28}$$

When $k = 0$, its roots will be:

$$0, \quad \pm\sqrt[4]{\frac{\alpha^2}{K}}\,\frac{1\pm\sqrt{-1}}{\sqrt{2}}.$$

We assume that k is small enough that, as in this case, two of the five roots are complex with negative real parts, and set β equal to one of those two roots in (27); the velocity will then be zero at infinity. W will then be complex, but the real part of the expression for W that is presented in (27) will also satisfy equations (23) and (26). We choose that real part for W. If we then set:

$$\beta = -a + b\sqrt{-1},$$

calculate ϑ from W with the help of (24) and (25), let C denote a new, real, arbitrary constant, and choose the time origin, then we will find that:

$$\vartheta = C\; e^{(a^2-b^2)t} \sin 2abt.$$

This equation determines the oscillation that the ball experiences. If one lets T refer to the period of a simple oscillation and lets d refer to the *logarithmic decrement* of the oscillation – i.e., the natural logarithm of the ratio of two successive oscillation arcs – then one will get:

$$T = \frac{\kappa}{2ab},$$

$$\delta = (b^2 - a^2)\, T = \frac{b^2-a^2}{2ab}\,\pi.$$

It is easy to find a and b when one assumes that k is infinitely small and considers only the terms of lowest order in the ones that are influenced by the value of k. To that end, one denotes the value that β takes on for $k = 0$ by β_0, sets:

$$\beta = \beta_0 + \varepsilon,$$

writes equation (28) as:

$$F(\beta) = 0,$$

and makes:

$$F'(\beta) = \frac{dF(\beta)}{d\beta};$$

one then has to calculate ε from the equation:

$$F(\beta_0) + \varepsilon\, F'(\beta_0) = 0.$$

One then gets:

$$F(\beta_0) = \frac{8\pi}{3} R^5 \sqrt{k\mu}\;\beta_0^4, \qquad F'(\beta_0) = -\,4RK\;\beta_0^4,$$

so

$$\varepsilon = \frac{2\pi}{3} R^4 \sqrt{k\mu}\,\frac{1}{K}.$$

If one then denotes the period of oscillation for the ball for the case in which the fluid exerts no effect upon it by T_0 – i.e., one sets:

$$T_0 = \frac{\pi}{\alpha}\sqrt{K}\,,$$

then one will get:

$$a = \sqrt{\frac{\pi}{2T_0}} - \varepsilon, \qquad b = \sqrt{\frac{\pi}{2T_0}}$$

and

$$T = T_0\left(1 + \varepsilon\sqrt{\frac{2T_0}{\pi}}\right), \qquad \delta = \varepsilon\sqrt{2\pi T_0}\,.$$

The particular solution of equations (23) and (26) that we are now discussing assumes a certain initial state of the fluid. We would like to look for other particular solutions of those equations that refer to other initial states. One solution of equation (23) is:

$$W = e^{\beta^2 t}\,\frac{1}{r}\left(C\,e^{\beta\sqrt{\frac{\mu}{k}}r} + C'\,e^{-\beta\sqrt{\frac{\mu}{k}}r}\right),$$

in which C, C', and β mean arbitrary complex constants. It will satisfy the condition (26) when the equation:

$$\begin{aligned} 0 = {} & C\left\{\left(\sqrt{\frac{k}{\mu}} - R\beta\right)(\alpha^2 + K\beta^4) + \frac{8\pi}{3}R^3\mu\sqrt{\frac{k}{\mu}}\beta^2\left(3\frac{k}{\mu} - 3\sqrt{\frac{k}{\mu}}R\beta + R^2\beta^2\right)\right\}e^{\beta\sqrt{\frac{\mu}{k}}R} \\ & + C'\left\{\left(\sqrt{\frac{k}{\mu}} + R\beta\right)(\alpha^2 + K\beta^4) + \frac{8\pi}{3}R^3\mu\sqrt{\frac{k}{\mu}}\beta^2\left(3\frac{k}{\mu} + 3\sqrt{\frac{k}{\mu}}R\beta + R^2\beta^2\right)\right\}e^{-\beta\sqrt{\frac{\mu}{k}}R} \end{aligned}$$

exists between those constants. This equation determines the ratio $C : C'$ when β is assumed to be arbitrary. The expression for W that one gets in this way is complex; its real part also satisfies equations (23) and (26). If one chooses that real part for W then the velocity will be infinite at infinity, in general. An exception to this can exist and the velocity can vanish at infinity only when either one of the two constants C and C' are equal to zero or the constant β is purely imaginary. The first case is the one that we have considered up to now, and to which equation (27) refers; the second one leads to new solutions that we would like to study.

§ 6.

The following considerations will lead us to the oscillations of a ball that is found in a fluid with friction, and whose center lies on a straight line and moves along it.

A particular solution to equations (8) is:

$$u = \frac{\partial^2 P}{\partial x \, \partial z}, \qquad v = \frac{\partial^2 P}{\partial y \, \partial z}, \qquad w = \frac{\partial^2 P}{\partial z^2},$$

$$p = -\mu \frac{\partial^2 P}{\partial z \, \partial t},$$

when:

$$\Delta P = 0;$$

a second one is:

$$u = \frac{\partial^2 W}{\partial x \, \partial z}, \qquad v = \frac{\partial^2 W}{\partial y \, \partial z}, \qquad w = -\frac{\partial^2 W}{\partial x^2} - \frac{\partial^2 W}{\partial y^2},$$

$$p = 0,$$

when

$$k \, \Delta W = \mu \frac{\partial W}{\partial t}.$$

The aforementioned equations will also be fulfilled by:

$$u = \frac{\partial^2 (P+W)}{\partial x \, \partial z}, \qquad v = \frac{\partial^2 (P+W)}{\partial y \, \partial z}, \qquad w = -\frac{\partial^2 (P+W)}{\partial x^2} - \frac{\partial^2 (P+W)}{\partial y^2},$$

$$p = -\mu \frac{\partial^2 P}{\partial z \, \partial t},$$

when:

$$\Delta P = 0, \qquad k \, \Delta W = \mu \frac{\partial W}{\partial t}. \tag{29}$$

We now assume that P and W are functions of only the two variables r and t; we will then get:

$$\begin{aligned}
u &= \frac{xz}{r} \frac{\partial}{\partial r}\left(\frac{1}{r} \frac{\partial (P+W)}{\partial r}\right), \\
v &= \frac{yz}{r} \frac{\partial}{\partial r}\left(\frac{1}{r} \frac{\partial (P+W)}{\partial r}\right), \\
w &= -\frac{x^2+y^2}{r} \frac{\partial}{\partial r}\left(\frac{1}{r} \frac{\partial (P+W)}{\partial r}\right) - \frac{2}{r} \frac{\partial (P+W)}{\partial r},
\end{aligned} \tag{30}$$

$$p = -\mu\,\frac{z}{r}\frac{\partial^2 P}{\partial r\,\partial t}.$$

We further assume that one has:

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(P+W)}{\partial r}\right) = 0 \tag{31}$$

for $r = R$. One will then have:

$$u = 0, \quad v = 0, \quad w = -\frac{2}{r}\frac{\partial(P+W)}{\partial r}, \tag{32}$$

and the equations that were developed represent a possible motion in the case in which a ball is found in the fluid whose outer surface is $r = R$, and which moves in the direction of the z-axis with a velocity that is equal to the value of:

$$-\frac{2}{r}\frac{\partial(P+W)}{\partial r} \qquad \text{for} \qquad r = R.$$

Let Z be the sum of the z-components of the pressure that the ball exerts upon the fluid; equation (21) will then be true – i.e.:

$$Z = \int \frac{ds}{r}(x\,Z_x + y\,Z_y + z\,Z_z)\,.$$

If one ponders the fact that from (1):

$$x\,Z_x + y\,Z_y + z\,Z_z = zp - k\left(x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} + z\frac{\partial w}{\partial z}\right) - k\left(x\frac{\partial u}{\partial z} + y\frac{\partial v}{\partial z} + z\frac{\partial w}{\partial z}\right),$$

so one will have:

$$\int x^2 ds = \int y^2 ds = \int z^2 ds = \frac{4\pi}{3}r^4,$$

morever, and employs equation (31) then one will find from (30) that:

$$Z = \frac{4\pi}{3}r^2\left\{2kr\frac{\partial^2}{\partial r^2}\left(\frac{1}{r}\frac{\partial(P+W)}{\partial r}\right) - \mu\frac{\partial^2 P}{\partial r\,\partial t}\right\} \qquad \text{for } r = R.$$

Now, let ζ be the displacement of the ball from a certain position at the time t, such that:

$$\frac{d\zeta}{dt} = w \qquad \text{for } r = R, \tag{33}$$

let m be the mass of the ball, and let Z' be the force that acts on the ball in the direction of the z-axis, if one ignores the pressures that the fluid exerts upon it; one will then have:

$$m\,\frac{d^2\zeta}{dt^2} = Z' - Z.$$

Set:

$$Z' = -\,\alpha^2\,\zeta,$$

in which α means an arbitrarily-given constant; corresponding to equation (26), one will then get:

$$\frac{2\alpha^2}{r}\frac{\partial(P+W)}{\partial r} - \frac{4\pi}{3}r^2\frac{\partial}{\partial t}\left\{2kr\frac{\partial^2}{\partial r^2}\left(\frac{1}{r}\frac{\partial(P+W)}{\partial r}\right) - \mu\frac{\partial^2 P}{\partial r\,\partial t}\right\}$$
$$+\ \frac{2m}{r}\frac{\partial^3(P+W)}{\partial r\,\partial t^2} = 0 \qquad \text{for} \qquad r = R. \tag{34}$$

One satisfies the equations for P and W that were presented in (29) by setting:

$$P = Be^{\beta^2 t}\,\frac{1}{r}, \qquad W = Ce^{\beta^2 t}\,\frac{1}{r}e^{\beta\sqrt{\frac{\mu}{k}}\,r}, \tag{35}$$

in which B, C, and b are arbitrary constants. Conditions (31) and (34) give two equations for those constants that are linear and homogeneous in B and C, and from which one can compute the ratio $B : C$, as well as β. With the help of the differential equations (29), one easily finds from (31) that for $r = R$:

$$\frac{\partial(P+W)}{\partial r} = \frac{1}{3}\cdot\frac{\mu}{k}\beta^2 rW,$$
$$\frac{\partial^2}{\partial r^2}\left(\frac{1}{r}\frac{\partial(P+W)}{\partial r}\right) = \frac{\mu}{k}\beta^2\,\frac{1}{r}\frac{\partial W}{\partial r},$$

and then, from (34):

$$0 = (\alpha^2 + m\,\beta^4)\,W - \frac{2\pi}{3}r^2\beta^2\left(9k\frac{\partial W}{\partial r} - \mu\beta^2 rW\right),$$

or, since one has:

$$\frac{\partial W}{\partial r} = \left(-\frac{1}{r} + \beta\sqrt{\frac{\mu}{k}}\right)W$$

for every value of r, one will have:

$$0 = \alpha^2 + m\,\beta^4 + \frac{2\pi}{3}R\,\beta^2\,(\mu R^2\,\beta^2 - 9\,\sqrt{k\mu}\,R\,\beta + 9k). \tag{36}$$

If one sets:

$$\frac{4\pi}{3} R^3 \,\mu = m',$$

and then lets m' denote the mass of the fluid that is displaced by the ball then this equation will go to:

$$0 = \alpha^2 + \left(m + \frac{m'}{2}\right) \beta^4$$

for $k = 0$. It then follows from this that when k is sufficiently small (as we have assumed), its four roots will lie in the vicinity of the values:

$$\pm \sqrt[4]{\frac{\alpha^2}{m + \frac{m'}{2}}} \frac{1 \pm \sqrt{-1}}{\sqrt{2}} .$$

We choose β to be one of the two roots whose real part is negative; the velocity will then become zero at infinity. Therefore, the expressions for P and W that were presented in (35) be complex. The real parts of these expressions will satisfy equations (29), (31), and (34), and we will now think that P and W have been set equal to these real parts. If we further make:

$$\beta = -\,a + b\,\sqrt{-1},$$

calculate ζ from P and W with the help of (32) and (33), let C denote a new, real, arbitrary constant, and fix a time origin then we will get:

$$\zeta = Ce^{(a^2 - b^2)t} \sin 2abt ,$$

from which, the equations:

$$T = \frac{\pi}{2ab}, \qquad \delta = (b^2 - a^2)\, T$$

will again follow for the period of oscillation T and the logarithmic decrement δ. If one assumes that k is infinitely small then one will find from this and equation (36) by a calculation that is similar to the one that was performed in the previous § that:

$$T = T_0 \left(1 + \varepsilon \sqrt{\frac{2T_0}{\pi}}\right), \qquad \delta = \varepsilon \sqrt{2\pi T_0} ,$$

in which:

$$T_0 = \frac{\pi}{\alpha} \sqrt{m + \frac{m'}{2}} ,$$

$$\varepsilon = \frac{9}{8} \frac{1}{R} \sqrt{\frac{k}{\mu}} \frac{m'}{m + \frac{m'}{2}} .$$

It is not difficult to give other particular solutions to equations (29), (31), and (34) than the ones that were discussed, and which would correspond to other initial states of the fluid. They have an especial interest because they allow one to judge very closely the influence that air exerts upon the oscillations of a pendulum that consists of a ball and a thin rod. In regard to that, we refer to a treatise of Stokes (Transactions of the Cambridge Philosophical Society, vol. IX, part 2, pp. 8) and one of Emil Meyer (Borchardt's Journal, Bd. 73).