"Ueber die Transformation der allgemeinen Gleichung des zweiten Grades zwischen Linien-Coordinaten auf eine canonische Form," Math. Ann **23** (1884), 539-578.

On the transformation of the general second-degree equation in line coordinates into a canonical form.

By

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Translated by D. H. Delphenich

A line complex of degree *n* encompasses a triply-infinite number of straight lines that are distributed in space in such a manner that those straight lines that go through a fixed point define a cone of order *n*, or – what says the same thing – that those straight lines that lie in a fixed planes will envelope a curve of class *n*.

Its analytic representation finds such a structure by way of the coordinates of a straight line in space that **Pluecker** introduced into science (**). According to **Pluecker**, the straight line has six homogeneous coordinates that fulfill a second-degree condition equation. The straight line will be determined relative to a coordinate tetrahedron by means of it. A homogeneous equation of degree *n* between these coordinates will represent a complex of degree *n*.

(*) In connection with the republication of some of my older papers in Bd. XXII of these Annals, I am once more publishing my Inaugural Dissertation (Bonn, 1868), a presentation by Lie and myself to the Berlin Academy on Dec. 1870 (see the Monatsberichte), and a note on third-order differential equations that I presented to the sächsischen Gesellschaft der Wissenschaften (last note, with a recently-added Appendix). The Mathematischen Annalen thus contain the totality of my publications up to now, with the single exception of a few that are appearing separately in the book trade, and such provisional publications that were superfluous to later research. Supplementary remarks that I have added to the republication will again be denoted (as in Bd. XXII) by putting the date in square brackets; e.g., [January 1884]. In the republication of my dissertation, some inaccuracies were omitted that **Segre** (Turin) was kind enough to bring to my attention.

Klein [Jan. 1884]

(**) Proceedings of the Royal Soc. (1865); Phil. Transactions (1865), pp. 725, translated in Liouv. Journal, 2 Séries, t. XI; Les Mondes, par Moigno (1867), pp. 79; Annali di matematica, Ser. II, t. i; *Neues Geometrie des Raumes, gegründet auf die Betrachtung der geraden Linie als Raumelement*, 1st ed., Leipzig 1868, by B. G. Teubner. (On that subject, cf. the historical notice on the origins of line geometry that **Clebsch** gave in the Göttinger Abhandlungen of 1872 [in remembrance of **Julius Pluecker**]. **Pluecker** published his first ideas on that subject in 1846 in *Systeme der Geometrie des Raumes* [no. 258]; however, the coordinates of spatial lines had already appeared earlier in **Grassmann's** *Lineale Ausdehnungslehre* (first edition, 1844). One further confers **Cayley**: "On a new analytical representation of curves in space" (Quarterly Journal, t. III, 1859).

[Jan. 1884]

In the sequel, our goal will be to transform the second-degree equation in the line coordinates into a canonical form by a corresponding conversion of the coordinate tetrahedron. We first give the general formulas that come to be applied to such transformations. On the basis of them, the problem is treated algebraically as the simultaneous linear transformation of the equation of the complex into a canonical form and the transformation of the second-degree condition equation that line coordinates must satisfy into itself. By performing that transformation, we arrive, in particular, at a classification of second-degree complexes into different types.

I. On line coordinates in general.

1. If we denote the homogeneous coordinates of two points that are chosen arbitrarily from a given straight line by:

$$x_1, x_2, x_3, x_4$$

and

$$y_1, y_2, y_3, y_4,$$

resp., then the given straight line, which is determined geometrically as the connecting line between the two points (x) and (y), takes on *the following six – likewise homogeneous – coordinates:*

$$(1) \qquad \begin{cases} p_1 = x_1 y_2 - x_2 y_1, & p_4 = x_3 y_4 - x_4 y_3, \\ p_2 = x_1 y_3 - x_3 y_1, & p_5 = x_4 y_2 - x_2 y_4, \\ p_3 = x_1 y_4 - x_4 y_1, & p_6 = x_2 y_3 - x_3 y_2. \end{cases}$$

These are the six determinants of degree two that are formed from the elements:

$$\begin{matrix} x_1, x_2, x_3, x_4, \\ y_1, y_2, y_3, y_4, \end{matrix}$$

taken with a sign that emerges by distinguishing a column of elements (the first, by our assumption).

As a result of the form of the determinants, the six chosen coordinates will preserve the same values when we choose any other two points of the given straight line in place of the chosen two points (x) and (y). The coordinates of any such point can then be brought into the form:

$$\lambda x_1 + \mu y_1 , \ldots, \lambda x_4 + \mu y_4 ,$$

where λ, μ denote constants that are to be determined, and the substitution of such quantities in place of the x and y into the expressions that were given for the coordinates

p will yield, as one finds immediately, multiples of the values of p that were obtained originally.

The six coordinates p satisfy the following second-degree relation identically:

$$P \equiv p_1\, p_4 + p_2\, p_5 + p_3\, p_6 = 0,$$

which we can also write:

$$\sum_\kappa p_\kappa \cdot p_{\kappa+3} = 0,$$

in which we let the index κ run from 1 to 3, or also 1 to 6, and thereby understand $\kappa + 3$ to mean the number that assumes the ($\kappa + 3$) location in the continuing sequence:

$$1, 2, \ldots, 5, 6, 1, 2, \ldots$$

Four constants will necessarily enter into the determination of a straight line as a result of these relations that the *six* homogeneous coordinates p satisfy.

We obtain the following equations for those four planes (projection planes) that can be laid through the straight line that is determined by the two points (x) and (y), and the four vertices of the coordinate tetrahedron:

$$(2) \qquad \begin{cases} p_4 z_2 + p_5 z_3 + p_6 z_4 = 0, \\ p_4 z_1 - p_3 z_3 + p_2 z_4 = 0, \\ p_5 z_1 + p_3 z_2 - p_1 z_4 = 0, \\ p_6 z_1 - p_2 z_2 + p_1 z_3 = 0, \end{cases}$$

respectively, where we have denoted the running point coordinates by $z_1, \ldots, z_4$. Thus, they are the four constants that enter into the equations of the coordinates p of the four projection-planes. The equation:

$$P = 0$$

expresses the idea that the four planes in question cut the same straight line. It is therefore not only the *necessary*, but also the *sufficient* condition for six arbitrarily-chosen quantities:

$$p_1, p_2, \ldots, p_6$$

to be regarded as line coordinates. The geometric construction of the straight line that is determined by them will be mediated by any two of the planes (2).

The coordinate determination (1) is based upon the principle that one should consider the constants that enter into the equations of the straight line in point coordinates (2) as its determining data, and that they are to be represented by the coordinates of a number of points of the straight line that is necessary and sufficient to define the latter geometrically.

2. In the foregoing, we have determined the straight line by two of its points. In this manner of determination, we consider a straight line to be a locus of points – i.e., a *ray*.

In a completely analogous way, we can determine a straight line by two of its planes and then consider it to enveloped by planes – i.e., an *axis* (*).

Let two arbitrary planes (t) and (u) of the given straight line be determined by the coordinates:

$$t_1, t_2, t_3, t_4$$

and

$$u_1, u_2, u_3, u_4 .$$

In complete correspondence to what we said before, we will then obtain the *following six expressions for the coordinates of the given line:*

$$(3) \qquad \begin{cases} q_1 = t_1u_2 - t_2u_1, & q_4 = t_3u_4 - t_4u_3, \\ q_2 = t_1u_3 - t_3u_1, & q_5 = t_4u_2 - t_2u_4, \\ q_3 = t_1u_4 - t_4u_1, & q_6 = t_2u_3 - t_3u_2, \end{cases}$$

which satisfy the following equation:

$$Q \equiv \sum_\kappa q_\kappa \cdot q_{\kappa+3} = 0$$

identically. Corresponding to the four equations (2), we obtain the following four equations for the intersection points of the straight line that is determined by the planes (t) and (u) with the four faces of the tetrahedron:

$$(4) \qquad \begin{cases} q_4v_2 + q_5v_3 + q_6v_4 = 0, \\ q_4v_1 - q_3v_3 + q_2v_4 = 0, \\ q_5v_1 + q_3v_2 - q_1v_4 = 0, \\ q_6v_1 - q_2v_2 + q_1v_3 = 0, \end{cases}$$

where $v_1, \ldots, v_4$ mean the running plane coordinates.

If the ray coordinates p and the axis coordinates q refer to *the same* straight line then one will have the following proportions between them:

$$(5) \qquad \frac{p_1}{q_4} = \frac{p_2}{q_5} = \frac{p_3}{q_6} = \frac{p_4}{q_1} = \frac{p_5}{q_2} = \frac{p_6}{q_3}.$$

The validity of these relations is implied immediately when we form the quantities q from the coordinates of two planes (2) or the quantities q from the coordinates of two points (4).

The coordinates p then differ from the coordinates q only by their ordering. Corresponding to their double geometric meaning, straight lines will be represented by

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(*) Cf., **Pluecker's** *Neue Geometrie*, pp. 1.

the same six quantities. This is not the least of the advantages of **Pluecker's** choice of coordinates.

3. We would like to denote the four vertices of the coordinate tetrahedron by:

$$O_1, O_2, O_3, O_4,$$

and the four faces opposite to them by:

$$E_1, E_2, E_3, E_4 .$$

The six edges of the tetrahedron will then be determined by the following couplings of the symbols O (E, resp.):

$$\begin{array}{cccccc} O_1 O_2, & O_1 O_3, & O_1 O_4, & O_3 O_4, & O_4 O_2, & O_2 O_3, \\ E_1 E_2, & E_1 E_3, & E_1 E_4, & E_3 E_4, & E_4 E_2, & E_2 E_3 . \end{array}$$

Five of the *six* coordinates of an edge of the coordinate tetrahedron will vanish, and only the sixth one will keep a finite value. That is implied immediately when we substitute two vertices (faces, resp.) of the tetrahedron into the expression (1) [(3), resp.]. In the present sequence, we would like to denote the edges of the coordinate tetrahedron by:

$$P_1, P_2, P_3, P_4, P_5, P_6,$$

or

$$Q_1, Q_2, Q_3, Q_4, Q_5, Q_6 .$$

All coordinates of an arbitrarily-chosen edge ($P_\kappa \equiv Q_{\kappa+3}$) will then vanish, up to the ones that we have denoted by $p_\kappa \equiv q_{\kappa+3}$.

The grouping of the edges of the tetrahedron amongst themselves is determined by the fact that P_1, P_2, P_3 (Q_4, Q_5, Q_6, resp.) intersect at a point, while P_4, P_5, P_6 (Q_1, Q_2, Q_3, resp.) lie in a plane.

For the sake of brevity, in the sequel we will make use of only the independent representation of the line coordinates through point coordinates, and the completely analogous (i.e., reciprocal) arguments that are connected with their representation by plane coordinates in each case will not always be expressly emphasized. We will thus avail ourselves of only the notation p for line coordinates in what follows, although preserving the coordinates q along with the coordinates p will allow many formulas to be written clearly.

4. In order for two given straight lines (p) and (p') to *intersect*, their coordinates must satisfy the following equation:

(6) $$\sum_\kappa p_\kappa \cdot p'_{\kappa+3} = 0.$$

Let the two straight lines (p) and (p') be determined by the pairs of points (a), (b) and (c), (d), respectively. If we then replace the coordinates p, p' in the present equation with the values from (1) for the coordinates of these points then we will obtain:

$$\sum \pm a_1 b_2 c_3 d_4 = 0.$$

The vanishing of this determinant is the condition for the four points (a), (b), (c), (d) to lie in a plane, and therefore, for the two straight lines (a, b) and (c, d) to intersect (*).

If we consider the $p'_{\kappa+3}$ in equation (6):

$$\sum_{\kappa} p_\kappa \cdot p'_{\kappa+3} = 0$$

to be fixed and the p_κ to be variable then they will represent the totality of all of those lines that cut the fixed line (p'). In particular, the coordinates of all of those straight lines that cut the coordinate edges P_κ will then satisfy the equation:

$$p_{\kappa+3} = 0.$$

When all coordinates of the edge P_κ itself vanish up to the one p_κ then that will immediately express the idea that it will cut all tetrahedral edges up to the one that is opposite to it.

If three straight lines (p), (p'), (p'') intersect each other mutually then an equation of the form (6) will exist between the coordinates of any two of them. The three straight lines will then either go through a point or lie in the same plane. The criterion for the first or second case is defined by the vanishing of the second or first factor, resp., of the product:

$$\sum \pm p_1 \, p'_2 \, p''_3 \cdot \sum \pm p_4 \, p'_5 \, p''_6 ,$$

which will always vanish under the assumptions that were made, and the similar product that is obtained from the latter one by permuting that indices 1, 2, 3 with the corresponding ones 4, 5, 6 in every case.

The proof is provided by considering equations (2) and (4). If three lines intersect at a point then the three planes that go through a vertex of the coordinate tetrahedron and one of the given straight lines, in any case, will have a straight line in common, and conversely, when three lines lie in a plane then those three points at which one face of the coordinate tetrahedron will be intersected by the given straight line will be in a straight line.

5. We can assign *imaginary values* to the six variable p, always under the assumption that the condition equation:

(*) Cf., the article by **Lüroth**: "Zur Theorie der windschiefen Flächen," **Crelle's** Journal, LXII, p. 130.

$$\sum_{\kappa} p_{\kappa} \cdot p_{\kappa+3} = 0$$

is fulfilled. Thus, let:

$$p_{\kappa} = p'_{\kappa} + i p''_{\kappa}.$$

We consider the quantities p_{κ} to be *the coordinates of an imaginary straight line.* This purely formal definition leads to the following geometric one: From equation (6) of number 4, the given imaginary straight line, as well as the conjugate imaginary one, will be cut by all real lines whose coordinates satisfy the following two linear condition equations:

$$\sum_{\kappa} p'_{\kappa} \cdot p_{\kappa+3} = 0, \qquad \sum_{\kappa} p''_{\kappa} \cdot p_{\kappa+3} = 0.$$

The two equations are determined by any four lines whose coordinates satisfy these two equations (*), or by the two-parameter group that is defined by them, moreover:

$$\sum_{\kappa} (\lambda p'_{\kappa} + \mu p''_{\kappa}) p_{\kappa+3} = 0$$

(in addition, when the chosen four straight lines of it belong to the generator of a hyperboloid). An imaginary straight line and its conjugate are thus given geometrically *as the two rectilinear transversals to four real straight lines.*

That therefore agrees with the definition that the new synthetic geometry gives for an imaginary straight line in space.

In general, an imaginary straight line possesses no real point and no real plane. Only when the given imaginary line and its conjugate intersect will both of them have a real point and a real plane in common. The imaginary straight line will then be intersected by all real lines that go through that point (plane, resp.). It will no longer be determined by four of its real, rectilinear transversals. One defines it geometrically by the real point, the real plane, and a second-order cone that emanates from the real point or curve of class two that lies in the real plane.

II. Transformation of line coordinates that corresponds to a conversion of the coordinate tetrahedron.

6. In the following, we will next exhibit those transformations of line coordinates that correspond to a conversion of the coordinate tetrahedron, or – what says the same thing – the linear transformation of point or plane coordinates (**).

(*) One will find the system of such straight lines considered, in particular, in the article of **O. Hermes:** "Ueber Strahlensysteme der ersten Ordnung und der ersten Classe," **Crelle**'s Jounral, LXII, pp. 153.

(**) Confer the two articles of **Battaglini**:

"Intorno ai sistemi di rette di primo ordine," Rendiconti della R. Accademia di Napoli, 6 Giugno 1866.

"Intorno ai sistemi di rette di secondo ordine," Rendiconti della R. Accademia di Napoli, III, 1866.

These transformation formulas will be *linear*. They would lose their linear character if one were to take five independent homogeneous coordinates, as would succeed in determining a straight line, instead of six homogeneous coordinates that satisfy a condition equation. We arrive at the result that *the linear substitutions that we spoke of are the general ones that take the expression:*

$$P \equiv \sum_{\kappa} p_{\kappa} \cdot p_{\kappa+3}$$

to a multiple of itself.

The latter theorem admits the following determination:

Let a linear substitution be given that takes the expression P to a multiple of itself. We can freely choose one of the six new variables, which we would like to give the name p_1. The variable p_4 is then determined likewise. We can, moreover, choose p_2 from the remaining four variables with no further assumptions; p_5 will then be given. However, which of the two remaining variables p_3 and p_4 we can take is no longer arbitrary. Any edge of the new tetrahedron that refers to the new p_3 will cut the two edges that correspond to the new p_1 and p_2 at a point, so it will be determined uniquely (no. 3). The expressions for the line coordinates in terms of the coordinates of two points (planes, resp.) that were given by (1) and (3), resp., will be valid only under the assumption that p_3 is chosen accordingly.

If we understand x_{κ}, y_{κ} to mean point coordinates then let:

$$\left\{ \begin{aligned} x_{\kappa} &= \sum_{\kappa} \alpha_{\kappa,\lambda} \cdot x'_{\kappa}, \\ y_{\kappa} &= \sum_{\kappa} \alpha_{\kappa,\lambda} \cdot y'_{\kappa} \end{aligned} \right. \tag{7}$$

be a general linear substitution, as would correspond to an arbitrary conversion of the coordinate tetrahedron. The substitution coefficients $\alpha_{\kappa,\lambda}$ then represent the coordinates of the faces of the previous tetrahedron relative to the new ones, such as the ones that are given when we let x_{κ} (y_{κ}, resp.) vanish.

We will obtain the desired formulas by substituting these values for x_{κ}, y_{κ} into the expression for the line coordinates p that is given by (1). The substitution coefficients that enter into them will take on the determinant form:

$$\alpha_{\kappa,\mu} \cdot \alpha_{\lambda,\nu} - \alpha_{\kappa,\nu} \cdot \alpha_{\lambda,\mu} ,$$

and will thus represent, geometrically, the *coordinates of the edges of the previous tetrahedron relative to the new one*, taken in terms of quantities like the ones that established formula (3) for the coordinates $\alpha_{\kappa,\lambda}$ of the faces of the previous tetrahedron relative to the new one. If we denote them by $a_{\kappa,\lambda}$ when they belong to an edge P_{κ} and occupy the λ^{th} plane, under the coordinates of this edge, when we write them in the

Both articles are found to be reprinted in: Giornale di Matematiche, Napoli, Anno VI, 1868.

sequence that was established by (1), then the desired transformation formulas will become:

$$p_\kappa = \sum_\lambda a_{\kappa+3,\lambda+3} \cdot p'_\lambda \,. \tag{8}$$

We then arrive at the fact that p_κ vanishes – that is, that the straight line (p, p') cuts the edge $P_{\kappa+3}$ – so from number four, the condition for this is the vanishing of the expression:

$$\sum_\lambda a_{\kappa+3,\lambda+3} \cdot p'_\lambda \,.$$

7. Under the substitution (8), the identically-vanishing expression:

$$P \equiv \sum_\kappa p_\kappa \cdot p_{\kappa+3}$$

will go to a multiple of the corresponding one:

$$P' \equiv \sum_\kappa p'_\kappa \cdot p'_{\kappa+3} \,.$$

If we form the first expression from (8) and compare it with the second one then we will obtain a sequence of relations for the coefficients a that they will satisfy identically by means of their representation in terms of the coefficients α.

The actual development of the expression P from the p' will yield a polynomial in these variables of degree two and 21 terms. The coefficients of 18 of these terms must vanish, while those of the remaining three will be equal to each other. The 36 quantities a are thus subject to 20 conditions, and for that reason, will be representable in terms of the 16 independent quantities α. With these numerical ratios, it will come down to the same thing, whether we base the expressions for line coordinates upon point (or plane) coordinates and transform the latter linearly or whether we immediately transform the line coordinates themselves linearly and then require that the expression:

$$P \equiv \sum_\kappa p_\kappa \cdot p_{\kappa+3}$$

must go to a multiple of itself. We will find the complete confirmation of this statement in the geometric interpretation of the conditions that the substitution coefficients a will be subjected to as a result of the latter restriction. However, from the previous number, the naming of the new variables must observe a fixed rule.

8. Therefore, let:

$$p_\kappa = \sum_\lambda b_{\kappa+3,\lambda+3} \cdot p'_\lambda \tag{9}$$

be a linear substitution, under which the expression:

$$P \equiv \sum_{\kappa} p_{\kappa} \cdot p_{\kappa+3}$$

will go to a multiple of itself. The following relations will then be valid for the coefficient b:

(10) $$\sum_{\kappa} b_{\kappa+3,\lambda} \cdot b_{\kappa,\lambda+\mu} = 0 \quad (\mu = 1, 2, 4, 5, 6),$$

(11) $$\sum_{\kappa} b_{\kappa+3,\lambda} \cdot b_{\kappa,\lambda+3} = d,$$

where d denotes an arbitrarily-determined constant.

As a result of conditions (10), the following two products will vanish:

$$\sum \pm b_{11}\, b_{22}\, b_{33} \cdot \sum \pm b_{41}\, b_{52}\, b_{63}$$

and

$$\sum \pm b_{14}\, b_{25}\, b_{36} \cdot \sum \pm b_{44}\, b_{55}\, b_{66} .$$

The development of this product according to the multiplication theorem for determinants will yield a new three-parameter determinant whose element $(\kappa; \lambda)$ will obey the rule:

$$(\kappa; \lambda) + (\lambda, \kappa) = 0.$$

We now add the further condition to (10) and (11) that the last two products will vanish due to the fact that the two factors:

$$\sum \pm b_{41}\, b_{52}\, b_{63}\,, \qquad \sum \pm b_{14}\, b_{25}\, b_{36}$$

are equal to zero. Moreover, corresponding to that, the similarly-constructed determinants should vanish that are derived from the latter ones by permuting two of the first or second indices 1, 2, 3 with the corresponding 4, 5, 6, each time. These conditions do not at all restrict the relative magnitudes of the coefficients b, but only the arbitrariness in their sequence.

The solution of the substitutions (9) will be, when one appeals to the condition equations (10), the following:

(12) $$\sum_{\kappa} b_{\kappa,\lambda+3} \cdot b_{\kappa+3,\lambda} \cdot p'_{\lambda} = \sum_{\kappa} b_{\kappa,\lambda} \cdot p_{\kappa}\,,$$

or, with consideration to equations (11):

(13) $$d \cdot p'_{\lambda} = \sum_{\kappa} b_{\kappa,\lambda} \cdot p_{\kappa}\,.$$

When we return from (13) to (9), that will yield the following formula, corresponding to (10):

(14) $$\sum_{\lambda} b_{\kappa,\lambda+3} \cdot b_{\kappa+\mu,\lambda} = 0 \quad (\mu = 1, 2, 4, 5, 6).$$

If we denote the substitution determinant $\sum \pm b_{1,1} b_{2,2} \cdots b_{6,6}$ by D, that of the sub-determinant that belongs to an arbitrary element $b_{\kappa,\lambda}$ of it – such as, above all, the sub-determinants in what follows – through the two indices κ; λ ($D_{\kappa,\lambda}$), and correctly determine the sign in it – viz., $((-1)^{(\kappa+\lambda)})$ – then it will follow from the solutions (12) to equations (9) that:

$$D_{\kappa,\lambda+3} \cdot \sum_{\kappa} b_{\kappa,\lambda+3} \cdot b_{\kappa+3,\lambda} = b_{\kappa+3,\lambda} \cdot D.$$

This formula will remain valid for any index κ and any index λ. One then finds, up to a factor:

(15) $$D = \prod_{(\lambda=1,2,3)} \sum_{\kappa} b_{\kappa+3,\lambda} \cdot b_{\kappa,\lambda+3},$$

or, as a result of (11):

(16) $$D = d^3.$$

The comparison of two terms in the development of D with the product that enters (15) will give the proof that the factor in question is equal to unity.

It follows from equations (10) that the columns of the substitution coefficients (9) represent the line coordinates of the edges of the new tetrahedron relative to those of the previous one. Once we set $\mu = 6$, that equation will then say that the coefficients in a column of the substitutions (9) have the meaning of line coordinates, and then, corresponding to the other four values of μ, that each of the six lines that are determined by the substitution coefficients will cut four of the five remaining ones, so the six straight lines that are represented will form a tetrahedron.

We would like to denote the six edges of this tetrahedron that correspond to the coordinates $b_{\kappa,\lambda}$ by P'_{λ}. The conditions that we have added to equations (10) and (11) for the sequence of coefficients $b_{\kappa,\lambda}$ then say nothing but the facts that the three edges P'_1, P'_2, P'_3 intersect at a point and that the three edges P'_4, P'_5, P'_6 lie in a plane. What is excluded from these conditions (in case the substitution determinant $\sum \pm b_{1,1} \cdots b_{6,6}$ does not vanish) are the possibilities that P'_1, P'_2, P'_3 are contained in a plane and that P'_4, P'_5, P'_6 go through a point. One of these two possibilities must take place. In connection with these conditions, the three equations (11) say that the ratios of the coordinates of these six straight lines can be chosen to have magnitudes that would come from the coordinates of the four vertices (faces, resp.) of the tetrahedron that they define on the basis of formulas (1), (3).

With that, the proof that the chosen transformation corresponds to the conversion of the given coordinate tetrahedron into another one is completed.

9. We imagine that the substitution coefficients b are represented independently in terms of the coefficients b of one of the linear transformations of point coordinates that correspond to the coordinate conversion:

(17) $$x_\kappa = \sum_\lambda \beta_{\kappa,\lambda}\, x'_\kappa .$$

We then obtain the following relation:

(18) $$d = \sum \pm\, \beta_{1,1} \cdots \beta_{4,4} .$$

In order to show the validity of this, we will satisfy ourselves with direct calculation, when we start with one of the formulas (11), but then make the remark that the determinant *D*, since it is formed from the second sub-determinants of the four-term determinant $\sum \pm\, \beta_{1,1} \cdots \beta_{4,4}$, is equal to the third power of that determinant.

The constant *d* can assume any positive or negative values, although it cannot vanish. If it did, then, as a result of equations (11), it would cut the opposite edges of the new tetrahedron, and the coordinate determination would then be impossible. This would correspond to four vertices or four faces of the tetrahedron coinciding, which would find its expressions in the vanishing of the determinant $\sum \pm\, \beta_{1,1} \cdots \beta_{4,4}$.

In what follows, we will assume that the constant *d* is equal to positive unity, such that the expression *P* must go to itself under the linear substitution, into which only 15 independent coefficients would enter.

The transition from the substitution (9) to the substitution (17) takes the following form: We can choose the coefficient *b* from the rows and three columns in such a way that if we think of *b* as being introduced into the quantities β, and we let λ denote a running index, and let κ, μ denote two indices that are well-defined in each individual case, then either terms of the form $\beta_{\kappa,\lambda}$ or $\beta_{\lambda,\mu}$ must enter in. The determinant of the coefficients *b* thus-chosen will then be composed of the sub-determinants of the determinant $\beta_{\kappa,\mu}$, and will consequently have the absolute value $d^3_{\kappa,\mu}$. The determinants $d_{\kappa,\mu}$ will be precisely those coefficients that enter into the solutions of the equations (17).

10. The problem of transforming a given expression in line coordinates into a given form by a linear substitution can lead to imaginary substitution coefficients, and thus to tetrahedra with imaginary edges. We might call such a tetrahedron simply an *imaginary tetrahedron*.

In general, an imaginary tetrahedron is associated with a conjugate one. Both tetrahedra will then always appear together.

However, in particular, the imaginary edges of an imaginary tetrahedron can be conjugate to each other. If all of the faces (vertices, resp.) are imaginary then the tetrahedron will possess two real, non-intersecting edges, while the four remaining edges will either contain a real point or a real plane, and will be pair-wise conjugate to the opposite edges.

By contrast, if only two faces (vertices, resp.) are imaginary then, as in the foregoing case, only two opposite edges will be real. However, two real planes of the tetrahedron will intersect along the one of them, while two real vertices of it will lie on the other one as intersection points with these faces. The remaining four edges of the tetrahedron will be pair-wise conjugate. Any two conjugate edges will run inside one of the real faces and intersect in it at the corresponding real vertex. Two such imaginary straight lines will be of the type that was considered in the conclusion of number five.

Thus, if the imaginary edges of a tetrahedron are conjugate then two opposite edges will always be real, and we will be dealing with a tetrahedron of the one type or the other, according to whether the remaining four edges do or do not intersect the conjugate one, respectively. Tetrahedra of the one kind or the other can appear in isolation, as long as they are self-conjugate.

Such imaginary tetrahedra that are not conjugate can possess two real, mutually-opposite edges. They will then be common to the given tetrahedron, as well as the conjugate one.

III. On line complexes in general.

11. A homogeneous equation in line coordinates determines a triply-infinite system of straight lines. According to **Plücker**, such a structure is called a *line complex* (*). If we substitute a complex of degree *n* for the line coordinates in the expressions (1) or (3) then we will obtain the following two identical geometric definitions of such a complex (**):

In a complex of degree n, the straight lines that go through a fixed point will define a cone of order n.

In a complex of degree n, the straight lines that lie in a fixed plane will be define a curve of class n.

Therefore, if the complex is linear, in particular, then each point will correspond to a plane that goes through it, while every plane will correspond to a point that lies in it. Such a complex will be defined by the totality of all straight lines that cut a given straight line (no. 4).

The totality of all tangents to a surface of order or class *n* can be regarded as the distinguished case of a complex of degree *n* (*n* – 1). If one specializes the surface in such a way that it degenerates into a developable surface with associated edges of regression then the complex will envelop all of those straight lines that contact the first one or cut the second one.

12. The general equation of degree *n* involves $(n + 5)^5$ different terms. As long as $n > 1$, a complex can depend upon a number of independent constants that is less than *n* minus one only if one is free to extend a series of terms in its equation by means of the relation:

(*) The general concept of *line complex*, it would seem, is entirely associated with **Plücker.** However, that of linear complex had already been thoroughly examined, first of all by **Moebius** in his celebrated treatise: "Ueber eine besondere Art dualer Verhältnisse im Raume" (Crelle's Journal, t. X, 1833).

[January 1884]

(**) **Pluecker**, *Neue Geometrie,* no. 19.

$$P \equiv \sum_{\kappa} p_\kappa \cdot p_{\kappa+3} = 0.$$

We can add to the equation P of a given complex, when multiplied by an arbitrary function of degree n – 2, without changing the complex. Such a function will contain $(n + 3)^5$ undetermined constants. We can thus also arbitrarily assume that there is an equal number of constants in the equation of the complex, except that they must be coupled with terms that have one of the three factors:

$$p_1\, p_4, \quad p_1\, p_4, \quad p_1\, p_4\,,$$

and are all different from each other, except for these factors.

The reduction in number of the independent constants drops out as long as we do not write the equation of the complex n-fold linearly in the coordinates p_κ, but in the $6n$ coordinates:

$$p'_\kappa,\ p''_\kappa,\ \ldots,\ p^{(n)}_\kappa .$$

The expression P is then written bilinearly:

$$\sum_{\kappa} p'_\kappa \cdot p''_{\kappa+3}\,,$$

and is then no longer equal to zero, except when the two straight lines (p') and (p'') intersect, such that it can no longer be added to the equation of the given complex with no further assumptions.

From the foregoing, *a complex of degree two* depends, not on 21 – 1 = 20 independent constants, but on only 19. By contrast, there is a simply-infinite family of associated polar systems (viz., bilinear systems), each of which are determined by 20 constants. In such a polar system, an arbitrarily-chosen straight line will correspond to a linear complex (*). Those lines that correspond to themselves will be the same in all polar systems: viz., the lines of the associated second-degree complex.

The theory of complexes is entirely analogous to the theory of curves that lie in a surface of second order or the theory of developable surfaces that envelope a surface of class two. The *individual* surface, that determines a curve by its intersection with the given second-order surface does not come under consideration at all in the discussion of these intersection curves, but only the family that is determined by it and the given second-order surface. By contrast, a point of the given second-degree surface is associated, relative to the intersection curve in question, with another structure that lies

(*) **Pluecker**, *loc. cit.* – This is not the place to further pursue the reciprocity that was suggested in the text between straight lines and complexes of first degree, which, under a subsequent manner of treatment, would lead to a classification of complexes of first degree in six independent, homogeneous coordinates. (cf., **Pluecker**'s *Neue Geometrie*, no. 19) In this way of looking at things, the straight line seems to be a linear complex whose coordinates satisfy the equation (no. 4):

$$P = 0.$$

on that surface, according to the choice of the second surface that is determined by the curve.

Those straight lines that are common to two complexes define a *congruence*. The congruence is said to be of degree mn when the two complexes that determine it are of degrees m and n, respectively. All lines of a linear complex cut two fixed straight lines, which can be real or imaginary: viz., the *directrices* of the congruence.

Those straight lines that simultaneously belong to three complexes that have degree m, n, p, respectively, define a *ruled surface* (skew surface) of order and class $2mnp$. In particular, three linear complexes determine a second-degree surface by the lines of the one generator of it (*).

IV. Transformation of the second-degree equation in line coordinates into a canonical form.

13. Let:

(19) $$\Omega = 0$$

be the general equation of the second-degree complex, and let:

$$P = 0$$

denote the condition:

$$\sum_{\kappa} p_{\kappa} \cdot p_{\kappa+3} = 0.$$

Our problem is to determine a tetrahedron that has a distinguished relationship with the complex (19), and to give the form that the equation of the complex will assume when it is referred to this tetrahedron as a coordinate tetrahedron.

This problem is treated algebraically as the simultaneous linear transformation of the form P into itself and the form Ω into a canonical form. We thus define the canonical form of the form Ω as the simplest one into which it can be converted by means of such a transformation. A certain arbitrariness will also exist in this choice, and the way in which we will arrive at such a form in what follows is not a necessary one, but one that is chosen as desired. – The algebraic formulation of this problem is more general than the geometric one, in that P and Ω appear as individual forms in it, while in the geometric investigation, along with P, only the two-parameter group:

$$\Omega + \lambda P,$$

where λ means an arbitrary constant, comes under consideration (**).

(*) Cf., **Pluecker**'s *Neue Geometrie, loc. cit.*

(**) The algebraic treatment is coupled with the aforementioned extension of the geometric interpretation of six variables. Corresponding to a conversion of the coordinate tetrahedron, the coordinates of a first-

Since we have 15 arbitrary constants at our disposal in the linear transformation of the form P into itself, the canonical form of the form Ω will contain 6 constants. If we divide through by one of them and add the expression P to it, when multiplied by a suitable constant, then we can still take away two constants from it. The canonical form of the equation of the complex will then contain only four essential constants.

A specialization of the complex will be required when less that four constants should enter into its equation, or when it should be possible to transform it into the same form with four constants in an infinitude of ways.

In connection with the most recent paper of **Weierstrass** on quadratic forms (*), we begin with a singular reformation of the two forms P and Ω that is *always* applicable in our case. As a special case, it includes the transformation of the two forms P and Ω into two that contain only squares of the variables, which is a transformation that is known to not be possible in all cases.

P and Ω will go to two new forms P' and Ω' under the reformation that we spoke of. We will then return to a simple linear transformation of P' to P and thus transform Ω' into a new form Ω'' that we will refer to as *canonical*. By employing the results that were obtained in the cited treatise, we will thus arrive, in the shortest way, at the exhibition of the canonical form that corresponds to each case, and thus, at the classification of second-degree complexes.

We next repeat the results to which **Weierstrass** arrived in the aforementioned article in a form that will correspond to the case that is present here. **Weierstrass** considered the simultaneous transformation of two arbitrarily-given quadratic (or bilinear) forms, and corresponding to the case, special precautionary measures must be observed in order for one of the two forms to vanish. In our case, the one form P is given and has the non-vanishing determinant (– 1).

14. Let:

$$\Phi, \Psi$$

denote two quadratic forms of the same n variables $x_1, x_2, \ldots, x_n$. We make the assumption that the determinant of Φ does not vanish. The determinant of the form:

$$s\Phi + \Psi,$$

which we would briefly like to denote by S, is then an entire function of degree n of s, and can always be represented as the product of n factors that are linear functions of s.

Let $(s - c)$ be any of these factors, under the assumption that the coefficient of the highest power of s that is contained in S is equal to unity, or that it can be extracted as the constant factor in the product of those n factors. We let l denote the exponent of the

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degree complex will transform linearly in such a way that the expression P, which does not vanish, will go to itself.

(*) "Zur Theorie der quadratischen und bilinearen Formen," Monatsberichte d. Berl. Akad., May, 1868, pp. 310-338. Cf., an earlier article on the same situation: Monatsberichte, 1858, pp. 207-220. (The principle of the elementary divisors that will find application in the sequel was indeed first known by **Sylvester**; see his paper: "Enumeration of the Contacts of Lines and Surfaces of the Second Order," in Philosophical Magazine of 1851, v. 1, pp. 119-140 [Jan., 1884].

highest power of those factors that is realized in S. Furthermore, $l^{(\alpha)}$ means the exponent of the highest power of $(s - c)$ by which *all* of the partial determinants of order $(n - k)$ that are formed from the elements of S are divisible. As **Weierstrass** showed, one then has the following inequalities:

$$l > l' > l'' > \ldots > l^{(\nu-1)} > 0,$$
$$l^{(\alpha-1)} - l^{(\alpha)} > l^{(\alpha)} - l^{(\alpha+1)}.$$

If one then sets:

$$e = l - l', \qquad e' = l' - l'', \ldots, e^{(\nu-1)} = l^{(\nu-1)}$$

then $e, e', \ldots, e^{(\nu-1)}$ are positive numbers that are ordered by their magnitudes, such that:

$$e^{(\alpha)} \geq {}^{(\alpha+1)}.$$

Each of the v factors of $(s - c)^l$ thus defined:

$$(s-c)^e, (s-c)^{e'}, \ldots, (s-c)^{e^{(\nu-1)}}$$

is called *an elementary divisor of the determinant S* (*). We say that an elementary divisors has *order e* when e is the highest power of s that is contained in it.

One now has the general theorem that no matter how one also might transform the two forms Φ, Ψ into Φ′, Ψ′ by linear substitutions, the associated elementary divisors will remain the same. Conversely, if two pairs of forms Φ, Ψ and Φ′, Ψ′ possess the same elementary divisors then they can be taken to each other by a linear substitution with a non-vanishing determinant (**).

We now let $S^{(\kappa)}$ denote that sub-determinant of the determinant S that arises from it by omitting the first κ rows and columns. Furthermore, under the assumption that α, β are both greater than κ:

$$(-1)^{(\alpha+\beta)} S_{\alpha\beta}^{(\kappa)}$$

means the $(n - \kappa - 1)^{\text{th}}$-order determinant whose elementary system emerges from that of $S^{(\kappa)}$ by omitting the $(\alpha - \kappa)^{\text{th}}$ row and $(\beta - \kappa)^{\text{th}}$ column, but will be set to zero when one of the two numbers $\alpha, \beta \leq \kappa$.

The functions:

$$S, S', S'', \ldots$$

are divisible by:

$$(s-c)^l, (s-c)^{l'}, (s-c)^{l''}, \ldots$$

(*) The elementary divisors to which the two forms Φ, $\lambda\Phi + \Psi$ lead are among the elementary divisors that belong to the forms Φ, Ψ, but differ from them in that s is replaced with $s + \lambda$ in them.

(**) We remark that this theorem is only true in general when one also admits those linear substitutions whose substitution coefficients possess values that are imaginary, but not mutually conjugate.

respectively. We assume that no power of ($s - c$) that is higher than the given ones is included as a factor in any of these functions. Should that be the case, then one would let $S^{(\nu)}$ be the first in the sequence of functions:

$$S, S', S'', \ldots$$

that contains a higher power of ($s - c$) then $l^{(\nu)}$. We can then let a linear transformation of the form:

$$\bar{x}_\nu = x_\nu + h_{\nu+1}\, x_{\nu+1} + \ldots + h_n\, x_n\,,$$
$$\bar{x}_\alpha = x_\alpha \qquad \text{when } \alpha \neq \nu$$

enter the picture beforehand, and arrange the constants $h_{\nu+1}, \ldots, h_n$ in it in such a way that the new $\bar{S}^{(\nu)}$ will contain only the $l^{(\nu)}$ power of ($s - c$). Since the substitution in question has the determinant (+ 1), the functions:

$$S, S', S'', \ldots, S^{(\nu-1)}$$

will remain unchanged by it. We can thus proceed in the given way such that we will always have that no higher power of ($s - c$) than $l^{(\kappa)}$ is contained in the determinants $S^{(\alpha)}$.

Assuming that this auxiliary transformation has been performed, let:

$$(20)\qquad \left\{\begin{aligned} X &= S_{11}\frac{d\Phi}{dx_1} + S_{12}\frac{d\Phi}{dx_2} + \cdots + S_{1n}\frac{d\Phi}{dx_n},\\ X' &= \qquad\qquad\;\; S'_{22}\frac{d\Phi}{dx_2} + \cdots + S'_{2n}\frac{d\Phi}{dx_n},\\ X'' &= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}\right.$$

Moreover, e_λ will denote the κ^{th} number in the sequence of numbers e that is associated with the divisor ($s - c$). We may likewise write c_λ instead of c, such that the same root c of the equation $S = 0$ can take on different indices according to its associated elementary divisors. One then develops the functions:

$$\frac{X^{(\kappa-1)}}{\sqrt{S^{(\kappa-1)}\cdot S^{(\kappa)}}}$$

in increasing powers of ($s - c_\lambda$). The development begins with the power $-e_\lambda / 2$ of ($s - c_\lambda$), and has the form:

$$\sum X_{\lambda,\mu}\cdot(s-c_\lambda)^{\mu-\frac{e_\lambda}{2}}\,, \qquad \mu = 0, 1, \ldots, \infty.$$

Therefore:

(21) $$X_{\lambda\mu} = \frac{1}{\sqrt{C_\lambda}}\left(C_{\kappa\lambda\mu}\frac{d\Phi}{dx_\kappa} + \cdots + C_{n\lambda\mu}\frac{d\Phi}{dx_n} \right),$$

where C_λ and all of the coefficients of the quantities $d\Phi / dx$ are entire functions of c_λ and the coefficients of the forms Φ, Ψ.

The coefficient C_λ, whose sign is given weight in the case where c_λ is a real quantity, is written, when developed:

(22) $$C_\lambda = \left\{ \frac{(s - c_\lambda)^{2l_\lambda^{(\kappa)} + e_\lambda}}{S^{(\kappa-1)} \cdot S^{(\kappa)}} \right\}, \qquad s = c_2,$$

where $\lambda_\lambda^{(\kappa)}$ has the previous given meaning of $l^{(\kappa)}$, and the index λ refers to only the common identity of e_λ and c_λ .

If e means an arbitrary whole number then one will now denote:

$$\sum X_{\lambda\mu} \cdot X_{\lambda\nu} \quad \text{with } (X_\lambda X_\lambda)_e \qquad (\mu + \nu = e - 1).$$

One then obtains the following conversions:

(23) $$\begin{cases} \Phi = \sum_\lambda (X_\lambda X_\lambda)_{e_\lambda}, \\ \Psi = \sum_\lambda c_\lambda (X_\lambda X_\lambda)_{e_\lambda} + (X_\lambda X_\lambda)_{e-1}, \end{cases}$$

where the summation extends over the various elementary divisors that correspond to λ, and $(X_\lambda X_\lambda)_{e_\lambda - 1}$ is to be set equal to zero when e_λ has the value 1.

These are the reformations of the forms Φ, Ψ in question. It can be verified that the n new variables:

$$\begin{array}{l} X_{1,0}, X_{1,1}, \ldots, X_{1,e_\lambda - 1}, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ X_{\lambda,0}, X_{\lambda,1}, \ldots, X_{\lambda,e_\lambda - 1}, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \end{array}$$

in terms of which Φ and Ψ are presently expressed, are derived from the variables X (20), and thus, from the original variables x, through a substitution whose determinant does not vanish.

Corresponding to a given system of elementary divisors, from formulas (23), we can write down a system of two forms with no further assumptions. In particular, if all elementary divisors are of first order then Φ and Ψ will be represented by the squares of the new variables.

15. Before we go on to the application of the present reformations of the two forms P, Ω that we were given, we might investigate to what extent the variables $X_{\lambda,\mu}$ that were introduced in (21) can be replaced with other ones that are on an equal footing such that Φ and Ψ can likewise be represented in the form (23).

Let μ_ν times ν of the elementary divisors of the determinant *S* be equal to each other. It is then possible to give a linear substitution that contains:

$$\sum_\nu \mu_\nu \cdot \frac{\nu(\nu-1)}{1\cdot 2}$$

arbitrary constants and possess the property that Φ *and* Ψ, *in the form that was given* (23), *will transform into themselves.*

Let ν elementary divisors of order *e* be given that are all the same, and then let *e* > 1. We denote the divisors in sequence by the indices 1, 2, …, ν, and in general by the index α. In the representation of the forms Φ and Ψ, each of these elementary divisors corresponds, in Φ, to a function of the *e* variables:

$$X_{\alpha,0},\ X_{\alpha,1},\ \ldots,\ X_{\alpha,e_\alpha-1}$$

that we have denoted by $(X_\alpha X_\alpha)_{e_\alpha}$, and in Ψ, to the same function of the same *e* variables, multiplied by one of the constants that is independent of the index *a*, minus a function $\left((X_\alpha X_\alpha)_{e_\alpha-1}\right)$ that is a function of only the variables:

$$X_{\alpha,0},\ X_{\alpha,1},\ \ldots,\ X_{\alpha,e_\alpha-2}\,.$$

The variables $X_{\alpha,e_\alpha-1}$ are present in only the first function, and in it, according to the meaning of the symbol $(X_\alpha X_\alpha)_{e_\alpha}$, only in the combination:

$$2\,X_{\alpha,0}\ X_{\alpha,e_\alpha-1}\,.$$

Thus, they are present in Φ and Ψ only in the following expression:

$$2\,X_{1,0}\ X_{1,e_1-1} + 2\,X_{2,0}\ X_{2,e_2-1} + \ldots + 2\,X_{\nu,0}\ X_{\nu,e_\nu-1}\,.$$

The form of Φ and Ψ will then remain unchanged when we transform the variables $X_{\alpha,e_\alpha-1}$ by the following linear substitution:

$$X_{\alpha,e_\alpha-1} = X_{\alpha,e_\alpha-1}\,,$$

minus a linear function of:

$$X_{1,0}\,,\ \ldots,\ X_{\nu,0}\,,$$

and thus demand that the expression:

$$X_{1,0}\ X_{1,e_1-1} + X_{2,0}\ X_{2,e_2-1} + \ldots + X_{\nu,0}\ X_{\nu,e_\nu-1}$$

must go to itself under this substitution. For such a substitution, we have ν^2 constants at our disposal, and $\frac{\nu(\nu+1)}{1\cdot 2}$ conditions to satisfy. Therefore:

$$\nu^2 - \frac{\nu(\nu+1)}{1\cdot 2} = \frac{\nu(\nu-1)}{1\cdot 2}$$

constants will remain arbitrary.

If we assume that $e = 1$ then we will get the same number. The function:

$$X_{1,0}^2 + X_{2,0}^2 + \cdots + X_{\nu,0}^2$$

will then be transformed into itself.

In this way, we can proceed with the each system of equal divisors that are contained in the sequence of elementary divisors of the determinant *S*, and thus obtain the number that was given above:

$$\sum_\nu \mu_\nu \cdot \frac{\nu(\nu-1)}{1\cdot 2}.$$

This number refers to the value that the forms (23) possess on the basis of the system of variables for these forms.

16. A further examination is linked to the sign of the constants C_λ that are determined by the equation (22).

As is known, one classifies the quadratic forms in *n* variables with non-vanishing determinants into classes according to the excess that the number of positive squares over the number of negative squares yields when one transforms the given form into a form that only includes the squares of the variables by means of any *real* linear substitution with a non-vanishing determinant. Let *m* denote the excess that belongs to the given function Φ. The following theorem is then true, independently of the choice of the form Ψ:

When one divides the constants C_λ that belong to real elementary divisors of an odd order into two groups according to their signs, *the group of positive* C_λ *will contain m terms more than the group of negative ones.*

This implies the theorem that the determinant *S*, independently of the choice of the form Ψ, *must contain at least m real elementary divisors of odd order* (*).

(*) I would like to draw special attention to this general theorem, which seems to have be rarely noticed up to now. When translated into the usual manner of speaking, it says that of the roots of the equation $| s a_{i\kappa} - b_{i\kappa} | = 0$, *at least m of them* will be real when $\sum a_{i\kappa} x_i x_\kappa$ or $\sum b_{i\kappa} x_i x_\kappa$ is converted into a sum of real squares such that the squares of one sign prove to be *m* more than those of the other sign. If the number *m* agrees with the number of variables *x* then the theorem will revert naturally to a well-known theorem.

[Jan. 1884].

If $(s-c)^{e_\lambda}$ denotes a real elementary divisor, and ε_λ denotes positive or negative unity, according to whether C_λ is positive or negative, then we (along with **Weierstrass**) would like to set:

$$X_{\lambda_\mu} = + \sqrt{\varepsilon_\lambda} \cdot \mathfrak{X}_{\lambda_\mu} ,$$

and therefore:

$$(X_\lambda X_\lambda)_{e_\lambda} = \varepsilon_\lambda (\mathfrak{X}_\lambda \mathfrak{X}_\lambda)_{e_\lambda} .$$

The $\mathfrak{X}_{\lambda_\mu}$ will then be linear functions of the original variables x with *real* coefficients. By contrast, if $(S-c_\lambda)^{e_\lambda}$ is an imaginary divisor then one will find a second one $(S-c_{\lambda'})^{e_{\lambda'}}$ that is conjugate to it, where $e_\lambda = e_{\lambda'}$. If we then assign conjugate values to the roots $\sqrt{C_\lambda}$, $\sqrt{C_{\lambda'}}$ and set:

$$X_{\lambda\mu} = \mathfrak{X}_{\lambda\mu} + i\,\mathfrak{X}'_{\lambda\mu} ,$$
$$X_{\lambda'\mu} = \mathfrak{X}_{\lambda'\mu} - i\,\mathfrak{X}'_{\lambda'\mu}$$

then $\mathfrak{X}_{\lambda\mu}$ and $\mathfrak{X}'_{\lambda\mu}$ will be linear functions of the variables x with likewise real coefficients, and one will have:

$$(X_\lambda X_\lambda)_{e_\lambda} + (X_{\lambda'} X_{\lambda'})_{e_{\lambda'}} = 2(\mathfrak{X}_\lambda \mathfrak{X}_\lambda)_{e_\lambda} - 2(\mathfrak{X}'_\lambda \mathfrak{X}'_\lambda)_{e_{\lambda'}} .$$

After these substitutions, Φ will be represented in terms of n real variables. We now have to transform Φ into the squares of n new variables by some sort of real substitution. The excess of positive squares over the number of negative squares must then amount to m.

Any two conjugate imaginary obviously contribute nothing to this excess m. $(\mathfrak{X}_\lambda \mathfrak{X}_\lambda)_{e_\lambda}$ will then likewise yield four squares of the one sign, as will $(\mathfrak{X}'_\lambda \mathfrak{X}'_\lambda)_{e_{\lambda'}}$.

The expression that corresponds to an odd real elementary divisor yields the excess of one square with the sign ε_λ. The expression $(\mathfrak{X}_\lambda \mathfrak{X}_\lambda)_{e_\lambda}$ then contains one square and $\frac{e_\lambda - 1}{2}$ products of each two variables. Such a product will involve one positive square and one negative one.

By contrast, if the real elementary divisor is of even order then the expression $(\mathfrak{X}_\lambda \mathfrak{X}_\lambda)_{e_\lambda}$ will include only two products of the variables, and will thus yield an equal number of positive and negative squares.

With that, the present two theorems are proved. Conversely, it is clear from formulas (21) that one can determine a form Ψ with real coefficients for a given Φ that corresponds to an arbitrary system of elementary divisors, as long as among the squares that correspond to the odd real divisors in the representation (21) of Φ, m more positive than negative ones are present. One then imagines Φ as being transformed by some real, linear substitution into a form that contains only the squares of variables. Under the assumption that was made, one can always find linear substitution that take Φ from this form into the one that was given by (23), for which, the new variables will be expressed in terms of the earlier ones as either real or pair-wise conjugate imaginary, according to the type of elementary divisor that they correspond to. It will then suffice to furnish Ψ in (23) with coefficients that correspond to the various elementary divisors. Back-substitution will then lead to a form Ψ in the original variables that has real coefficients. There is then a means of writing down, with no further assumptions, all of the cases that can appear for a given Φ under the transformation of the forms Φ, Ψ into the form (23).

17. We return to the forms *P* and Ω that we were given. When we let *P* enter in place of Φ as a form with non-vanishing determinant, and Ω in place of Ψ, from (23), we will obtain the following representation of the forms *P* and Ω:

$$\text{(24)} \qquad \left\{ \begin{aligned} P &= \sum_\lambda (X_\lambda X_\lambda)_{e_\lambda}, \\ \Omega &= \sum_\lambda c_\lambda (X_\lambda X_\lambda)_{e_\lambda} + (X_\lambda X_\lambda)_{e_{\lambda-1}}. \end{aligned} \right.$$

As in the general case, the new variables are determined by the formulas (20), (21), (22). The number of variables *n* must be replaced with 6 everywhere in them. We only remark that for the given form of *P* these formulas will simplify due to the fact that the variables *x* will appear in place of the quantities $d\Phi / dx$, but in an altered sequence.

The discussion on the multiplicity of the transformation into the form (23) that was made in number 15 will also preserve its validity. Because of that, we might keep the notation μ_ν for the number of systems of ν elementary divisors among them that are equal.

The theorems that were given in number 16 on the number of positive and negative real squares that are contained in the representation (23) of the form Φ will be modified as follows, according to the special form of *P*:

If we transform the form *P* into a form that contains only squares of the variables by some real substitution with a non-vanishing determinant then we will find *just as many* positive and negative squares. The number *m*, which gave the excess of positive over negative squares in the general case, will then be equal to zero in the case of the form *P*.

One will then always *find an equal number of positive and negative real squares* in the representation (24) of the form *P*. We might denote this number by σ. The number of real elementary divisors of an odd order will then be 2σ. At least *m* such elementary divisors will be present in the general case of the form Φ, while the number of real elementary divisors of odd order will be arbitrary. Conversely, the form Ψ can be chosen so that only *m* real elementary divisors will be present that are indeed of odd order. Since *m* has the value zero for the form *P*, *arbitrarily many elementary divisors of the determinant of the form sp* + Ω *can then be imaginary.* We might denote the number of elementary divisors of an odd order by 2ρ in the sequel.

We now have enough material for a *classification of second-degree complexes.* The order of the elementary divisors that are associated with Ω will determine the type of the forms (24). When we collect the numbers that give the orders of the individual elementary divisors then we will obtain a classification of all second-degree complexes into *eleven* different types by the following table:

	Order of the elementary divisor					
I	1	1	1	1	1	1
II	1	1	1	1	2	
III	1	1	1	3		
IV	1	1	2	2		
V	1	1	4			
VI	1	2	3			
VII	2	2	2			
VIII	1	5				
IX	2	4				
X	3	3				
XI	6					

The number 11 refers to the number of possible ways of expressing the number of variables – viz., 6 – as a summand.

The number of equal elementary divisors, as well as the number of negative ones, will give a further basis for the classification, and then the signs of the terms that correspond to the real, elementary divisors on the representation (24) of P. We refrain from individually enumerating the various cases that can thus exist, as well as from proving that they can be related to each other continuously as transitional cases between extreme terms.

18. We would like to transform the type of the form P that was given in (24) as follows: We leave all of those terms that correspond to elementary divisors unchanged. By contrast, we introduce new variables in place of the variables $X_{\lambda,0}$, …, $X_{\lambda,e_\lambda-1}$, which are associated with a real elementary divisor according to the sign of the constant C_λ (22). In the case where C_λ is positive, we preserve the original variables. In the opposite case, we set:

$$X_{\lambda\beta} = \pm\, \mathfrak{X}_{\lambda\beta}\,,$$

and thus determine the sign of the square root in such a way that each of the doubled products $2X_{\lambda,\beta} \cdot X_{\lambda,e_\lambda-\beta-1}$ will enter the new represent of the form P as $2\mathfrak{X}_{\lambda,\beta} \cdot \mathfrak{X}_{\lambda,e_\lambda-\beta-1}$ with the positive sign.

The form P is then represented in terms of the squares of 2ρ variables, among which, one will find 2σ real ones and $(3 - \rho)$ doubled products of each two of the remaining $6 - 2\rho$ variables. Thus, those doubled products that involve real variables will have the positive sign. From this representation of the form P, by means of a new linear substitution, we must return to the originally-given form:

$$\sum_{\kappa} p_{\kappa} \cdot p_{\kappa+3} ,$$

in which only doubled products of each two of the six variables will be present, and which will all have the positive sign.

To that end, we will, with no further assumptions, keep those $6 - 2\rho$ variables that are already coupled to the doubled products of two variables in the given representation of the form. By contrast, we will divide the 2ρ squares into ρ groups of two, and resolve each individual group into the doubled product of two new variables. We thus decompose:

$$Y_\alpha^2 + Y_\beta^2 ,$$

where Y_α^2, Y_β^2 mean two such squares, into the product of the two linear factors:

$$\lambda \cdot \frac{Y_\alpha + iY_\beta}{\sqrt{2}}, \quad \frac{1}{\lambda} \cdot \frac{Y_\alpha - iY_\beta}{\sqrt{2}} ,$$

where λ means a new arbitrary constant. If Y_α, Y_β are not conjugate imaginaries of each other then it will be preferable to set λ equal to simply positive unity. In the opposite case, we choose λ to be equal to $1 - i$, and thus obtain new variables that are composed of sums and differences of the real and imaginary components of Y_α (Y_β, resp.).

The type and manner of partitioning of the 2ρ squares into ρ groups of two is arbitrary. As long as the elementary divisors, which correspond to the individual squares, are all different, any system of new variables that is obtained from an arbitrary grouping of the 2ρ will be equally justified. We then have to choose between:

$$(2\rho - 1)(2\rho - 3)\ldots$$

different systems. This is then the number of possible ways of grouping 2ρ by twos. For the 11 cases that were enumerated in the previous section, this number will assume the following values:

15, 3, 3, 1, 1, 1, 1, 1, 1, 1, 1,

respectively.

Things are different when equal elementary divisors are found among the elementary divisors that belong to the 2ρ squares. We will then always group those squares that correspond to equal elementary divisors into two, and with the rest of the squares that remain under that operation, we will proceed in the same way that we did above with the 2ρ squares that are generally present.

In number 15, we let μ_ν denote the number that gives how often one will find ν equal elementary divisors. Corresponding to it, we will let $\mu'_{2\nu}$ ($\mu'_{2\nu+1}$, resp.) denote those numbers that express how often one will find 2ν ($2\nu + 1$, resp.) equal elementary divisors of odd order. Finally, we introduce the notation $\mu''_{2\nu}$ for the sum $\mu'_{2\nu} + \mu'_{2\nu+1}$.

Any subdivision of 2ν associated (i.e., corresponding to equal elementary divisors) squares will yield:

$$(2n - 1)(2n - 3)\ldots$$

different systems of new variables.

From any subdivision of $2\nu + 1$ associated squares, we must next arbitrarily single out a square, which can happen in $(2\nu + 1)$ ways, and then combine the remaining 2ν ones by twos. We will then obtain the number:

$$(2\nu + 1)(2\nu - 1)(2\nu - 3)\ldots$$

Ultimately, $\sum \mu'_{2\nu+1}$ individual squares will remain. They will admit:

$$\left(\sum \mu'_{2\nu+1} - 1\right)\left(\sum \mu'_{2\nu+1} - 3\right)\ldots$$

groupings.

We then obtain the product:

$$R = \left(\sum \mu'_{2\nu+1} - 1\right)\left(\sum \mu'_{2\nu+1} - 3\right)\ldots \prod_{\nu} (2\nu+1)^{\mu'_{2\nu+1}} \cdot [(2\nu-1)(2\nu-3)\cdots]^{\mu''_{\nu}}$$

as the total number of systems of equivalent variables.

The expression above:

$$(2\rho - 1)(2\rho - 3)\ldots$$

is derived from this general expression as a special case.

If P is again assumed to have its previous form when it was introduced then we will give the six new variables thus determined *the meaning of line coordinates.* By substituting them into the form Ω (24), it will go over into a new form *that we will refer to as canonical*. It will take a different form according to the number and order of the elementary divisors. We shall refrain from writing down the eleven different types of complexes that correspond to them. When equal elementary divisors of odd order appear, a number of the constants that appear in the associated canonical form will take on the value zero.

19. The transformation of the form Ω to canonical form is a *multi-valued* one. The number R in the previous section will determine the degree of this multi-valuedness. We shall now investigate to what extent one should find transformations among these different possibilities that would lead to *real* new variables.

In order for that to be true, one must first fulfill the condition that one should find no imaginary roots among the multiple roots of the equation in s that expresses the idea that the determinant S of the form $sP + \Omega$ vanishes (no. 13). Such roots will then correspond to either one sequence of equal elementary divisors or, when that is not the case, to at least one elementary divisor of order higher than one. In both cases, we will obtain imaginary canonical variables. Moreover, the assumption that a system of real canonical variables is possible demands the condition that ν positive and ν negative squares will be present amongst the 2ν ($2\nu + 1$, resp.) squares that belong to equal elementary divisors.

If this condition is fulfilled consistently for values of n that are greater than zero, and we combine only squares of opposite signs by twos then, from the discussion in number 17, we will find just as many positive and negative squares among the individual squares that ultimately remain. With the present assumptions, we will obtain the following number of real transformations. Any group of 2ν ($2\nu + 1$, resp.) associated squares will give $\nu!$ [$(\nu + 1)!$, resp.] different systems of new real variables. A square must be selected from the number of $2n + 1$ squares that has a sign such that there are ν other ones, in addition to it. This is then possible in $(\nu + 1)$ ways, and then n positive elements must be combined with ν negative ones by twos in such a way that each group contains one positive element and one negative one. There are $\nu!$ such combinations. Finally, there are still $\sum \mu'_{2\nu+1}$ individual squares to be grouped. We assumed above that the number of all real squares in the representation of Φ amounted to 2σ. Among the individual squares, there thus still remain:

$$2\left(\sigma - \sum \nu \cdot \mu''_\nu\right)$$

real ones. If we then take the conjugate imaginary individual squares together and combine the real squares each time with one positive and one negative then we will obtain $\left(\sigma - \sum \nu \cdot \mu''_\nu\right)!$ systems of new variables. A real transformation of the given form Ω to the canonical form is therefore possible in R' ways, where R' denotes the following product:

$$R' = \left(\sigma - \sum \nu \cdot \mu''_\nu\right)! \cdot \prod_\nu (\nu + 1)^{\mu'_{2\nu+1}} \cdot (\nu!)^{\mu'_\nu} .$$

In particular, if all elementary divisors are different then this number will be equal to $\sigma!$. For example, in the case that was denoted by I above:

0, 2, 4, 6

of the elementary divisors can be imaginary, and thus, of the 15 different systems of linear substitutions that transform Ω into the canonical form under the assumption of distinct divisors in this case:

6, 2, 1, 1

of them will be real, respectively.

It is obviously also possible to transform Ω into a simple form by a real substitution in those cases in which all systems of canonical variables prove to be imaginary, such as when we consider the real and imaginary parts of the imaginary variables that we spoke of to be new variables. However, we cannot refer to such a form as "canonical," because it is derived from the representation (24) of the form Ω by a different method from the one that was applied in all of the other cases.

25. In summation, we have arrived at the following result:

Let a second-degree complex be given:

$$\Omega = 0,$$

and let:

$$P = 0$$

denote the second-degree condition equation that line coordinates must satisfy.

Furthermore, let $(s - c_\lambda)^{e_\lambda}$ *be an arbitrary elementary divisor of the determinant of the form sP + Ω, and let* μ_ν *mean the number that gives how often one finds* ν *equal elementary divisors.* $\mu'_{2\nu}$ *and* $\mu'_{2\nu+1}$ *might denote those numbers that express how often* 2ν ($2\nu + 1$, *resp.*) *one finds equal elementary divisors of odd order, while* μ''_ν *means the sum* $\mu'_{2\nu} + \mu'_{2\nu+1}$. *Finally, let* 2σ *be the number of real elementary divisors of odd order.*

P and Ω can then be transformed simultaneously, by a substitution with a non-vanishing determinant that includes:

$$\sum_\nu \mu_\nu \cdot \frac{\nu(\nu-1)}{1\cdot 2}$$

arbitrary constants, into the following forms:

$$P = \sum_\lambda \sum_{(\mu+\nu=e_\lambda-1)} X_{\lambda\mu} \cdot X_{\lambda\nu} \, ,$$

$$\Omega = \sum_\lambda \left\{ c_\lambda \sum_{(\mu+\nu=e_\lambda-1)} X_{\lambda\mu} \cdot X_{\lambda\nu} + \sum_{(\mu+\nu=e_\lambda-2)} X_{\lambda\mu} \cdot X_{\lambda\nu} \right\},$$

where $X_{\lambda,0}, \ldots, X_{\lambda,e_\lambda-1}$ *mean the new variables, and the sum:*

$$\sum_{(\mu+\nu=e_\lambda-2)} X_{\lambda\mu} \cdot X_{\lambda\nu}$$

is set to zero when e_λ *has the value of unity.*

From this representation of the form Ω, we can go to its canonical form by means of:

$$R = \left(\sum \mu'_{2\nu+1} - 1\right)\left(\sum \mu'_{2\nu+1} - 3\right)\cdots \prod_\nu (2\nu+1)^{\mu_{2\nu+1}} \cdot [(2\nu-1)(2\nu-3)\cdots]^{\mu''_\nu}$$

different systems of linear substitutions with non-vanishing determinants. In the favorable case, the system of new variables can be chosen in:

$$R' = \left[\sigma - \sum \nu \cdot \mu''_\nu\right]! \cdot \prod_\nu (\nu+1)^{\mu'_{2\nu+1}} \cdot (\nu!)^{\mu''_\nu}$$

different ways such that the transformation will be real. For this to be true, it is requisite that no multiple imaginary roots of the determinant of sP + Ω be found when it is set equal to zero, and then that the number of real, positive squares that belong to the equal *elementary divisors in the form that was just given must be different from the number of real, negative ones by at most* 1.

V. Geometric meaning of the transformation to canonical form; in particular, in the case where all elementary factors are linear and different.

21. From number 8, the coefficients of the substitution that transforms Ω to the canonical form immediately give the edges of the distinguished coordinate tetrahedron, relative to which the equation of the complex will be written in canonical form. Corresponding to the different substitutions, we will obtain an:

$$\sum_{\nu} \mu_{\nu} \cdot \frac{\nu(\nu-1)}{1\cdot 2} \text{-fold}$$

infinite family of R coordinate tetrahedra, among which R' of them will be real. These tetrahedra will all have the same distinguished relationship with the complex.

We thus understand a simple, two-fold, …, m-fold infinite family of coordinate tetrahedra to mean the totality of all of them that possess edges whose coordinates can be derived from the coordinates of the edges of one of them, with the assistance of 1, 2, …, m arbitrary constants. If we then understand P_κ, $P_{\kappa+3}$ to mean two opposite edges of the coordinate tetrahedron, and then apply the transformation:

$$p_\kappa = \lambda p'_\kappa, \qquad p'_{\kappa+3} = p_{\kappa+3},$$

under which the edges themselves, and thus, the tetrahedron will not change, while only the unit that the coordinate system is based upon will change to another, then we must consequently speak of a simply-infinite family of tetrahedra, corresponding to the various values of the arbitrary constant λ.

22. In the sequel, we shall restrict ourselves to the geometric discussion of the results that were obtained in only the case where all of the elementary divisors are linear and distinct. In this case, P and Ω will be represented in the following forms:

$$\begin{cases} P = \sum_{\lambda} X_\lambda^2, \\ \Omega = \sum_{\lambda} c_\lambda X_\lambda^2, \end{cases} \tag{25}$$

where the summation must go from 1 to 6, and $c_1, \ldots, c_6$ mean distinct quantities. We can transform Ω from this representation into the canonical form in 15 different ways. According to whether:

0, 2, 4, 6

of the elementary divisors are imaginary:

6, 2, 1, 1,

respectively, of the 15 distinguished tetrahedra will be real. *With respect to any of these latter tetrahedra, the form* Ω *can be written in the following form:*

$$\begin{aligned} \Omega &= A_{1,4}\,(p_1^2 \pm p_4^2) + 2\,B_{1,4}\,p_1\,p_4 \\ &+ A_{2,5}\,(p_2^2 \pm p_5^2) + 2\,B_{2,5}\,p_2\,p_5 \\ &+ A_{3,6}\,(p_3^2 \pm p_6^2) + 2\,B_{3,6}\,p_3\,p_6\,, \end{aligned} \tag{26}$$

where $p_1, \ldots, p_6$ denote the new real variables, and A denote the coefficients of the one group, while the B denote those of the other (no. 12). Of the three signs that remain undetermined, one must choose a pair of imaginary elementary divisors that correspond to – 1.

This is the canonical form of Ω *in the case of linear, distinct elementary divisors, whose derivation was our problem.*

23. We now address the grouping of the 15 distinguished tetrahedra amongst themselves. We might briefly refer to them as the *fundamental tetrahedra* of the complex.

Two opposite edges of one of the fifteen fundamental tetrahedra are common to two other ones. If we choose any two of the six squares, by means of which P was represented in (23), then the four remaining ones can be split into two groups of two in three ways. The system of fifteen fundamental tetrahedra then encompasses thirty edges. Their sixty faces intersect six of these edges, and their sixty vertices are likewise distributed over six of them. Any of the thirty edges will then cut twelve of the remaining ones. The same twelve edges will be cut by a second edge that has an exclusive relationship to the first one. Therefore, the thirty edges will divide into fifteen groups of two that belong together.

The variables X_λ that enter into (25) will represent linear complexes when they are each set equal to zero. If we choose two of them, X_1, X_2, arbitrarily then the two equations:

$$X_1 + i\,X_2 = 0, \qquad X_1 - i\,X_2 = 0$$

will represent two associated edges of the thirty edges of the fundamental tetrahedra. All straight lines that cut the one or the other of these two edges will satisfy the two given equations, and will thus belong to the two complexes X_1, X_2 . The two edges that we speak of will then be the directrices of the congruence that is defined by the two complexes X_1, X_2 (no. 12). That will give the geometric interpretation of the variables X_λ in terms of the system of fundamental tetrahedra.

Three of the complexes X_λ – say, X_1, X_2, X_3 – determine a second-degree surface (viz., a hyperboloid) as a skew surface (no. 12). This surface is associated with the following six of the system of edges of the fundamental tetrahedron as lines of a generator of it:

$$X_1 + i\, X_2 = 0, \qquad X_2 + i\, X_3 = 0, \qquad X_3 + i\, X_1 = 0,$$
$$X_1 - i\, X_2 = 0, \qquad X_2 - i\, X_3 = 0, \qquad X_3 - i\, X_1 = 0$$

as directrices of the congruences of any two of the three complexes (*), and because of the way that the thirty edges in question of the tetrahedra are grouped, the following six edges:

$$X_4 + i\, X_5 = 0, \qquad X_5 + i\, X_6 = 0, \qquad X_6 + i\, X_4 = 0,$$
$$X_4 - i\, X_5 = 0, \qquad X_5 - i\, X_6 = 0, \qquad X_6 - i\, X_4 = 0,$$

will be lines of the other generator.

The six symbols X_λ can be combined three at a time in $\frac{6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3} = 20$ ways. The thirty edges of the fundamental tetrahedra thus separate into twenty groups of six that pair-wise belong together. The six edges of one group are lines of the same general of a second-degree surface, while the six edges of the associated group are lines of the generator of the same surface. The system of thirty edges then has a distinguished relationship with ten different second-degree surfaces, of which four of them will cut an edge.

24. If all elementary divisors are real then six of the fifteen fundamental tetrahedra will be real. The remaining nine tetrahedra are such that they possess two real, opposite edges and are conjugate to the remaining opposite edges. Two associated real edges are common to two real and one imaginary tetrahedron. Of the thirty edges of the fifteen tetrahedra, eighteen of them are then real, and the other twelve are imaginary, such that conjugate imaginary ones belong together.

If two of the six elementary divisors are imaginary then only two of the fifteen tetrahedra are real. Like the nine tetrahedra in the previous case, one of them is conjugate to itself. The remaining twelve imaginary tetrahedra are pair-wise conjugate to each other. Of the thirty edges, only ten of them will be real, while the remaining twenty will be imaginary. Of these twenty edges, twice two edges will be conjugate and likewise associated, while the remaining sixteen edges will divide into two groups that are conjugate, and each of which will include eight edges that are pair-wise associated.

Finally, in the third and fourth cases, where four or six of the elementary divisors, respectively, are imaginary, only one of the fifteen tetrahedra will be real. Of the thirty edges, six of them will be real, while the remaining 24 will be imaginary. They divide into two mutually conjugate groups, each of which involves twelve edges that are pair-wise associated. Among the imaginary fundamental tetrahedra, none of them will have only two imaginary vertices (no. 10).

In the case where six of the fifteen fundamental tetrahedra are real, we get from one real tetrahedron to a second one, and indeed to the ones that have the two edges P_3, P_6 in common with the given one, when we set:

$$p_1 + p_4 = x_1, \qquad p_2 + p_5 = x_2\,,$$

(*) Cf., **Pluecker**'s *Neue Geometrie*, no. 101.

$$p_1 - p_4 = i x_4, \qquad p_2 - p_5 = i x_5 ,$$

and

$$x_1 - i x_5 = 2 p_1', \qquad x_2 + i x_4 = 2 p_2',$$
$$x_1 + i x_5 = 2 p_4', \qquad x_2 - i x_4 = 2 p_5'.$$

The following direct transformation will emerge from this:

$$\begin{aligned} 2p_1' &= \;\; p_1 + p_4 - p_2 + p_5 , \\ 2p_4' &= \;\; p_1 + p_4 + p_2 - p_5 , \\ 2p_2' &= \;\; p_1 - p_4 + p_2 + p_5 , \\ 2p_5' &= - p_1 + p_4 + p_2 + p_5 , \\ 2p_3' &= - 2p_3 , \\ 2p_6' &= \;\; 2p_6 , \end{aligned}$$

and this will correspond to the following transformation of the point coordinates ($z_1, \ldots, z_4$):

$$\sqrt{2}\, z_1' = \;\; z_1 + z_4 , \qquad \sqrt{2}\, z_2' = z_2 - z_3 ,$$
$$\sqrt{2}\, z_4' = - z_1 + z_4 , \qquad \sqrt{2}\, z_3' = z_2 + z_3 .$$

Since we can also apply exactly the same conversion in the case of imaginary tetrahedra, we will obtain the theorem that the four faces of two fundamental tetrahedra that intersect along an edge, as well as the four vertices that lie on an edge, will be harmonically conjugate to each other.

25. We thus go on to the examination of the geometric meaning of the form of equation (26). For the sake of simplicity, we thus assume that all elementary divisors are real, so only positive signs will appear in (26).

Let a line of the complex be known whose coordinates are:

$$p_1, p_2, p_3, p_4, p_5, p_6 .$$

The same complex then belongs to a sequence of other straight lines whose coordinates are given by the same six quantities, but in a different sequence. We can switch p_1 with p_4, p_2 with p_5, p_3 with p_6 . We then obtain seven new straight lines that are likewise lines of the complex:

$$\begin{aligned} &p_4, p_2, p_3, p_1, p_5, p_6 , \\ &p_1, p_5, p_6, p_4, p_2, p_3 , \\ &p_4, p_5, p_6, p_1, p_2, p_3 , \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

We will obtain further lines of the complex when we change the signs of p_1 and p_4, p_2 and p_5, and p_3 and p_6 . Corresponding to each of the given eight lines, we will obtain three new ones that likewise belong to the complex. In all, when *one* line is given there will thus be 32 of them that are determined.

This number can be easily derived, as follows: $+ p_4$, $- p_1$, $- p_4$ can appear in place of $+ p_1$; likewise, $+ p_5$, $- p_2$, $- p_5$ ($+ p_6$, $- p_3$, $- p_6$, resp.) can appear in place of $+ p_1$ ($+ p_3$, resp.). We then obtain:

$$4 \cdot 4 \cdot 4 = 64$$

combinations, and 64 / 2 = 32 straight lines that correspond to them, because a change of sign in all coordinates of the straight line that they represent will change nothing.

The relationship between these 32 straight lines is reciprocal. They can be ascertained geometrically by means of those ten second-degree surfaces that each contain twelve edges of the system of thirty edges of the fundamental tetrahedron (no. 23). Any of these surfaces is represented by the equation:

$$z_1 z_2 + z_3 z_4 = 0,$$

relative to a suitably-chosen fundamental tetrahedron, where $z_1, \ldots, z_4$ might mean point coordinates. The arbitrarily-chosen straight line with the coordinates:

$$p_1, p_2, p_3, p_4, p_5, p_6$$

then corresponds to a second one that is polar to it, relative to this surface, and whose coordinates are:

$$p_4, p_2, - p_3, p_1, p_5, - p_6 ,$$

and this is one of the currently specified 32 lines. Each of these 32 lines thus yields a straight line as its polar, relative to each of the ten surfaces, which is itself included in the sequence of 32 lines.

We can say that *the given complex is reciprocal to itself with respect to each of these ten second-degree surfaces*. That is: Each line of the complex corresponds to other lines in it as its polars with respect to each of these surfaces, or in other words, those of the complex lines that go through a fixed point will define a cone that corresponds to a plane curve with respect to each of the aforementioned surfaces, which will itself be enveloped by lines of the complex.

The determination of these ten surfaces is independent of the constants that enter into equation (26).

26. Equation (26) for the form gives a geometric construction of the second-degree complex. We can first bring this form into the following form:

$$2 P_1 P_4 + 2 P_2 P_5 + 2 P_3 P_6 = 0, \tag{27}$$

where $P_1, \ldots, P_6$ denote linear complex. To that end, we need only to resolve the aggregate of terms:

$$A_{\kappa,\kappa+3}(p_\kappa^2 + p_{\kappa+3}^2) + 2B_{\kappa,\kappa+3}\, p_\kappa \cdot p_{\kappa+3}$$

into a product of two linear factors:

$$2(\alpha p_\kappa + \beta p_{\kappa+3}) \cdot (\beta p_\kappa + \alpha p_{\kappa+3}),$$

where α, β are determined by the equations:

$$2\alpha\beta = A_{\kappa,\kappa+3} , \qquad \alpha^2 + \beta^2 = B_{\kappa,\kappa+3} .$$

The complexes $P_1, \ldots, P_6$ can be constructed linearly (*) by means of the coordinate tetrahedra and a straight line that belongs to them, or more generally, from five of their lines.

One determines the six planes that correspond to an arbitrary point in space relative to these six linear complexes. We might likewise denote these planes with the symbols $P_1, \ldots, P_6$. Those two edges, along which the second-order cone that was defined by the lines of the desired complex at the given point will cut any one of these planes – say, P_1 – can be constructed as the intersection of this plane with the cone:

$$P_2 P_5 + P_3 P_6 = 0,$$

which is given by two projective pencils of planes, say:

$$P_2 + \lambda P_6 , \qquad P_3 - \lambda P_5 .$$

A three-fold repetition of this construction will yield six edges of the complex cone in question. Five of them will be sufficient to determine it.

27. In conclusion, we might investigate the type of distinguished relationship that the fundamental tetrahedra have to the complexes.

According to **Pluecker**, any line is associated with a second one as its polar with respect to a second-degree complex (**). It will have the double relationship with the former that it is, on the one hand, the geometric locus for the poles of the first straight line with respect to all curves that will be enveloped by lines of the complex in the planes that go through them. On the other hand, they will be enveloped by the polar planes to the initial straight line with respect to all cones that were defined at the points of the lines of the complexes. This relationship between the two lines is not reciprocal. The second line will be associated with a third, etc. Except for the lines of the complexes that are conjugate to themselves, only a finite number of straight lines will be given that are themselves again the polars of their polars.

If we three make arbitrary coordinates in equation (26) vanish, which refer to three edges of the fundamental tetrahedra that intersect at a point or lie in a plane, then we will

(*) **Pluecker**'s *Neue Geometrie*, no. 29.

(**) **Pluecker**'s *Neue Geometrie*, no. 172. – The relationship of a straight line to its polars can be represented with the help of the simply-infinite family of linear polar systems that was mentioned in number 12, which correspond to the given straight line relative to the given second-degree complex. The given straight line and its polars are the directrices of the congruence that is determined by the *one*-parameter group of linear polar complexes. (**Pluecker,** *loc. cit.*)

obtain an equation for the representation of the cone that emanates from the vertex, relative to the curve that lies in the face, that only involves the squares of the remaining three variables. The cone and the curve are thus referred to a three-edge and a three-corner, resp., that are self-conjugate relative to it. It follows from this that the polar that is associated with an arbitrary edge of a fundamental tetrahedron by the complex is the opposite edge of that fundamental tetrahedron, so the fundamental tetrahedron is chosen in such a way that *any two opposite edges of it will be mutually conjugate with respect to the complex.*

28. One can now show that except for the thirty edges of the fifteen fundamental tetrahedra, no other lines possess the property that they correspond to themselves relative to the given complex.

Let two such lines be given then. We choose them to be opposite edges of a coordinate tetrahedron and refer the complex to them. If we denote the two give edges by P_1 and P_4 then its equation will be missing the eight terms with the double products:

$$p_1\,p_2,\ p_1\,p_3,\ p_1\,p_5,\ p_1\,p_6,$$
$$p_4\,p_2,\ p_4\,p_3,\ p_4\,p_5,\ p_4\,p_6,$$

so p_1 and p_4 will appear only in the combination:

$$a_{11}p_1^2 + 2a_{14}p_1p_4 + a_{44}p_4^2\,.$$

If we then transform the two forms:

$$P = \sum_\kappa p_\kappa \cdot p_{\kappa+3}$$

and

$$\Omega(p_1, p_2, p_3, p_4, p_5, p_6),$$

as we would like to write instead of Ω, corresponding to equations (25), into two forms that only involve the squares of the variables then it will be permissible to carry out this transformation with the two pairs of forms:

$$2p_1\,p_4 \qquad \text{and} \qquad a_{11}p_1^2 + 2a_{14}p_1p_4 + a_{44}p_4^2$$

and

$$2p_2\,p_5 + 2p_3\,p_6 \qquad \text{and} \qquad \Omega(0, p_2, p_3, 0, p_5, p_6)$$

individually. With that, we have proved that the edges P_1, P_4 of the given coordinate tetrahedron belong to the system of thirty edges of the fundamental tetrahedra. We thus have the theorem:

Let a complex be given whose associated elementary divisors are all linear and distinct from each other. There are then thirty straight lines that are mutually conjugate to each other with respect to the complex. According to whether 0, 2, *or more of the* *linear elementary divisors are imaginary,* 18, 10, *or* 6 *of these thirty straight lines will be real, respectively* (*).

(*) My own papers in Bd. II of these Annals (pp. 198, *et seq.* and 366, *et seq.*, 1869) are a continuation of my dissertation.

Furthermore, one should perhaps confer the following publications:

First, as far as *linear transformations of line coordinates* are concerned: a publication of **Hemming** in the Züricher Veirteljahrsschrift, Bd. 16 (1871), and then, in particular, **Fiedler**'s descriptive geometry (2nd ed., 1875), or the same author's treatment of **Salmon**'s *Space Geometry* (v. I, 3rd ed., 1879).

Furthermore, as far as the *classification of second-degree complexes* is concerned: The treatise of **Weiler** in volume VII of Math. Ann. (1873), and then the more recent investigations of **Segre** (Memorie della R. Accademia di Torino, ser. II, t. 36, 1883). Also linked with this are the special papers of **Hirst** (Collectanea mathematica in memoriam Chelini, 1881, or the Proceedings of the London Mathematical Society, v. 10, 1879), and of **Segre** and **Loria** (**Segre**, resp.) in volume XXIII of these Annalen (1883).

Finally, on the *geometric generation of second-degree complexes:* The doctoral disseration of **Schur** (Berlin, 1879, or also Math. Ann., Bd. XV), **Fiedler**, in volume 24 of Züricher Veirteljahrsschrift (1879), or in volume 2 of **Salmon**'s *Space Geometry* (3rd ed., 1880), **W. Stahl**, in volume 93 of the Jour. f. Math. (1882), **Weiler** in volume 27 of the Zeitschrift für Math. und Physik (1882), as well as volume 95 of the Jour. f. Math. (1883). In particular, the *polar theory* of second-degree complexes was treated from the geometric viewpoint by **Bertini** (Giornale di matematiche, vol. 17, 1879), and then **W. Stahl** (Jour. f. Math., Bd. 94, 1883).