"Über die Integralform der Erhaltungssätze und die Theorie der räumliche-geschlossen Welt," Nachr. d. Kgl. Ges. d. Wiss. zu Göttingen (1918). *Gesammelte mathematische Abhandlungen*, v. I, art. XXXIII.

On the integral form of conservation laws and the theory of the spatially-closed world

By F. Klein

(Presented at the session on 6 December 1918) ([1])

Translated by D. H. Delphenich

In my note on 19 July 1918, I attempted to give an overview of the different forms that one can give to the differential laws for the conservation of impulse and energy in **Einstein**'s theory of gravitation. My problem today shall be, above all, to address the integral form of the conservation laws that **Einstein** presented for the form of the differential laws that he preferred. In connection with that, I will treat **Einstein**'s theory of the spatially-closed universe and the variation on it that **de Sitter** found ([2]). The physical questions will only be touched upon, while the goal is to clarify the *mathematical* issues completely; I feel a certain satisfaction that, in that way, my old ideas from 1871-72 are shown to be valid in a decisive way ([3]). The reader himself might decide how far things have progressed by a comparison with the presentations of the other authors.

I shall next recall the following results: The conservation law, in the form that I referred to as the **Lorentz** form [formula (12) of the previous Note ([†])], reads:

([1]) To be submitted to print at the end of January 1919.

([2]) The publications that come under consideration are:

Einstein. 1. "Kosmologische Betractungen zur allgemeinen Relativitätstheorie," Sitz. d. Berliner Akad. on 8 February 1917.

2. "Kritisches zu einer von Herrn de Sitter gegeben Lösing de Gravitationsgleichungen, *ibidem*, 7 March 1918.

3. "Der Energiesatz in der allgemeinen Relativitätstheorie," *ibidem*, 16 May 1918.

de Sitter. In various articles published in Verslag der Amsterdamer Akademie, 1917, as well as in a series of review articles in the Monthly Notices of the Royal Astronomical Society: "On Einstein's theory of gravitation and its astronomical consequences." (In particular, see the concluding Part III on November 1917.)

([3]) See, in particular:

1. "Über die sogennante Nicht-Euklidische Geometrie," Math. Ann. **4** (1871). [Abh. XVI of this collection.)
2. The Erlanger Program: "Vergleichende Betrachtungen über neuere geometrische Forschungen," Erlangen, 1872. [Abh. XXVII of this collection.]

([†]) Translator: Art. XXXII in *Ges. math. Abh.*

(1) $$\frac{\partial\left(\mathfrak{T}_\tau^\sigma + \frac{1}{\kappa}\mathfrak{A}_\tau^\sigma\right)}{\partial w^\sigma} = 0.$$

If we write:

(2) $$\frac{1}{\kappa}\mathfrak{A}_\tau^\sigma = \mathfrak{t}_\tau^\sigma$$

then we will get the **Einstein** form of the conservation law:

(3) $$\frac{\partial\left(\mathfrak{T}_\tau^\sigma + \mathfrak{t}_\tau^\sigma\right)}{\partial w^\sigma} = 0.$$

[Formula (44) of the previous note ([4])]

Now, it would correspond to **Einstein**'s basic assumption if I were to refer to:

the $\frac{1}{\kappa}\mathfrak{A}_\tau^\sigma$ (the $\frac{1}{\kappa}\mathfrak{A}_\tau^{*\sigma}$, resp.)

briefly as the *gravitational components of the energy (qualified by a fortuitous choice of coordinates the relevant Ansatz).* Furthermore, I would like to denote the components of the "total energy" that one gets in that way briefly by the symbol V ($\mathfrak{V}$, resp.):

(4) $$\mathfrak{T}_\tau^\sigma + \frac{1}{\kappa}\mathfrak{A}_\tau^\sigma = \mathfrak{V}_\tau^\sigma, \qquad \mathfrak{T}_\tau^\sigma + \frac{1}{\kappa}\mathfrak{A}_\tau^{*\sigma} = \mathfrak{V}_\tau^{*\sigma}.$$

I would like to mention in advance a peculiarity of my following presentation, namely, that I will always consider $\mathfrak{A}$ and $\mathfrak{A}^*$ (or also $\mathfrak{V}$ and $\mathfrak{V}^*$) together, since both of them have their advantages. One will then see more clearly the extent to which the conservation laws have acquired a subjective momentum in the integral forms that are presented.

For the convenience of the reader, I shall once more present the basic definitions of the corresponding Latin symbols according to formulas (16), (55) of the previous Note. One has:

(5) $$2U_\tau^\sigma = K\delta_\tau^\sigma - \frac{\partial K}{\partial g_\sigma^{\mu\nu}} g_\tau^{\mu\nu} - \frac{\partial K}{\partial g_{\rho\sigma}^{\mu\nu}} g_{\rho\tau}^{\mu\nu} + \frac{1}{\sqrt{g}} \frac{\partial\left(\frac{\partial\sqrt{g}\,K}{\partial g_{\rho\sigma}^{\mu\nu}}\right)}{\partial w^\rho} g_\tau^{\mu\nu},$$

([4]) [This entire section can be abbreviated essentially once one carries out the sign changes at this point in the first publication that are necessary in the present Note and been incorporated already in the republication of this article. **Vermeil** has directed my attention to the necessity of those sign changes, and he has also assisted me by doing the calculations that are needed for many of the following considerations, for which I am grateful. K.]

(6) $$2U_\tau^* = G^* \delta_\tau^\sigma - \frac{\partial G^*}{\partial g_\sigma^{\mu\nu}} g_\tau^{\mu\nu}.$$

I have employed the square root $\sqrt{g}$ in (5), as I did throughout my previous Note, in connection with **Hilbert**'s original notation. If one would like to employ **Einstein**'s notations throughout then one would have to take $\sqrt{-g}$ everywhere. That change will have no effect on the ultimate formulas (1) to (3). However, it is still preferable if the underlying quantities of direct observation are to have real components throughout; from now on, that shall then be assumed in any case.

I. The integral laws for closed systems in the usual theory.

§ 1.

The vectorial notation for multiple integrals. First introduction of the I_τ (I_τ^*, resp.).

The usual notation – e.g., $\iint f(x, y)\, dx\, dy$ – is not expedient whenever one is dealing with the transformation of multiple integrals. The increments dx, dy are to be thought of as pointing in different directions, so they will belong to two different vectors, such that one would have already gained something if one wrote $\iint f(x, y)\, d'x\, d''y$. The notation will become even clearer when one chooses the vectors d', d'' to be not exactly parallel to the two coordinate axes, but arbitrary, and correspondingly replaces the product $d'x\, d''y$ with the area of the parallelogram that is enclosed between the two vectors. We will then come to the notation:

(7) $$\iint f(x, y) \begin{vmatrix} d'x & d'y \\ d''x & d''y \end{vmatrix},$$

which I like to call the **Grassmann** notation, since it corresponds to the sphere of ideas that was found in **Grassmann**'s *Ausdehnungslehre* of 1861. The formula is better suited to the mobility that we expect of the concept of a multiple integral.

With an eye towards its specialized evaluation, one can obviously return to the usual notation from formula (7) at any moment. However, (7) is to be preferred for all considerations that relate to transformations. For example, if we set $x = \varphi(\xi, \eta)$, $y = \psi(\xi, \eta)$ then it will be immediately clear from (7) why the transformation formula of the integrals includes the **Jacobian** functional determinant. One will then have:

$$f(x, y) \cdot \begin{vmatrix} d'x & d'y \\ d''x & d''y \end{vmatrix} = f(\varphi, \psi) \cdot \begin{vmatrix} \varphi_\xi & \varphi_\eta \\ \psi_\xi & \psi_\eta \end{vmatrix} \cdot \begin{vmatrix} d'x & d'y \\ d''x & d''y \end{vmatrix},$$

identically.

Assuming that, we will now consider certain triple integrals in what follows that can be written thus:

(8) $$I_\tau = \iiint \begin{vmatrix} \mathfrak{V}_\tau^{I} & \cdots & \mathfrak{V}_\tau^{IV} \\ d'w^I & \cdots & d'w^{IV} \\ d''w^I & \cdots & d''w^{IV} \\ d'''w^I & \cdots & d'''w^{IV} \end{vmatrix},$$

or also the other one, which I call:

(9) $$I_\tau^*,$$

which can be obtained from the foregoing one when one replaces the $\mathfrak{V}_\tau^\sigma$ replaces with $\mathfrak{V}_\tau^{*\sigma}$. These integrals are extended over any piece of a "hypersurface" that lies in the four-dimensional world. d', d'', d''' denote three mutually-independent vectors that extend from the individual points of the hypersurface in tangential directions.

One can conclude from the outset from the differential laws (1), (3) that the $\mathfrak{V}_\tau^\sigma$ ($\mathfrak{V}_\tau^{*\sigma}$, resp.) satisfy [while assuming the usual continuity (single-valuedness, resp.) of the $\mathfrak{V}$] that *this I_τ (I_τ^*, resp.) will be zero when one assume that their domain of integration is closed in such a way that it bounds a well-defined subset of the universe.* In fact, I_τ will then be converted in a known way into the fourfold-extended integral over the bounded subset of the universe:

(10) $$I_\tau = \iiint\int \left(\frac{\partial V_\tau^\sigma}{\partial w^\sigma}\right) \begin{vmatrix} dw^I & \cdots & dw^{IV} \\ d'w^I & \cdots & . \\ d''w^I & \cdots & . \\ d'''w^I & \cdots & . \end{vmatrix},$$

and similarly for the I_τ^*, for which the integrands themselves will now vanish with no further assumptions, due to the conservation laws.

However, our particular interest in the subject of how the I_τ, I_τ^* will behave under *affine* transformations of the w, so when one subjects the w to linear transformations with constant coefficients:

(II) $$\bar{w}^\rho = a_1^\rho w^I + \ldots + a_4^\rho w^{IV} + c^\rho.$$

This is where our vectorial notation proves itself. We know from the developments of the previous Note that the V_τ^σ ($V_\tau^{*\sigma}$, resp.) behaves like a mixed tensor under the transformations (11); the $\mathfrak{V}_\tau^\sigma$ ($\mathfrak{V}_\tau^{*\sigma}$, resp.) will increase under them by multiplication by $\sqrt{g}$ ($\sqrt{-g}$, resp.). Hence, it will be clear with no further assumptions that the integrands dI_τ (dI_τ^*, resp.) will transform like "contragredient" vectors. We shall say that they suffer the *homogeneous* linear substitutions:

$$dI_\tau = a_\tau^I d\bar{I}_1 + \ldots + a_\tau^{IV} d\bar{I}_4$$

that are derived from (11). However, the coefficients a in (11) are constant, by assumption. We will then have corresponding substitution formulas for our integral I_τ themselves:

(12) $$I_\tau = a_\tau^I \overline{I}_1 + \dots + a_\tau^{IV} \overline{I}_4$$

(and naturally a similar expression for the I_τ^*), and with that, the result that was to be derived has already been achieved.

However, the conceptual advance that is linked with this formula (12) can be expressed as follows: The dI_τ, dI_τ^*, like all vectors in the general theory of transformations, are intrinsically coupled with a certain world-point w as the starting point, so they are *bound* vectors (or, if we would like to be still more precise: four-vectors). Now, this coupling to a special point will lead back to the transformation formulas for the I_τ, I_τ^* completely. *Better yet, one can refer to the I_τ, I_τ^* as **free** contragredient four-vectors;* i.e., as four-vectors that have only a direction and an intensity ($= \sqrt{\sum g^{\mu\nu} I_\mu I_\nu}$), but no particular position in the four-dimensional world.

Naturally, this concept of free four-vectors is entirely rooted in the fact that we based it upon the group (11) of affine transformations of the w. In physics (mechanics, resp.), things are precisely as I described in my Erlanger Programm for geometry: Namely, that one can first speak of the differentiation of distinct types when one knows the transformation group in which one measures the basic concepts. For decades, I have already been saying that the physicists should consciously try to adopt the basic notion in that, which clarity alone would suggest ([5]). In particular, in 1910, I expressly remarked in my talk on the geometric foundations of the Lorentz group ([6]) that one should never speak casually of the "theory of relativity," but only of the theory of invariants that relate to a group. – There are as many kinds of relativity theories as there are groups ([7]).

The concept that one formulates in that way is in complete contradiction to the opinions that are often propagated currently in connection with **Einstein**'s general explanations, but not to **Einstein**'s own far-reaching detailed developments, upon which I place great value. Moreover, as I have commented in the present Note, **Einstein**'s papers show that in the individual cases, he appealed to precisely the very same freedom in forming ideas that I recommended in my Erlanger Programm, and without systematically fixing the train of thought.

([5]) Cf., *inter alia*, my treatise "Zur Schraubentheorie von Sir Robert Ball" in vol. 47 of this Zeit. f. Mat. u. Phys. (1902) [reprinted in vol. **62** (1906) of Math. Ann. with some extensions.] [Cf., Abh. XXIX of this collection.] (As in this article, no relatively new physical concepts are introduced in it, but only the concepts that many people discuss in more thorough treatments of the individual problems that relate to a clear mathematical principle.]

([6]) Jahresbericht der Deutschen Mathematiker-Vereinigung, **19** (1910), published in the Phys. Zeit. 12th yearly issue, 1911. [Cf., Abh. XXX in this collection.]

([7]) Also confer **Noether**'s notice on "Invariante Variationsprobleme" in the yearly issue of 1918 of the Göttinger Nachrichten (esp., its concluding remarks).

§ 2.

The integrals I_τ, I_τ^* for closed systems.

In the article that was cited in ([3]), **Einstein** understood "closed" system to mean one that "swims," so to speak, in a **Minkowski** space; i.e., a system whose individual parts run through a world-tube, *outside of which* a ds^2 of vanishing **Riemannian** curvature reigns. One can write that ds^2 with constant coefficients [without being required to put it into the typical form $dt^2 - c^{-2}(dx^2 + dy^2 + dz^2)$, moreover]: **Einstein** then spoke of *Galilean* coordinates. As such, the w^2 outside of the world-tube must be chosen once and for all, while it can vary arbitrarily insider of it, assuming that there is a continuous transition. Nothing in particular can be said correspondingly about the $\mathfrak{V}_\tau^\sigma$, $\mathfrak{V}_\tau^{*\sigma}$ inside of the tube, but outside of it, they must be zero in any event. Not only will all $\mathfrak{T}_\tau^\sigma$ vanish there then, but also, as a glance at the defining formulas (5), (6) will show, all $\mathfrak{U}_\tau^\sigma$ ($\mathfrak{U}_\tau^{*\sigma}$, resp.), due to the constancy of the $g_{\mu\nu}$.

We imagine that the interior of the world-tube is naturally filled with a continuous family of world-lines, all of which have been given a positive sense, that correspond to the points of the system. Any world-line will have tangent vectors with the same sense and whose components might be $dw^I, \ldots, dw^{IV}$.

The question now arises of which three-dimensional manifolds (hypersurfaces) one might refer to as "cross-sections" Q of the world-tube. In order to be able to express that more conveniently, in what follows, we will focus exclusively on those cross-sections that are cut by each world-line at only *one* point. Three mutually-independent vectors d', d'', d''' might then be chosen in such a way that the determinant:

$$\begin{vmatrix} dw^I & \cdots & dw^{IV} \\ d'w^I & \cdots & \cdot \\ d''w^I & \cdots & \cdot \\ d'''w^I & \cdots & \cdot \end{vmatrix}$$

keeps a fixed sign. When we link that with the example:

$$\begin{aligned} d &= 0, \ 0, \ 0, \ dt, \\ d' &= dx, \ 0, \ 0, \ 0, \\ d'' &= 0, \ dy, \ 0, \ 0, \\ d''' &= 0, \ 0, \ dz, \ 0, \end{aligned}$$

we choose that sign to be negative, for the sake of convenience.

Assuming that, we define the four integrals:

$$I_\tau \quad (I_\tau^*, \text{resp.})$$

of the previous paragraph for the cross-section.

In close connection with **Einstein**'s development, we will then assert that *those integrals are independent of the choice of cross-section, as well as the choice of coordinates that we might introduce inside the tube.* From the standpoint of the affine transformations of the w that encompass the entire universe, the I_τ (I_τ^*, resp.) likewise define a free contragredient vector. The new theorems that **Einstein** posed say that *these vectors depend upon only the material system as such, but not upon the peculiarities of the analytical representation.*

In order to prove the new theorems, it will suffice in any event to place two cross-sections Q and $\bar{Q}$, one after the other, such that they bound a unified piece of the world-tube (so they do not intersect each other). The general case in which Q and $\bar{Q}$ cross each other can be dealt with easily when one adds a third cross-section (Q) that meets either Q or $\bar{Q}$, and one first combines Q with (Q) and then (Q) with $\bar{Q}$.

Furthermore, that will divide the proof (always in connection with **Einstein**) into two parts:

a) We first imagine that the coordinate system of w inside and outside of the world-tube has been chosen by any sort of prescription. We then imagine that the piece of the tube that is found between Q and $\bar{Q}$ is externally complemented continuously in such a way that it seems to be bounded by a unitary hypersurface that runs through the interior of the tube at Q and $\bar{Q}$. From the previous paragraph, the integrals I_τ, I_τ^*, when they are extended appropriately over that closed hypersurface, will all be zero. However, that part of our hypersurface that overhang the world-tube will contribute nothing at all to those integrals, since the integrands $\mathfrak{V}_\tau^\sigma$, $\mathfrak{V}_\tau^{*\sigma}$ will themselves vanish on it. What will remain are the contributions from the two cross-sections Q and $\bar{Q}$, but when we calculate them according to the aforementioned sign convention, the integrals over the closed hypersurface will have opposite signs. Since their sum will be zero, the aforementioned contributions will be equal to each other. Q. E. D.

b) We now come to see that the I_τ, I_τ^* that pertain to the individual cross-sections will actually remain unchanged under all changes of the w^ρ that vanish outside of the world-tube. We shall do that in such a way that we first think of two kinds of coordinate systems w and $\bar{w}$ as being given inside the tube that both merge into the same external (**Galilean**) coordinate system in a continuous manner. We shall make use of the first one in order to calculate the integrals I_τ, (I_τ^*, resp.) over the cross-section Q, and the other one in order to calculate the integrals over $\bar{Q}$, which might yield the values $\bar{I}_\tau$ ($\bar{I}_\tau^*$, resp.). We need to show that $I_\tau = \bar{I}_\tau$ ($I_\tau^* = \bar{I}_\tau^*$, resp.), and that verification will be achieved when we succeed in introducing a third coordinate determination $\bar{\bar{w}}$ that connects with the w closely enough along Q and similarly connects with the $\bar{w}$ closely enough along $\bar{Q}$, while they yield the **Galilean** coordinates that reign along the surface of the tube and outside of it, as before. In that statement, "closely enough" means that the calculation of the V_τ^σ ($V_\tau^{*\sigma}$, resp.) from the $\bar{\bar{w}}$ over the cross-section Q will yield the same result as when one uses the w, and correspondingly, it will yield the same result over $\bar{Q}$ as when one employs the $\bar{w}$. Due to the presence of the differential quotients of the $g_{\mu\nu}$ in formulas (5), (6) for the definition of the $V_\tau^\sigma, V_\tau^{*\sigma}$, resp., it will suffice in that regard (from an estimate that **Vermeil** made) that the $\bar{\bar{w}}$ should coincide with the w along Q, along with their first three differential quotients, and similarly for $\bar{w}$ along $\bar{Q}$. Now one can obviously satisfy all such conditions that one imposes upon the coordinate determination $\bar{\bar{w}}$ by the following example: One introduces the *equations* that the cross-section Q ($\bar{Q}$, resp.) satisfies in the $\bar{w}$ and the w. Let $f(\bar{w}) = 0$ be the first of those equations, while $\bar{f}(w)$ is the second one. I will then write simply:

$$\bar{\bar{w}} = \frac{[\bar{f}(w)]^4 \cdot w + [f(\bar{w})]^4 \cdot \bar{w}}{[\bar{f}(w)]^4 \cdot w + [f(\bar{w})]^4}, \tag{13}$$

and will have then, in fact, fulfilled all conditions. Our second verification is then achieved, and with it, the proofs of the new theorems are accomplished.

§ 3.

Ultimate definition of free impulse-energy vectors for the closed system.

The I_τ, (I_τ^*, resp.) define the natural foundation for the impulse-energy vectors that are ascribed to the closed system. However, in order to completely establish the latter, it will still be necessary to focus upon the dimensions of the types of quantities that are coupled to each other. On page 569 of my previous Note, it was agreed that ds^2 would have the dimension s^2. We would correspondingly like to assume that the w^ρ that are employed will all have the dimensions s^{+1}. The $g^{\mu\nu}$, $g_{\mu\nu}$, and g are dimensionless, while K, U_τ^σ, $\mathfrak{U}_\tau^\sigma$, $U_\tau^{*\sigma}$, $\mathfrak{U}_\tau^{*\sigma}$ will coincide with the dimension s^{-2}. Since the gravitational constant κ possesses the dimension $g^{-1}\,cm^{+1}$, $\mathfrak{U}_\tau^\sigma/\kappa$, $\mathfrak{U}_\tau^{*\sigma}/\kappa$ will take on the dimension $g^{+1}\,cm^{-1}\,s^{-2}$; i.e., the dimension of a "specific" (viz., referred to the volume unit) energy. That says that they will combine with the $\mathfrak{T}_\tau^\sigma$ additively in the $\mathfrak{V}_\tau^\sigma$, $\mathfrak{V}_\tau^{*\sigma}$.

Now, these $\mathfrak{V}_\tau^\sigma$, $\mathfrak{V}_\tau^{*\sigma}$ will be multiplied by three-rowed determinants under the integral signs I_τ, I_τ^*, and from our convention on the dimension of w, those determinants will themselves have a dimension of s^{+3}. Obviously, I must add a factor of c^3 (c = speed of light) to the I_τ, I_τ^* if I am to arrive at the dimension of an actual energy. *Consistent with that, the ultimate quadruple quantities:*

$$J_\tau = c^3\, I_\tau, \qquad J_\tau^* = c^3\, I_\tau^* \tag{14}$$

shall be referred to as the free impulse-energy vectors of the closed system in question. Numerical factors that might still be dubious shall not be further introduced; this sign convention shall also be fixed.

I can glimpse the proof of the validity of this Ansatz in the fact that **Einstein**'s own definition of the impulse-energy vector is included in the definition of our J_τ^*. In order to see that, we will once more drop the factor c^3 from our definition of the J_τ^* (namely, since **Einstein** based his own definition upon units of measurement that would make c = 1, which is merely a superficial detail for the comparison in question). However – and this is a true specialization – we must then choose the cross-section Q such that it can be represented by the equation $w^{IV} = 0$ by the freedom of coordinate choice that we are allowed. In order to see the significance of this restriction, one argues that the **Galilean** coordinates outside of the world-tube are established up to an affine transformation. The new condition also emerges from the fact that the cross-section Q must be chosen such that it will go through the surface of the world-tube of our system in a structure that will be represented by a *linear* equation when it is viewed from the outside in **Galilean** coordinates that are initially assumed to be arbitrary.

If we would, in fact, like to assume that w^{IV} vanishes along the cross-section then $d'w^{IV}$, $d''w^{IV}$, $d'''w^{IV}$ will vanish *eo ipso.* Our integral I_τ^* will then reduce (when we set c = 1) to:

$$\iiint \mathfrak{B}_\tau^{*4} \begin{vmatrix} d'w^I & \cdots & d'w^{III} \\ d''w^I & \cdots & d''w^{III} \\ d'''w^I & \cdots & d'''w^{III} \end{vmatrix},$$

so when we revert to the usual notations, it will reduce to:

$$J_\tau^* = \iiint \mathfrak{B}_\tau^{*4} dw^I dw^{II} dw^{III}, \tag{15}$$

which is precisely **Einstein**'s formula, up to the choice of symbols.

This formula is indeed undoubtedly simpler than the one that I used as a basis, at least superficially. However, the vector character of the J_τ^*, which **Einstein** asserted, but did not prove rigorously, is harder to see in it. In a lengthy correspondence with **Einstein**, I did not, in fact, originally want to succeed in establishing that vector character until I took up the **Grassmann** notation for the integral that I started with above. However, that also gave me the generalization of the concept of cross-section that I chose.

The essential difference between my representation and **Einstein**'s is that I gave the vector J_τ the same status as the vector J_τ^* by putting the **Lorentzian** $\mathfrak{U}_\tau^\sigma / \kappa$ under the integral sign, instead of **Einstein**'s $\mathfrak{t}_\tau^\sigma = \mathfrak{U}_\tau^{*\sigma} / \kappa$. *We will see directly that the J_τ and the J_τ^* are generally different in an example.* I would even be able to assign infinitely-many different impulse-energy vectors to the closed system then if I were to, e.g., replace $\mathfrak{t}_\tau^\sigma$ with the aggregate $\mathfrak{t}_\tau^\sigma + \lambda\left(\mathfrak{U}_\tau^\sigma / \kappa - \mathfrak{t}_\tau^\sigma\right)$, in which we understand λ to mean any numerical constant. Above all, I would be allowed to replace $\mathfrak{t}_\tau^\sigma$ with any $\mathfrak{U}_\tau^\sigma$ that differs from $\mathfrak{t}_\tau^\sigma$ by only a term of the required dimension that defines a mixed tensor under affine transformations, and which vanishes identically outside of the world-tube, but has a vanishing divergence inside of it. Which of these infinitely-many vectors is to be preferred will remain undecided as long as I demand only the validity of the integral theorem. A decision can be made only when one introduces new grounds for preferring precisely one of the infinitude of forms for the differentials that determine it.

II. Einstein's spatially-closed universe (cylinder universe).

§ 4.

Closed space of constant positive curvature.

In **Einstein**'s note in February 1917, he first suggested only the possibility of a *spherical* space that he immediately cut out of a four-dimensional manifold (= ξ, η, ζ, ω) whose arc length element is given by the equation:

(16) $$d\sigma^2 = d\xi^2 + d\eta^2 + d\zeta^2 + d\omega^2$$

by way of the "sphere equation" ([8]):

(17) $$\xi^2 + \eta^2 + \zeta^2 + \omega^2 = R^2 .$$

For those who are familiar with geometric literature, it is probably self-evident that I refer to **Einstein**'s investigations into non-Euclidian geometry at the same time as my own older work from 1871, according to which, along with the spherical space form, there was another closed space form of constant positive curvature, namely, *elliptic* space (as it was called in connection with my other considerations at the time). One obtains it from the spherical space when one simply combines any two diametrically-opposite points on the sphere with contacting linear space by central projection. We might accordingly set:

(18) $$x = R\frac{\xi}{\omega}, \qquad y = R\frac{\eta}{\omega}, \qquad z = R\frac{\zeta}{\omega}.$$

Inversely, one then has:

(19) $$\xi = \frac{Rx}{\sqrt{x^2+y^2+z^2+R^2}}, \quad \eta = \ldots, \quad \zeta = \ldots, \quad \omega = \frac{R^2}{\sqrt{x^2+y^2+z^2+R^2}}.$$

The elliptic space is simpler than the spherical one in that one represents its geodetic lines roughly as *straight* lines [which always intersect at only *one* point when they meet

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([8]) From now on, "space" shall consistently mean a three-dimensional domain (that is contained in the four-dimensional "world").

at all ([9])]. The length of such a geodetic line is $R\pi$, the total volume of space is $R^3\,\pi^2$ (instead of $2R\pi$ and $2\,R^3\,\pi^2$, resp., in the spherical case).

Naturally, the difference between the two space forms does not emerge merely by being given the arc-length element ([10]). I can use the value of $d\sigma^2$ that is given by (16) and (17) for elliptic space, as well as its value:

$$d\sigma^2 = \frac{R^2}{(x^2+y^2+z^2+R^2)^2}\{R^2\,(dx^2+dy^2+dz^2)+(y\,dz-z\,dy)^2 + (z\,dx-x\,dz)^2+(x\,dy-y\,dx)^2\} \tag{20}$$

that is calculated in terms of x, y, z in spherical space, or also the value that can be expressed in polar coordinates in both cases by:

$$d\sigma^2 = R^2\,(d\vartheta^2+\sin^2\vartheta\cdot d\varphi^2+\sin^2\vartheta\sin^2\varphi\cdot d\varphi^2). \tag{21}$$

§ 5.

Einstein's "cylinder world" and its group.

Furthermore, as in the previous paragraphs (and also without specifying the coordinate system), $d\sigma^2$ shall briefly mean the square of the arc-length element of a closed space of constant curvature $1 / R^2$, which might be assumed to be spherical or elliptical now. The ascent to **Einstein**'s spatially-closed world will then be accomplished simply by setting:

$$ds^2 = dt^2 - \frac{d\sigma^2}{c^2}, \tag{22}$$

([9]) For that reason, elliptic space emerges when one starts with the basic concepts of projective geometry, which I did in 1871. It will then be placed directly alongside hyperbolic space (the space of **Bolyai** and **Lobachevski**) and parabolic space (viz., Euclidian space), and it is a fundamental misunderstanding when one refers to formulas (18) as a "map" of spherical space to "Euclidian" space, as many authors still do. The totality of all systems of values of three variables x, y, z will first become "Euclidian" when we add the differential form $dx^2 + dy^2 + dz^2$, or in terms of group theory, when we replace the totality of projective transformations of the x, y, z (whose invariant theory is projective geometry) with the subgroup of those transformation that leave the stated differential form unchanged. I shall speak of all these things in the present article, although they are sufficiently well-known elsewhere, since this article is also intended for physicists, and these ideas still seem to have been disseminated only slightly in physics circles, which is backlash from the one-sided **Helmholtz** tradition that goes back to 1868.

([10]) In fact, the "connectivity" that the associated space form exhibits in the large is still not determined when one is given the $d\sigma^2$. That fact is often still not observed in the contemporary literature. I have treated the relevant behavior for spaces of constant curvature thoroughly in a treatise from 1890 [Math. Ann., v. 37 (see Article XXI of this collection)]. For textbooks on that subject, the text of **Killing** (*Einführung in die Grundlagen der Geometrie*, Part I, 1893), in particular, goes into that. I would also like to cite the recent publications of **Hadamard** and **Weyl**.

and letting t vary from $-\infty$ to $+\infty$ (exclusively of the limits). If we then formally calculate the curvature scalar for the space t = const. then we will get $-c^2 / R^2$. That negative sign naturally corresponds to just the fact that the ds for the aforementioned space that was introduced in (22) was pure imaginary. No contradiction exists then with the previous paragraphs, in which we referred to space briefly as one of constant positive curvature.

Our main problem is to find what the largest continuous group of coordinate transformations that takes the ds^2 in (22) to itself would be.

It is clear from the outset that, at the very least, a G_7 of such transformation exists. There is already a continuous G_6 that takes $d\sigma^2$ to itself. In order to link that with (16), it is the totality of orthogonal transformations of the ξ, η, ζ, ω that have determinant + 1. One must then include the G_1 that corresponds to a lengthening of t by an arbitrary constant. The G_7 that is obtained in that way is certainly transitive; i.e., one take any world-point to any other one by means of it – for example, to the point $t = 0$, $\vartheta = 0$ [in order to make use of the polar coordinate system that was introduced in (21)]; one might briefly call that point O. A continuous G_3 of spatial rotations around O is still possible.

We now assert that *there is also no continuous group of coordinate transformations that takes ds^2 to itself that is larger than our G_7* . To that end, it will suffice to show that for a fixed O, only the aforementioned G_3 of rotations will exist. In order to prove that, we introduce "Riemannian normal coordinates" that emanate from O. We can accomplish that, for example, when we regard t as variable and introduce:

$$(23) \qquad y_1 = \frac{R}{c}\,\vartheta\cdot\cos\varphi, \qquad y_2 = \frac{R}{c}\,\vartheta\cdot\sin\varphi\cos\psi, \qquad y_3 = \frac{R}{c}\,\vartheta\cdot\sin\varphi\sin\psi,$$

in place of the polar coordinates ϑ, φ, ψ. If we write y_4 for t, for the sake of consistency, then we will get ds^2 in the form:

$$(24) \qquad ds^2 = (dy_4^2 - dy_1^2 - dy_2^2 - dy_3^2) + \frac{c^2}{3R^2}\sum_{1,2,3}(y_i\,dy_\kappa - y_\kappa\,dy_i)^2$$

$$+ \text{ terms of higher order in the } y_1, y_2, y_3,$$

which shows that we are, in fact, dealing with normal coordinates. Now, as far as the transformations of ds^2 to itself are concerned, since O is fixed, according to the general theory of normal coordinates, we will only have ask what the largest continuous group of *homogeneous linear* substitutions of the y would be that would convert this ds^2 into itself. The two terms of ds^2 that were written down must then each go to itself in its own right, for the sake of dimensions. It will then be clear that y_4 must remain unchanged, while y_1, y_2, y_3 can be subject to at most the continuous group of ternary orthogonal substitutions of determinant 1. However, with that, we have already reached our goal.

From the theorem that was proved in that way, it will perhaps be permissible to refer to **Einstein**'s spatially-closed world briefly as the *cylinder world*, since it possesses the symmetry of a cylinder of rotation, so-to-speak: viz., arbitrary displacements along the t-axis and arbitrary rotations around O for a fixed t. Naturally, the analogy is not complete, since one can just as well rotate around any other point (than O). I would also not like to introduce a permanent term, but only have a brief, *ad hoc*, expression that might exhibit the contrast with **de Sitter**'s hypothesis B that will be treated in the next section.

Furthermore, we might say that in the present case, once we have agreed upon the unit of time and the starting point for the time direction, *the concept of time will no longer contain any arbitrariness* ([11]), or if one would prefer, that inside of the four-dimensional world, *the triply-extended spaces t = const. are manifolds, sui generis.* Hence, they are a remarkable approximation to the way of describing things in classical mechanics.

That should be obvious from the outset when one ponders the physical argument by which **Einstein** introduced the cylinder world. Namely, in order to comprehend the totality of mass distributions and events in world from a higher standpoint, **Einstein** initially fabricated a mean state in which the totality of masses in the space that is assumed to be closed is *incoherent* and *uniformly distributed*, and inside of that space, it is *at rest* while t runs from $-\infty$ to $+\infty$. The actual mass distribution and events shall be regarded as deviations from that mean state. Time (or more precisely the time difference between two world-points when measured in the agreed-upon unit) is then something absolute *eo ipso* when measured in that mean state, while space is intrinsically homogeneous ([12]). However, this concept finds its precise mathematical expression in the invariant theory of our G_7.

It is particularly interesting to see how our G_7 can be extended to the *Lorentz group* G_{10}, so one will come to *picture of the "special" theory of relativity* when one takes the curvature scalar of our space to vanish; i.e., one sets $R = \infty$. Our ds^2 (22) will then, in fact, reduce to only its first term ([13]): $dy_4^2 - dy_1^2 - dy_2^2 - dy_3^2$, and will then remain unchanged by all homogeneous linear substitutions of the dy_1, dy_2, dy_3, dy_4 that transform this *individual* quadratic form into itself. *In that way, $y_4 = t$ ceases to be a variable that stands alone, and will combine with the y_1, y_2, y_3 under allowable substitutions, since that is precisely the essence of the special theory of relativity.*

§ 6.

The field equations of the cylinder world.

We must still confirm that the assumption of matter at rest filling up all of space uniformly – say, with a constant density ρ – is, in fact, compatible with the **Einstein** field equations that are posed for our ds^2. Naturally, we mean by that the field equations "with the λ term," which I spoke of already in my previous Note [formula (57)]:

$$K_{\mu\nu} - \lambda\, g_{\mu\nu} - \kappa T_{\mu\nu} = 0. \tag{25}$$

([11]) This was also noted in **de Sitter**, *loc. cit.*

([12]) **Einstein**, e.g., accepted the fact that space can then be assumed to be spherical or elliptical as one desires with no further assumptions. Moreover, **de Sitter** also always treated those two assumptions together, and similarly **Weyl**'s new book (*Raum, Zeit, Materie*).

([13]) Not only does the second term drop out, but so do all higher terms.

Since the distribution of matter in space should be everywhere uniform, it will suffice to verify that compatibility at the point O. Since we are dealing with a relation between tensor components, we can also base our verification upon the ds^2 (24) that is written in normal coordinates from the outset.

However, when one starts from that point, one will find, without any detailed calculation, [cf., the note of **Vermeil** in the Göttinger Nachrichten of 26 October 1917, "Notiz über das mittlere Krümmungsmass einer n-fach ausgedehnten **Riemannian** Mannigfaltigkeit"):

$$K_{11} = K_{22} = K_{33} = -\frac{c^2}{R^2}, \qquad K_{44} = \frac{3c^2}{R^2}, \tag{26}$$

while all other $K_{\mu\nu}$ will vanish.

Now, when one bases the calculation upon normal coordinates, one will have:

$$\text{all } T_{\mu\nu} = 0, \qquad \text{up to } T_{44} = c^2 \rho \tag{27}$$

at the point O. The field equations (25) will then yield:

$$-\frac{c^2}{R^2} + \lambda = 0, \quad \frac{3c^2}{R^2} - \lambda - k\, c^2 \rho = 0\,;$$

i.e.:

$$\lambda = \frac{c^2}{R^2}, \qquad \rho = \frac{2}{\kappa R^2}, \tag{28}$$

which agrees with the result that **Einstein** himself gave (as long as one sets $c^2 = 1$).

In regard to that, we remark that we calculate the following constant value for K itself:

$$K = \frac{6c^2}{R^2}. \tag{29}$$

Naturally, in order to apply this to the universe, given our current knowledge of stellar astronomy, it still remains for us to estimate the corresponding value of R with any likelihood. **De Sitter** did that in his oft-cited article. I would like to quote his result so that one can see that **Einstein**'s cosmological consideration, whose mathematical content is all that we shall deal with here, is not left hanging, physically speaking. From **de Sitter**, one must take:

$$R = 10^{12} \text{ to } 10^{13} \text{ radii of Earth's orbit.}$$

The density ρ is so slight that only, perhaps, 10^{-26} grams of mass are found in a cubic centimeter; i.e., one will find the mass of one hydrogen molecule in about 100 cubic centimeters. However, constant λ will be incidentally 10^{-30} s^{-2}.

§ 7.

The integral laws for the cylinder world.

If one takes the field equations with the λ term then one must replace U_τ^σ and $t_\tau^\sigma = U_\tau^{*\sigma} / \kappa$ with:

(30) $$\bar{U}_\tau^\sigma = U_\tau^\sigma + \lambda\,\delta_\tau^\sigma, \qquad \bar{t}_\tau^\sigma = t_\tau^\sigma + \frac{\lambda}{\kappa}\delta_\tau^\sigma,$$

resp., as I did in § 7 of my previous Note in connection with **Einstein**'s analysis, in order for the conservation law to remain true. We will correspondingly take the integral $\bar{I}_\tau$ ($\bar{I}_\tau^*$, resp.), instead of I_τ (I_τ^*, resp.), and we will be certain from the outset that this integral will vanish when it is taken over a closed hypersurface that bounds a piece of the cylinder world.

Furthermore, the concept of *cross-section*, which we employ for the "world-tube" that is considered in that case, must be adapted. As such, we would like for that to refer to an otherwise-arbitrary closed hypersurface that cuts every world-line – i.e., every parallel to the t-axis – exactly once. The simplest example is given by the "space" t = constant.

As before, we will then have the double theorem:

1. When the integral $\bar{I}_\tau$ ($\bar{I}_\tau^*$, resp.) is taken over an arbitrary cross-section, it will have a value that is independent of that choice.

2. That value does not depend upon which coordinates one employs for evaluating that integral of the cross-section, either.

The only thing that will change is the fact that it will no longer be true that set of integrals $\bar{I}_\tau$ ($\bar{I}_\tau^*$, resp.) can be referred to as a (free) four-vector. The reason for that breakdown has its roots in group theory, according to the nature of our G_7.

$\bar{I}_4$ ($\bar{I}_4^*$, resp.) stands alone innately. We might refer to its value, when multiplied by c^3, as the *total energy of the cylinder world.*

However, we do not need to worry very much about the classification of the quantities $\bar{I}_1$, $\bar{I}_2$, $\bar{I}_3$ ($\bar{I}_1^*$, $\bar{I}_2^*$, $\bar{I}_3^*$, resp.), since one can convince oneself in various ways that *they are all zero.*

First of all, (as **Einstein** also showed relative to $\bar{I}_\tau^*$) that results on the grounds of symmetry. When we fix the normal coordinates y of the ∞^6 continuous transformations that take the space $y_4 = 0$ to itself, naturally, only ∞^3 of them represent homogeneous linear substitutions of the y_1, y_2, y_3 that produce rotations of space about O. However, for our purposes, it will also suffice for us to consider the subgroup that they define. Relative to it, the $\bar{U}_1^\sigma$, $\bar{U}_2^\sigma$, $\bar{U}_3^\sigma$ (and likewise, the $\bar{U}_1^{*\sigma}$, $\bar{U}_2^{*\sigma}$, $\bar{U}_3^{*\sigma}$) will behave like the components of a three-dimensional tensor, so the $\bar{I}_1$, $\bar{I}_2$, $\bar{I}_3$ ($\bar{I}_1^*$, $\bar{I}_2^*$, $\bar{I}_3^*$, resp.) will behave like the components of a three-vector that is based at *O*. However, as we know, the cylinder world is spatially-isotropic around *O*. The aforementioned three-vector must then remain unchanged under an arbitrary spatial relation about *O*, and that can happen only when all of its components vanish.

Secondly, we might go down the path of direct calculation. We choose the cross-section over which our integral is extended to be any manifold y_4 = const. Inside of it, one might think of introducing any coordinates w^I, w^{II}, w^{III}. According to the explanation in § **3**, one can then write the integrals $\bar{I}_\tau$ ($\bar{I}_\tau^*$, resp.) as:

(31) $$\bar{I}_\tau = \iiint \left(T_\tau^4 + \frac{1}{\kappa} \bar{U}_\tau^4 \right) \sqrt{-g} \cdot dw^I \, dw^{II} dw^{III}$$

or

(31*) $$\bar{I}_\tau^* = \iiint \left(T_\tau^4 + \bar{t}_\tau^4 \right) \sqrt{-g} \cdot dw^I \, dw^{II} dw^{III} ,$$

resp. Now, direct calculation will imply that the T_τ^σ, $\bar{U}_\tau^4$, $\bar{t}_\tau^4$ will all vanish for τ = 1, 2, 3.

We have obtained the expression for the *total energy of the cylinder world* in these formulas:

(32) $$\bar{J}_4 = c^3 \iiint \left(T_4^4 + \frac{1}{\kappa} \bar{U}_4^4 \right) \sqrt{-g} \cdot dw^I \, dw^{II} dw^{III}$$

or

(32*) $$\bar{J}_\tau^* = c^3 \iiint \left(T_\tau^4 + \bar{t}_\tau^4 \right) \sqrt{-g} \cdot dw^I \, dw^{II} dw^{III} ,$$

resp. The energy content is then represented as the sum of two summands in either of the two cases. We might refer to the summand that corresponds to T_4^4 as the *mass energy* and the other as *gravitational energy*.

One can now calculate the mass energy with no further assumptions. Namely, T_4^4 will equal $c^2\rho$, no matter how we might choose w^I, w^{II}, w^{III}, and $c^3\sqrt{-g}\ dw^I\, dw^{II}\, dw^{III}$ will be nothing but the volume element *dV* of our space y_4 = const. *The mass energy will then be simply* $c^2\rho\, V$, in which we understand *V* to mean the total volume of space, so it will be $2\pi^2R^3$ or π^2R^3, according to whether we would like to assume the spherical hypothesis or the elliptic one.

*However, in **Einstein**'s case, he found that the gravitational energy was zero,* and thus, when one starts with formula (32*) and employs spatial polar coordinates. In that case, one will have $dV = \sin^2 \vartheta \sin \varphi \cdot d\vartheta\, d\varphi\, d\psi$, $\bar{t}_4^4$ will be (when I assemble **Einstein**'s terms) $\cos 2\vartheta / \sin^2 \vartheta$, and the result of the integration will be zero, since $\int \cos 2\vartheta \cdot d\vartheta$ is taken from 0 to π. That result is certainly very remarkable. Since it must be independent of the choice of w^I, w^{II}, w^{III}, one asks whether one might introduce more preferable coordinates in place of the polar coordinates, which bring with them a lengthy mechanical calculation (which **Einstein** only suggested). I would like to propose that one should operate exclusively with the supernumerary coordinates ξ, η, ζ, ω of § 4 (between which, the dependency $\xi^2 + \eta^2 + \zeta^2 + \omega^2 = R^2$ will then exist). Naturally, one must therefore generalize the basic formulas of tensor analysis to the case of dependent coordinates, but that would mean all Ansätze in the literature. I suspect that after one has carried out that conversion, not only will the integral of the gravitational energy over all spatial volume elements vanish, but also the differentials that correspond to the individual volume elements, which might bring about a better insight into the simplicity of **Einstein**'s results.

So much for $\bar{t}_4^4$. The new idea that I now have to develop is that *we will get an entirely different result* (and with no complicated calculations) *when we choose* $\bar{U}_4^4 / \kappa$ *instead of* $\bar{t}_4^4$, *and therefore choose* $\bar{J}_4^*$ *in place of* $\bar{J}_4$. As we know, $\bar{U}_4^4 = U_4^4 + \lambda$. If we once more go back to the formula for U_4^4 that was quoted in (5) above then that will show that in the case of the cylinder world, all terms except for the first one will drop out for an arbitrary choice of w^I, w^{II}, w^{III}. U_4^4 will be simply $= K / 2$, so:

$$\bar{U}_4^4 = \tfrac{1}{2}K + \lambda = \frac{4c^2}{R^2}. \tag{33}$$

It will then have a constant, but non-vanishing, value. As a result, if we base that value upon the U_τ^σ then *the gravitational energy of the cylinder world will not be roughly zero, but twice as large as the mass energy.*

The state of affairs that is established in that way is obviously meaningful enough in the case of the cylinder world. It gives an example in which the energy components U_τ^σ / κ will generally give different results than the t_τ^σ, even for the *integral forms* of the conservation laws. That is what I called the interplay of a subjective moment in the establishment of energy balance in the introduction, and whose importance for closed systems I explained in more detail at the end of § **3**. The result is in no way wonderful in itself, but it still contradicts the impression that one gets from a first reading of **Einstein**'s note that the t_τ^σ have some exclusive legal title to the claim of leading to simple integral theorems.

III. On de Sitter's hypothesis *B*.

In his oft-cited publications – in particular, in Note 3 of the Monthly Notices – **de Sitter** modified the assumption of a cylinder world, which he referred to as Hypothesis *A*, *inter alia*, in such a way that he posed a world of *constant curvature* instead of the cylinder world (which preserving the characteristic sign for ds^2). That was the hypothesis that he called *B* ([14]). I pose the problem of cogently describing the behavior that comes

([14]) **de Sitter** remarked that this assumption (which the mathematicians recommend, due to its symmetry) was first proposed by **Ehrenfest**. In my talk in early 1917 (a small number of exemplars of whose write-up have been distributed), in which I referred to **Einstein**'s "Kosmologische Betrachtungen" that had just appeared at the time (although the formulas were not precisely comparable), I myself made the same Ansatz arbitrarily, and then later when I wrote about the physical consequences, I wondered whether the result might naturally agree with the one that **Einstein** gave for his cylinder world.

about in that way by means of the simplest-possible formulas. Moreover, one will already find the essence of my arguments in the *Protokollen über die Sitzungen der Göttinger Mathematischen Gesellschaft* in the Summer of 1918, which were published in the October 1918 volume of the *Jahresberichts der Deutschen Mathematiker-Vereinigung* (oblique pages pp. 42-44). Cf., also an article by the Amsterdamer Akademie (published on 29 Sept. 1918).

§ 8.

The geometric foundations of the universe of constant curvature.

We are justified in assuming that the world is a manifold of constant curvature in a simple way when we write down the usual equation for a sphere in five variables *with one sign changed*, and measure things in a Euclidian way on this "pseudo-sphere" ([15]). As a result, in order to comply with the previous conventions in regard to dimension, we will, however, call R / c the radius, not R. For the sake of consistency, we shall likewise invert the usual sign of ds^2. I shall then write the equation of the pseudo-sphere as:

(34) $$\xi^2 + \eta^2 + \zeta^2 - \upsilon^2 + \omega^2 = \frac{R^2}{c^2}$$

and the associated ds^2 as:

(35) $$-ds^2 = d\xi^2 + d\eta^2 + d\zeta^2 - d\upsilon^2 + d\omega^2.$$

Due to the minus sign that the ds^2 is affected with, the *pseudo-spherical world* (ξ, η, ζ, υ, ω) that is given by that will have the constant (**Riemannian**) curvature scalar $-c^2 / R^2$. Moreover, it will go to itself under a continuous G_{10} of "pseudo-orthogonal" substitutions – i.e., linear homogeneous substitutions of the ξ, η, ζ, υ, ω – but not, as one can easily verify, under a more extensive group.

Along with it, we will then likewise define a *pseudo-elliptical world* when we write:

(36) $$x = \frac{R}{c}\cdot\frac{\xi}{\omega}, \qquad y = \frac{R}{c}\cdot\frac{\eta}{\omega}, \qquad z = \frac{R}{c}\cdot\frac{\zeta}{\omega}, \qquad u = \frac{R}{c}\cdot\frac{\upsilon}{\omega},$$

while preserving the ds^2 that was given in (35), which inverts to:

(37) $$\xi = \frac{Rx}{c\sqrt{x^2+y^2+z^2-u^2+R^2/c^2}}, \qquad \eta = \ldots, \qquad \zeta = \ldots, \qquad \upsilon = \ldots,$$

$$\omega = \frac{R^2}{c\sqrt{x^2+y^2+z^2-u^2+R^2/c^2}}.$$

([15]) The prefix "pseudo" shall always refer to the appearance of an altered sign.

We can use these ξ, η, ζ, υ, ω to homogenize the equations (which we shall do many times) in the treatment of the pseudo-elliptical world. We remark that as long as we restrict ourselves to real values of the original coordinates ξ, …, ω, which is obvious:

$$x^2 + y^2 + z^2 - u^2 + \frac{R^2}{c^2} = \frac{R^2}{c^2}\frac{(\xi^2 + \eta^2 + \zeta^2 - \upsilon^2 + \omega^2)}{\omega^2} = \frac{R^2}{c^4\omega^2} \tag{38}$$

will always be positive.

For the sake of brevity, we will speak of only this pseudo-elliptical world (and thus drop the pseudo-spherical one), and I must already beg the reader to allow me to appeal to only *projective* notions, which are the only ones that will be justified for the relationships that come under consideration. In that regard, I will present a series of statements that should be obvious to the trained geometer:

1. In the pseudo-elliptical world, one deals with a *projective metric*, whose fundamental structure is given by:

$$x^2 + y^2 + z^2 - u^2 + \frac{R^2}{c^2} = 0, \tag{39}$$

which can henceforth be referred to briefly as a (two-sheeted) hyperboloid, by analogy. From the sign convention in (38), we will find ourselves *between* the sheets of that hyperboloid (i.e., in the part of the world that runs along the real tangent cones to the hyperboloid), which agrees with the indefinite character of our ds^2. When written in homogeneous coordinates ξ, …, the equation of the hyperboloid will read:

$$\xi^2 + \eta^2 + \zeta^2 - \upsilon^2 + \omega^2 = 0, \tag{40}$$

so the hyperboloid will be the intersection of the *asymptotic cone* of our pseudo-sphere with our domain of x, y, z, u.

2. The continuous family of pseudo-orthogonal substitutions of the ξ, η, … yields the largest continuous group of collineations of the x, y, z, u that take our hyperboloid to itself.

3. New structures that are represented by a single linear equation in the x, y, z, u (the corresponding homogeneous equation in the ξ, η, …, resp.) are called simply *spaces*.

4. Spaces that cut the fundamental hyperboloid only in imaginary points (such as, e.g., $u = 0$) will exhibit simply elliptic metrics, and will thus have finite extent. One can then refer to our world as "spatially closed" and put it directly alongside **Einstein**'s cylinder world.

5. Along with those spaces, one will also find spaces that contact the hyperboloid at a point as limiting cases; e.g., the spaces:

(41) $$u = \pm\frac{R}{c}, \qquad \text{or, what amounts to the same thing,} \qquad \upsilon \mp \omega = 0.$$

Such spaces might be briefly called "tangential spaces."

6. Any two tangential spaces bound a connected subset of the world from the projective standpoint, into whose interior the hyperboloid does not penetrate, and that one prefers to call a *double wedge,* from its form in the projective context. That double wedge protrudes on two sides into the still-two-dimensional domain that is common to the two tangential spaces and that one can therefore appropriately call the *double edge* (of the wedge).

7. One can glimpse this state of affairs most simply when one considers the two tangential spaces of no. 5 (which one can take to be any pair of tangential spaces in ∞^4 ways by means of the G_{10} of our collineations). The double wedge will then subsume the points for which:

(42) $$-\frac{R}{c} < u < +\frac{R}{c}, \qquad \text{i.e.,} \qquad -1 < \frac{\upsilon}{\omega} < +1.$$

The edge will be defined by those points for which u is undetermined, so for the υ and ω that vanish simultaneously (so x, y, z will become infinite).

8. According to the theory of projective metrics, any such double wedge will give one the right to introduce a real *pseudo-angle* for any two elliptical spaces that are contained within its edge.

9. For the sake of clarity, I will relate this to the example (41), (42). Two associated (viz., their entire extent is contained in the double wedge) elliptic spaces will then be given by equations:

(43) $$u = u_1, \quad u = u_2, \qquad \left(\frac{\upsilon}{\omega} = \frac{\upsilon_1}{\omega_1}, \frac{\upsilon}{\omega} = \frac{\upsilon_2}{\omega_2}, \text{resp.}\right)$$

(in which u_1 and u_2 lie between $\pm R / c$ and υ_1 / ω_1, υ_2 / ω_2 lie between ± 1). They define two mutually-inverse double ratios with the *sides* of the double wedge – i.e., the two tangential spaces (42) – of which, we would like to pick, say:

(44) $$Dv = \frac{u_1 + R/c}{u_1 - R/c}\cdot\frac{u_2 - R/c}{u_2 + R/c} = \frac{\upsilon_1 + \omega_1}{\upsilon_1 - \omega_1}\cdot\frac{\upsilon_2 - \omega_2}{\upsilon_2 + \omega_2}.$$

One will then define the logarithm of this double ratio, multiplied by any real constant A, to be the pseudo-angle between the two elliptic spaces (43).

10. In regard to **de Sitter**'s analysis, we would like to take $A = R / 2c$ and set $u_2 = 0$, moreover; i.e., to make the pseudo-angle begin from $u = 0$. If we now drop the index on u_1, υ_1, ω_1 then *we will get the defining formula for the pseudo-angle:*

(45) $$\varphi = \frac{R}{2c}\log\frac{R/c+u}{R/c-u} = \frac{R}{2c}\log\frac{\omega+\upsilon}{\omega-\upsilon},$$

and see clearly how it increases from $-\infty$ to $+\infty$ when u goes from $-R/c$ to R/c; i.e., it ranges over the entire double wedge.

11. Naturally, φ will be completely undetermined for the points of the edge itself, where ω and υ will vanish simultaneously. For the general analytical picture, one will have no other singularity to deal with than the one in the polar angle φ at the origin of an ordinary planar (polar) coordinate system. However, the two absolute directions that are the basis for the angle determination (in the sense of projective theory), which are imaginary in the usual case, will be real from (45) ([16]).

§ 9.

Introduction of matter and time.

We now imagine that our ds^2 (35) is expressed in terms of four independent, temporarily-arbitrary, parameters w (which we can take to be our x, y, z, u):

(46) $$ds^2 = \sum g_{\mu\nu}\, dw^\mu dw^\nu .$$

Since we know that this ds^2 has constant **Riemann** scalar curvature, we can write down the associated $K_{\mu\nu}$ directly using the analysis of **Herglotz** ([17]):

(47) $$K_{\mu\nu} = \frac{3c^2}{R^2}\cdot g_{\mu\nu}.$$

We will then satisfy **Einstein**'s field equations with the λ term:

(48) $$K_{\mu\nu} - \lambda\, g_{\mu\nu} - \kappa T_{\mu\nu} = 0$$

when we set:

(49) $$\lambda = \frac{3c^2}{R^2} \qquad \text{and all} \qquad T_{\mu\nu} = 0;$$

([16]) The reader who wishes to go further into the matters in nos. 8-11 might confer my older presentations in vol. 4 of Math. Ann. [see Abd. XVI in this collection] (in which the relationships and arguments that come under consideration are described in full rigor).

([17]) Sächsiche Berichte of 1916, pp. 202.

i.e., *we assume that there is no matter at all.* Later on, we will also see that we will be necessarily led to that assumption when we start from the assumption of a world that is uniformly filled with incoherent matter "at rest" for a suitable introduction of a time t. In fact, **de Sitter** also came to that result, except that he expressed it somewhat differently, as one might see in the cited place.

Naturally, we do not at all diverge from **Einstein**'s original intention with this formula (49), which start from the idea that one was provided with a mean structure of the world in which one had a uniform distribution of matter in space. However, we also find ourselves contradicting another basic law of **Einstein**, at least formally, according to which, equations (48) should give no solutions besides zero unless one assumes that there is matter (cf., **Einstein**'s note on March 1918 that was cited above). That basic law of **Einstein** undoubtedly grew out of physical arguments originally, but it has an intrinsically mathematical nature, so it will be contradicted by the very existence of our ds^2 (46) (which **Einstein** himself occasionally brought to my attention in correspondence). Generally, one can remark that the $g_{\mu\nu}$ of this ds^2 (one performs the calculations for, say, x, y, z, u) will become infinite along the fundamental hyperboloid, which can be regarded as equivalent to the absence of matter at the non-singular points of the world.

We shall now address the introduction of a suitable "time" t (which we can choose to be w^{IV}). According to **Einstein**'s way of looking at things, the starting point for that must be the idea that the world that we seek shall be capable of being regarded as a *static* system; i.e., that ds^2 shall remain unchanged when one increases $w^{IV} = t$ by an arbitrary constant, while leaving w^I, w^{II}, w^{III} fixed. Hence, it shall be included in the one-parameter group:

$$\bar{w}^I = w^I, \qquad \bar{w}^{II} = w^{II}, \qquad \bar{w}^{III} = w^{III}, \qquad \bar{w}^{IV} = w^{IV} + C \tag{50}$$

in the ten-parameter group that takes our ds^2 to itself. Some geometric conclusions will suffice to see that such a one-parameter group must mean the same thing as an advancing rotation of our pseudo-elliptical world around a fixed, two-dimensional axis, so (with a suitable choice of time unit) *t must coincide with the pseudo-angle of a double wedge, as it was defined in* (45), *up to an additive constant.* When we understand $\upsilon = 0$, $\omega = 0$ to mean, as it previously did, any two tangential spaces to the fundamental hyperboloid and assign no value to the additive constant, we will then have to take:

$$t = \frac{R}{2c}\log\frac{\omega+\upsilon}{\omega-\upsilon}. \tag{51}$$

Now, there are ∞^6 such pairs of tangential spaces. *We will then have* ∞^6 *ways of introducing t according to* (51), in contrast to the cylinder world, in which t is established completely up to an additive constant, and also in contrast to the special theory of relativity (viz., the Lorentz group), in which t contains three arbitrary parameters (always while fixing the time unit and the starting point).

We shall next confirm that we will come to precisely the ds^2 that **de Sitter** based his Hypothesis B upon with (51). Namely, with the use of spatial polar coordinates, **de Sitter** (as long as I likewise employ the symbols that I used before and also take ds^2 to have the sign that was suggested previously) wrote:

(52) $$-ds^2 = \frac{R^2}{c^2}(d\vartheta^2 + \sin^2\vartheta \cdot d\varphi^2 + \sin^2\vartheta\,\sin^2\varphi \cdot d\psi^2) - \cos^2\vartheta \cdot dt^2,$$

and that ds^2 will arise from the one that was placed at the forefront in (35):

$$-ds^2 = d\xi^2 + d\eta^2 + d\zeta^2 - d\upsilon^2 + d\omega^2,$$

when I set simply:

(53) $$\begin{cases} \xi = \dfrac{R}{c}\sin\vartheta\cos\varphi, & \eta = \dfrac{R}{c}\sin\vartheta\sin\varphi\cos\psi, \\ \zeta = \dfrac{R}{c}\sin\vartheta\sin\varphi\sin\psi, & \upsilon = \dfrac{R}{c}\cos\vartheta\sinh\dfrac{ct}{R}, \\ \omega = \dfrac{R}{c}\cos\vartheta\cosh\dfrac{ct}{R}. \end{cases}$$

In this, sinh and cosh mean hyperbolic functions in the usual way. One will then have:

(54) $$\tanh\frac{ct}{R} = \frac{\upsilon}{\omega},$$

which does, in fact, agree with formula (51).

I will call the part of our pseudo-elliptical world that behaves according to (53) when one lets ϑ, φ, ψ vary within the usual limits, but t varies from $-\infty$ to $+\infty$, a *de Sitter* world. According to (54), υ / ω will then vary only between the values – 1 and + 1. Obviously, that **de Sitter** world is nothing but the *double wedge* of the previous paragraph. Its two "sides" (viz., $\upsilon - \omega = 0$ and $\upsilon + \omega = 0$) seem to be the infinitely-distant future and infinitely-distant past, resp. However, its edges (which consist of nothing but ordinary points in the general conception of the pseudo-elliptical world) seem to be somewhat singular, namely, the loci of world-points for which t assumes the value 0/0.

I have touched upon this behavior already in the aforementioned place in the *Jahresberichts der Deutschen Mathematiker-Vereinigung* (talk presented to the Göttinger Mathematischen Gesellschaft on 11 June 1918). In order to allow the paradoxical relationships that are present in the physical picture emerge clearly as such, I said at the time: "Two astronomers that both live in a **de Sitter** world and are equipped with different **de Sitter** clocks could operate in very interesting ways depending upon the real or imaginary character of any world-events." That means that the double wedge that would be cut from the pseudo-elliptical world by distinct pairs of tangential spaces to the fundamental hyperboloid would always have only pieces in common that have other pieces above them.

Moreover, whoever so desires can easily orient themselves more deeply in the details of the **de Sitter** world. That world reaches only the two points $\xi = 0$, $\eta = 0$, $\zeta = 0$, $\upsilon \mp \omega = 0$ on the fundamental hyperboloid. All world-lines are conic sections that contact the hyperboloid at those two points (whose plane then contains the one-dimensional axis $\xi = 0$, $\eta = 0$, $\zeta = 0$). There is only one continuous G_4 that transforms the **de Sitter** world into itself that corresponds to the substitution $\bar{t} = t + C$, combined with the continuous G_3 of unimodular orthogonal substitutions of ξ, η, ζ. In that way, $\xi^2 + \eta^2 + \zeta^2$ will be invariant, so the group of the **de Sitter** world is no longer transitive. The "axis" $\xi = 0$, $\eta = 0$, $\zeta = 0$, and the "edge" $\upsilon = 0$, $\omega = 0$ are invariant structures.

In conclusion, we convince ourselves that the density ρ of the incoherent matter at rest that fills up the **de Sitter** world uniformly should, in fact, necessarily be set to zero. Namely, we stay with our "static" coordinates. We then have:

$$\lambda\, g_{\mu\nu} = \frac{3c^2}{R^2}\, g_{\mu\nu}$$

for all other combinations of indices μ, ν, and:

$$\lambda\, g_{44} = \frac{3c^2}{R^2}\, g_{44} + \kappa c^2 \rho$$

for only $\mu = 4$, $\nu = 4$, from which, it will follow uniquely that:

$$\lambda = \frac{3c^2}{R^2}, \qquad \rho = 0,$$

as we have assumed already in formula (49).

All of these results are in complete agreement with the ones that **de Sitter** gave. However, they contradict the objection that **Einstein** had raised against **de Sitter** in his article on March 1918, and which **Weyl** supported by thorough calculations in his book ([18]), as well as more recently in a special review in the *Physikalischen Zeitschrift* ([19]). Both authors found that matter must be present along the edge of the double wedge (for the sake of brevity, I shall continue to use my terminology). I have not checked the validity of **Weyl**'s calculations, but I would rather adopt the concept that **Einstein** expressed to me that the difference between the two results must be based upon the difference between the coordinates that were employed. What I referred to as the individual points of the edge when one uses the ξ, η, ζ, υ, ω will become a simply-extended region when one employs the ϑ, φ, ψ, t (due to the still-undetermined value of t). It should not be difficult to succeed in clarifying this completely.

However, my concluding verdict about what **de Sitter** said is that mathematically everything is in order (in any event, except for that one still-not-completely-clarified point [which I would gladly like to see explained in a general way]), but one will be led to physical consequences that contradict our usual way of thinking and in any event the very reason that led **Einstein** to introduce the spatially-closed world.

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([18]) *Raum, Zeit, Materie*, pp. 225.
([19]) 1919, no. II (on 15 January 1919).