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XXXI. On Hilbert's first note on the foundations of physics ([1]).

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By F. Klein

Translated by D. H. Delphenich

I. From a letter from F. Klein to D. Hilbert.

…When I studied your Note carefully, I remarked that one could essentially shorten the intermediate calculations that you carried out by the use of the ordinary *Lagrange* variational theorem, and in connection with that, gain a more precise insight into the meaning of the conservation theorem that you have posed for your energy vector. In the following presentation of my arguments, I shall, as much as possible, adopt your notation, except that, for the sake of consistency, I will distinguish the world-parameters w by *upper* indices:

$$w^I,\ w^{II},\ \ldots,\ w^{IV},$$

and denote the undetermined indices by Greek symbols throughout. In that way, I shall ease the comparison with the parallel developments of Einstein, about which, I have, at the same, made a few remarks.

1. I thus begin with pp. 404 of your note, in which, you introduced the two integrals that I shall call I_1 and I_2:

(1) $$I_1 = \int K\,d\omega, \qquad I_2 = \int L\,d\omega,$$

and one understands $d\omega$ to mean the invariant space element:

$$d\omega = \sqrt{g}\cdot dw^I\ldots dw^{IV}.$$

In this, K is the fundamental position invariant of the basic ds^2, which I write, with the use of the *Riemann* four-index symbol:

(2) $$K = \sum_{\mu,\nu,\rho,\sigma}(\mu\nu,\rho\sigma)(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}),$$

([1]) Göttinger Nachrichten, Math.-phys. Klasse, (1915), 395-407 (Communicated on 20 November 1915).

but for L, since it does not seem possess enough generality as a physical assumption, I would like to write the simplest allowed value on pp. 407 of your Note:

(3) $$L = \alpha Q = -\alpha \sum_{\mu,\nu,\rho,\sigma} (q_{\mu\nu} - q_{\nu\mu})(q_{\rho\sigma} - q_{\sigma\rho})(g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho}).$$

In this, according to *Einstein's* way of looking at things, α is taken to be equal to the gravitational constant, multiplied by $8\pi / c^2$, so it will be a very small number:

$$-\alpha = 1.87 \cdot 10^{-27},$$

in the units that are useful to physicists; I quote this numerical value expressly in order for one to see that the old *Maxwellian* theory of electron-free space, which sets $\alpha = 0$ and does not speak of K at all, can be regarded as an adequate approximation to the new Ansätze that will be discussed here for the usual measurements. (Cf., no. **5**, below.)

2. I shall next construct the variations of the integrals I_1, I_2 that correspond to arbitrary variations of the $g^{\mu\nu}$, q_ρ by $\delta g^{\mu\nu}$, δq_ρ ([2]), resp., in a purely formal way, and write them briefly as:

(4a) $$\delta I_1 = \int \sum_{\mu,\nu} K_{\mu\nu} \delta g^{\mu\nu} d\omega,$$

(4b) $$\delta I_2 = \alpha \int \left(\sum_{\mu,\nu} Q_{\mu\nu} \delta g^{\mu\nu} + \sum_{\rho} Q^\rho \delta q_\rho \right) d\omega.$$

In this, $K_{\mu\nu}$, $Q_{\mu\nu}$ denote the well-known tensors that are contragredient to the products $dw^\mu\, dw^\nu$:

(5a) $$K_{\mu\nu} = \left(\frac{\partial \sqrt{g} K}{\partial g^{\mu\nu}} - \sum_\rho \frac{\partial \left(\frac{\partial \sqrt{g} K}{\partial g_\rho^{\mu\nu}} \right)}{\partial w^\rho} + \sum_{\rho,\sigma} \frac{\partial \left(\frac{\partial \sqrt{g} K}{\partial g_{\rho\sigma}^{\mu\nu}} \right)}{\partial w^\rho\, \partial w^\sigma} \right) : \sqrt{g},$$

(5b) $$Q_{\mu\nu} = \left(\frac{\partial \sqrt{g} Q}{\partial g^{\mu\nu}} \right) : \sqrt{g},$$

while Q^ρ is the vector that is cogredient to the dw^ρ:

([2]) [Here, we make the assumption that $\delta g^{\mu\nu}$, $\delta g_\rho^{\mu\nu}$, and δq_ρ vanish on the boundary of the domain of integration.]

$$Q^\rho = -\left(\sum_\sigma \frac{\partial\left(\dfrac{\partial\sqrt{g}Q}{\partial q_{\rho\sigma}}\right)}{\partial w^\sigma}\right) : \sqrt{g}\,. \tag{6}$$

The equations:

$$Q^\rho = 0, \tag{7}$$

when written in the coordinates w, are the Maxwell equations that correspond to your physical assumptions; on the other hand, as you remarked on pp. 407 of your note, the $Q_{\mu\nu}$ *are the components of the energy of the electromagnetic field.*

3. For the sake of clarity, I would still like to distinguish between the *scalar divergence* of a "vector p^ρ" and the *vectorial divergence* of a "tensor $t_{\mu\nu}$." As is well-known, in our general coordinates w^ν, one expresses the former by the sum:

$$\sum_\nu \frac{\partial\left(\sqrt{g}\,p^\nu\right)}{\partial w^\nu} : \sqrt{g}\,, \tag{8}$$

but the latter is somewhat more complicated; its components read:

$$\left(\sum_{\mu,\nu} \frac{\partial\left(\sqrt{g}\,t_{\mu\alpha}\,g^{\mu\nu}\right)}{\partial w^\nu} + \tfrac{1}{2}\sqrt{g}\sum_{\mu,\nu} t_{\mu\nu}\,g^{\mu\nu}_\sigma\right) : \sqrt{g} \tag{9}$$

for $\sigma = 1, 2, 3, 4$.

4. I will now develop *the four simple partial differential equations* that I_1 (I_2, resp.) must satisfy (because both of them are invariants of arbitrary transformations of the w). Naturally, to that end, as *Lie* in particular has done in his numerous, incisive publications, one determines the formal variations that an arbitrary infinitesimal transformation:

$$\delta w^I = p^I,\ \ldots,\ \delta w^{IV} = p^{IV} \tag{10}$$

will produce. (We understand p^σ to mean an infinitesimal vector whose higher powers can be neglected.) You have done this for the integral I_1 on pp. 398-400 of your Note in such a way that you next directed your attention to the relatively complicated variations of K, in order to rise to the variation of I_1 by integration. My entire simplification of the argument consists in the fact that in connection with formula (4a) – i.e., the variation of I_1 – I have calculated it directly from the *Lagrange* derivative. *The variation of I must be zero when* I_1 *substitutes the values of* $\delta g^{\mu\nu}$ (*in* 4a) *that correspond to the infinitesimal* *transformation* (10). Since the $g^{\mu\nu}$ are cogredient to the $dw^{\mu}\, dw^{\nu}$, one will find simply that ([3]):

(11) $$\delta g^{\mu\nu} = \sum_{\sigma} (g_{\sigma}^{\mu\nu} p^{\sigma} - g^{\mu\sigma} p_{\sigma}^{\nu} - g^{\nu\sigma} p_{\sigma}^{\mu}) .$$

We will then have [when we set the p^{σ} equal to zero on the boundary]:

$$\int \sum_{\mu,\nu} K_{\mu\nu} \left(\sum_{\sigma} g_{\sigma}^{\mu\nu} p^{\sigma} - \sum_{\sigma} g^{\mu\alpha} p_{\sigma}^{\nu} - \sum_{\sigma} g^{\nu\alpha} p_{\sigma}^{\mu} \right) d\omega = 0.$$

In this, we have rearranged the terms in the p_{σ}^{ν}, p_{σ}^{μ} by partial integration, in a well-known way, in which we subject the otherwise arbitrary p^{σ} to the condition that it must have vanishing first and second differential quotients on the boundary of the integration. We then get:

$$\int \sum_{\sigma} p^{\sigma} \left(\sqrt{g} \sum_{\mu,\nu} K_{\mu\nu} g_{\sigma}^{\mu\nu} + 2 \sum_{\mu,\nu} \frac{\partial\left(\sqrt{g}\, K_{\mu\sigma} g^{\mu\nu}\right)}{\partial w^{\nu}} \right) dw^{I} \ldots dw^{IV} = 0,$$

and from this, due to the arbitrariness in the p^{σ}, I get *the four differential equations that are true for the tensor* $K_{\mu\nu}$:

(12) $$\sqrt{g} \sum_{\mu,\nu} K_{\mu\nu} g_{\sigma}^{\mu\nu} + 2 \sum_{\mu,\nu} \frac{\partial\left(\sqrt{g}\, K_{\mu\sigma} g^{\mu\nu}\right)}{\partial w^{\nu}} = 0 \qquad (\sigma = 1, 2, 3, 4)$$

that you (Einstein, resp.) posed, and which we can obviously interpret by saying that: *The vectorial divergence of the tensor* $K_{\mu\nu}$ *vanishes.*

One can treat the integral I_2 in precisely the same way. Along with the increment (11) in the $g^{\mu\nu}$, only the following increments of the q_{σ} will occur ([4]):

(13) $$\delta q_{\sigma} = \sum_{\sigma} (q_{\rho\sigma} p^{\sigma} + q_{\sigma} p_{\rho}^{\sigma}) .$$

We then get the following four differential equations for the $Q_{\mu\nu}$, Q^{ρ}:

(14) $$\sum_{\mu,\nu} \left(\sqrt{g}\, Q_{\mu\nu} g_{\sigma}^{\mu\nu} + 2 \frac{\partial\left(\sqrt{g}\, Q_{\mu\sigma} g^{\mu\nu}\right)}{\partial w^{\nu}} \right) + \sum_{\rho} \left(\sqrt{g}\, Q^{\rho} q_{\rho\sigma} - \frac{\partial\left(\sqrt{g}\, Q^{\rho} q_{\sigma}\right)}{\partial w^{\rho}} \right) = 0,$$

for s = 1, 2, 3, 4.

([3]) [This will be explained more precisely in § 1 of the following Abh. XXXII.]

([4]) My $\delta g^{\mu\nu}$ (11) and δq_{ρ} (13) are nothing but the quantities that you denoted by $p^{\mu\nu}$ and p_{ρ} in your Note on pp. 398.

It is quite unnecessary to put this into words. However, it is probably worthwhile to convert it in such a way that it will admit our Q in that special form (and which likewise appeared in various places, *mutatis mutandis*, in your Note). Q depends upon only the differences $q_{\rho\sigma} - q_{\sigma\rho}$, and therefore, as a glimpse at (6) will show, it will have a vanishing scalar divergence:

$$\sum_{\rho} \frac{\partial\left(\sqrt{g}\, Q^{\rho}\right)}{\partial w^{\rho}} = 0.$$

As a result of this, we can put the differential equations (14) *into an other form:*

$$(14') \qquad \sum_{\mu,\nu}\left(\sqrt{g}\, Q_{\mu\nu} g_{\sigma}^{\mu\nu} + 2\frac{\partial\left(\sqrt{g}\, Q_{\mu\sigma} g^{\mu\nu}\right)}{\partial w^{\nu}}\right) + \sum_{\rho}\left(\sqrt{g}\, Q^{\rho}(q_{\rho\sigma} - q_{\sigma\rho})\right) = 0,$$

for $\sigma = 1, 2, 3, 4$.

5. Only now do I introduce the basic assumption of Einstein's theory. Naturally, it would be best to introduce it in the form that you chose in your Note, which I shall express here by saying that *the variation should satisfy:*

$$(15) \qquad \delta I_1 + \delta I_2 = 0$$

for arbitrary $\delta g^{\mu\nu}$, δq_{ρ} .

According to (4a), (4b), this gives the 14 well-known *field equations*, namely, the ten equations:

$$(16a) \qquad K_{\mu\nu} + \alpha\, Q_{\mu\nu} = 0$$

and the four equations:

$$(16b) \qquad Q^{\rho} = 0.$$

In your Note, you remarked that four dependencies must exist between these 14 equations, and on pp. 406, you showed, by explicit calculation, the connection that exists between the four equations (16b) – viz., the Maxwell equations – and the ten equations (16a). Naturally, for me, that is already contained in the formulas of the previous number. In fact, one needs only to add equations (14′), when multiplied by α, to equations (12) in order to immediately read off the fact that the vanishing of Q^{ρ} will follow from equations (16a).

At the same time, what was said about the character of the older *Maxwell* theory as a limiting case of the new theory becomes entirely clear. If we treat the older *Maxwell* theory in arbitrary curvilinear coordinates $w^{I}, \ldots, w^{IV}$, then we will also have to deal with a ds^2 whose *Riemann* curvature vanishes identically, so the $K_{\mu\nu}$ will also be simply zero. On the other hand, one takes $\alpha = 0$. *The ten equations* (16a) *are then fulfilled by themselves; the energy components* $Q_{\mu\nu}$ *of the electromagnetic field are then subject to no further constraints from there on.* All that remains are the four equations (16b); i.e., just *Maxwell's* equations. As a consequence of them, according to (14), the $Q_{\mu\nu}$ will have a vanishing vectorial divergence.

Naturally, before *Einstein,* the rest of us introduced curvilinear coordinates w into physics only in such a way that the three space coordinates were transformed arbitrarily, but the t remained essentially unchanged. The fact that t was included with the same status in the coordinate transformation seems to be one of *Einstein's* great achievements. The other one is then self-explanatory, i.e., that the calculation of gravitation can be carried out when one replaces the ds^2 with a vanishing *Riemann* curvature with a more general ds^2. On the other hand, we should also emphasize that the mathematical background for dealing with these new physical thoughts has already been long-established, since for us, spaces of arbitrarily many dimensions with arbitrary arc-length elements have been commonplace since the time of *Riemann*. This is not the place to digress into a historical excursion that would have to begin with the methods of *Lagrange's Mécanique analytique;* otherwise, one would have to discuss not only the aforementioned papers of *Christoffel*, but also those of *Beltrami* and *Lipschitz.*

6. I would now like to add equations (12), (14) together, without employing the field equations (16), and then multiply the latter by α. For $\sigma = 1, 2, 3, 4$, that will give the identities:

(17)
$$\sum_{\mu,\nu} \sqrt{g}\,\left(K_{\mu\nu} + \alpha Q_{\mu\nu}\right) g_\sigma^{\mu\nu} + \sum_\rho \alpha \sqrt{g}\, Q^\rho q_{\rho\sigma}$$
$$= -2 \sum_{\mu,\nu} \frac{\partial \left[\sqrt{g}\,(K_{\mu\sigma} + \alpha Q_{\mu\sigma}) g^{\mu\nu} - \frac{\alpha}{2}\sqrt{g}\, Q^\nu q_\sigma \right]}{\partial w^\nu} .$$

I then multiply these equations by p^σ (where one understands that p^σ means the vector that is cogredient to dw^σ) and sum over σ. Here, I can insert the p^σ under the differentiation sign in the right-hand side, while I put corresponding extended terms into the left-hand side. Thus, I would like to switch the symbols σ and ν in the left-hand side, and also replace $2\, g^{\mu\sigma} p_\sigma^\nu$ with the symmetric expression $(g^{\mu\nu} p_\sigma^\nu + g^{\nu\sigma} p_\sigma^\mu)$, which is equivalent in the context of this argument. In that way, it will arise that:

(18)
$$\sum_{\mu,\nu,\sigma} \sqrt{g}\;(K_{\mu\nu} + \alpha\, Q_{\mu\nu})\,((g_\sigma^{\mu\nu} p^\sigma - g^{\mu\sigma} p_\sigma^\nu - g^{\nu\sigma} p_\sigma^\mu)$$
$$+ \sum_{\rho,\sigma} \alpha \sqrt{g}\, Q^\rho (q_{\rho\sigma} p^\sigma + q_\sigma p_\rho^\sigma)$$
$$= -2 \sum_{\mu,\nu,\sigma} \frac{\partial \left\{ \left[\sqrt{g}\,(K_{\mu\sigma} + \alpha Q_{\mu\sigma}) g^{\mu\nu} - \frac{\alpha}{2}\sqrt{g}\, Q^\nu q_\sigma \right] p^\sigma \right\}}{\partial w^\nu},$$

which is naturally just another way of writing (17). In view of the special value that I have chosen for your H (viz., $H = K + \alpha Q$), all that is on the left-hand side here is precisely what you gave as the value of scalar divergence of your *energy vector* e^ν, when multiplied by $\sqrt{g}$ (pp. 402 of your Note), namely:

$$\sum_\nu \frac{\partial \sqrt{g}\, e^\nu}{\partial w^\nu}.$$

It then follows that *your energy vector* e^ν *differs from:*

$$-2\sum_{\mu,\sigma} \left[(K_{\mu\sigma} + \alpha Q_{\mu\sigma}) g^{\mu\nu} + \frac{\alpha}{2} Q^\nu q_\sigma \right] p^\sigma$$

by just one term whose scalar divergence vanishes identically.

If we now add the 14 field equations (16a), (16) then e^ν will reduce to that extra term, and the statement on pp. 402 of your note that:

$$\sum_\nu \frac{\partial \sqrt{g}\, e^\nu}{\partial w^\nu} = 0 \tag{19}$$

will appear to be an identity. That statement cannot, by any means, be then regarded as an analogue of the law of conservation of energy that prevails in ordinary mechanics. If we write the latter as:

$$\frac{d(T+U)}{dt} = 0$$

then this differential relationship will not be an identity, but only as a result of the differential equations of mechanics.

7. Naturally, it would be desirable to give the extra term explicitly, in order to distinguish your e^ν from the terms that vanish as a result of the field equations. However, I find your formulas so complicated that I have not attempted to duplicate them. It only seems clear that they subsume components that are linear in the p^σ, as well as other ones that include the p_μ^σ linearly, and perhaps only ones that include the $p_{\mu\nu}^\sigma$ linearly. It might actually not be difficult to give the most general vector of that form whose scalar divergence vanishes identically. One obtains nothing but vectors X^ν of vanishing divergence when one starts with any six-tensor (i.e., a skew-symmetric tensor) $\lambda^{\mu\nu}$ and sets:

$$\sqrt{g}\, X^\nu = \sum_\mu \frac{\partial \lambda^{\mu\nu}}{\partial w^\mu}. \tag{20}$$

If one would wish that the X^ν should be linear in the p^σ and p_μ^σ then one can choose, for example:

$$\lambda^{\mu\nu} = \left(\left(\sum_\rho g^{\mu\rho} q_\rho\right) p^\nu - \left(\sum_\rho g^{\nu\rho} q_\rho\right) p^\mu\right). \tag{21}$$

8. Here, I must interject an essential remark. One knows that Nöther has continued to advise me in my research and that I was actually only introduced into the present topic by her. When I now ultimately spoke to Nöther about my results concerning your energy vector, she wrote me that she had already derived the same thing from the developments in your Note (and thus, not with the simplified calculations in my own no. **4**) a year ago, and wrote that down in a manuscript at the time (which I then looked into). However, she had not proved it as decisively as I did recently in the Mathematischen Gesellschaft (22 January).

9. In conclusion, I would like to make the remark in regard to that, that it is self-explanatory that the same thing will be true for your theorem (19) on the "conservation laws" that *Einstein* had formulated in 1916 ([5]). He himself actually expressed it quite clearly. I would not like to go into the details of his calculations here, but only refer to his final result, which he wrote thus:

$$\sum_\nu \frac{\partial}{\partial w^\nu}(\mathfrak{T}_\sigma^\nu + \mathfrak{t}_\sigma^\nu) = 0 \qquad (\sigma = 1, 2, 3, 4), \tag{22}$$

in which he denoted the "mixed" energy components of the electromagnetic (gravitational, resp.) by $\mathfrak{T}_\sigma^\nu$ ($\mathfrak{t}_\sigma^\nu$, resp.). In that way, he got that this $\mathfrak{T}_\sigma^\nu + \mathfrak{t}_\sigma^\nu$ could be expressed as:

$$\mathfrak{T}_\sigma^\nu + \mathfrak{t}_\sigma^\nu = -\sum_{\mu,\rho}\left(\frac{\partial}{\partial w^\rho}\left(\frac{\partial \mathfrak{G}^*}{\partial g_\rho^{\mu\sigma}} g^{\mu\nu}\right)\right) \tag{23}$$

when one invokes the field equations and introduces a function $\mathfrak{G}^*$ that he defined more precisely that is independent of the coordinate system, and that the *identity* equation:

$$\sum_{\mu,\nu,\rho} \frac{\partial}{\partial w^\nu \partial w^\rho}\left(\frac{\partial \mathfrak{G}^*}{\partial g_\rho^{\mu\sigma}} g^{\mu\nu}\right) = 0 \tag{24}$$

exists for that $\mathfrak{G}^*$ independently of the value of σ. That is precisely what we arrive at here.

([5]) Cf., the self-sufficient paper: "Die Grundlagen der allgemeinen Relativitätstheorie" (Leipzig 1916), as well as the communication to the Berlin Academy on 20 Oct. 1916, "Hamiltonsches Prinzip und allgemeine Relativitätstheorie" (Sitzungsberichte, pp. 1111-1116).

In order to present the connection with the notation that I have used, I remark that *Einstein's* $\mathfrak{T}_\sigma^\nu$ is the same thing as my $\sum_\mu \sqrt{g}\, Q_{\mu\sigma} g^{\mu\nu}$, but *Einstein's* $\mathfrak{t}_\sigma^\nu$ differs from the corresponding expression $\frac{1}{\alpha}\sum_\mu \sqrt{g}\, K_{\mu\sigma} g^{\mu\nu}$ by a summand that one gets when one compares equations (23) with the field equations:

$$K_{\mu\nu} + \alpha\, Q_{\mu\nu} = 0.$$

II. From D. Hilbert's response.

… I am actually in complete agreement with your explanation of the energy theorem. *Emmy Nöther*, whose help I called upon for the clarification of some questions of an analytical nature regarding my energy theorem over a period of more than a year, found, at that time, that the energy components that I presented formally (like those of *Einstein*) by means of the *Lagrange* differential equations (4), (5) in my first communication could be converted into expressions whose divergences vanished *identically*; i.e., without the use of the *Lagrange* equations (4), (5). On the other hand, since the energy equations of classical mechanics, the theory of elasticity, and electrodynamics are fulfilled only as a consequence of the *Lagrange* differential equations of the problem in question, it is justified if you do not therefore perceive the analogues to any of those theories in my energy equations. Admittedly, I then assert that for *general* relativity – i.e., in the case of the *general* invariance of *Hamilton's* function – energy equation that correspond to the energy equations of the orthogonally-invariant theory, in your sense, do not exist at all. Indeed, I would like to refer to that situation as even a characteristic feature of the general theory of relativity. The mathematical proof of my assertion is forthcoming.

Please allow me, on this occasion, to show briefly how I treated the question of the energy equations of the orthogonally-invariant theories of physics (viz., electrodynamics, hydrodynamics, and the theory of elasticity) in a lecture last winter.

For the sake of brevity, we choose the world-function H, which depends upon only the components of an electrodynamical four-potential q_s and its first derivatives q_{sl} with respect to w_k (s, l = 1, 2, 3, 4) to be orthogonally-invariant (the method is true just the same, whether H perhaps includes the four-density r and its derivatives or some other physical parameter and its derivatives). *Hamilton's* principle:

(1) $$\delta \int H\, d\omega = 0$$

then leads to the system of four *Lagrange* differential equations:

(2) $$[H]_s = 0 \qquad (s = 1, 2, 3, 4),$$

in which we generally mean:

$$[H]_s = \frac{\partial H}{\partial q_s} - \sum_k \frac{\partial}{\partial w_k} \frac{\partial H}{\partial q_{sk}}.$$

In order to arrive at the energy equations of this problem, we embark upon the path that explains the statements in my first paper, namely, the path of the theory of gravitation.

Let $\bar{H}$ be the general invariant with the arguments:

$$q_s\,,\; q_{sl}\,,\; g^{\mu\nu},\; g_l^{\mu\nu}$$

that goes to H for:

(3) $$g^{\mu\nu} = g_{\mu\nu} = \delta_{\mu\nu}, \qquad g_l^{\mu\nu} = 0.$$

We will be dealing with the same thing when we replace the q_{sl} in H with the covariant derivatives:

$$\bar{q}_{sl} = q_{sl} - \sum_h \begin{Bmatrix} s\,l \\ h \end{Bmatrix} q_k\,,$$

and at the same time, raise the indices with the $g^{\mu\nu}$. If H takes on, for example, a term in the form of the orthogonal-invariant expression:

(4) $$-Q = \sum_{m,n} (q_{mn} - q_{nm})^2 = \tfrac{1}{4} \sum_{m,n} M_{mn}^2$$

then it will have to be replaced with:

$$-\bar{Q} = \tfrac{1}{4} \sum_{m,n,k,l} M_{mn} M_{kl}\, g^{mk} g^{ml}\,.$$

The expression:

$$T = \sum_{s,h} q_{sh}^2$$

will get converted into:

$$\bar{T} = \sum_{s,h,m,n} q_{sh} \bar{q}_{mn}\, g^{sm} g^{hn}\,,$$

etc.

Now, an identity is verified by every general invariant that was proved in my first paper (Theorem III) only in the case for which the invariant depended upon $g^{\mu\nu}$ and its derivatives; however, the method of proof that was followed there is also true for our general invariant $\bar{H}$. If we employ the notations of my first paper then, in place of the equation in it, namely:

$$\int P_g\left(J\sqrt{g}\right) d\omega = 0,$$

we will get the equation:

$$\int \left\{P_g\left(\bar{H}\sqrt{g}\right) + P_q\left(\bar{H}\sqrt{g}\right)\right\} d\omega \equiv \int \left\{P\left(\bar{H}\sqrt{g}\right)\right\} d\omega = 0$$

in our present case, which is an identity that has the immediate consequence that:

$$\int \left\{ \sum_{\mu,\nu} \left[\sqrt{g}\, \bar{H} \right]_{\mu\nu} p^{\mu\nu} + \sum_{\mu} \left[\sqrt{g}\, \bar{H} \right]_{\mu} p_{\mu} \right\} d\omega = 0.$$

After introducing $p^{\mu\nu}$, p_μ and applying partial integration, we can bring the integral on the left-hand side into a form in which the integrand is multiplied by p^s. However, since p^s is an arbitrary vector, the other factor under the integral sign must vanish identically, and that will imply the identities (s = 1, 2, 3, 4) ([6]):

$$\text{(5)} \qquad \sum_{\mu,\nu} \left[\sqrt{g}\, \bar{H} \right]_{\mu\nu} g_s^{\mu\nu} - 2 \sum_{m} \frac{\partial}{\partial w_m} \left\{ \sum_{\mu} \left[\sqrt{g}\, \bar{H} \right]_{\mu s} g^{\mu m} \right\} + \sum_{\mu} \left[\sqrt{g}\, \bar{H} \right]_{\mu} q_{\mu s} - \sum_{\mu} \frac{\partial}{\partial w_\mu} \left(\left[\sqrt{g}\, \bar{H} \right]_{\mu} q_s \right) = 0.$$

These four identities are, at the same time (precisely as you have remarked above) the ones between the 14 *Lagrange* equations of our problem whose existence was asserted in Theorem I that I presented.

If we now return to the original problem, in which we eliminate the gravitational potentials by means of (3) and consider the *Lagrange* differential equations (2), then the identities (5) will go to:

$$\text{(6)} \qquad \sum_{m} \frac{\partial}{\partial w_m} \left\{ \left[\sqrt{g}\, \bar{H} \right]_{ms} \right\}_{g_{\mu\nu} = \delta_{\mu\nu}} = 0.$$

If we then refer to the bracket expressions:

$$\text{(7)} \qquad \varepsilon_{ms} = 2 \left\{ \left[\sqrt{g}\, \bar{H} \right]_{ms} \right\}_{g_{\mu\nu} = \delta_{\mu\nu}}$$

as the *components of the energy tensor* then we will obtain the desired energy equations of the physical problem (1) from the divergence equations (6).

If we take H to be the invariant Q in (4), in particular, then ε_{ms} will become the components of the known electromagnetic energy tensor, and the *Maxwell* equations:

$$\left\{ \left[\sqrt{g}\, \bar{H} \right]_{m} \right\}_{g_{\mu\nu} = \delta_{\mu\nu}} = \mathrm{Div}_m\, M = r_m$$

(we understand r to mean the electric four-density) will yield our identities (5) in this case:

$$\mathrm{Div}_s\, \varepsilon - \sum_{m} r_m q_{ms} + \sum_{m} \frac{\partial}{\partial w_m} (r_m q_s) = 0,$$

or, since $\mathrm{Div}_s\, r = 0$:

$$\mathrm{Div}_s\, \varepsilon = - r_s \cdot M\,;$$

([6]) [For this, cf., my formula (14), which agrees with this one term-by-term. K] i.e., they produce the well-known divergence expression for the ponderomotive force.

It is only in the case of general relativity – i.e., when the original invariant H is already a general invariant – that the given path fails to exhibit the energy equations for the problem (1). In the general theory of relativity, we have a substitute for the missing energy equations in your sense in the fact that the *Lagrange* equations (Theorem I in my first paper) are four-fold superfluous, as was expressed in the four identities (5). Conversely, the energy theorem of the orthogonal theory seems to be the residue of those four identities of the theory of gravitation.

It should be remarked that as one sees immediately, the energy tensor (7) possesses not only the properties of orthogonal invariance and symmetry, but in addition, it also fulfills the requirements of the special physical theory: The same thing will be true in the case of electrodynamics, in which H contains the q_{sl} only in the combinations:

$$M_{ks} = q_{sk} - q_{ks} ,$$

as well as upon only these components of the six-vector M, and on the other hand, in the case of the theory of elasticity, in which it also depends upon only the actual distortion quantities that enter into the question of elasticity…

III. From a further writing of F. Klein.

…It is important for me to characterize the difference between the orthogonal-invariant theory of electrodynamics and that of gravity that I have been considering in a few words.

In that regard, things become much clearer when, as I already suggested above (no. **5**), one appeals to the treatment of classical electrodynamics in arbitrary ("curvilinear") world-coordinates as an intermediate step.

Your main theorem that the energy components of the electrodynamical field can be represented simply by the $Q_{\mu\nu}$ then already comes to the foreground in all of its significance; I would then prefer to not speak of the modern theory of gravitation in this theorem.

I also find it useful to distinguish the integrals $\int K\, d\omega$ and $\int Q\, d\omega$ in the representation, and not fuse them together into one integral $\int H\, d\omega$ from the outset.

We then have *four* identities for the $K_{\mu\nu}$ and the $Q_{\mu\nu}$ [viz., the equations (12) and (14) – or (14′) – in my first paper], so *eight*, in all, and the opposite of the earlier and present theory can then be put into precise words as follows:

1. In both cases, we have 14 "field equations", along with the eight identities, for the comparison that comes under consideration here.

2. In the previous theory, they read ([7]):

([7]) [As a consequence of the 20 equations that come from the *Riemann* curvature vanishing identically. K]

$$a)\ K_{\mu\nu} = 0, \qquad\qquad b)\ Q^{\rho} = 0.$$

The four identities (12) are fulfilled by themselves by means of the ten equations *a*), but the identities (14) – or (14′) – reduce to the four statements that one calls the four conservation theorems (impulse-energy theorem) by means of the four equations *b*).

3. In return, one has the field equations in the new theory:

$$a')\qquad K_{\mu\nu} + \alpha\, Q_{\mu\nu} = 0, \qquad \text{with } \alpha \neq 0 \qquad b')\qquad Q^{\rho} = 0.$$

The equations $Q^{\rho} = 0$ now appear to be a consequence of the ten equations *a′*) by means of the eight identities.

When one drops the Q^{ρ}, "conservation laws" for the $Q_{\mu\nu}$ will still follow from the identities (14). However, they will now no longer have any autonomous (physical) meaning, since they will reduce to the four identities (12) by means of the ten equations *a′*); they are even already included in the ten field equations.

All of this is actually in complete agreement with the presentation in your letter. However, it would be very interesting to me to see the details of the mathematical proof that you promised at the end of the first paragraph.

[In the meantime, the stated proof was provided by E. Nöther (see her Note on "Invariante Variationsprobleme" in the Göttinger Nachrichten of 26 July 1918.) I shall return to that at the conclusion of XXXII.

Moreover, I would like add the following remarks in order to make the relationship with the articles XXXI to XXXIII on the Erlanger Programm clearer:

1. The invariant theory of the Lorentz group that was treated in XXX is precisely what the modern physicists refer to as "the special theory of relativity."

2. Therefore, the Lorentz group can apparently be defined to be the largest continuous family of the most general continuous transformations of finite values of *x*, *y*, *z*, *t* that take the quadratic differential form:

$$ds^2 = dt^2 - \frac{dx^2+dy^2+dz^2}{c^2}$$

to itself.

3. One now imagines that instead of the *x*, *y*, *z*, *t*, one introduces any real, everywhere continuous at finite points, differentiable sufficiently often, single-valued, invertible functions:

$$w^{\rho} = \varphi^{\rho}(x, y, z, t) \qquad\qquad (\rho = 1, 2, 3, 4).$$

The ds^2 that we just wrote down will then go to a more general quadratic form in the *dw*, which we would like to write in the *Einstein* manner as:

$$ds^2 = \sum g_{\mu\nu}\, dw^{\mu}\, dw^{\nu}.$$

4. Naturally, this new ds^2, like the one that was given in 2, has inertia character $+ - - -$. Its coefficients $g_{\mu\nu}$ are continuous, differentiable sufficiently often, real functions of the w that are naturally specialized only by the fact that the *Riemann* curvature that is constructed from ds^2 must vanish identically.

5. From the basic laws of the Erlanger Programm, we can now also treat the special theory of relativity in such a way that we base it upon the total group of all real, continuous, differentiable sufficiently often, uniquely invertible transformations of the w^ρ, but then *adjoin* the ds^2 of 3; i.e., we append the conversions that the $g_{\mu\nu}$ experience under the respective transformations of the w. One then obtains uniquely-determined linear transformations of the $g_{\mu\nu}$, since the relations that couple the $g_{\mu\nu}$ as the coefficients of a form with vanishing curvature have too advanced a character to have any influence. In addition, one must observe that not only the coefficients of the substitution, but also the $g_{\mu\nu}$ themselves are functions of the x, y, z, t (the w^2, resp.). The conversions that differential quotients of the $g_{\mu\nu}$ experience from the respective transformations then follow from that. Furthermore, the group that is "extended" by all of them is the one that all considerations should be based upon.

6. When we do that, we will have taken a decisive step into the "general theory of relativity." A further step might be when we introduce the most general (for real w) everywhere-real, continuous, differentiable sufficiently often function of the w as the coefficients $g_{\mu\nu}$ of the ds^2. The *Riemann* curvature and the invariant that is derived from it that *Hilbert* denotes by K will then no longer be identically zero.

Moreover, one will choose the "group" exactly as we did in 5.

Incidentally, this also raises the question of the connection to the world as a whole, which analogous in to the consideration on Abh. XXI in relation to the case of the geometry of the plane. This question still seems to have not been dealt with very much. That particular case presents possibilities that will emerge in Abh. XXXIII. The entire question naturally goes away for the special theory of relativity, in which we let x, y, z, t run from $-\infty$ to $+\infty$ inclusive in order to obtain all world-points.

7. The general theory of relativity of the pure gravity field thus results from *Einstein's* fundamental Ansatz [which was formulated almost simultaneously by *Einstein* and *Hilbert* ([8])] when one subjects the $g_{\mu\nu}$ to the equations $K_{\mu\nu} = 0$, which are ten, in all, and invariant under the group that we speak of. (Here, for the sake of brevity, I am using the notation (5a) of my own Note.)

8. We might now direct our attention to any other sort of physical phenomena, along with gravitation, or furthermore, we might, as we did in the present article in connection with *Hilbert's* first Note, restrict ourselves to electromagnetic processes in empty space, along with gravitation.

9. One will consider this most simply, also in the case of the special theory of relativity (which was, unfortunately, not mentioned in Abh. XXX), when one poses the linear form:

$$\sum q_\rho dw^\rho ,$$

along with our ds^2, in which the real, everywhere continuous, differentiable sufficiently often, functions q_ρ define the so-called four-potential of the electromagnetic field.

([8]) *Einstein* "Zur allgemeinen Relativitätstheorie" in the Sitzungsberichten der Berliner Akademie of 11 and 25 Nov. 1915 (pp. 799 to 801, pp. 844 to 847, resp. of the year's issue), *Hilbert* in his (presently critiqued) first Note on the "Grundlagen der Physik" in the Göttinger Nachrichten of 20 Nov. 1915. One cannot speak of any question of priority then, since both authors pursued completely different lines of thought (and indeed, for that reason, the consistency of the results did not seem certain at first). *Einstein* proceeded *inductively* and considered arbitrary material systems. *Hilbert* proceeded *deductively*, in that he let the restriction on electrodynamics that was mentioned in this article under 8 enter in as a prescribed main variational principle, moreover. *Hilbert* then also connected up with Mie, in particular. *Einstein* first presented the connection between the two kinds of Ansätze in his aforementioned (pp. 8) communication to the Berlin Academy on 29 Oct. 1916.

10. The fundamental group is now extended in comparison to 5 in that, along with the transformations of the $g_{\mu\nu}$ and their differential quotients, which come about (i.e., are "induced") by the transformations of the w, now, those of the q_ρ and their differential quotients also enter in.

11. However, the $g_{\mu\nu}$, q_ρ are now subjected to the 14 equations (16a), (16b) of this article:

$$K_{\mu\nu} + \alpha\, Q_{\mu\nu} = 0, \qquad Q_\rho = 0,$$

which are again invariant under the extended group. In connection with 10, this is the core of the general theory of relativity in physics, as far as it concerns us here.

Obviously, these formulations can only be expressed in another language, which is what *Einstein* and *Hilbert* said, anyway. Here, I would like to refer to *Hilbert's* second communication on the foundations of physics (in the Göttinger Nachrichten 1917, pp. 53-76) ([9]). There, on pp. 61, it was expressly stated that only those consequences that one could deduce from the differential equations 11 that had *physical meaning* could, like the differential equations themselves, possess an *invariant character* (N. B., under the group that was defined in 10). That is, *mutatis mutandis*, the same thing that was concluded in the Erlanger Programm from the statements about any geometry (that is characterized arbitrarily by a group).

It hardly needs to be said that similarly the further study of *Einstein's* theory, such as what *Weyl* has done, can be connected with the schema of the Erlanger Programm.

There is even an especially close relationship to the individual explanations that are present there (Note VI of Abh. XXVII, pp. 491-492), insofar as it was not a form ds^2, but an *equation* $ds^2 = 0$, that was fundamental there. K.]

([9]) Sumitted on 23 Dec. 1916.