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LXXVII. On stress surfaces and reciprocal diagrams, with special consideration of the Maxwell papers.

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[1]) After a lecture of F. Klein presented to the Göttinger Mathematischen Gesellschaft on 7 July 1903 and further developed by K. Wieghardt.

The following is a free reference to J. C. Maxwell's treatises on frames, reciprocal figures, and diagrams [2]) and indeed, to essentially the last one cited. We will very much deviate from the presentation of Maxwell itself, and results will also be announced that were not found by him and which, insofar as nothing special is remarked, originate with F. Klein.

The inducement for the present treatise was the encyclopedia reference of Herrn Henneberg: *Über die graphische Statik der starren Körper* [3]). Naturally, the Maxwell papers find only a brief mention in it, while it still seems desirable to deduce the essential content of them – which are difficult to read – in a thorough fashion and likewise discuss some ideas that are organically connected with those of Maxwell.

§ 1.

On the Airy stress surfaces of planar continua

1. When a planar continuum carries *stresses* with the components P¸Q, U (Fig. 1), it is known that static equilibrium at each place is obtained from the existence of the two differential equations:

$$(1) \qquad \begin{cases} \dfrac{\partial P}{\partial x} + \dfrac{\partial U}{\partial y} = 0, \\[2ex] \dfrac{\partial U}{\partial x} + \dfrac{\partial Q}{\partial y} = 0, \end{cases}$$

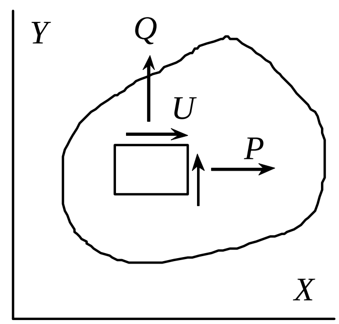

Fig. 1.

as long as external forces act only on the boundary of the continuum, but not on the interior. Airy [4]) has already remarked that these equations say nothing more than that P, Q, U can be expressed by the equations:

$$(2) \qquad P = \frac{\partial^2 F}{\partial y^2}, \qquad Q = \frac{\partial^2 F}{\partial x^2}, \qquad U = -\frac{\partial^2 F}{\partial x\,\partial y}$$

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[2]) *The scientific papers of J. C. Maxwell:*

a) V. I, pp. 514-525: On reciprocal figures and diagrams of forces. London, Edinburgh, and Dublin Phil. Mag. v. 27 (4); pp. 250 (1864).
b) V. I, pp. 598-604: On the calculation of the equilibrium and stiffness of frames. Phil. Mag., v. 27 (4); pp. 294 (1864).
c) V. II, pp. 102-104: On reciprocal diagrams in space and their relation to Airy's function of stress, Proc. London Math. Soc., v. 2.
d) V. II, pp. 492-497: On Bow's method of drawing diagrams in graphical statics, etc. Camb. Phil. Soc. Proc., v. 2, pp. 407 (1876).
e) V. II, pp. 647-659: Diagrams. Encyclopaedia Brittanica.
f) V. II, pp. 161-207: *On reciprocal figures, frames, and diagrams of forces.* Trans. Royal Soc. Edinburgh, v. 26, pp. 1 (1872).

[3]) Encyclopädie d. mathematischen Wissenschaften IV. 1 (printed 1903).

[4]) Airy: On the strains in the interior of beams. Phil. Trans. 1863 (appeared 1864), v. 153.

as the second partial differential quotients of a function $F(x, y)$; we will thus call such a function $F(x, y)$ with this meaning an *"Airy function"* or *"stress function."* From now on, it will be suspected and also shown that the stress function for a planar stress problem plays a central role.

In order to orient ourselves to this situation somewhat, we, along with Maxwell, would like to immediately make the connection between *resultant stresses* along an arc segment (ab) in our continuum with the stress functions. On the arc segment ds (Fig. 2), one has stresses with components:

$$(3)\qquad \begin{cases} X\,ds = \quad P dy - U dx = d\left[\dfrac{\partial F}{\partial y}\right], \\ Y\,ds = -Q dy - U dx = -d\left[\dfrac{\partial F}{\partial x}\right], \\ (yX - xY)\,ds = \quad d\left[x\dfrac{\partial F}{\partial x} + y\dfrac{\partial F}{\partial y} - F\right], \end{cases}$$

and thus, on the arc segment (ab) one has the resultant stress:

$$(4)\qquad \begin{cases} X_r = \left(\dfrac{\partial F}{\partial y}\right)_b - \left(\dfrac{\partial F}{\partial y}\right)_a, \quad Y_r = -\left(\dfrac{\partial F}{\partial x}\right)_b + \left(\dfrac{\partial F}{\partial x}\right)_a, \\ M_r = \left(x\dfrac{\partial F}{\partial x} + y\dfrac{\partial F}{\partial y} - F\right)_b - \left(x\dfrac{\partial F}{\partial x} + y\dfrac{\partial F}{\partial y} - F\right)_a. \end{cases}$$

Thus, the *sign* is chosen in such a way that in the coordinate system chosen by us in Fig. 2 the X_r, Y_r, M_r represent the resultant stress that acts on each part of the continuum that lies on the *left-hand* side of an advance from a to b on the arc segment (ab).

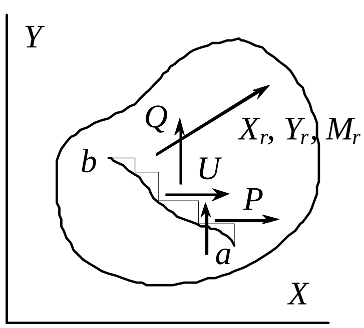

Fig. 2.

A further aspect of the importance of the stress function is the fact that its existence is independent of the special physical properties of the continuum since they now reflect back into the nature of F, and indeed, in such a way that the physical properties of the continuum are due to properties of the stress function, and conversely.

It is therefore almost self-explanatory that we illustrate such an important function by the geometric consideration of the surface:

$$(5)\qquad z = F(x, y);$$

the Z-axis is perpendicular to the plane of the continuum. We naturally call this surface an *"Airy surface"* or *"stress surface"*; the fact that it is independent of the coordinate system is easy to confirm, but will not be mentioned expressly.

2. We now first consider a *homogeneous, elastically-isotropic plate*, so there exist the well-known relations:

$$(6)\qquad \begin{cases} P = (\lambda + 2\mu)\dfrac{\partial u}{\partial x} + \lambda \dfrac{\partial v}{\partial y}, \\ Q = \lambda \dfrac{\partial u}{\partial x} + (\lambda + 2\mu)\dfrac{\partial v}{\partial y}, \\ U = \mu\left(\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}\right), \end{cases}$$

between the stress components and the elastic deformations ("strain"), where λ, μ are two constants that are individual to the material of the continuum. If one eliminates the deformation magnitudes from these equations by differentiation and takes into account (2) then the following condition remains for the stress function:

$$(7)\qquad \Delta\Delta F = 0,$$

where, in the usual way, one has:

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Therefore, if a surface z = *F*(*x*, *y*) *is to be an Airy surface for a homogeneous, elastically-isotropic plate then it must, in any case, satisfy the differential equation* $\Delta\Delta z = 0$ [5]).

Secondly, we consider a special *electrostatic stress state* in the ether, in which we specialize the function Ψ that appears in the formula on page 147 of Maxwell's "Electricity" [6]) in such a way that it depends upon only x and y, but not z. We then have the stress state:

$$(8)\qquad \begin{cases} P = \dfrac{1}{2}\left[\left(\dfrac{\partial\Psi}{\partial x}\right)^2 - \left(\dfrac{\partial\Psi}{\partial y}\right)^2\right], \\ Q = \dfrac{1}{2}\left[\left(\dfrac{\partial\Psi}{\partial y}\right)^2 - \left(\dfrac{\partial\Psi}{\partial x}\right)^2\right], \\ U = \dfrac{\partial\Psi}{\partial x}\cdot\dfrac{\partial\Psi}{\partial y}, \end{cases}$$

in an ether plane, where Ψ is a function that satisfies the condition $\Delta\Psi = 0$. Since $P + Q = 0$, it follows immediately upon consideration of equation (2): *If a surface z* = *F*(*x*, *y*) *is the stress surface that goes with the electrostatic stress state in equations* (8) *then it must*

[5]) This result seems to have been first found by Herr Michell: J. H. Michell, *On the direct calculation of stress in an elastic solid,* etc. Proc. London Math. Soc., v. 31 (1900).

[6]) J. C. Maxwell, A treatise of electricity and magnetism. v. 1, 2nd ed., Oxford 1881.

likewise satisfy the differential equation $\Delta z = 0$. (Incidentally, F and Ψ are connected by the formula: $F = -\iint \frac{\partial \Psi}{\partial x}\frac{\partial \Psi}{\partial y}\,dx\,dy$.)

Finally, we would like to investigate what sort of stress distribution exists in a planar continuum when the associated stress surface is a piece of a *developable surface*, which we assume, for the sake of simplicity, is single-valued and singularity-free. If $z = F(x, y)$ is its equation and we let the XY-plane go through one of its generators then $\partial z/\partial x$ and $\partial z/\partial y$ are naturally constant along this generator. Therefore, along the X-axis, as the projection of the generator onto the XY-plane, one has:

$$P = \frac{\partial^2 z}{\partial y^2} = \text{function of } x, \qquad Q = \frac{\partial^2 z}{\partial x^2} = 0, \qquad U = \frac{\partial^2 z}{\partial x\,\partial y} = 0.$$

Thus, if $(\partial z/\partial x)'$ and $(\partial z/\partial y)'$ are the values of the first differential quotients for an infinitely close generator then from equations (4) one obtains, on an infinitely narrow *generating strip* between two generators, the constant resultant normal stress:

$$P\,dy = \frac{\partial^2 z}{\partial y^2}\,dy = \left(\frac{\partial z}{\partial y}\right)' - \left(\frac{\partial z}{\partial y}\right). \tag{9}$$

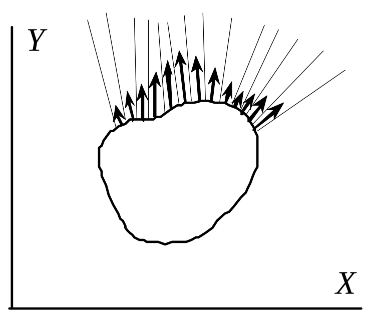

Fig. 3.

A developable surface, as a stress surface, thus corresponds to a stress distribution in the form of a sort of "sequence of strips," namely, the strips in the XY-plane of the continuum that come about as projections from the surface. Along each strip there reigns a certain normal stress that generally varies from strip to strip, while no stress at all is carried over from one strip to another. We can think of the stress system of this sequence of strips as best realized mechanically by neighboring threads – i.e., a *sequence of threads* – when we replace each strip with a middle thread, which we stress in such a way that its stress is equal to the stress on the strip when it is replaced (Fig. 3).

3. Of particular interest now is the consideration of a continuum that is composed of a sequence of strips or a corresponding sequence of threads and a homogeneous, elastically-isotropic plate, since that is closest to the problem that appears in the applications of the theory of elasticity of finding the stresses in such a plate under the influence of an equilibrium system of external forces that act on the boundary of the plate.

We think of the boundary of the plate – which we assume to be *simply-connected*, in order to avoid complications – as being given in such a way that the coordinates of its points are given by two functions of a parameter t that runs from 0 to T, and in a completely analogous way, we give the external forces:

$$X\,dt, \qquad Y\,dt, \qquad M\,dt = (yX - xY)\,dt$$

that act on an element *dt* by means of three equations:

$$X = \varphi(t), \qquad Y = \psi(t), \qquad M = \chi(t),$$

where, for the sake of simplicity, the functions φ, ψ, χ may be chosen such that the sequence of strips (11) covers the part of the plane outside of the plate only once. Since the system of forces is an equilibrium system, one has the equations:

$$\text{(10)} \qquad \int_0^T \varphi\,dt = 0, \qquad \int_0^T \psi\,dt = 0, \qquad \int_0^T \chi\,dt = 0.$$

If we now extend our elastic plate to one that is extended over the entire plane by setting:

$$\text{(11)} \qquad -\psi(t)\cdot x + \varphi(t)\cdot x - \chi(t) = 0$$

outside of the sequence of strips (threads, resp.) then we will be led to essentially the solution of the aforementioned problem of elasticity of finding a stress surface for the continuum thus composed for which along any piece t_0, t of the plate boundary the resultant strip-stress has the components:

$$\int_{t_0}^{t} \varphi\,dt\,, \qquad \int_{t_0}^{t} \psi\,dt\,, \qquad \int_{t_0}^{t} \chi\,dt\,.$$

Next, we construct those parts of the desired stress surface that exist over the sequence of strips. There is an associated developable surface that admits the parametric representation:

$$\text{(12)} \qquad z = \Phi(x, y, t) = A(t)\,x + B(t)\,y + C(t), \qquad \Phi' = A'(t)\,x + B'(t)\,y + C'(t) = 0,$$

where the prime symbol means differentiation with respect to the parameter and A, B, C are three unknown functions; they may be determined immediately. For an arbitrary advance on the surface, one has:

$$dz = A\,dx + B\,dy + \Phi'\,dt = A\,dx + B\,dy;$$

hence, one has:

$$\frac{\partial z}{\partial x} = A, \qquad \frac{\partial z}{\partial y} = B.$$

Due to the boundary conditions, if we recall equations (4) then we must have:

$$\left[\frac{\partial z}{\partial x}\right]_{t_0}^{t_1} = -\int_{t_0}^{t_1} \psi\,dt \qquad \text{and} \qquad \left[\frac{\partial z}{\partial y}\right]_{t_0}^{t_1} = \int_{t_0}^{t_1} \varphi\,dt\,.$$

Thus, it follows that:

$$A = -\int_0^t \psi\, dt + a, \qquad B = \int_0^t \varphi\, dt + b,$$

where *a* and *b* are integration constants. Since, from equation (11), one must have:

$$A' : B' : C' = -\psi : \varphi : -\chi,$$

one then has, analogously:

$$C = -\int_0^t \chi\, dt + c,$$

where *c* is a new integration constant. *The desired developable surface is then given by the equations:*

$$\text{(13)} \qquad \begin{cases} z = -\int_0^t \psi\, dt \cdot x + \int_0^t \varphi\, dt \cdot x - \int_0^t \chi\, dt \cdot x + by + c, \\ -\psi \cdot x + \varphi \cdot y - \chi = 0. \end{cases}$$

The developable part of the desired stress surface is then uniquely established, up to the addition of an arbitrary plane, which naturally must influence the stresses on the sequence of strips. Outside of that, it is closed. (From (10), the three integrals from 0 to T in (13) vanish.)

With the developable part, we now proceed to likewise say something about the still-missing piece of the desired stress surface, namely, the one with coordinates and tangential planes along the space curve that the developable surface has in common with the vertical cylinder above the boundary of the plate. They must then equal the corresponding quantities for the developable surface in the event that we would like to exclude the singularity here that appears as *finite* stresses in the boundary of the plate itself, hence, an element without breadth. If the entire stress surface possesses a notch somewhere on the space curve in question then, from equations (4), one would find a finite resultant stress for an ever so small arc segment (*ab*) that intersects the plate boundary at the corresponding place. This is not in itself absurd, except that the plate must then be surrounded by a particular stressed thread, which we shall not assume here. It thus still remains for us to solve the problem: *To find a function* $F(x, y)$ *that satisfies the differential equation* $\Delta\Delta F = 0$ *in the interior of the plate and on its boundary assumes the prescribed values for F and the differential quotient with respect to the normal* $\partial F/\partial n$. The solution to this problem cannot be multi-valued, as was proved by Mathieu [7]). We thus arrive at the end result (F. Klein):

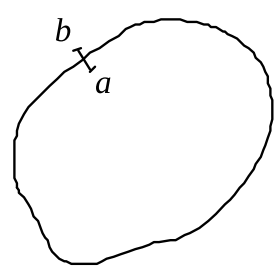

Fig. 4.

[7]) E. Mathieu: *Mémoire sur l'équation aux différences partielles du quatrième order* $\Delta\Delta u = 0$, *etc.*, in Liouville's Journal, Ser. 2, v. 14 , pp. 378 (1869).

In order to find the stress distribution that is produced in a simply-connected, homogeneous, elastically-isotropic plate by an equilibrium system of forces that act on the boundary one first constructs a developable surface – which is completely determined up to the addition of an arbitrary plane – as the stress surface that is defined by sequence of strips for the system of forces, and thus is a developable surface that connects to the plate boundary everywhere without a notch and satisfies the differential equation $\Delta\Delta z = 0$ *everywhere in the interior of the plate. If* $z = F(x, y)$ *is this surface then the desired stresses themselves are given by the equations:*

$$P = \frac{\partial^2 F}{\partial y^2}, \qquad Q = \frac{\partial^2 F}{\partial x^2}, \qquad U = -\frac{\partial^2 F}{\partial x\,\partial y}.$$

If the plate is *multiply connected* and the external forces are in equilibrium at every point of its boundary then the stated elasticity problem can be solved in a completely analogous way by a surface $\Delta\Delta z = 0$ that is connected with as many closed developable surfaces without notches as the plate boundaries possess. Now, since however *each* of these developable surfaces is determined by the external forces only up to an arbitrary plane one thus obtains *essentially different* surfaces $\Delta\Delta z = 0$ here for different choices of these arbitrary planes. Herr Michell, who incidentally, as it seems, was the first to recognize the connection between the differential equation $\Delta\Delta F = 0$ with the stated elasticity problem [8]), found in the treatise cited on page 4 the necessary additional condition for the stress surface when he considered the circumstance that the stresses arising from the shifting of the points of the plate must be unique.

If the external forces are in equilibrium on not just each individual boundary, but only for all of the boundaries collectively, then the stress surface shows another inessential multi-valuedness (affine periodicity) relative to the stresses themselves, whose analogues for discontinuous systems of stresses we will examine thoroughly in § 2.

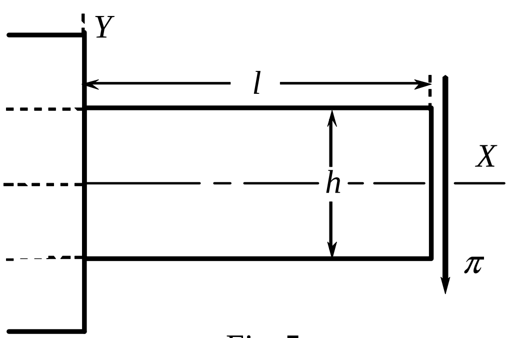

Fig. 5.

4. The ordinary, static *beam problem* provides beautiful and simple examples of stress functions $\Delta\Delta F = 0$.

We first consider (Fig. 5) a one-sided, anchored, horizontal, perpendicular to our plane, infinitely narrow beam of finite height h and length l that is loaded at the free end in such a way that the resultant of all forces is a force π that is directed vertically downwards. By a suitable assumption on the distribution of unit forces over the cross-section, the associated stress function is:

$$F(x, y) = \frac{\pi}{2h^3}(l - x)(4y^3 - 3h^2 y).$$

[8]) Michell, loc. cit.

It leads to the stresses that are given for this problem in all of the textbooks. *Thus, under the assumption that the beam can be regarded as a homogeneous, isotropic, elastic plate the stress distribution in its interior that is given in the textbooks is precisely correct.* (The same is not true for calculation of the deformation that the beam suffers under the influence of the stress system; here, the usual theory introduces approximations that one can, by the way, avoid in connection with equations (6) with no extra effort. Naturally, we cannot go into this here, but we would like to argue that one generally should separate the determination of the stresses from the possibility of determining the deformations.) Of particular interest is the construction of the stress surface that is associated with our example! Its developable part is, however, too complicated to be described without a model. In order to still have an example for which this is easily possible, one considers, for the anchored beam of Fig. 5, the case of the so-called "pure shear," which then corresponds to the stress surface:

$$z = \frac{2M}{h^3} \cdot y^3,$$

which satisfies the differential equation $\Delta\Delta z = 0$ inside of the beam, where M is the shear moment that acts on the free end. This surface – obviously, a cylinder whose generator is parallel to the X-axis – now likewise defines the developable part of the stress surface of the beam.

Maxwell gave a further example in the last treatise that was cited on page 2. The beam has a height h and extends from $x = -l$ to $x = +l$. It is loaded along its upper boundary with the load K per unit length, and it further has a weight k per unit length. A null pressure acts on the center of the end surfaces at $x = \pm l$; a corresponding simplest-possible distribution of positive and negative pressure remains preserved on the individual elements of the end surfaces. Maxwell found:

$$F(x, y) = \frac{k+K}{2h^3}\left[(l^2 - x^2)(3hy^2 - 2y^3) + hy^4 - \frac{2y^5}{5} - h^3y^2\right]$$

for the stress function [9]). He also gave an interesting suggestion as to how the stress distribution given here could be realized experimentally in a clever way.

Precisely this example – and a large number of other examples – had already been treated by Airy himself in his treatise, as well as presenting some illustrations – namely, the stress trajectories – in which he introduced what he called the stress function for just that purpose [10]). He always assumed F to be a polynomial in x, y and thus took as many lower-order terms as possible that he could satisfy the boundary conditions. Thus, it is noteworthy that he completely ignored the fact that F must fulfill a partial differential equation in the interior of the beam that depends upon the elastic properties of the beam

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[9]) Since the weight is assumed to act on the elements of the beam interior, the stress components themselves assume the form:

$$P = \frac{\partial^2 F}{\partial y^2}, \qquad Q = \frac{\partial^2 F}{\partial x^2} - gy, \qquad U = -\frac{\partial^2 F}{\partial x \partial y}.$$

[10]) Citation on page 2.

(which is just the equation $\Delta\Delta F = 0$ if the beam is elastically isotropic). Maxwell already criticized this in his treatise, but likewise showed in the case just mentioned, which was examined closely by him, that errors of the sort that were present in Airy did not affect the numerical values of the stress components essentially.

§ 2.

On stress surfaces for plane discontinua (frames)

1. Equations (1), from which Airy concluded the existence of the stress function for a planar continuum that carries stresses, immediately have no sense when one is treating the stress distribution in a discontinuum, such as a planar frame. However, the stress function or the stress surface is somewhat more general than the equations that it was first obtained from; it exists just as well for discontinua as for continua. One must then (with Maxwell) simply use the formulas (4) that were obtained by integration as the basis, as will be discussed further in the sequel. In what follows, it will be our problem to discuss the special circumstances that arise from this for frames.

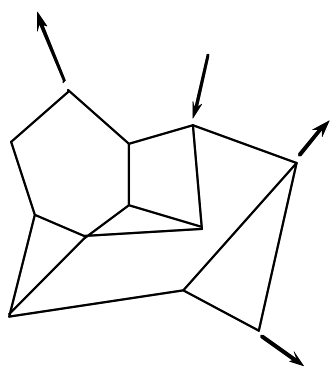

Fig. 6

In order to avoid unnecessary complications in the presentation we will throughout treat only those planar frames that define the image of a polygonal net whose elements nowhere cross and overlap, but all lie flat next to each other (Fig. 6). For them, we will assume, until later, that external forces act only on the junctions of the encircling polygon. At the conclusion of this paragraph, we will then also briefly treat some frames that do not fit into this schema. Naturally, we think of the junctions as *frictionless links*, such that only stresses in the normal direction of the rods – so-called "basic stresses" or principal stresses" – can be present.

K. Wieghardt set the goal for himself in what follows of starting with the complete analogy to the Maxwell Ansätzen and working out the connection that exists in a continuum between the Airy stress function and the Airy stress surface: *It shall be shown that the existence of equilibrium conditions for a frame is completely equivalent with the existence of an as-yet-to-be-defined stress surface.*

One achieves this objective in two steps: First, one shows that one can always give a stress surface, by means of which, a stress system can be defined, and which, by a simultaneous definition of a notion of force, is in equilibrium with our frame, and second, one shows the converse, that such a stress surface can be constructed from any stress distribution on the frame that is in equilibrium for the given notion of force.

We construct a surface over our frame that is composed of nothing but adjacent planar polygonal surfaces such that their edges, when projected onto the plane of the frame, yield precisely the rods of the frame. The construction of such a surface, which we would like to call a "faceted surface," is always possible; in the simplest case, it represents a single planar surface polygon. On this faceted surface we now fix a *"polyhedral zone"* as follows: Through each edge of the encircling polygon we lay a plane that is, however, not completely arbitrary, but subtends an angle different from 90^{o} with the plane of the frame and furthermore the edges created wherever two consecutive planes intersect possess the following properties: a) Of the two half-rays, into which each such edge will be divided by the vertex of the faceted surface periphery that it lies on, one of them – when projected onto the frame – shall always intersect the domain of the frame, but not the other. b) All half-rays that – when projected onto the plane of the frame – do not intersect the domain of the frame shall never intersect. Any two of these latter consecutive half-rays then intersect in a strip with the side of the faceted surface periphery that lies between them and the three planes that they have in common, and the surface that is composed of these strips is the desired polyhedral zone; it is the analogue of the developable surface of § 1. The denser the edges of the polyhedral zone are joined together, the more they will be comparable to the generators of a developable surface, while simultaneously our strips approach the generating strips of a developable surface.

Having constructed the total surface in this way from faceted surfaces and polyhedral zones such that it is continuous and covers the plane once, we must now consider it in more detail; we would like to see which stress distributions on the frame it gives rise to when we regard it as a stress surface. In any case, one thing is clear from the outset: that it gives us no information about *specific* stresses – i.e., stresses per unit length or area – if they are given by equations (2) by means of the *second* differential quotients of the stress surface, since the second differential quotients for our surface are either null, namely, in the interior of the individual facets and strips, or infinitely large, namely, on the edges. By contrast, the *first* differential quotients are finite everywhere, but also discontinuous on the edges. Accordingly, we define the *resultant* stresses on our surface by means of Maxwell equations (4) on any arc segment *ab* in the plane of the network. If we now first assume the formulas (4) for an arc segment that lies completely inside of a facet or a strip and then for an arbitrarily small arc segment *ab* that intersects a (projected) edge of the stress surface then we find that our surface, when regarded as a stress surface, mediates a system of stresses that is found to be in equilibrium and acts on the projected edges. If we then replace the stresses in the projected edges of the polyhedral zone with forces that act upon the vertices of the encircling polygon then we have arrived at a first result: *Any surface that is composed in the manner described of faceted surfaces and polyhedral zones defines for us an equilibrium system of external forces that act upon the corresponding frame and a stress distribution that is under the influence of these forces when in equilibrium.*

With that, the first step towards attaining our goal is completed, and we now take the second one. Any line of action of the force system will be divided into two half-rays by the vertex point of the encircling polygon that lies on it. We expect these half-rays to fulfill conditions a) and b), just like the corresponding half-rays in space. We now choose any junction of the frame and, for the sake of simplicity, make it the starting point of an *XYZ*-coordinate system, as shown in Fig. 7 (on pp. 12). The edges 1, 2, …, *n* and the angle-spaces I, II, …, *N* can collide at this junction. For one of the angle-spaces – say, I – we assume only an arbitrary facet (strip, resp.):

$$z = \alpha x + \beta y + \gamma.$$

We then proceed through the facets (strips, resp) II, III, …, N by going through our junction cyclically, while making sure that whenever we apply equations (4) to two consecutive planes I, II, …, N this always gives the same stress on the common edge. If X_i, Y_i are the components of the stress on the edge $J - 1$, J, as in Fig. 7, then the equations for these planes read [11]):

$$(14)\qquad \begin{cases} (1) & z = \alpha x + \beta y + \gamma, \\ (2) & z = -Y_2 \cdot x + X_2 \cdot y + \alpha x + \beta y + \gamma, \\ (3) & z = -(Y_2 + Y_3)x + (X_2 + X_3)y + \alpha x + \beta y + \gamma, \\ & \vdots \\ (i) & z = -\left(\sum_2^i Y_i\right) \cdot x + \left(\sum_2^i X_i\right) \cdot y + \alpha x + \beta y + \gamma, \\ & \vdots \\ (n) & z = -\left(\sum_2^n Y_i\right) \cdot x + \left(\sum_2^n X_i\right) \cdot y + \alpha x + \beta y + \gamma. \end{cases}$$

Now, since the n stresses on the junction are in equilibrium, the last equation is identical with:

$$z = Y_1 x - X_1 y + \alpha x + \beta y + \gamma,$$

and if we assume the points x, y are on the projected edge 1 then we would have: $z = \alpha x + \beta y + \gamma$, i.e., the entire cyclic sequence of n facets (facets and strips, resp.) is closed in itself. *Therefore, the fact that equilibrium reigns between the stresses at any junction of our frame implies the existence of a piece of the stress surface at the junction that surrounds it and is completely determined up to an arbitrary plane.*

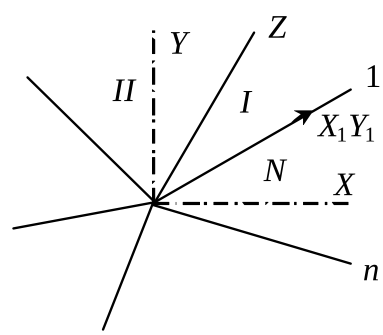

Fig. 7

We will seek to unite all of these pieces that are associated with the various junctions of the frame into a continuous, nowhere branching surface by a suitable choice of the arbitrary plane. Beginning with any junction, we shade the stress surface piece (Fig. 8, left) that was constructed around it (with an arbitrary plane). We enumerate the vertex points of the shaded polygon (which might possibly extend to infinity) cyclically by 1, 2, …, m. We can now identify the plane, which is arbitrary for the stress surface piece around 1, with one of the shaded surfaces that contact 1 (perhaps I); then M also belongs to this surface piece, since in such surface pieces the opposite position to two consecutive planes is completely determined by the stress in their common (projected) edge. Hence, the stress surface piece around 1 is smoothly connected with the

[11]) Formulas (14) and (15), which will be operated with here and on page 14, were first presented by F. Klein in a lecture during the Summer of 1896; cf., the Henneberg Enzyklopädie reference. They correspond to the formulas that were presented in (13) of § 1 for the developable surface that was considered there (which is a limiting case of the polyhedral zone considered here).

shaded polygon. We can then smoothly connect the pieces of the stress surfaces 2, 3, …, $m - 1$ on the shaded surfaces II, III, …, $M - 1$ one after the other into our shaded polygon, with the presently arbitrary planes identified. The question still remains of whether the last surface piece constructed Z (facet or strip) smoothly connects with the first surface piece constructed A. However, this must be the case if we choose (Fig. 8, right) the plane A to be the arbitrary plane at the junction m and construct the piece of the stress surface around m, since then we can arrive at no other facets (strips) than M, $M - 1$, …, Z since the opposite location to two neighboring planes of this sequence is completely determined by the stress in the common (projected) edge. The sequence A to Z – as the piece of the stress surface around m – then likewise closes on itself. If one now shades all of the stress surface pieces constructed up to now and repeats the construction that we just described at the vertices of the now-shaded larger polygon then one ultimately arrives, in fact, at a continuous, closed in itself, *"single-valued"* stress surface that covers the entire plane simply. With that, our goal is achieved; in summation, we say:

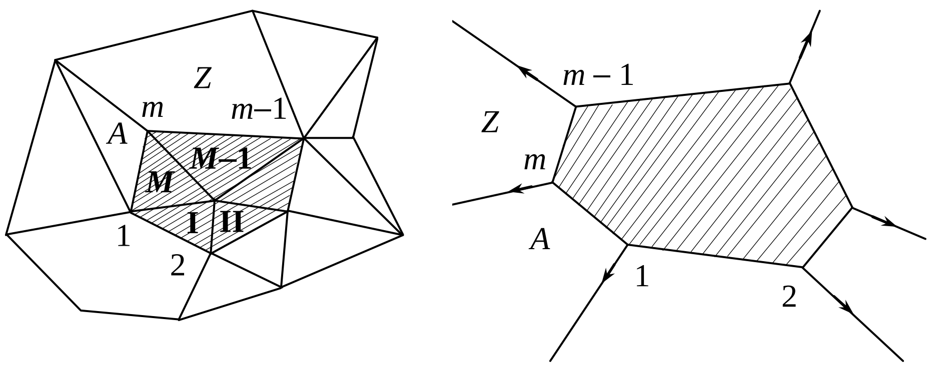

Fig. 8.

For a planar frame that is composed of nothing but smoothly neighboring polygons, and whose junctions are frictionless links, and on which external forces of the described type act only upon the junctions of the encircling polygon, the existence of the equilibrium conditions is completely equivalent to the existence of a continuous and everywhere single-valued stress surface. The stress surface consists of a polyhedral zone whose edges, when projected, yield the lines of action of the system of forces, and a facetted surface whose edges, when projected, deliver the rods of the frame.

If the external forces do not satisfy both of the restrictions that we made then complications can appear that hardly affect the understanding of the problem, but do affect the two-dimensional representation, insofar as the polyhedral zone can be become very complicated (similar to the developable surface in the first beam example of § 1, pp. 8); we shall therefore not go into them further.

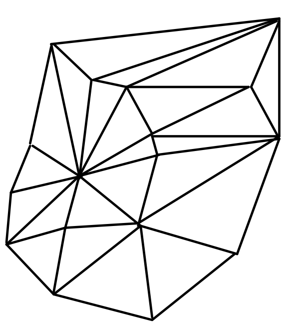

Fig. 9.

2. A lovely application of the theorem just derived is the following one: Let the frame be composed of nothing but triangles (Fig. 9); it is subjected to the influence of some equilibrium system of forces that act on the junctions of the encircling polygon. The question is this: How many stress surfaces are there for this given system of forces; in other words, how statically indeterminate is the frame? We construct – in direct reproduction of the developable surface for a continuum (equation (13)) – the polyhedral zone that is defined by our system of forces X_i, Y_i, M_i, when we set down the following sequence of strips [12]):

$$(15)\qquad \begin{cases} z = \alpha x + \beta y + \gamma, \\ z = -Y_2 x + X_2 y - M_2 + \alpha x + \beta y + \gamma, \\ z = -(Y_2 + Y_3)x + (X_2 + X_3)y - (M_2 + M_3) + \alpha x + \beta y + \gamma, \\ \quad\vdots \\ z = -\left(\sum_2^i Y_i\right)\cdot x + \left(\sum_2^i X_i\right)\cdot y - \left(\sum_2^i M_i\right) + \alpha x + \beta y + \gamma, \\ \quad\vdots \\ z = -\left(\sum_2^m Y_i\right)\cdot x + \left(\sum_2^m X_i\right)\cdot y - \left(\sum_2^m M_i\right) + \alpha x + \beta y + \gamma, \\ z = -\left(\sum_1^m Y_i\right)\cdot x + \left(\sum_1^m X_i\right)\cdot y - \left(\sum_1^m M_i\right) + \alpha x + \beta y + \gamma = \alpha x + \beta y + \gamma. \end{cases}$$

The faceted surfaces on the periphery of our frame are likewise established by means of this closed polyhedral zone, which is completely determined up to the arbitrary plane $z = \alpha x + \beta y + \gamma$. However, the coordinates of the faceted surface over any junction inside of the encircling polygon can be chosen in a completely arbitrary way since a plane can indeed be defined by three completely arbitrary points. *Hence, the degree of the static indeterminacy of our triangular frame is simply equal to the number of its "internal" junctions.*

Thus, should the problem of determining the stress surface of a plane triangular frame possess a unique solution one would have to make special assumptions about the physical nature of the rods of the frame that are analogous to their behavior as continua. We therefore do not go into this here; the case in which the rods are elastic, in the sense of Hooke's law, was examined by K. Wieghardt in a special treatise [13]).

3. The stress surfaces that we obtained up to now were always single-valued surfaces; a point x, y always corresponded to just a single value z. One can, with little effort, define examples of multi-valued stress surfaces that likewise lead to the stress systems of planar frames. Thus, there is a spatial polyhedron that is composed of planar polygons and closed in itself that can be regarded as the stress surface of a *system of self-stresses* in that frame that comes about as its orthogonal projection. There is no

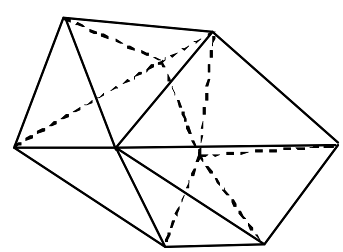

Fig. 10.

[12]) Cf., the footnote on pp. 12.

[13]) [Appearing in the Verhandlungen des Vereins zur Förderung des Gewerbefleisses in Preussen, 85 (1906)].

difficulty in seeing this: Formulas (4) are also valid here, only one must pay attention to the fact that the frame surface now covers the surface of its encircling polygon *twice*, and therefore at a point x, y the differential quotients $\partial z / \partial x$, $\partial z / \partial y$ have different values according to whether one finds them in the upper "sheet" or the lower one. One must further observe that for the lower sheet the signs in formulas (4) are inverted. The detailed behavior of these figures was treated quite thoroughly by Maxwell himself [14]).

By the way, such a self-covering frame can also give rise to the existence of a *single-valued* stress surface; namely, if one knows the stress system then one can conversely construct a single-valued stress surface. One simply regards all of the geometric intersection points of the rods as actual junctions and then constructs the stress surface that they generate, with the encircling polygon corresponding to the simply-covering frame of the given stress system. However, since any stress surface of this frame does not conversely have meaning for our self-covering frame this is of only secondary interest. An interesting question is the following: What is the general situation for the single or multi-valued stress surfaces of such self-covering frames that lead to the so-called "one-sided" surfaces, when regarded spatially? (Fig. 11.) [15])

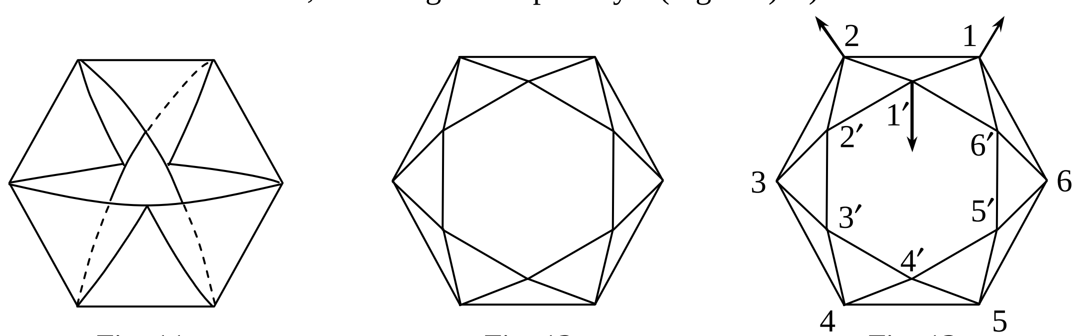

Fig. 11. Fig. 12. Fig. 13.

Conversely, there are, however, also *multi-valued* stress surfaces for our frames that do *not* cover them. For example, if no external forces act then all of the strips of the polyhedral zone lie in a plane, and we regard precisely that part of this plane as the polyhedral zone that extends the faceted surface to a closed spatial polyhedron. We consider the frame of Fig. 12 as an example. We ask: How many self-stresses are possible in it? The associated spatial polyhedron is obviously composed of two hexagons and twelve triangles. Had we established the two hexagons arbitrarily then it would be completely determined. Now, since the opposite location (?) to two planes includes essentially three arbitrary parameters it follows easily from our construction that there are ∞^3 self-stresses in our frame (of the form: $S = aS_1 + bS_2 + cS_3$).

Another interesting example of multi-valued stress surfaces is provided by the *"multiply connected"* frames. We call a frame that does not cover itself multiply connected when not all of the internal junctions are force-free. Thus, the frame that we just mentioned is multiply connected for the loading of Fig. 13. We seek to construct a stress surface for it! We regard the entire frame as one such with two encircling polygons – an outer and an inner hexagon – and then begin to fasten a polyhedral zone for the

[14]) See reference a) on pp. 2.

[15]) [I have answered this, in any case, theoretically important question. See the treatise LXXVIII that follows this one. K]

corresponding external forces to both vertical cylinders over these two encircling polygons. Let the forces on the outer hexagon be X_i , Y_i , M_i and on the inner one X_i', Y_i', M_i'. In order for all of them to be in equilibrium it is only necessary that:

$$\text{(16)} \qquad \sum_1^6 (X_i + X_i') = 0, \qquad \sum_1^6 (Y_i + Y_i') = 0, \qquad \sum_1^6 (M_i + M_i') = 0,$$

while the individual sums $\sum_1^6 X_i$, $\sum_1^6 X_i'$, etc., can very well be non-zero. If we now construct the external polyhedral zone, by enumerating the strips:

$$\text{(17)} \qquad z = -\left(\sum_0^\nu Y_i\right)\cdot x + \left(\sum_0^\nu X_i\right)\cdot y - \left(\sum_0^\nu M_i\right), \qquad (\nu = 0, 1, 2, \ldots)$$

just as in the prescription of equations (15) – where an arbitrary plane is considered through the three quantities X_0, Y_0, M_0 – then this sequence does not close under a complete circuit, and moreover, the z of the polyhedral zone increases with each circuit by the period:

$$\text{(18)} \qquad z_0 = -\left(\sum_1^6 Y_i\right)\cdot x + \left(\sum_1^6 X_i\right)\cdot y - \left(\sum_1^6 M_i\right),$$

so we do not have a *closed polyhedral zone*, but an *"affine-periodic" one, in the sense of these formulas*. We obtain corresponding formulas for the internal polyhedral zone, only we must invert all signs when applying formulas (15) in order that under the application of equations (4) to two neighboring strips the correct stress is associated with the common edge. We thus have the sequence of planes:

$$\text{(19)} \qquad z' = \left(\sum_0^\nu Y_i'\right)\cdot x - \left(\sum_0^\nu X_i'\right)\cdot y + \left(\sum_0^\nu M_i'\right), \qquad (\nu = 0, 1', 2', \ldots)$$

where the three quantities X_0', Y_0', M_0' again represent the arbitrariness in a plane. (The ∞^3 self-stresses of the frame are expressed by the three quantities: $X_0 - X_0'$, $Y_0 - Y_0'$, $M - M_0'$!) We thus obtain the period:

$$\text{(20)} \qquad z_0' = \left(\sum_1^6 Y_i\right)\cdot x - \left(\sum_1^6 X_i\right)\cdot y + \left(\sum_1^6 M_i\right),$$

and, from formulas (16), this is equal to the period z_0 above. Now, since the faceted surface is also given for the two polyhedral zones, as is immediately obvious, we have the result: *The entire stress surface has a period determined by formula* (18) *or* (19) *corresponding to a cyclic circuit of our frame; it is composed of two unclosed polyhedral zones and an unclosed faceted surface that winds upwards like a spiral staircase.*

Naturally, none of this infinite winding upwards of the spiral staircase is noticed in the projection of the stress surface onto the plane of the frame; it covers the projection simply and will thus be unknowable.

The aforementioned behavior has nothing surprising to say about the things that one trusts in the integration of exact differentials of first order, perhaps from function theory. If:

$$df = p\,dx + q\,dy$$

is such a differential and one integrates f over a ringlike-connected domain then f takes on an additive period under a circuit of the ring. Precisely the same thing is true for the stress function that is defined by the second differential:

$$d^2F = Q \cdot dx^2 - 2U \cdot dx\,dy + P \cdot dy^2,$$

except that the additive period is not a constant, as in the previous case, but a linear entire rational function of x and y. – It would be interesting to reconstruct the behavior that was discussed here only in an abstract, analytical fashion for numerous examples *in concreto.*

§ 3.

On reciprocal figures and diagrams.

1. a) Any rectilinear *line segment* (in a plane) with the endpoints a (x_a, y_a) and b (x_b, y_b) possesses, *ab initio,* a certain *length* and a certain *direction*, but no definite *sense.* A line segment thus defines for us, in the simplest form, two vectors that are parallel to it, namely, the two vectors with the components:

$$\Xi = x_b - x_a\,, \qquad \mathrm{H} = y_b - y_a\,, \qquad \text{on the one hand,}$$

and

$$\Xi = -(x_b - x_a), \qquad \mathrm{H} = -(y_b - y_a), \qquad \text{on the other,}$$

as well as two vectors that are perpendicular to them:

$$\Xi = -(y_b - y_a), \qquad \mathrm{H} = x_b - x_a), \qquad \text{on the one hand,}$$

and

$$\Xi = y_b - y_a, \qquad \mathrm{H} = -(x_b - x_a), \qquad \text{on the other.}$$

We would like to call the first two *polar vectors* to our line segment, and the last two the associated *transversal vectors.* When we now assign our line segment a definite *sense of motion* we call it "the line segment (a b)" or the "the line segment (b a)," according to whether we think of ourselves as moving from a to b or from b to a. For the sake of illustration, one will interpret the sense of motion as saying that one understands the line segment to be endowed with an arrowhead. We can now associate each of the two "line segments with arrowheads" with one of the two polar vectors, and likewise, one of the two transversal vectors, such that when one of the two possibilities is established once and for all each of the two line segments with arrowheads is associated with a completely well-defined polar vector, as well as a completely well-defined transversal vector. We now deal with the arbitrariness in the following way:

By the phrase "the polar vector that belongs to the line segment (a b)" – or briefly "the *polar* vector (a b)" – we understand this to mean the vector with the components:

$$\Xi = x_b - x_a\,, \quad \mathrm{H} = y_b - y_a\,;$$

and by the phrase "the transversal vector that belongs to the line segment (a b)" – or, briefly, "the *transversal* vector (a b)" – we understand this to mean the vector with the components:

$$\Xi = y_b - y_a\,, \quad \mathrm{H} = -\,(x_b - x_a).$$

With these abbreviations, one obtains the transversal vector (a b) in the coordinate system that we have always used, when one rotates the polar vector (a b) 90^{o} clockwise.

b) Any simply-connected *planar region* with a boundary curve that does not intersect itself possesses a certain *surface area* and a certain *normal direction*. A planar region of this type thus defines, in the simplest way, two vectors whose lengths are equal to the surface area and whose directions are equal to the normal directions. When we assign our planar region one or the other *sense of circulation* and each of these senses of circulation is again one of the two previously-defined vectors through the planar region, once we have eliminated the arbitrariness once and for all, each "planar region with sense of circulation" – or briefly, each "planar magnitude – is associated with a completely well-defined vector.

A *curved surface region* with a definite sense of circulation also defines a completely well-defined vector; it may be, as we can say, "regarded as a planar magnitude," namely: One divides it into infinitely many, infinitely small planar regions. One assigns each planar region with a sense of circulation such that the boundary of the surface region preserves its original sense of circulation and each edge between two neighboring planar regions is associated with both senses of circulation. If one associates each planar region with the vector that was described above and sums over all of these infinitely many, infinitely small vectors then one obtains a completely well-defined resultant vector that is, moreover, defined by the surface region with the sense of circulation.

c) All of these ideas, which have been more or less well-known since Grassmann's "Ausdehnungslehre," take on a practical meaning when one treats the graphical representation of the stresses in a continuous or discontinuous medium (frame). For example, a plate is governed by an equilibrium system of stresses. If we cut the plate along any arc segment then we perturb the equilibrium, insofar as we eliminate the stresses that prevail along the arc segment. Any two of these forces, which act on the two lips, but at the same separation locus, are then equal and opposite and completely measure the stress that occurs on the arc element of the separation locus. (We can also sum all individual forces along each lip and thus obtain two resultants that are equal and opposite and measure the stresses that occur on the entire arc segment.)

From the previous remarks, it is clear that *a line segment* $\alpha\beta$ *– and indeed, a simple line segment with no sense of motion – is quite sufficient for the graphical representation of the stress that belongs to a definite cross-section.* Its length and direction then provide the length and direction of the stress, with nothing further (the stress direction is either parallel to the direction of the line segment or perpendicular to it). *However, by a certain abbreviation, it also provides a sense – i.e., the sign – for the stress. Two* senses of motion are then assigned to it; a certain (polar or transversal) vector belongs to each of these two senses of motion, although we can, in an abbreviated way that is fixed once and for all, assign each of the two senses of motion with one of the two lips of the cross-section locus on which the stress acts. Hence, through the intermediary of our line segment, each lip of the cross-section locus in question relates to a definite vector and this assignment may naturally be arranged so that this vector directly represents the stress that acts on this lip, from which the sense of the stress is established. One can proceed, e.g.: After the notations $\alpha\beta$ and ab are introduced on the cross-section, one thinks of α as corresponding to a, β, to b, and thus, the vector $(\alpha\beta)$, to the vector (ab). On the other hand, one assigns the vector (ab) to perhaps the lip of the cross-section that lies to the left of an advance of the cross-section in the direction from a to b. In the given case, the vector $(\alpha\beta)$ then represents the force that acts on this lip and the vector $(\beta\alpha)$, the one on the right.

Naturally, in a completely analogous way, a *planar region (surface region, resp.) with no definite sense of circulation* is a very suitable means for the geometric representation of the stress that acts on an associated surface element.

We will learn some examples of these general developments in § 4.

2. If one could "designate" planes in space with the same facility as one does for straight lines in the plane then one would, by the means of the stresses on a planar frame, plausibly direct one's primary attention to the stress surface, since it indeed represents all of the stress phenomena simply and clearly. We do not actually possess this capability now; one must therefore initially endeavor to find *planar* figures that exhibit the same behavior as stress surfaces. One calls these figures "*force planes*"; we will see that they are closely connected with stress surfaces.

We would like to regard the force plane of a frame, as Maxwell himself did, as a special case of Maxwell's *"reciprocal planar diagrams;"* we will therefore arrive at the m as the simplest ones. Maxwell's definition is the following: In the xy-plane, one finds any continuum or discontinuum with the stress surface: $z = F(x, y)$. We let it correspond to a definite continuum or discontinuum in the $\xi\eta$-plane that is defined by the equations:

$$\xi = \frac{\partial F}{\partial x}, \qquad \eta = \frac{\partial F}{\partial y}, \tag{21}$$

and has the stress surface:

$$\zeta = \Phi(\xi, \eta), \tag{22}$$

where Φ is defined by the equation:

$$F + \Phi = x\xi + y\eta. \tag{23}$$

The stress surfaces thus defined in xyz-space and $\xi\eta\zeta$-space have a reciprocal relationship to each other that one can express in the sense of projective geometry when one says: Each of the two stress surfaces is always the *polar image* of the other one relative to the paraboloid:

$$2z = x^2 + y^2. \tag{24}$$

Insofar as the interpretation of the surfaces F and Φ as stress surfaces can also be overlooked, since their reciprocity obviously does not depend upon it, we would like to speak of them more generally as *"reciprocal (spatial) figures."* The two plane figures that one obtains by projection of the reciprocal figures onto the xy-plane and the $\xi\eta$-plane, which we would like to call "diagrams," for the sake of brevity, are also reciprocal, if, along with equations (21), the reciprocal equations:

$$x = \frac{\partial \Phi}{\partial \xi}, \qquad y = \frac{\partial \Phi}{\partial \eta}, \tag{25}$$

are also valid, as one confirms by differentiating equation (23). We thus speak of the diagrams as *"reciprocal (planar) diagrams."*

What does the $\xi\eta$-diagram now tell us if we would like to instruct ourselves in the stresses that were elicited by the stress function F in the xy-diagram? In the xy-plane, we denote an arc element by ds and in the $\xi\eta$-plane, by $d\sigma$ (with the components $d\xi$, $d\eta$). From equations (3), one has:

$$X\,ds = d\eta, \qquad Y\,ds = -\,d\xi. \tag{26}$$

Hence, for a finite arc segment ab, which might correspond to the arc segment $\alpha\beta$:

$$X_r = \eta_\beta - \eta_\alpha, \qquad Y_r = -\,(\xi_\beta - \xi_\alpha). \tag{27}$$

The connecting line segment of the two endpoints α and β of a finite or infinitely small arc segment in the $\xi\eta$-plane provides, when regarded as a transversal vector, the magnitude, direction, and sense of the resultant stress that appears on the corresponding arc segment ab in the xy-plane; indeed, the transversal vector ($\alpha\beta$) provides the action of the stress on that lip of the cross-section that lies on the left-hand side when one goes from a to b, and the transversal vector ($\beta\alpha$) provides the action on the right-hand side. The rule is valid only as long as the xyz-figure is single-valued, and in the other case, where the xy-diagram thus doubly covers the plane, one must switch left and right in the rule above for the lower leaf.

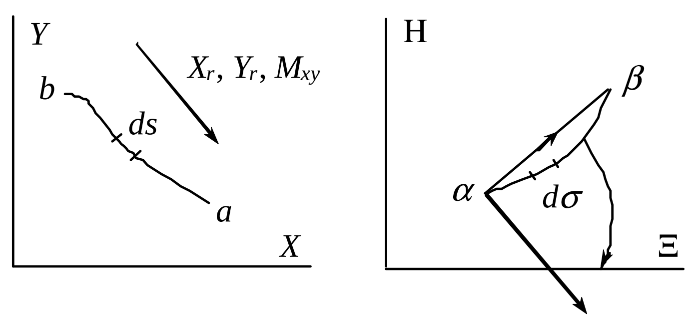

Fig. 14.

One finds the case of *continuous diagrams* discussed and illustrated by examples in Maxwell [16]); there, he treated the example from beam theory that was mentioned above (pp. 8). Here, in order to arrive at the behavior for frames, we would like to consider the case where the xyz-figure is a closed spatial polyhedron composed of planar polygons.

[16]) In the treatise of reference f), pp. 2 (Table XIV).

Due to its polar relationship relative to the paraboloid, the $\xi\eta\zeta$-figure is also like it, and indeed each polygon of the one polyhedron mutually corresponds to a vertex of the other, each vertex, to a polygon, and each edge, to an edge. Corresponding statements are then true for two reciprocal diagrams, and in addition, corresponding edges are perpendicular to each other and the edges of one diagram – when interpreted in the most beautifully described way as transversal vectors – yield stresses on the corresponding edges on the other diagram that are in equilibrium for that diagram. Since the spatial polyhedron is a closed surface, each diagram covers the surface of its encircling polygon at least doubly (Fig. 15.)

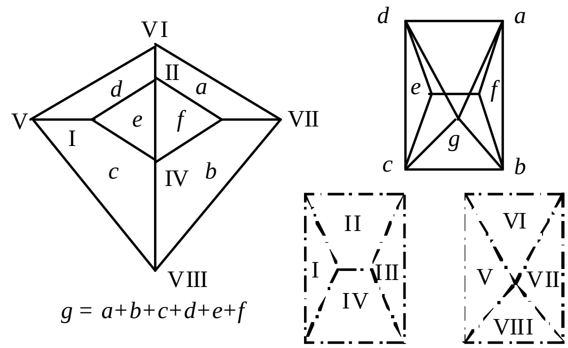

Fig. 15.

3. This Ansatz has a different meaning for the statics of frames. We can, e.g. (cf. § 2), immediately regard either of the two diagrams as a frame, and the other one provides a possible system of self-stresses in this frame; however, we can also proceed as follows: We single out any polygon of one of the two spatial polyhedra and assert: The polygon is the stress surface that belongs to a frame acted upon by external forces, and whose polyhedral zones is cut from a plane – viz., the plane of the distinguished polygon. In the projection – into the diagram – we then must regard the edges that intersect the vertices

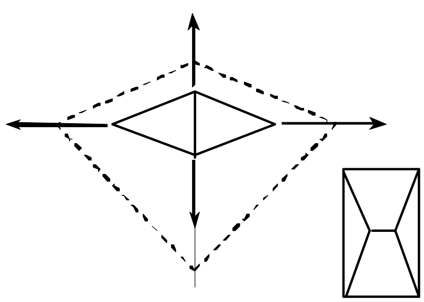

Fig. 16.

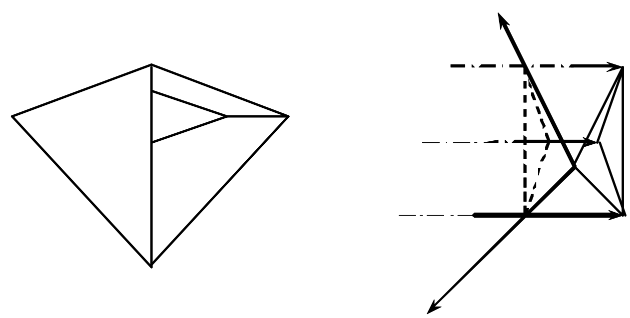

Fig. 17.

of the distinguished polygon as the line of action of an equilibrium system of forces, the projection of the distinguished polygon as a so-called wire polygon (Seilpolygon) of this force system, and the remaining edges as the rods of a frame that is under the influence of this system of forces. We then obtain a *force plane* of the thus-defined frame using these forces when we omit the superfluous lines in the reciprocal diagrams – they are the corresponding edges of the wire polygon. We then obtain, e.g., when we distinguish the polygon *g* on the left in Fig. 15, the assignment of Fig. 16, and the assignment of Fig. 17 when we distinguish the polygon I on right in Fig. 15.

The force plane of a frame naturally includes just as many essential indeterminacies as the stress surface of the frame. This fact, that there are just as many force planes for a frame as solutions to the stress problem, is usually not very clearly emphasized in the textbooks on graphical statics, which might imply that one is essentially occupied at such a time with statically determinate frames, since the force plane naturally allows no essential arbitrariness.

Of particular interest are the force planes that belong to the multiply-connected frames of § 2. Corresponding to the circumstance that the stress surface of such a frame is affine-periodic, the force plane is no longer a closed figure, but consists of the parallel, congruent repetition of one and the same basic figure.

A noteworthy fact must still be mentioned here, namely, that one, as is known in practice, mostly does not operate on the *Maxwellian force plane*, but on the so-called *Cremona force plane*. This is nothing but the Maxwell force plane rotated through a right angle; in fact, when Cremona gave the self-sufficient basis for his theory, he expressly referred to Maxwell [17]). As is known, he used, in place of the Maxwell formulas (21) and (23), the following ones to define the reciprocal figures:

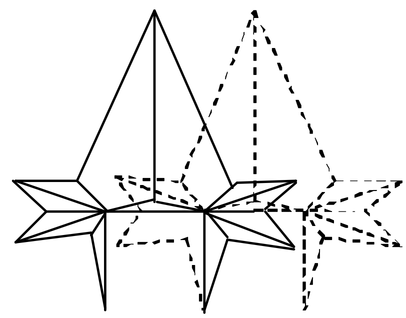

Fig. 18.

$$\xi = -\frac{\partial F}{\partial y}, \qquad \eta = \frac{\partial F}{\partial x}, \qquad z - \zeta = \eta x - \xi y\,.$$

This means the same thing as: In place of the polar correspondence relative to the paraboloid of eq. (24), one has a polar correspondence relative to a *Möbius null system* by means of the Cremona equations. If Maxwell associated the plane:

$$z = \alpha x + \beta y + \gamma$$

with the point:

$$\xi = \alpha, \qquad \eta = \beta, \qquad (\zeta = -\gamma),$$

then Cremona associated it with the point:

$$\xi = -\beta, \qquad \eta = \alpha, \qquad (\zeta = \gamma);$$

thus, for Cremona the edges of the force plane run parallel to the corresponding rods of the frame, while, for Maxwell, they are perpendicular; the stresses will thus no longer be represented by *transversal* vectors, but by *polar* ones. If one is now inclined to use the Cremona association in practice then this certainly lies partly in the fact that regarding a line segment as a *polar* vector is familiar in general mechanics, while regarding the line segment as a *transversal* vector is somewhat foreign, and also partly in the fact that one might find it more convenient to draw the parallels to a given line as directed perpendicular to it. *Theoretically, the Maxwell association serves the purpose, in any case, because it alone allows a generalization to space* (which we will likewise do).

[17]) See, especially: L. Cremona: Les figures réciproques en statique graphique (transl. by Bossut), Paris 1885, pp. 7 and 8. Then: L. Cremona, Le figure reciproche nella statica grafica, Mailand 1872; 3rd ed., with introduction by G. Jung, Mailand 1879.

Incidentally, the entire chapter "Reciprocal diagrams" in Maxwell might be more interesting to read than the one in Cremona. Overlooking the fact that Cremona restricted himself to the consideration of discontinua (frames), the theory to him seemed trivial, since the idea of stress surface is not emphasized, and that is the quintessence of the entire theory.

4. Henceforth, we, along with Maxwell, would like to spatially generalize formulas (21) to (25), and then introduce *reciprocal spatial diagrams* in a purely geometric way. In *xyz*-space, let any figure be given – a "spatial diagram, – and furthermore, a function $F(x, y, z)$, which we will, however, later interpret as a "stress function" that belongs to this diagram. With the help of the equations:

$$\xi = \frac{\partial F}{\partial x}, \qquad \eta = \frac{\partial F}{\partial y}, \qquad \zeta = \frac{\partial F}{\partial z}, \tag{28}$$

we associate this diagram with a second diagram in a $\xi\eta\zeta$-space. The relationship between both diagrams is then a reciprocal one: If we would like to define a function $\Phi(\xi, \eta, \zeta)$ through the equation:

$$F + \Phi = x\xi + y\eta + z\zeta, \tag{29}$$

then one has:

$$x = \frac{\partial \Phi}{\partial \xi}, \qquad y = \frac{\partial \Phi}{\partial \eta}, \qquad z = \frac{\partial \Phi}{\partial \zeta}, \tag{30}$$

which one easily confirms by differentiating eq. (29).

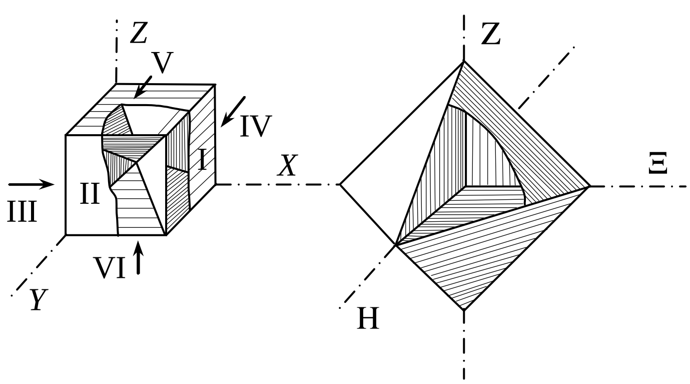

Fig. 19.

In particular, we now define "*reciprocal cell systems,*" just as in the Maxwell process, and indeed, as a spatial analogue of the frame diagrams of Figures 10, 12, 15, which doubly cover the surface of a certain encircling polygon. We think of there being a configuration of adjacent polyhedra (cells) in *xyz*-space that doubly fills up the volume of a certain closed encircling polyhedron. We thus obtain a first spatial cell system, whose cells, faces, edges, and vertices we shall speak of (Fig. 19 left). We then define a continuous function $F(x, y, z)$ of a sort that, inside of the individual cell J, it always agrees with a linear function $a_i\, x + b_i\, y + c_i\, z + d_i$, and that when J and K come together in the face JK the equation of this face will be represented through:

$$(a_i - a_k)\, x + (b_i - b_k)\, y + (c_i - c_k)\, z + d_i - d_k = 0.$$

This cell system then corresponds to a second cell system in $\xi\eta\zeta$-space by means of formulas (28). Both cell systems have the following reciprocal relationship: *Each cell of a diagram corresponds to a vertex of the other one, each vertex, to a cell, each face, to an edge, and each edge, to a face. Each edge of one system is perpendicular to the corresponding face of the other system.*

Inspired by the example that K. Wieghardt constructed in Fig. 19, the relationship is as follows. We have:

Left:		Right:	
7 cells	(The cube itself and the 6 pyramids into which it decomposes when one intersects it along the twelve triangles that run from the center of the cube to the edges)	7 vertices	(The 6 octahedral vertices and the midpoint)
18 faces	(The 6 cube faces and the 12 isosceles triangles with the 12 cube edges as bases and its midpoint as vertex)	18 edges	(The 12 octahedral edges and the 6 line segments that extend from its midpoint to the octahedral vertices)
20 edges	(The 12 cube edges and the 8 line segments that extend from its midpoint to the cube vertices)	20 faces	(The 8 octahedral faces and the 12 isosceles right triangles with the 12 octahedral edges as hypotenuses and the midpoint as vertices)
9 vertices	(The 8 corners of the cube and its midpoint)	9 cells	(The octahedron itself and the 8 tetrahedra into which it is divided when one intersects it along the three planes that go through each four octahedral vertices)

The function $F(x, y, z)$ has the following values in the individual cells:

In the cubic cells:		Zero
In the pyramidal cells:	I:	$x - 1$
" "	II:	$y - 1$
" "	III:	$-x$
" "	IV:	$-y$
" "	V:	$z - 1$
" "	VI:	$-z.$

We will give an application of this reciprocal cell system for the purposes of mechanics in the next paragraph.

§ 4.

Comments on spatial stress systems and associated stress functions

1. The thought certainly suggests itself that we might connect the equilibrium conditions for the stresses in a *spatial* continuum (Fig. 20):

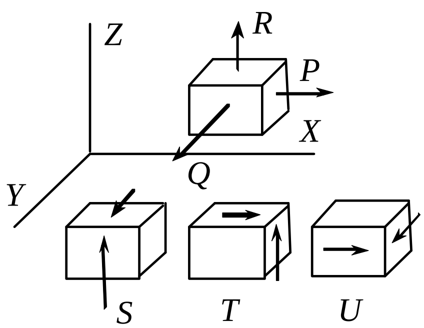

Fig. 20.

$$(31)\qquad \begin{cases} \dfrac{\partial P}{\partial x} + \dfrac{\partial U}{\partial y} + \dfrac{\partial T}{\partial z} = 0, \\[2mm] \dfrac{\partial U}{\partial x} + \dfrac{\partial Q}{\partial y} + \dfrac{\partial S}{\partial z} = 0, \\[2mm] \dfrac{\partial T}{\partial x} + \dfrac{\partial S}{\partial y} + \dfrac{\partial R}{\partial z} = 0, \end{cases}$$

(in the event that only outer surface forces act on it)

with a function $F(x, y, z)$ that would be analogous to the Airy Ansatz. In that regard, Maxwell found that, as one may also assume this to be true, one does not arrive at *all possible* stress systems for the continuum in this way, since the equations above must be related to three different functions, moreover. He examined this move closely.

There are thus special, but interesting, spatial stress distributions that we obtain when we, with Maxwell, now make the following two different spatial extensions of the Airy formulas (2):

The *first Ansatz* is:

$$(32)\qquad \begin{cases} P = \Delta F - \dfrac{\partial^2 F}{\partial x^2}, \quad Q = \Delta F - \dfrac{\partial^2 F}{\partial y^2}, \quad R = \Delta F - \dfrac{\partial^2 F}{\partial z^2}, \\ S = -\dfrac{\partial^2 F}{\partial y\,\partial z}, \qquad T = -\dfrac{\partial^2 F}{\partial z\,\partial x} \qquad U = -\dfrac{\partial^2 F}{\partial x\,\partial y}, \end{cases}$$

[where we now let $\Delta = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$].

The *second Ansatz* consists of assuming that the P, Q, U of the Airy Ansatz can be defined by the following formula, in which α, β are understood to mean arbitrary quantities:

$$P \cdot \alpha^2 + 2U \cdot \alpha\beta + Q \cdot \beta^2 = -\begin{vmatrix} \dfrac{\partial^2 F}{\partial x^2} & \dfrac{\partial^2 F}{\partial x\,\partial y} & \alpha \\ \dfrac{\partial^2 F}{\partial y\,\partial x} & \dfrac{\partial^2 F}{\partial y^2} & \beta \\ \alpha & \beta & 0 \end{vmatrix},$$

which yields the following spatial generalization:

$$(33)\qquad \begin{cases} P \cdot \alpha^2 + Q \cdot \beta^2 + R \cdot \gamma^2 \\ +2S \cdot \beta\gamma + 2T \cdot \alpha\gamma + 2U \cdot \alpha\beta \end{cases} = -\begin{vmatrix} \dfrac{\partial^2 F}{\partial x^2} & \dfrac{\partial^2 F}{\partial x\,\partial y} & \dfrac{\partial^2 F}{\partial x\,\partial z} & \alpha \\ \dfrac{\partial^2 F}{\partial y\,\partial x} & \dfrac{\partial^2 F}{\partial y^2} & \dfrac{\partial^2 F}{\partial y\,\partial z} & \beta \\ \dfrac{\partial^2 F}{\partial z\,\partial x} & \dfrac{\partial^2 F}{\partial z\,\partial y} & \dfrac{\partial^2 F}{\partial z^2} & \gamma \\ \alpha & \beta & \gamma & 0 \end{vmatrix},$$

where α, β, γ mean arbitrary quantities.

One easily confirms that the stress components of (32) and (33) satisfy equations (31) for an arbitrary F.

2. The content of formulas (32) and (33) may be better described when we link them with the idea of reciprocal diagrams. The statement of the question is then: For a given function $F(x, y, z)$, there is given, on the one hand, a reciprocal relationship between an xyz-diagram and a $\xi\eta\zeta$-diagram according to formulas (28), and, on the other hand, a stress distribution in the xyz-diagram according to formulas (32) or (33). Of what use to us is the knowledge of the $\xi\eta\zeta$-diagram if we would like to learn about this stress distribution?

Let do be a surface element in xzy-space with the normal n, and let $d\varpi$ be the corresponding surface element in $\xi\eta\zeta$-space. *We then find, by an application of the first*

Maxwell Ansatz (equations (32)) *to do, first, a normal stress of magnitude* ΔF *per unit area, and second, a stress [in the surface element itself] with the components* $\partial\xi / \partial n$, $\partial\eta / \partial n$, $\partial\zeta / \partial n$ *per unit area. An application of the second Ansatz* (33), *however, gives us simply* $d\varpi$, *regarded as a planar quantity, which gives the stress on do as a magnitude, direction, and sense.* If we are concerned with the resultant stress on a *finite* surface piece *o* then we find, by an application of the first Ansatz, its components are determined by means of the equations:

$$(34)\quad \begin{cases} X_r = \iint \left(\Delta F \cdot \cos nx - \dfrac{\partial\xi}{\partial n}\right) do, \quad Y_r = \cdots, \quad Z_r = \cdots, \\ M_{yz} = \cdots, \quad M_{zx} = \cdots, \quad M_{xy} = \iint \left\{ x\left(\Delta F \cdot \cos nx - \dfrac{\partial\eta}{\partial n}\right) - y\left(\Delta F \cdot \cos nx - \dfrac{\partial\xi}{\partial n}\right)\right\} do, \end{cases}$$

in which we convert the surface integral into an integral over the boundary curve by means of the equations:

$$(35)\quad \begin{cases} X_r = \int \eta\, dz - \zeta\, dy, \quad X_r = \cdots, \quad X_r = \cdots, \\ M_{yz} = \cdots, \quad M_{zx} = \cdots, \quad M_{xy} = \int \zeta (x\,dx + y\,dy + z\,dz) - (x\xi + y\eta + z\zeta - F)dz. \end{cases}$$

An application of the second Ansatz, however, tells us the resultant stress on *o* as a magnitude, direction, and sense (we shall not bother with the three rotational moments), the surface piece *o* corresponding to the surface piece ϖ, when regarded as planar quantities.

The three opposite relations thus found between a stress function $F(x, y, z)$ for an associated stress system and its reciprocal diagram are completely analogous to the corresponding relations in two dimensions. This is especially clear upon application of the second Maxwell Ansatz; the surface element $d\varpi$, regarded as a planar quantity, is the direct spatial generalization of the arc element *do* of the plane, regarded as a transversal vector. However, also by the first Ansatz, the analogy is easy to find. We first have, e.g., for two dimensions, the formula (cf., eq. (3)) for X_r:

$$(36)\qquad X_r = \int P\,dy - U\,dx = \int (P \cos nx + U \cos ny)\, ds\,,$$

and that is nothing but:

$$\int \left(\Delta F \cdot \cos nx - \frac{\partial\xi}{\partial n}\right) ds\,,$$

[where $\Delta = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$], which is analogous to the first spatial equation (34); likewise, the first equation in (35) has its planar analogue, namely, the equation that arises from equations (27) on pp. 20:

$$X_r = \eta_\beta - \eta_\alpha\,,$$

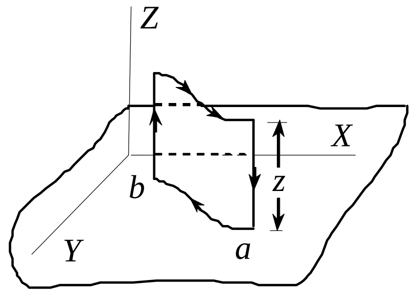

Fig. 21.

if $z = \partial F / \partial z$ is null for two dimensions, hence, we obtain for the stress component on the vertical cylinder of unit height over the arc segment *ab* (Fig. 21):

$$\int \eta \, dz - \zeta \, dy = \eta_\beta - \eta_\alpha . \tag{37}$$

As a consequence, one can carry out the analogy when one introduces a *"four-dimensional space"* as an aid, and constructs an *"Airy manifold"* in it, corresponding to the formula:

$$t = F(x, y, z).$$

With the formulas:

$$\xi = \frac{\partial F}{\partial x}, \qquad \eta = \frac{\partial F}{\partial y}, \qquad z = \frac{\partial F}{\partial z}, \qquad t + \vartheta = x\,\xi + y\,\eta + z\,\zeta,$$

one could "polarize" this Airy manifold relative to the "paraboloid":

$$2\,t = x^2 + y^2 + z^2,$$

and the "perpendicular" projection of this polar image onto the $\xi\eta\zeta$-space yields the diagram that is reciprocal to the stress system in it. Such consequences would lead us into four-dimensional relationships that are very beautiful and convincing, but the majority of readers would then encounter unnecessary difficulties.

3. After the development that was just given, we may now, with only minor effort, arrive at the following mechanically-described meaning for our previously considered reciprocal cell system:

With both Maxwell Ansätzen, we obtain, on the one hand, *stresses on the faces of the diagram* that are given by the lengths of the corresponding edges of the other diagram (homogeneous stresses, as might occur in the cell faces of soapsuds). On the other hand, we obtain *stresses on the edges of the diagram* that are quantities given by the surface areas of the corresponding faces of the other diagram.

Hence, only the second spatial Ansatz leads to the understanding of stress systems in spatial *frames*. For that reason, the Henneberg reference also speaks only of this second Ansatz (no. 41), and also only more casually, since the general study of Airy stress functions could not be assumed in it.

The statics of stress states for arbitrary loads that one obviously achieves when one grasps the entire Maxwell line of thought has an interesting aspect to it; however, its various parts appear in a wonderful connection that has been understood only slightly up to now. For that reason, in the current presentation the task of working out that connection was considered to be an actual goal, from which the basis for new developments of the theory is likewise arrived at.

Göttingen, 10 February 1904.