"Über die BIlding von Wirbeln in reibungslosen Flüssigkeiten," Zeit. Math. Phys. **58** (1910); *Ges. math. Abhand.* LXXX, 710-713.

On the formation of vortices in frictionless fluids

By Felix KLEIN

Translated by D. H. Delphenich

At the conclusion of his celebrated treatise on vortex motions ([1]), **Helmholtz** described a simple method of generating vortices that anyone could conveniently explore with his cup of coffee on any given day. One moves the point of a spoon along the surface of the fluid and then suddenly takes it away. What will remain in the fluid is a vortex filament whose form corresponds to the outline of the immersed point of the spoon and which advances in the direction of the velocity that was assigned to the spoon in the fluid originally. Naturally, all that one observes are the two points at which these vortex filaments cut the free surface of the fluid. They appear to be flat or – for a faster forward velocity of the spoon – cone-shaped indentations into the free surface around which the fluid circulates. It hardly needs to be said that these indentations can be explained, as such, by the collective action of gravity and the centrifugal force that acts upon the individual fluid particles during their circulation.

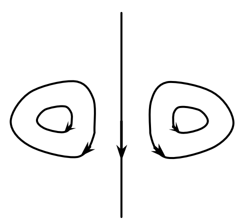

Figure 1.

One observes the same phenomenon on a larger scale while rowing: After each stroke of the oars, two indentations that correspond to the outer edges of the immersed oars will drift across the surface of the water. The midpoints of these indentations should be regarded as the intersections of the surface of the water with a vortex filament that one must think of as running along the entire outline of the immersed oar.

How does one explain that phenomenon? In most cases, one will first think of the effect of friction. However, as far as I can see, such an effect will enter in as at most a secondary one. One may assume that the phenomenon would take place in completely frictionless fluids in essentially the same way that was described. However, the actual basis for the fact that the usual statements in frictionless hydrodynamics obviously contradict experience seems to be something else entirely.

([1]) "Über Integrale der hydrodynamischen Gleichungen, welche den Wirbelbewegungen entsprechen," Crelles Journal **55** (1858). – *Ges. Wissenschaftliche Abhandlungen*, Bd. I.

For the sake of convenience of presentation, I would like to think of the entire process as being two-dimensional. I will immerse an infinitely-wide, flat oar that is bounded by a horizontal line into an infinite, frictionless, volume of water that is bounded by a horizontal plane (which is subject to only gravity) and move it forwards perpendicular to its plane and in the midst of that motion, remove it instantaneously. Fig. 2 gives a schematic picture of that experimental arrangement.

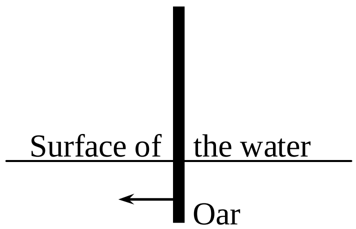

Figure 2.

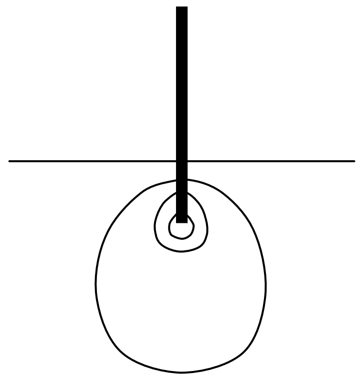

Figure 3.

The successive processes of motion can then be described as follows:

1. As long as the oar is immersed to its full depth and displaced forwards, the known potential motion will arise whose velocity curves are drawn in Fig. 3. Those velocity curves are not precisely perpendicular to the oar, but define an acute angle with it downwards that will become smaller when one goes on to the lower boundary line of the oar.

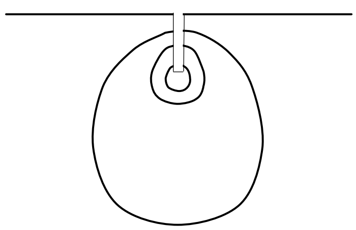

Figure 4.

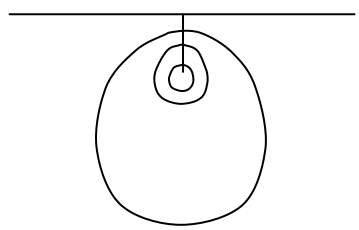

Figure 5.

2. One now suddenly draws the oar out of the water vertically. That will not have any sort of immediate effect on the motion of the water particles, since friction was expressly ignored. The single instantaneous change is that one will now find a vertical slit in the volume of water where the oar was. (Fig. 4)

3. However, we now come to gravity (the fluid pressure that results from it, resp.), whose effect will be to *make the parts of the water on the left and right of the slit flow together immediately.* One now has a *discontinuity surface* for the velocities that are assigned to the individual water particles in the volume of water that is bounded by the

horizontal plane where the oar was before; i.e., a *vortex layer*. The intensity of the vorticity thus increases from the upper end of the layer towards the lower one.

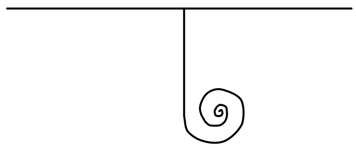

Figure 6.

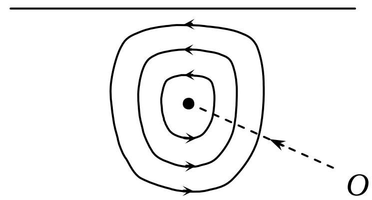

Figure 7.

4. The development of this vortex layer seems to be such that it very rapidly rolls up from the lower end in a spiral, such that after some time the motion noticeably takes place as if an almost point-like vortex region *O* were found in the vicinity of the lower end of the original layer. (The piece of the vortex layer that extends from *O* spreads out in a continuous deformation of the branch of the spiral, which becomes ever-longer until it reaches the surface of the water and then loses any further meaning in the fluid motion.) The vortex region in our Fig. 7 is naturally the intersection of our reference plane with a *vortex filament* that runs perpendicular to it (and thus, parallel to the lower boundary edge of the oar), and the emergence of such a vortex filament is explained by what we just said.

It remains for us to carry over this entire argument from two dimensions to three (when we replace the infinitely-extended oar with one of finite width). The rectilinear vortex filament that was found will be replaced with one that (more or less exactly) follows the contour of the immersed oar, such as the experiment that we had in mind when we began.

Naturally, one can demand that the process of motion that was formulated here only qualitatively should be formulated quantitatively (be derived from the differential equations of hydrodynamics, resp.) While I shall leave that to other mathematicians, I shall respond to only the next question that emerges of whether the theory that was put forth here is compatible with the well-known theorem that **Helmholtz** himself presented in his aforementioned treatise that vortices can never arise during the motion of a rigid body in a frictionless fluid that is subject to only gravity. Obviously, that compatibility will rest upon the fact that we have directed our attention to two originally-separate fluid parts that flow together, while in order to define the basis for the aforementioned theorem, it was assumed that the fluid particles that were on the surface of the fluid at one point in time would always remain on the surface.

Langeoog, 20 August 1909.