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On the dynamics of the general theory of dislocations when moment stresses are considered

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Abstract – A Lagrange function for the general theory of dislocations is presented. The stress functions are then varied, which fulfill the equations of motion of the linear Cosserat continuum identically. The invariance requirement of the theory under gauge transformations of the stress functions leads to conservation laws (balance principles, resp.) for the dislocation density and foreign matter. The energy balance equation and the field balance equation are derived, and the forces that act upon the dislocation density and the foreign matter are presented.

1. Introduction

Since the continuum theory of static dislocations is as good as complete [**1-3**], the problem now remains of extending the static theory to the dynamical theory of moving dislocations. Basic equations for a continuum theory of moving dislocations were given previously by, e.g., Holländer [**4**], Kosevich [**5**], Mura [**6**, **7**], and Bross [**8**]. All of these theories have in common that in order to determine the stress state and the velocity field a dislocation flux density (the time variation of the plastic deformation, resp.) must be present along with the dislocation density. There thus exists a balance equation that couples the temporal variation of the dislocation density with the dislocation flux density, in complete analogy to the charge conservation law of electrodynamics. Along with the presentation of basic dynamical equations, a basic kinematical equation was also derived [**9**, **10**] with the aid of a velocity vector for the dislocations.

In this paper, we link up with the work of Holländer and Kosevich; i.e., we first solve the equations of motion by means of stress functions. We then define a Lagrange density that consists of the Lagrange density for the elastic body plus interaction terms. Since we start from the Cosserat continuum, and thus consider moment stresses, we will thus be inescapably led to introduce new density and flux density tensors in the interaction terms of the Lagrange density. These tensors describe foreign matter, so we deal with the general dislocation theory in the sense of Kröner. We derive the field equations from Hamilton's principle. An examination of the gauge invariance of the theory leads us to the conservation law (balance principle, resp.) for the dislocation density and the foreign matter. Finally, we derive the energy balance equation and the field impulse balance equation by the method of classical field theory. We thus also obtain the forces that act on the dislocations and the foreign matter in the stress and velocity fields.

2. The equations of motion and the material equations

The starting point of our reasoning is defined by the linear equations for the Cosserat continuum [**12**]. They are balance equations for the dynamical impulse density and the rotational impulse density:

$$\rho \dot{v}_i = \sigma_{ki,k} , \tag{2.1}$$

$$\theta_{ik} \dot{s}_k = \mu_{ki,k} + \varepsilon_{irs}\, \sigma_{ki} . \tag{2.2}$$

Here, ρ means the mass density, v_i , the velocity of the mass element, σ_{ki} , the stress tensor – which is asymmetric here – μ_{ki} is the moment stress tensor, and ε_{ikl} is the totally anti-symmetric unit tensor. For the partial derivatives, we set $\partial u / \partial t = \dot{u}$, $\partial u / \partial x_l = u_{,j}$. We associate the individual mass elements with a spin degree of freedom, in the sense of the Cosserat theory, that is described by the angular velocity s_i and the moment of inertia θ_{ik} .

For the elastic deformation tensor ε^E_{ik}, we assume that it is coupled to the stress tensor by Hooke's law:

$$\sigma_{(ik)} = C_{iklm}\,\varepsilon^E_{lm} , \qquad \sigma_{(ik)} = \tfrac{1}{2}(\sigma_{ik} + \sigma_{ki}). \tag{2.3}$$

Since we are dealing with an incompatible theory, the deformation tensor may no longer be derived from a displacement field. An analogous material law is true for the moment stress tensor:

$$\mu_{ik} = b_{iklm}\,\chi^E_{lm} , \tag{2.4}$$

where χ^E_{lm} is the elastic curvature tensor, which can be computed from the anti-symmetric part of the elastic distortion only in the absence of foreign matter. One can also consider cross-coupling in the material equations (2.3) and (2.4). The cross-coupling will play no role in what follows, since it first becomes important when one looks for solutions of the field equations thus obtained.

A certain geometric quantity is lacking for us in the material law for the anti-symmetric part of the stress tensor. We will thus employ no material law for $\sigma_{[ik]}$, whether or not this practice is entirely satisfactory.

3. The solution of the equations of motion by the use of stress functions

In elastostatics, one often uses the stress function φ_{ik} , which makes $\sigma_{ik} = \varepsilon_{irs}\, \varphi_{sk,r}$ satisfy the equations $\sigma_{ik,i} = 0$. We extend this solution Ansatz by introducing additional stress functions:

$$\sigma_{ik} = \varepsilon_{irs}\, \varphi_{sk,r} + \dot{\psi}_{lik,l} , \tag{3.1}$$

$$\rho\, v_i = \psi_{kli,k,l} , \tag{3.2}$$

$$\mu_{ik} = \varepsilon_{rim}\, \varepsilon_{rkl}\, \varphi_{ml} - \varepsilon_{rim}\dot{\psi}_{irs} + \varepsilon_{irs}\, \phi_{sk,r} + \chi_{ik} , \tag{3.3}$$

$$\theta_{ik}\, s_k = \chi_{li,l} . \tag{3.4}$$

Eq. (2.1) and (2.2) are fulfilled identically with this Ansatz. One must now arrive at equations for the stress functions, for which we appeal to Hamilton's principle. Let it be remarked that the stress functions for the solution of eq. (2.1) alone were already given by Holländer [**4**] and Kosevich [**5**] and by Günther [**13**] in the static problem with moment stresses.

4. The Lagrange density and the field equations

In classical linear elastomechanics, the Lagrange density has the following form: $L = \frac{1}{2}(\sigma_{ik}\varepsilon_{ik}^{E} - \rho v^2)$. We extend this expression by considering the elastic energy that is coupled to the curvature and the kinetic energy that is coupled to the spin of the mass element. We thus obtain the Lagrange density:

$$L = \tfrac{1}{2}(\sigma_{ik}\varepsilon_{ik}^{E} + \mu_{ik}\chi_{ik}^{E} - \rho v^2 - \theta_{ik}s_i s_k) - D_{ik}\varphi_{ik} + V_{ikl}\psi_{ikl} + B_{ik}\phi_{ik} - S_{ik}\chi_{ik}\,. \tag{4.1}$$

In regard to the physical meaning of the quantities D_{ik}, V_{ikl}, B_{ik}, and S_{ik}, we will first make a few remarks about the field equations. By analogy with electrodynamics, one can compare D_{ik} with the electric charge density, φ_{ik} with the scalar potential, V_{ikl} with the electric current density, and ψ_{ikl} with the vector potential [**4**, **5**]. In order to arrive at the field equations, we employ the Hamilton principle:

$$\delta\int_{t_1}^{t_2}\int_V L\,dV\,dt = 0, \tag{4.2}$$

in which we vary the stress functions, while the quantities D_{ik}, V_{ikl}, B_{ik}, and S_{ik} are left fixed by the variation. By varying the φ_{ik}, we then obtain:

$$\frac{\partial L}{\partial \varphi_{ik}} - \left(\frac{\partial L}{\partial \varphi_{ik,l}}\right)_{,l} = 0, \tag{4.3}$$

by varying the ψ_{ikl}, we get:

$$\frac{\partial L}{\partial \psi_{ikl}} - \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{\psi}_{ikl}}\right) + \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \psi_{ikl,m}}\right)_{,m} + \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \psi_{ikl,m,n}}\right)_{,m,n} = 0, \tag{4.4}$$

by varying the ϕ_{ik}, we get:

$$\frac{\partial L}{\partial \phi_{ik}} - \left(\frac{\partial L}{\partial \phi_{ik,l}}\right)_{,l} = 0, \tag{4.5}$$

and by the variation of χ_{ik}, we get:

$$\frac{\partial L}{\partial \chi_{ik}} - \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{\chi}_{ik}}\right) - \left(\frac{\partial L}{\partial \chi_{ik,l}}\right)_{,l} = 0. \tag{4.6}$$

These are the desired field equations, which assume the following form with the help of eq. (4.1):

$$\varepsilon_{ikl}\varepsilon^{E}_{ij,k} + \varepsilon_{rik}\varepsilon_{rji}\chi^{E}_{kl} = D_{ij}\,, \tag{4.7}$$

$$\varepsilon_{rkl}\dot{\chi}^{E}_{lr} - v_{l,i,k} + \dot{\varepsilon}^{E}_{kl,i} = -V_{ikl}\,, \tag{4.8}$$

$$\varepsilon_{ikl}\chi^{E}_{ij,l} = B_{ij}\,, \tag{4.9}$$

$$s_{k,i} - \dot{\chi}^{E}_{ik} = S_{ij}\,. \tag{4.10}$$

In order to facilitate the physical interpretation of the quantities D_{ik} , V_{ikl} , B_{ik} , and S_{ik} ,we first set $B_{ik} = 0$ and $S_{ik} = 0$. On the basis of (4.9), one can then write the elastic curvature tensor as:

$$\chi^{E}_{ji} = \tfrac{1}{2}\varepsilon_{krs}\beta^{E}_{[rs],i}\,, \tag{4.11}$$

and thus it follows from eq. (4.7) that:

$$\varepsilon_{ikl}\beta^{E}_{lj,k} = D_{ij}\,. \tag{4.12}$$

We then summarize the elastic distortion β^{E}_{ik} as follows:

$$\beta^{E}_{ik} = \varepsilon^{E}_{ik} + \beta^{E}_{[ik]}\,. \tag{4.13}$$

It follows immediately from eq. (4.12) that D_{ik} is the tensor of dislocation density. It depends upon the elastic curvature tensor according to eq. (4.7) in the well-known way [**7**]:

$$\chi^{E}_{ji} = \varepsilon_{ilk}\varepsilon^{E}_{kj,l} - D_{ij} + \tfrac{1}{2}D_{ll}\delta_{ij}\,. \tag{4.14}$$

Next, we consider eq. (4.8), for which we can write, with eq. (4.11) and $v_{i,k} = \beta^{E}_{ki} + \beta^{P}_{ik}$ (where β^{P}_{ik} is the plastic distortion):

$$(\dot{\beta}^{E}_{ki} - v_{k,i})_{,l} = -V_{lik} = -\dot{\beta}^{P}_{ik,l} = -I_{ik,l}\,. \tag{4.15}$$

Here, we have employed, e.g., the dislocation flux density tensor $I_{ik} = \dot{\beta}_{ik}$ that was introduced by Kosevich [**5**] or Mura [**7**]. In the case $B_{ik} = 0$ and $S_{ik} = 0$, we thus find that V_{ikl} is precisely the gradient of the dislocation flux density tensor:

$$V_{ikl} = I_{kl,i}\,. \tag{4.16}$$

We now again come back to the general case with $B_{ik} \neq 0$ and $S_{ik} \neq 0$. Due to eq. (4.9), χ^{E}_{ik} no longer represents a gradient, now. However, for the sake of further calculations (e.g., for the presentation of the field impulse balance) and for comparison with the relations that are known in the literature, we would like to split a gradient off from χ^E_{ik}. We thus write:

$$\chi^E_{ik} = \tfrac{1}{2}\varepsilon_{krs}\beta_{[rs],i} + K_{ik} . \tag{4.17}$$

Here, $\beta_{[ik]}$ does not mean the anti-symmetric part of an elastic distortion tensor, since, as we will show, B_{ik} describes foreign matter, in the sense of Kröner, so from [**2**], there exists no elastic distortion tensor, at all. Analogous to the decomposition (4.17), we also decompose the tensor V_{ikl} into two parts:

$$V_{ikl} = I_{kl,i} - \varepsilon_{rkl}\, K_{ir} , \tag{4.18}$$

where I_{kl} is again the tensor of the dislocation flux density. At present, we cannot ultimately decide whether the splittings (4.17) and (4.18) are only convenient for calculations or whether, as we assume, a deeper physical sense lies beneath this.

With eq. (4.17) and (4.18), we now obtain the field equations (4.7) to (4.10) in the following form:

$$\varepsilon_{ilk}\, \beta_{kj,l} = D_{ij} - \varepsilon_{rik}\, \varepsilon_{rji}\, K_{kl} , \tag{4.19}$$

$$\varepsilon_{ijk}\, K_{il,j} = B_{kl} , \tag{4.20}$$

$$v_{k,i} - \dot{\beta}_{ik} = B_{ik} , \tag{4.21}$$

$$s_{k,i} - \dot{\chi}^E_{ik} = S_{ki} . \tag{4.22}$$

We have thus combined the elastic deformation tensor ε^E_{ik} with the anti-symmetric tensor $\beta_{[ik]}$ from eq. (4.17) into the distortion tensor according to:

$$\beta_{ik} = \varepsilon^E_{ik} + \beta_{[ik]}, \qquad \varepsilon^E_{ik} = \beta_{(ik)} . \tag{4.23}$$

Only in the case of no foreign matter is this equal to the elastic distortion tensor, since it would then contain no plastic part.

We now consider eq. (4.19). This equation is identical with the relation that was given by Kröner [**3**]:

$$\text{rot}\, \beta = \alpha + \gamma \tag{4.26}$$

if one sets the tensor D_{ik} equal to the dislocation density tensor α and sets $-\varepsilon_{rik}\, \varepsilon_{rjl}\, K_{kl}$ equal to the tensor γ, which is a measure of the foreign matter that is present in the crystal. On the basis of eq. (4.21), we thus obtain a confirmation of our conjecture that B_{ik} represents the foreign matter – i.e., the density of point defects. We will now show that B_{ik} is identical with the tensor B_{ik} that Kröner [**2**] introduced. We first consider eq. (4.21). In the general case $B_{ik} \neq 0$, it can be employed for the interpretation of β_{ik}, since we regard the dislocation flux density I_{ik} as a given quantity in our theory. If we decompose eq. (4.21) into its symmetric and anti-symmetric part then it follows that:

$$I_{(ik)} = v_{(k,i)} - \dot{\varepsilon}^E_{ik} = \dot{\varepsilon}^P_{ik}, \quad I_{[ik]} = v_{[k,i]} - \dot{\beta}_{[ik]}. \tag{4.25}$$

We will first discuss S_{ik} later, since a relationship that is comparable to eq. (4.22) in the literature is not known to us. However, it will be shown that S_{ik} can be interpreted as the flux density for the foreign matter.

One can now solve special problems when one substitutes stress functions in the field equations (4.7) to (4.10) and prescribes the fields D_{ik} , V_{ikl} , B_{ik} , and S_{ik} . Naturally, the form of the material equations – e.g., whether or not they have cross-couplings – then plays a role, while the splittings (4.17) and (4.18) are not necessary. We would not like to examine any special solutions in this paper, but only concern ourselves with general properties of the theory.

5. The gauge transformations of the stress functions and the conservation law for the tensors of dislocation density and foreign matter density.

Precisely as in elastomechanics, where one can add a tensor of null stress functions $q_{k,i}$ to the stress functions φ_{ik} , we have also have the possibility here of transforming the stress functions φ_{ik} , ψ_{ikl} , ϕ_{ik} , and χ_{ik} , without changing the physical quantities s_{ik} , ρv_i, μ_{ik} , θ_{ik} s_k . We would like to call these transformations *gauge transformations*, as in electrodynamics. These gauge transformations have the form:

$$\varphi'_{ik} = \varphi_{ik} + a_{k,i} - b_{ik} , \tag{5.1}$$

$$\psi'_{ikl} = \psi_{ikl} + \varepsilon_{kis}\, b_{sl} + \varepsilon_{irj}\, d_{klj,r} , \tag{5.2}$$

$$\phi'_{ik} = \phi_{ik} + c_{k,i} - \varepsilon_{lik}\, a_l + \varepsilon_{klm}\, d_{lmi} - h_{ik} , \tag{5.3}$$

$$\chi'_{ik} = \chi_{ik} + \varepsilon_{irj}\, h_{jk,r} . \tag{5.4}$$

The "null stress functions" a_i , b_{ik} , d_{ikl} , c_i , and h_{ik} can be chosen arbitrarily. However, that means that one can impose a large number of auxiliary conditions on the stress functions, and one might hope that by that means one might arrive at a coupling of the field equations in terms of the stress functions.

We would now like to investigate the influence of the gauge transformations on the Lagrange density. Naturally, the field equations that one obtains by variation of the stress functions must always be the same independently of the gauge of the stress functions. However, that means that the gauge transformation may change the Lagrange density only by a divergence and a term that can be written as a time derivative. This requirement then leads to conditions that must be fulfilled by the tensors of dislocation density, foreign matter density, and the corresponding fluxes. The Lagrange density (4.1) has the form $L = L^E + L^W$. $L^E = \frac{1}{2}(\sigma_{ik}\varepsilon^E_{ik} + \mu_{ik}\chi^E_{ik} - \rho v^2 - \theta_{ik} s_i s_k)$ does not change under a gauge transformation, since it is constructed out of the physical quantities σ_{ik} , μ_{ik} , v_i , and s_i ,. Things are different for the L^W part. For it, one has:

$$\int_{t_1}^{t_2}\int_V (-D_{ik}\varphi'_{ik} + V_{ikl}\psi'_{ikl} + B_{ik}\phi'_{ik} - s_{ik}\chi'_{ik})\,dV\,dt \; = I_1 + I_2$$
$$= \int_{t_1}^{t_2}\int_V (-D_{ik}\varphi_{ik} + V_{ikl}\psi_{ikl} + B_{ik}\phi_{ik} - S_{ik}\chi_{ik})\,dV\,dt$$
$$+ \int_{t_1}^{t_2}\int_V \{-D_{ik}(a_{k,i} - b_{ik}) + V_{ikl}(\varepsilon_{ikl}b_{ik} + \varepsilon_{irj}d_{kljir})$$
$$+B_{ik}(c_{k,i} - \varepsilon_{lik}a_l + \varepsilon_{krs}\ddot{d}_{rsi} - h_{ik}) - S_{ik}\varepsilon_{irs}h_{jk,r}\}\,dV\,dt\,. \tag{5.5}$$

Here, the last integral must vanish, up to a boundary integral, as well as the terms that one can write as time derivatives. For that reason, we convert the last integral I_2 in eq. (5.5) into:

$$I_2 = \int_{t_1}^{t_2}\int_V \Big\{(-D_{ik}a_k + \varepsilon_{rij}V_{rkl}d_{klj} + B_{ik}c_k - \varepsilon_{rij}S_{rk}h_{jk})_{,i}$$
$$+ \frac{\partial}{\partial t}(D_{ik}b_{ik} + \varepsilon_{skl}d_{klj}B_{js} - B_{ik}h_{ik})\Big\}\,dV\,dt$$
$$+ \int_{t_1}^{t_2}\int_V \Big\{(-\varepsilon_{lik}B_{ik} + D_{il,i})a_l + (-\dot{D}_{sl} + V_{rkl}\varepsilon_{kis})b_{sl}$$
$$+ (-\varepsilon_{irj}V_{ikl,r} - \varepsilon_{skl}\dot{B}_{js})\,d_{klj} - B_{ik,i}\,c_k + (\dot{B}_{js} + \varepsilon_{irj}S_{ik,r})\,h_{jk}\}\,dV\,dt. \tag{5.6}$$

The first integral does not contribute to the field equations, so the second one must vanish. Since the gauge functions a_i, b_{ik}, d_{ikl}, c_i, and h_{ik} can be chosen arbitrarily, by and large, the factors in front of these functions must be zero. Upon consideration of eq. (4.18), we thus obtain the following conditions, which we would to refer to briefly as the *conservation laws* (*balance equations*, resp.) for the tensors D_{ik} and B_{ik} :

$$D_{il,i} = \varepsilon_{lik}\,B_{ik}\,, \tag{5.7}$$
$$\dot{D}_{sl} + \varepsilon_{sik}\,V_{ikl} = 0,$$

or

$$\dot{D}_{sl} + \varepsilon_{sik}\,I_{kl,i} = \varepsilon_{kis}\,\varepsilon_{kjl}\,\dot{K}_{ij}\,, \tag{5.8}$$
$$\dot{B}_{js} = \varepsilon_{irj}\,\dot{K}_{is,r}\,, \tag{5.9}$$
$$B_{ik,i} = 0, \tag{5.10}$$
$$\dot{B}_{js} = -\,\varepsilon_{irj}\,S_{is,r}\,. \tag{5.11}$$

In the case of no foreign matter ($B_{ik} = 0$, $K_{ik} = 0$, $S_{ik} = 0$), these are precisely the known relations for the dislocation density [**2**, **5**, **7**, **8**]:

$$D_{il,i} = 0, \qquad \dot{D}_{sl} + \varepsilon_{sik}\,I_{kl,i} = 0. \tag{5.12}$$

If one considers foreign matter, then from Kröner [**2**], one has $D_{il,i} = \varepsilon_{lik}\,B_{ik}$. However, that is precisely our equation (5.7), from which the interpretation of B_{ik} as the tensor of foreign matter is confirmed. As a consequence of the consideration of foreign matter, a source term $\varepsilon_{kis}\,\varepsilon_{kjl}\,\dot{K}_{ij}$ appears in the balance equation (5.8) for the dislocation density, which is connected with the temporal variation of the foreign matter. However, on the grounds of eq. (4.19), one can also interpret $D_{ij} - \varepsilon_{kli}\,\varepsilon_{kmj}\,K_{lm}$ as the effective dislocation density. The divergence of this effective dislocation density then vanishes once more. Eq. (5.10) is a condition on the tensor of foreign matter that Kröner [**2**] likewise derived from a geometric argument.

Ultimately, we have obtained a balance equation for the tensor of foreign matter that is completely analogous to the dislocation density (5.12). We can thus interpret S_{ik} as the tensor of foreign matter flux. Due to eq. (5.19) and (5.11), this tensor depends upon the tensor V_{ikl} (K_{ik}, resp.) by way of:

$$\varepsilon_{irj}\,(\varepsilon_{skl}\,S_{is} - V_{i[kl]})_{,\,r} = 0, \tag{5.13}$$

or:

$$\varepsilon_{irj}\,(\varepsilon_{skl}\,\dot{K}_{is} + S_{is}) = 0. \tag{5.14}$$

The foreign matter flux S_{ik} is then equal to the tensor (the temporal variation of the curvature tensor K_{ik}, resp.), up to a gradient field. After a brief calculation, it follows that:

$$\varepsilon_{skl}\,S_{is} - V_{i[kl]} = (\varepsilon_{skl}\,s_r - v_{[k,l]})_{,\,i} = -\dot{\vartheta}_{[kl],i}\,,$$

or:

$$S_{ik} + \dot{K}_{ik} = (s_k - \varepsilon_{krs}\,\dot{\beta}_{[rs]})_{,\,i} = \dot{\vartheta}_{k,i}\,.$$

$\dot{\vartheta}_{[kl]}$ describes the relative angular velocity between the total angular velocity and the angular velocity of the individual mass element. One might conjecture that the tensor S_{ik} is closely connected with the velocity of the foreign atoms (or point defects), and possibly with diffusion in crystals. Further investigation is needed in order to make more precise statements.

6. The field impulse theorem

In order to arrive at statements about the forces the act on the dislocation density, the dislocation flux density, the foreign matter, and the foreign matter flux density, we exhibit the field impulse balance law that belongs to the Lagrange density (4.1). For this, we employ the method that was given by Landau and Lifschitz [**14**]. We define:

$$\begin{aligned}\frac{\partial L}{\partial x_j} &= \frac{\partial L}{\partial \varphi_{ik}}\varphi_{ik,j} + \frac{\partial L}{\partial \varphi_{ik,l}}\varphi_{ik,l,j} + \frac{\partial L}{\partial \psi_{ikl}}\psi_{ikl,j} + \frac{\partial L}{\partial \dot{\psi}_{ikl}}\dot{\psi}_{ikl,j} \\ &+ \frac{\partial L}{\partial \dot{\psi}_{ikl,r}}\dot{\psi}_{ikl,r,j} + \frac{\partial L}{\partial \psi_{ikl,r,s}}\psi_{ikl,r,s,j} + \frac{\partial L}{\partial \phi_{ik}}\phi_{ik,j} + \frac{\partial L}{\partial \phi_{ik,l}}\phi_{ik,l,j} \\ &+ \frac{\partial L}{\partial \chi_{ik}}\chi_{ik,j} + \frac{\partial L}{\partial \dot{\chi}_{ik}}\dot{\chi}_{ik,j} + \frac{\partial L}{\partial \chi_{ik,l}}\chi_{ik,l,j} + \frac{\partial L}{\partial D_{ik}}D_{ik,j} \\ &+ \frac{\partial L}{\partial V_{ikl}}V_{ikl,j} + \frac{\partial L}{\partial B_{ik}}B_{ik,j} + \frac{\partial L}{\partial S_{ik}}S_{ik,j}\,.\end{aligned} \tag{6.1}$$

With the help of the field equations (4.3) to (4.6), this equation can be easily converted into:

$$\frac{\partial}{\partial t}\left\{\frac{\partial L}{\partial \dot{\psi}_{ikl}}\psi_{ikl} - \left(\frac{\partial L}{\partial \dot{\psi}_{ikl,m}}\right)_{,m}\psi_{ikl,j} + \frac{\partial L}{\partial \dot{\chi}_{ik}}\chi_{ik,j}\right\}$$

$$+\left\{\frac{\partial L}{\partial \varphi_{ik,r}}\varphi_{ik,j} + \frac{\partial L}{\partial \phi_{ik,r}}\phi_{ik,j} + \frac{\partial L}{\partial \chi_{ik,r}}\chi_{ik,j} + \frac{\partial L}{\partial \dot{\psi}_{ikl,r}}\dot{\psi}_{ikl,j}\right.$$

$$\left. + \frac{\partial L}{\partial \psi_{ikl,m,r}}\psi_{ikl,m,j} - \left(\frac{\partial L}{\partial \psi_{ikl,r,n}}\right)_{,n}\psi_{ik,j} - L\delta_{rj}\right\}_{,r}$$

$$= -\left(\frac{\partial L}{\partial D_{ik}}D_{ik,j} + \frac{\partial L}{\partial V_{ikl}}V_{ikl,j} + \frac{\partial L}{\partial B_{ik}}B_{ik,j} + \frac{\partial L}{\partial S_{ik}}S_{ik,j}\right). \tag{6.2}$$

This is the desired field impulse balance, but still in a form that one cannot interpret, and even when one substitutes the derivatives of the Lagrange function (4.1). We now convert eq. (6.2), giving consideration to the field equations, in such a way that only physical quantities, and not stress functions, enter into the field impulse balance. After some tedious computations, it then follows that:

$$\dot{p}_j + \Sigma_{rj,r} = k_j \,. \tag{6.3}$$

In this:

$$p_j = -\rho\, v_l\, \beta_{jl} - \theta_{lk}\, s_k\, \chi^E_{jl} \tag{6.4}$$

means the field impulse density, while:

$$\Sigma_{rj} = \beta_{jm}\,\sigma_{rm} + \chi^E_{jl}\,\mu_{rm} - L^E\delta_{rj} \tag{6.5}$$

is the "Maxwell tensor of elasticity" that is coupled with the field impulse density p_i, and:

$$k_j = \varepsilon_{jkl}\,\sigma_{lm}\,(D_{km} - \varepsilon_{rik}\,\varepsilon_{rsm}\,K_{is}) - \varepsilon_{jkl}\,\mu_{lm}\,B_{km} - \varepsilon_{lrs}\,\mu_{rs}\,K_{jl} + \varphi_{lk}\,s_k\,S_{jl}\,, \tag{6.6}$$

is the force density that acts on the dislocations, the foreign matter, and the corresponding flux in the continuum considered.

The field impulse density is constructed similarly to the energy flux density $-\sigma_{ik}\,v_k$ in classical elasticity theory, except that in the field impulse density p_i, in place of the stress tensor σ_{ik}, one has the distortion tensor β_{ik}, and in place of the velocity, one finds the dynamical impulse density. One then adds the corresponding part with the elastic curvature tensor to this.

The stress tensor Σ_{ik} corresponds to the Maxwell stress tensor in electrodynamics. This tensor and eq. (6.3) were already given by Eshelby **15**] for the static case with no moment stresses. The name "Maxwell tensor of elasticity" also goes back to Eshelby.

The most interesting thing is the force density k_l . In the case with no foreign matter, one obtains precisely the force density $\varepsilon_{ikl}\ \sigma_{lm}\ D_{km}$ that was given by Peach and Koehler [**16**], which acts on the dislocation density in the stress field, and the force density $\rho\, v_j\, I_{ij}$ that Kosevich [**17**] found, which acts on the dislocation flux density in the velocity field v_i . Here, we have found, in addition, the force density k_i^F , which acts on the foreign matter. It reads:

$$k_i^F = -\,\varepsilon_{jkl}\ \sigma_{lm}\ \varepsilon_{rik}\ \varepsilon_{rsm}\ K_{is} - \varepsilon_{mrs}\ \sigma_{rs}\ K_{jm} = \varepsilon_{jkl}\ \sigma_{lm}\ K_{ml} - \varepsilon_{jkl}\ \mu_{lm}\ B_{km}\ . \tag{6.7}$$

The first term originates in the effective dislocation density $D_{ik} - \varepsilon_{jkl}\ \mu_{lm}\ K_{lm}$ in the Peach-Koehler expression in eq. (6.6); i.e., the foreign matter acts partly like a dislocation density, and for that reason a force acts on the foreign matter in the stress field according to the formula of Peach and Koehler. The second term gives the force density that acts upon the foreign matter in the moment-stress field. This force density is, up to sign, analogous to the Peach-Koehler force, except that in place of the dislocation density D_{ik} one now finds the tensor of foreign matter B_{ik} . The last term in eq. (6.7) finally gives the part of the force density that is connected with only the anti-symmetric part of the stress tensor. One can combine the first and third terms such that it formally has almost the same appearance as the Peach-Koehler force. In addition, we have also obtained the force density $\theta_{jk}\ s_k\ S_{ij}$ that acts on the foreign matter flux in eq. (6.6); we find the spin impulse $\theta_{jk}\ s_k$ in place of the impulse ρv_i in the force density of Kosevich, here.

If one sets D_{ik} , V_{ikl} , B_{ik} , and S_{ik} equal to zero then one can introduce a displacement field and the field impulse (6.3) reads, when we set $\mu_{ik} = 0$ and $s_i = 0$:

$$\frac{\partial}{\partial t}(-\rho v_l u_{l,j}) + \left\{ u_{m,j}\sigma_{rm} - \frac{1}{2}(\sigma_{ik}u_{i,k} - \rho v_i u_i)\delta_{rj} \right\}_{,r} = 0. \tag{6.8}$$

Here, u_i is the displacement vector. The field impulse balance is to be distinguished from the equation of motion (2.1), which is to be interpreted as the impulse balance for the dynamical impulse density ρv_i . Along with this dynamical impulse, however, one must, on the grounds of eq. (6.3) ((6.8), resp.), one must ascribe a field impulse density (which is analogous to the impulse density [*DB*] in electrodynamics) to the solid body in question. In order to interpret this situation, we consider a plane harmonic sound wave, which, according to Brenig [**18**], one must associate with an impulse. However, the dynamical impulse that is coupled to plane harmonic sound waves vanishes in the time average, while the temporal mean of the impulse density in eq. (6.8) coincides with the impulse density that was given by Brenig. It seems that in most cases the field impulse density, as a quadratic – i.e., small – quantity, is not noticeable when compared to the dynamical impulse density, and only plays an essential role when, as in the example of sound, the time average of the dynamical impulse density vanishes.

The field impulse balance equation (6.3) can also be derived directly from the field equations (4.19) to (4.22). In order to do this, one must multiply eq. (4.19) by $\varepsilon_{rim}\ \sigma_{mj}$, eq. (4.20) by $\varepsilon_{rim}\ \sigma_{mj}$, eq. (4.21) by ρv_k , and eq. (4.22) by $\theta_{jk}\ s_k$, add them, and convert them accordingly.

7. The energy theorem

Starting from the Lagrange density, we arrive at the energy theorem precisely as we arrived at the impulse theorem, except that we now form $\partial L / \partial t$. With the help of the field equations (4.3) to (4.6), we then obtain:

$$\frac{\partial}{\partial t}\left\{\frac{\partial L}{\partial \dot{\chi}_{ik}}\dot{\chi}_{ik}+\frac{\partial L}{\partial \chi_{ikl}}\chi_{ikl}-\left(\frac{\partial L}{\partial \dot{\psi}_{ikl,m}}\right)_{,m}\dot{\psi}_{ikl}-L\right\}$$
$$+\left\{\frac{\partial L}{\partial \varphi_{ik,r}}\dot{\varphi}_{ik}+\frac{\partial L}{\partial \phi_{ik,r}}\dot{\phi}_{ik}+\frac{\partial L}{\partial \chi_{ik,r}}\dot{\chi}_{ik}-\left(\frac{\partial L}{\partial \dot{\psi}_{ikl,r,m}}\right)_{,m}\dot{\psi}_{ikl}+\frac{\partial L}{\partial \psi_{ikl,m,r}}\dot{\psi}_{ikl,m}+\frac{\partial L}{\partial \dot{\psi}_{ikl,r}}\ddot{\psi}_{ikl}\right\}_{,r}$$
$$= \varphi_{ik}\dot{D}_{ik}-\phi_{ik}\dot{B}_{ik}-\psi_{ikl}\dot{V}_{ikl}+\chi_{ik}\dot{S}_{ik}\,. \tag{7.1}$$

The stress functions will be eliminated from this equation with the help of the field equations and the conservation law, and it then follows that:

$$\frac{\partial}{\partial t}\left\{\frac{\rho}{2}v^2+\frac{\theta_{ik}}{2}s_i s_k+\frac{1}{2}\sigma_{ik}\varepsilon^E_{ik}+\frac{1}{2}\mu_{ik}\chi^E_{ik}\right\}+\{-\sigma_{ik}\,v_k-\mu_{ik}\,s_k\}_{,\,i}$$
$$= -\sigma_{ik}\,I_{ik}-\mu_{ik}\,S_{ik}-\varepsilon_{lrs}\sigma_{rs}\dot{\vartheta}_l\,. \tag{7.2}$$

Here, $1/2\,(Sv^2+\theta_{iksisk})$ is the total kinetic energy, $1/2\sigma_{ik}\,\varepsilon^E_{ik}$ is the energy density that is coupled with the elastic deformation tensor, and is the energy density (viz., elastic potential) that is coupled with an elastic curvature tensor. The energy flux density is given by $-\sigma_{ik}\,v_k-\mu_{ik}\,s_k$. Along with the energy flux that is connected with the stress tensor, an analogous part emerges that is connected with the moment-stress tensor.

Now, one has no conservation law for the sum of the kinetic and potential energy since the right-hand side of eq. (7.2) is not equal to zero. A mechanical field energy for the solid body will be removed through the interaction of the stress fields with – e.g. – the dislocation flux density. This can happen reversibly when no friction is coupled with the dislocation fluxes, while in the other case $D = -\sigma_{ik}\,I_{ik}-\mu_{ik}\,S_{ik}$ gives the part of mechanical field energy that is converted into heat by way of dissipative processes. Here, one can (by neglecting the cross-couplings) introduce two material equations that are analogous to Ohm's law [**4**]:

$$\sigma_{ik}=r^{(1)}_{iklm}I_{lm}, \qquad \mu_{ik}=r^{(2)}_{iklm}S_{lm}. \tag{7.3}$$

The third term $\varepsilon_{lrs}\,\sigma_{rs}\,\vartheta_l$ is connected with the angular velocity $\dot{\vartheta}_i$ that was introduced in eq. (5.14), which one can connect to $\sigma_{[ik]}$ by means of a material law.

The appearance of dissipative terms in the energy balance does not contradict the application of Hamilton's principle, since we indeed regard the currents I_{ik} and S_{ik} as given, so they do not vary. For this reason, we also do not need to assume anything about the nature of the forces that these currents create.

We would like to consider the term – $\sigma_{ik}\ I_{ik}$ for a moving singular dislocation line (with no foreign matter) [**5**, **10**]. I_{ik} then reads:

$$I_{ik} = \varepsilon_{iml}\ \tau_l\ b_k\ V_m\ , \qquad \tau_l\ b_k = D_{lk}\ . \tag{7.4}$$

τ_i is the unit vector in the direction of the dislocation line, b_i is the Burgers vector, and V_i is the velocity of the dislocation line. With eq. (7.4), it follows:

$$-\ \sigma_{ik}\ I_{ik} = -\ \varepsilon_{iml}\ \tau_l\ b_k\ \sigma_{ik}\ V_m = -\ k_m^{PK} V_m\ . \tag{7.5}$$

In other words, in this case, $\sigma_{ik}\ I_{ik}$ can be interpreted as the work that is done by the Peach-Koehler force on the moving dislocation line.

8. On the description of a single foreign atom

It is known from the literature [**1**, **3**] that one can describe a single foreign atom as an elastic dipole. The force that acts on such a displacement dipole Q_{ik} in the stress field σ_{ik} is given by:

$$K_r = Q_{jl}\ \sigma_{jl,r} \tag{8.1}$$

in the static case and for $\mu_{ik} = 0$. In eq. (6.7), we also possess a formula for the force density of the foreign matter. Here, we are interested in seeing how the tensors K_{ik} and B_{ik} look for a single foreign atom. We would thus like make the following proposal, which is suggested by the possibility of representing a single foreign atom as an elastic dipole. We set:

$$-\ \varepsilon_{rik}\ \varepsilon_{rjl}\ K_{ij} = \varepsilon_{srk}\ M_{sl}\ S_{,r}\ . \tag{8.2}$$

Here, the constant tensor M_{ik} describes the individual foreign atom (or better get: its displacement dipole), and δ is the well-known δ-function, so $\delta_{,r}$ is then the dipole function. Next, we calculate the force that that is exerted on the foreign atom (8.2) in the stress field σ_{ik} , $C\mu_{ik} = 0$. In order to do this, we must substitute eq. (8.2) in eq. (6.7) and then integrate over the entire domain:

$$K_r = \int k_r^E dV\ = \int \{M_{ik}\sigma_{ik}\delta_{,r} - M_{rk}\sigma_{ik}\delta_{,i}\}\, dV\ = M_{ik}\ \sigma_{ik,r} - M_{rk}\ \sigma_{ik,i}\ . \tag{8.3}$$

For the sake of simplicity, we consider the static case $\sigma_{ik,i} = 0$, for which the force (8.3) is identical with the force (8.1) that is given in the literature, so M_{ik} is equal to the displacement dipole Q_{ik} . In this case, the tensor of foreign matter has a quadrupole:

$$B_{ik} = \left(\varepsilon_{lsi}\varepsilon_{jrk}M_{jl} - \frac{1}{2}\varepsilon_{iks}\varepsilon_{jrl}M_{jl}\right)\delta_{,r,s}\ , \tag{8.4}$$

and here we have:

$$\varepsilon_{lik}\, B_{ik} = 0. \tag{8.5}$$

In other words, the tensor B_{ik} is symmetric for an individual foreign atom with the dipole character (8.2).

Our proposal for the structure of B_{ik} and K_{ik} delivers the correct force on the foreign atom, but it has the drawback that we can give no direct physical basis for the Ansatz (8.2).

9. Concluding remarks

In our theory, we started from the equations of motion for the Cosserat continuum. We fulfilled these equations by means of stress functions identically. We have then extended the Lagrange density for the Cosserat continuum by an interaction term, and by variation of the stress functions, we were led to the field equations of the theory of dislocations with foreign atoms. Thus, since the appearance of foreign matter is connected with the use of the Cosserat continuum, when one starts with only eq. (2.1), one obtains only the basic equations for the theory of dislocations without foreign matter [**5**]. One can then say that the ordinary theory of dislocations corresponds to a continuum whose mass elements possess the degrees of freedom of a rigid body. This result is connected with our interpretation of the individual foreign atom as being described by an elastic dipole which then possesses the same degrees of freedom as the mass element of the Cosserat continuum.

In conclusion, we would like to once more mention the most important open questions that remain. First, there is the question of the geometrical quantities that one can associate with the anti-symmetric part of the stress tensor. Furthermore, the question of the physical interpretation of the splitting of the elastic curvature tensor and the dislocation flux density V_{ikl}, and in connection with that, the question of the meaning of the not-purely-elastic distortion β_{ik}. Ultimately, the question remains of reaching a better understanding of the flux density for the foreign matter S_{ik}, and thus, e.g., its possible connection with the diffusion currents.

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