CHAPTER SIX

THE EXTREMUM FOR AN INTEGRAL THAT INCLUDES HIGHER-ORDER DERIVATIVES OF THE UNKNOWNS

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§ 48. – Conditions for the integral *J* along a curve to keep its value upon introducing a new parameter for *t*.

Let $\mathfrak{B}$ be a piece of a plane curve along which *x* and *y* can be represented as continuous functions of a parameter *t*, and their derivatives up to order *n* inclusive are likewise continuous functions of *t*. Let the function:

$$F(x, x', x'', \ldots, x^{(n)}, y, y', y'', \ldots, y^{(n)})$$

be regular for every system of values of its arguments that is defined by an element of the arc $\mathfrak{B}$, and let at least one of the quantities x', y' be non-zero at every location along the arc. We shall now consider only those integrals:

$$J = \int F\, dt$$

whose values are determined by the curve $\mathfrak{B}$ alone, but do not depend upon the special nature of the connection between *x*, *y*, and *t*. If *x* and *y* are functions along the arc $\mathfrak{B}$ of the parameter θ, which has the same properties as the parameter *t* that was introduced before, and 0 and 1 are any two points of the arc then the equation:

$$J_{01} = \int_0^1 F(x, x', \ldots, y^{(n)})\, dt = \int_0^1 F\left(x, \frac{dx}{d\theta}, \ldots, \frac{d^n y}{d\theta^n}\right) d\theta$$

must be valid. The values of the variables *t* and θ will be associated with each other by means of the points of the curves, such that, e.g., θ can be regarded as a function of *t*. If one lets the upper limit of the integral vary and differentiates with respect to the value of *t* that belongs to it then that will give:

$$F(x, x', \ldots, y^{(n)}) = F\left(x, \frac{dx}{d\theta}, \ldots, \frac{d^n y}{d\theta^n}\right),$$

or

$$(1) \qquad F\left(x, \frac{dx}{dt}, \ldots, \frac{d^n y}{dt^n}\right) dt = F\left(x, \frac{dx}{d\theta}, \ldots, \frac{d^n y}{d\theta^n}\right) d\theta .$$

In particular, one can set $\theta = x$ at any location for which x' does not vanish, i.e., consider y to be a function of x whose derivatives up to order n can be represented as entire functions of x, x', …, $x^{(n)}$, y, y', …, $y^{(n)}$, divided by powers of x', so they will be continuous of x. Now since:

$$\frac{dx}{d\theta} = 1, \qquad \frac{d^2 x}{d\theta^2} = \ldots = \frac{d^n x}{d\theta^n} = 0 ,$$

as a result of equation (1), one will have:

$$F(x, x', \ldots, y^{(n)})\, dt = F\left(x, 1, 0, \ldots, 0, y, \frac{dy}{dx}, \ldots, \frac{d^n y}{dx^n}\right) dx ,$$

or in the new notation:

$$F(x, x', \ldots, y^{(n)})\, dt = f\left(x, y, \frac{dy}{dx}, \ldots, \frac{d^n y}{dx^n}\right) dx , \qquad J = \int f\, dx .$$

Let an arbitrary representation of the arc $\mathfrak{B}$ with the given properties be given by the equations:

$$(2) \qquad x = \varphi(\theta) , \qquad y = \psi(\theta) .$$

One will then get another representation when one sets $\theta = t + \tau$ and understands τ to mean an arbitrary function of t that is linear in certain constants ε. The equations:

$$x = \varphi(t + \tau) , \quad y = \psi(t + \tau)$$

are obviously valid then, as well as the **Taylor** developments:

$$x = \varphi(t) + \varphi'(t)\,\tau + \ldots , \qquad x - \varphi(t) = \varphi'(t)\,\tau + \ldots,$$
$$y = \psi(t) + \psi'(t)\,\tau + \ldots , \qquad y - \psi(t) = \psi'(t)\,\tau + \ldots$$

It follows from this that:

$$\frac{d^{\mathfrak{a}} x}{dt^{\mathfrak{a}}} - \varphi^{(\mathfrak{a})}(t) = \frac{d^{\mathfrak{a}}[\varphi'(t)\,\tau]}{dt^{\mathfrak{a}}} + \cdots ,$$
$$\frac{d^{\mathfrak{a}} y}{dt^{\mathfrak{a}}} - \psi^{(\mathfrak{a})}(t) = \frac{d^{\mathfrak{a}}[\psi'(t)\,\tau]}{dt^{\mathfrak{a}}} + \cdots ,$$

in which the omitted terms are of order at least two in τ and the derivatives of those quantities, so they will also contain the quantities ε in the same way. On the other hand, one obviously has:

$$\frac{d^{\alpha} x}{d\theta^{\alpha}} = \varphi^{(\alpha)}(\theta) = \varphi^{(\alpha)}(t+\tau), \qquad \frac{d^{\alpha} y}{d\theta^{\alpha}} = \psi^{(\alpha)}(\theta) = \psi^{(\alpha)}(t+\tau) .$$

Equation (1) then gives:

$$\begin{aligned} &F[\varphi(t+\tau), \varphi'(t+\tau), \ldots]\left(1+\frac{d\tau}{dt}\right) \\ &\quad = F\left(\varphi(t)+\tau\,\varphi'(t)+[\varepsilon]_2, \varphi'(t)+\frac{d[\tau\,\varphi'(t)]}{dt}+[\varepsilon]_2, \varphi''(t)+\frac{d^2[\tau\,\varphi'(t)]}{dt^2}+[\varepsilon]_2, \ldots\right), \end{aligned}$$

or when one develops the left-hand side in powers of τ and the right-hand side in powers of the parts of the argument that include the quantities ε, and uses the indices $\varphi\psi$ to suggest that one has set:

$$x = \varphi(t), \qquad x' = \varphi'(t), \qquad \ldots, \qquad y = \psi(t), \qquad y' = \psi'(t), \quad \ldots,$$

one will have:

$$\left(F_{\varphi\psi} + \tau\frac{dF_{\varphi\psi}}{dt} + [\varepsilon]_2\right)\left(1+\frac{d\tau}{dt}\right) = F_{\varphi\psi} + \sum_{\alpha=0}^{n}\left\{\left(\frac{\partial F}{\partial x^{(\alpha)}}\right)_{\varphi\psi}\frac{d^{\alpha}[\tau\,\varphi'(t)]}{dt^{\alpha}} + \left(\frac{\partial F}{\partial y^{(\alpha)}}\right)_{\varphi\psi}\frac{d^{\alpha}[\tau\,\psi'(t)]}{dt^{\alpha}}\right\} + [\varepsilon]_2 .$$

If one then keeps only the terms that are linear in the constant ε then that will give:

$$\tau\frac{dF_{\varphi\psi}}{dt} + F_{\varphi\psi}\frac{d\tau}{dt} = \sum_{\alpha=0}^{n}\left\{\left(\frac{\partial F}{\partial x^{(\alpha)}}\right)_{\varphi\psi}\frac{d^{\alpha}[\tau\,\varphi'(t)]}{dt^{\alpha}} + \left(\frac{\partial F}{\partial y^{(\alpha)}}\right)_{\varphi\psi}\frac{d^{\alpha}[\tau\,\psi'(t)]}{dt^{\alpha}}\right\},$$

or when one now represents the arc $\mathfrak{B}$ by the equations:

$$x = \varphi(t), \qquad y = \psi(t),$$

which are equivalent to equations (2), and denotes differentiation with respect to t by a prime, as always, one will have:

$$(F\,\tau)' = \sum_{\alpha=0}^{n}\left\{\frac{\partial F}{\partial x^{(\alpha)}}(x'\,\tau)^{(\alpha)} + \frac{\partial F}{\partial y^{(\alpha)}}(y'\,\tau)^{(\alpha)}\right\},$$

or finally, with the indefinite integral sign:

(3) $$F\,\tau = \int dt \sum_{\mathfrak{a}=0}^{n} \left\{ \frac{\partial F}{\partial x^{(\mathfrak{a})}} (x'\tau)^{(\mathfrak{a})} + \frac{\partial F}{\partial y^{(\mathfrak{a})}} (y'\tau)^{(\mathfrak{a})} \right\} .$$

We transform the right-hand side of that equation by means of the identity:

$$\int v u^{(\mathfrak{a})} dt \;=\; v u^{(\mathfrak{a}-1)} - v' u^{(\mathfrak{a}-2)} + v'' u^{(\mathfrak{a}-3)} - \cdots + (-1)^{\mathfrak{a}-1} v^{(\mathfrak{a}-1)} u + (-1)^{\mathfrak{a}} \int v^{(\mathfrak{a})} u \, dt \;,$$

whose validity when u and v are arbitrary functions of t will become obvious when one differentiates both sides with respect to t. If one sets:

$$v = \frac{\partial F}{\partial x^{(\mathfrak{a})}} , \qquad u = x'\tau ,$$

in particular, then that will yield:

$$\int \frac{\partial F}{\partial x^{(\mathfrak{a})}} (x'\tau)^{(\mathfrak{a})} dt \;=\; \frac{\partial F}{\partial x^{(\mathfrak{a})}} (x'\tau)^{(\mathfrak{a}-1)} - \frac{d}{dt} \frac{\partial F}{\partial x^{(\mathfrak{a})}} (x'\tau)^{(\mathfrak{a}-2)} + (-1)^{\mathfrak{a}} \int \frac{d^{\mathfrak{a}}}{dt^{\mathfrak{a}}} \left(\frac{\partial F}{\partial x^{(\mathfrak{a})}} \right) x'\tau \, dt .$$

Naturally, each of the derivatives with respect to t that newly appear in that must exist and be integrable. In order to arrange that (to the extent that is necessary), we introduce the new assumption that the derivatives of x and y up to order $2n$ inclusive are finite and continuous.

One can then define the formulas that will be obtained for $\mathfrak{a} = 1, 2, \ldots, n$. One adds that the quantities $(x'\tau)^{(\mathfrak{b})}$ on the right-hand side will then appear with the factors:

$$\frac{\partial F}{\partial x^{(\mathfrak{b}+1)}} - \frac{d}{dt} \frac{\partial F}{\partial x^{(\mathfrak{b}+2)}} + \frac{d^2}{dt^2} \frac{\partial F}{\partial x^{(\mathfrak{b}+3)}} - \cdots .$$

If one then sets:

$$P_m = \sum_{a=0}^{n-m} (-1)^{\mathfrak{a}} \frac{d^{\mathfrak{a}}}{dt^{\mathfrak{a}}} \frac{\partial F}{\partial x^{(m+\mathfrak{a})}} , \qquad Q_m = \sum_{a=0}^{n-m} (-1)^{\mathfrak{a}} \frac{d^{\mathfrak{a}}}{dt^{\mathfrak{a}}} \frac{\partial F}{\partial y^{(m+\mathfrak{a})}} ,$$

(4) $$P_0 = P , \qquad Q_0 = Q , \qquad P_n = \frac{\partial F}{\partial x^{(n)}} , \qquad Q_n = \frac{\partial F}{\partial y^{(n)}} ,$$

$$P_m = \frac{\partial F}{\partial x^{(m)}} - \frac{dP_{m+1}}{dt} , \qquad Q_m = \frac{\partial F}{\partial y^{(m)}} - \frac{dQ_{m+1}}{dt}$$

in general, then equation (3) will assume the following form:

(5) $$0 = \int (P\,x' + Q\,y')\,\tau\, dt + \sum_{\mathfrak{a}=1}^{n} \{ P_{\mathfrak{a}} (x'\tau)^{(\mathfrak{a}-1)} + Q_{\mathfrak{a}} (x'\tau)^{(\mathfrak{a}-1)} \} - F\,\tau .$$

All of the terms that are outside the integral sign can now be converted into a linear combination of the quantities $\tau, \tau', \ldots, \tau^{(n-1)}$ whose coefficients are independent of τ. For example, let $\tau^{(k)}$ be the highest derivative of τ whose factor vanishes identically. The combination will then have the form:

$$T = M\,\tau^{(k)} + N\,\tau^{(k-1)} + \cdots,$$

so its derivative with respect to t will be:

$$T' = M\,\tau^{(k+1)} + (M' + N)\,\tau^{(k)} + \cdots,$$

and the omitted terms include only derivatives of τ whose order is less than k. When equation (5) is differentiated, it will then give:

$$0 = (P\,x' + Q\,y')\,\tau + M\,\tau^{(k+1)} + \cdots$$

Now since the quantities $\tau, \tau', \ldots, \tau^{(n)}$ can take on arbitrary values at any location on the arc $\mathfrak{B}$, the last equation will be possible only if the coefficient of each derivative that occurs in it vanishes by itself. In particular, since $\tau^{(k+1)}$ occurs in only one term, that would imply that:

$$M = 0\,,$$

which would contradict the assumption. The combination T must then vanish identically, i.e., one will have the identities:

(6) $$F\,\tau = \sum_{\mathfrak{a}=1}^{n} \{P_{\mathfrak{a}}\,(x'\,\tau)^{(\mathfrak{a}-1)} + Q_{\mathfrak{a}}\,(y'\,\tau)^{(\mathfrak{a}-1)}\},$$
$$P\,x' + Q\,y' = 0.$$

If one sets τ constant in the first one then it will follow that:

(7) $$F = \sum_{\mathfrak{a}=1}^{n} [P_{\mathfrak{a}}\,x^{(\mathfrak{a})} + Q_{\mathfrak{a}}\,y^{(\mathfrak{a})}].$$

If one compares the factors of $\tau^{(n-1)}$ on both sides then that will give the following identities for $n > 1$:

(8) $$P_n\,x' + Q_n\,y' = 0\,, \qquad x'\frac{\partial F}{\partial x^{(n)}} + y'\frac{\partial F}{\partial y^{(n)}} = 0\,.$$

§ 49. – Definition of δJ. Necessary conditions for an extremal.

Let the quantities δx, δy, and their derivatives up to order n inclusive be continuous functions of t. If the point (x, y) traverses the arc $\mathfrak{B}$ then the point $(x + \delta x, y + \delta y)$ will describe an arc $\mathfrak{B}^0$. We shall denote the increase in any quantity u when $\mathfrak{B}$ goes to $\mathfrak{B}^0$ by Δu, as we did in Chapter One. Let δu be the simplified expression into which Δu goes when one regards the quantities:

$$\delta x,\ (\delta x)',\ \ldots,\ (\delta x)^{(n)},\ \delta y,\ (\delta y)',\ \ldots,\ (\delta y)^{(n)}$$

as small and accordingly neglects all terms that include them to order at least two. That will obviously imply the following formulas then:

$$\delta x^{(\mathfrak{a})} = \frac{d^{\mathfrak{a}}\delta x}{dt^{\mathfrak{a}}}, \qquad \delta y^{(\mathfrak{a})} = \frac{d^{\mathfrak{a}}\delta y}{dt^{\mathfrak{a}}} \qquad (\mathfrak{a} = 1, 2, \ldots, n),$$

$$\delta F = \sum_{\mathfrak{a}=0}^{n}\left(\frac{\partial F}{\partial x^{(\mathfrak{a})}}\delta x^{(\mathfrak{a})} + \frac{\partial F}{\partial y^{(\mathfrak{a})}}\delta y^{(\mathfrak{a})}\right),$$

$$\Delta F = \delta F + [\delta x, \delta x', \ldots, \delta y^{(n)}]_2 ,$$

and as a special case of the latter equation when x' is non-zero, one will get:

$$\delta\frac{dy}{dx} = \frac{d\,\delta y}{dx} - \frac{dy}{dx}\frac{d\,\delta x}{dx},$$

$$\delta\frac{d^2y}{dx^2} = \frac{d}{dx}\delta\frac{dy}{dx} - \frac{d^2y}{dx^2}\frac{d\,\delta x}{dx}, \ \ldots$$

Likewise, one clearly has immediately that:

$$\Delta J_{01} = \int_0^1 \Delta F\,dt, \qquad \delta J_{01} = \int_0^1 \delta F\,dt,$$

(9)

$$\Delta J_{01} = \int_0^1 \delta F\,dt + \int_0^1 dt\,[\delta x, \delta x', \ldots, \delta y^{(n)}]_2 .$$

In order to convert the integral δJ by partial integration, we replace $x'\tau$ and $y'\tau$ with δx and δy in the calculation that was performed in the previous section that led from equation (3) to the formula (5). That will then yield:

(10) $$\delta J_{01} = \sum_{\mathfrak{a}=1}^{n} (P_{\mathfrak{a}}\,\delta x^{(\mathfrak{a}-1)} + Q_{\mathfrak{a}}\,\delta y^{(\mathfrak{a}-1)})\Big|_0^1 + \int_0^1 (P\,\delta x + Q\,\delta y)\,dt\,.$$

If one now varies a segment 23, in particular, and sets:

$$\delta x = \varepsilon\,(t_3 - t)^{2n+1}\,(t - t_2)^{2n+1}\,,$$
$$\delta y = \eta\,(t_3 - t)^{2n+1}\,(t - t_2)^{2n+1}\,,$$

but:

$$\delta x = \delta y = 0$$

outside of it, in which ε, η denote constants, then δx, δy will be continuous function of t along the entire arc $\mathfrak{B}$, along with their first $2n$ derivatives, and the variations of the quantities:

(11) $$x_0\,,\ x_0'\,,\ \ldots,\ x_0^{(n-1)},\ y_0\,,\ y_0'\,,\ \ldots,\ y_0^{(n-1)},\ x_1,\ \ldots,\ x_1^{(n-1)},\ y_1,\ \ldots,\ y_1^{(n-1)}$$

will vanish. Formula (9) will then imply that:

$$\Delta J_{01} = \varepsilon \int_2^3 P\,(t_3 - t)^{2n+1}(t - t_1)^{2n+1}dt + \eta \int_2^3 Q\,(t_3 - t)^{2n+1}(t - t_1)^{2n+1}dt + [\varepsilon]_2\,.$$

In general, that quantity will be positive, as well as negative, for arbitrarily small values of ε, η. Therefore, if the curve $\mathfrak{B}$ is to yield an extremum for the integral J from among all curves that connect 0 and 1 and have the same continuity properties and the same values of the quantities (1) then the factors of ε and η in the expression ΔJ_{01} must vanish. With the conclusion that was reached in § **8**, that will imply the equations:

$$P = 0,\quad Q = 0\,,$$

which are essentially equivalent to the identity (6). A curve that satisfies one of those two equations is called an *extremal*. In particular, if the quantities x_0', x_1' are non-zero then one can set $x = t$ in the vicinity of the endpoint, so an extremum will exist for prescribed values of:

$$x,\ y,\ \frac{dy}{dx},\ \ldots,\ \frac{d^{n-1}y}{dx^{n-1}}\Bigg|^{0,1}$$

only when the equation:

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} + \frac{d^2}{dx^2}\frac{\partial f}{\partial y''} + \cdots = 0$$

is valid. The left-hand of that will be denoted by Q (f).

If the integral J is to be extremized when the quantities (11) are not given, but only subject to a number of relations:

$$g_{\mathfrak{b}}[x_0, x_0', \ldots, y_0^{(n-1)}, x_1, \ldots, y_1^{(n-1)}] = 0 \,, \tag{12}$$

whose left-hand sides are regular for the system of values for their arguments that is being considered, then first of all, the desired curve $\mathfrak{B}$ must be an extremal, since one can vary them without violating the condition equations in such a way that the quantities (11) will keep their values. We then vary the arc $\mathfrak{B}$ along a segment 02 and a segment 31 that does not overlap with it and set δx on the first one equal to an entire rational function of t for which:

$$\left.\frac{d^{\mathfrak{a}}\delta x}{dt^{\mathfrak{a}}}\right|^0 = \delta x_0^{(\mathfrak{a})}, \qquad \left.\frac{d^{\mathfrak{e}}\delta x}{dt^{\mathfrak{e}}}\right|^2 = 0 \qquad \begin{pmatrix} \mathfrak{a} = 0, 1, \ldots, n-1 \\ \mathfrak{e} = 0, 1, \ldots, 2n \end{pmatrix} .$$

We take δy to be an entire function that satisfies the equations that arise by exchanging x for y in the ones that were written down and encounter analogous determinations for the segment 31. If we then set:

$$\delta x = \delta y = 0$$

for the arc 23 then δx, δy will have the same continuity properties as x and y along the entire arc 01 and be linear in the quantities:

$$\delta x_0 \,, \ldots, \; \delta y_0^{(n-1)} , \; \delta x_1, \; \ldots, \; \delta y_1^{(n-1)} .$$

When one recalls the equations of the extremals, formulas (9), (10) will then yield:

$$\Delta J_{01} = \sum_{\mathfrak{a}=1}^{n} [P_{\mathfrak{a}}\, \delta x^{(\mathfrak{a}-1)} + Q_{\mathfrak{a}}\, \delta y^{(\mathfrak{a}-1)}]\Big|_0^1 + [\delta x_0, \ldots, \delta y_1^{(n-1)}]_2 \;.$$

If that quantity is to preserve a fixed sign for all systems of variations $\delta x_0, \ldots, \delta y_1^{(n-1)}$ that satisfy the condition (12) then, from § **7**, the equation:

$$\sum_{\mathfrak{a}=1}^{n} [P_{\mathfrak{a}}\, \delta x^{(\mathfrak{a}-1)} + Q_{\mathfrak{a}}\, \delta y^{(\mathfrak{a}-1)}]\Big|_0^1 = 0$$

must be satisfied as long as one imposes the linear equations:

$$\sum_{\mathfrak{a}=0}^{n-1} \left[\frac{\partial g_{\mathfrak{b}}}{\partial x_0^{(\mathfrak{a})}} \delta x_0^{(\mathfrak{a}-1)} + \frac{\partial g_{\mathfrak{b}}}{\partial y_0^{(\mathfrak{a})}} \delta y_0^{(\mathfrak{a}-1)} + \frac{\partial g_{\mathfrak{b}}}{\partial x_1^{(\mathfrak{a})}} \delta x_1^{(\mathfrak{a}-1)} + \frac{\partial g_{\mathfrak{b}}}{\partial y_1^{(\mathfrak{a})}} \delta y_1^{(\mathfrak{a}-1)} \right] = 0 \;.$$

With that, one has a rule for deriving necessary conditions for the extremum that is characterized by the relations (12).

The entire development of the last two sections can be adapted to the case in which the function F includes further unknowns z, w, …, and their first n derivatives. One has only to add terms to the general formulas that emerge from ones in the symbol y when one replaces it with z, w, …

§ 50. – Cases in which the extremal equations admit direct integration.

From the forms of the expressions P and Q, it is clear that the equations:

$$P = Q = 0$$

can be integrated $(\mathfrak{a} + \mathfrak{b})$ times when the quantities $x, x', \ldots, x^{(\mathfrak{a}-1)}, y, y', \ldots, y^{(\mathfrak{b}-1)}$ do not occur in the function F. If, e.g., x and y are missing then as a result of the last relations (4), one will have the integral:

$$P_1 = \text{const.}, \qquad Q_1 = \text{const.} \tag{13}$$

If one sets:

$$x = t, \quad F\,dt = f\,[x, y, y', \ldots, y^{(n)}]\,dx, \tag{14}$$

in particular, and if x does not occur explicitly then one will next get the first of the integrals (13). However, since:

$$\frac{\partial F}{\partial x''} = \frac{\partial F}{\partial x'''} = \ldots = 0,$$

under the assumption in (14), equation (7) of § **48** will go to:

$$f = P_1 + \sum_{\mathfrak{a}=1}^{n} Q_{\mathfrak{a}}\, y^{(\mathfrak{a})}.$$

Therefore:

$$Q_{\mathfrak{a}} = \frac{\partial f}{\partial y^{(\mathfrak{a})}} - \frac{d}{dx}\frac{\partial f}{\partial y^{(\mathfrak{a}+1)}} + \frac{d^2}{dx^2}\frac{\partial f}{\partial y^{(\mathfrak{a}+2)}} - \cdots$$

The indicated integral can then be written:

$$f - \sum_{\mathfrak{a}=1}^{n} Q_{\mathfrak{a}}\, y^{(\mathfrak{a})} = \text{const.}, \tag{15}$$

and was already found in that form by **Euler**.

Example: Let the position of a system of mutually-interacting masses be determined by a parameter y, let x be time, let Y be a function of x, and let $-Y\,dy$ be the work done by given external forces under a displacement of the system. According to **Helmholtz**, the generalized **Hamilton** principle will then have the following form:

$$\delta\int (H + Y\,y)\,dx = 0 .$$

Therefore, H, viz., the kinetic potential, is a given function of y and the derivatives of that quantity with respect to time, and it will not include x explicitly. In the usual case of the older dynamics, H would be the difference between the potential and kinetic energies of the system. If one regards x and y as functions of a parameter t then one will have:

$$(16)\qquad \begin{gathered}(H + Yy)\,dx = F\,dt, \quad F = (H + Y\,y)\,x', \\ \frac{\partial F}{\partial x} = \frac{dY}{dx}\,y\,x' = \frac{d}{dt}(Y\,y) - Y\,y'.\end{gathered}$$

Furthermore, from the general equations (4), (7), one has:

$$P_1\,x' = F - \sum_{\alpha=1}^{n} Q_\alpha\,y^{(\alpha)} + \cdots, \qquad P = \frac{\partial F}{\partial x} - \frac{dP_1}{dt},$$

and the omitted terms include only the second and higher derivatives of x as a factor. If one then sets:

$$(17)\qquad x = t, \quad x' = 1, \qquad x'' = x''' = \ldots = 0$$

then it will follow that:

$$(18)\qquad P_1 = H + Y\,y - \sum_{\alpha=1}^{n} Q_\alpha\,y^{(\alpha)},$$

and for the extremal, one will get:

$$Q = Y + \frac{\partial H}{\partial y} - \frac{dQ_1}{dx} = 0, \qquad P = \frac{\partial F}{\partial x} - \frac{dP_1}{dx} = 0 .$$

From (16), (18), the last equation can be written:

$$(19)\qquad Y\frac{dy}{dx} + \frac{d}{dx}\left(H - \sum_{\alpha=1}^{n} Q_\alpha\,y^{(\alpha)}\right) = 0,$$

so according to the assumption (17), one will have:

$$y^{(\mathfrak{a})} = \frac{d^{\mathfrak{a}} y}{dx^{\mathfrak{a}}}, \quad Q_{\mathfrak{a}} = \frac{\partial H}{\partial \dfrac{d^{\mathfrak{a}} y}{dx^{\mathfrak{a}}}} - \frac{d}{dx} \frac{\partial H}{\partial \dfrac{d^{\mathfrak{a}+1} y}{dx^{\mathfrak{a}+1}}} + \cdots$$

for $\mathfrak{a} > 0$. Equation (19) shows that one has regarded the quantity:

$$\mathfrak{E} = H - \sum_{\mathfrak{a}=1}^{n} Q_{\mathfrak{a}}\, y^{(\mathfrak{a})}$$

as the energy of the system because one will have:

$$d\mathfrak{E} = -\, Y\, dy$$

for every time element, i.e., $d\mathfrak{E}$ is equal to the work done from the outside. The energy principle is a consequence of the **Hamilton**'s principle then. If external forces are not present then it will follow that:

$$\mathfrak{E} = \text{const.}$$

The energy principle will then appear to be a special case of **Euler**'s integral equation (15).

From the rule that was given at the end of § **49**, that development can be adapted directly to the case in which the mass-system depends upon several parameters *y*, *z*, … If no derivatives with respect to time of order higher than *n* occur in the kinetic potential then one will get the expression:

$$\mathfrak{E} = H - \sum_{\mathfrak{a}=1}^{n} (Q_{\mathfrak{a}}\, y^{(\mathfrak{a})} + R_{\mathfrak{a}}\, z^{(\mathfrak{a})} + \cdots)$$

for the energy, in which:

$$R_{\mathfrak{a}} = \frac{\partial H}{\partial \dfrac{d^{\mathfrak{a}} z}{dx^{\mathfrak{a}}}} - \frac{d}{dx} \frac{\partial H}{\partial \dfrac{d^{\mathfrak{a}+1} z}{dx^{\mathfrak{a}+1}}} + \frac{d^2}{dx^2} \frac{\partial H}{\partial \dfrac{d^{\mathfrak{a}+2} z}{dx^{\mathfrak{a}+2}}} - \cdots,$$

and analogous quantities are introduced for the remaining parameters.

Problem XII. – Find a curve 01 that encloses the smallest-possible volume along with its evolute and its normals that are drawn at the points 0 and 1.

If *r* is the radius of curvature and *ds* is the arc-length element then the surface that is defined will be the sum of all infinitely-thin triangles of area $\frac{1}{2} r\, ds$. Now, since:

$$\pm r = \frac{(x'^2 + y'^2)^{3/2}}{x' y'' - x'' y'}, \quad ds = \sqrt{x'^2 + y'^2}\, dt ,$$

one will, in turn, be dealing with the problem of extremizing the integral:

$$J = \int \frac{(x'^2 + y'^2)^2 dt}{x' y'' - x'' y'} .$$

The integrand is free of x and y, so the integral equations (13) will be valid, or:

$$P_1 = a , \qquad Q_1 = b .$$

Furthermore, the identity (7), or:

$$F = P_1 x' + Q_1 y' + P_2 x'' + Q_2 y''$$

will give, since:

$$P_2 = \frac{\partial F}{\partial x''} = \frac{y'(x'^2 + y'^2)^2}{(x' y'' - x'' y')^2}, \quad Q_2 = \frac{\partial F}{\partial y''} = \frac{-x'(x'^2 + y'^2)^2}{(x' y'' - x'' y')^2} ,$$

the result that:

$$F = a x' + b y' - F \qquad \text{or} \qquad 2F = a x' + b y' .$$

If one writes $x = t$, $y' = p$ then one can write:

$$2(1+p^2)^2 = (a + b\,p)\frac{dp}{dx} \qquad \text{or} \qquad dx = \frac{a + b\,p}{2(1+p^2)^2}\,dp$$

for that equation, and it will follow from this that:

$$dy = p\,dx = \frac{(a + b\,p)\,p\,dp}{2(1+p^2)^2} .$$

A rational integrable linear combination with constant coefficients can be easily defined by dx and dy. Namely, since one has the identity:

$$d\left(\frac{\alpha + \beta\,p + \gamma\,p^2}{1+p^2}\right) = \frac{\beta + 2(\gamma - \alpha)p - \beta\,p^2}{(1+p^2)^2}\,dp$$

when α, β, γ are constants, the expression:

$$2\,(b\,dx - a\,dy) = \frac{a\,b + (b^2 - a^2)\,p - a\,b\,p^2}{(1+p^2)^2}\,dp\,,$$

which will go to the previous one when one sets:

$$\beta = a\,b\,, \qquad \gamma = \frac{b^2}{2}\,, \qquad \alpha = \frac{a^2}{2}\,,$$

will be rationally integrable, and one will get:

$$4\,(b\,dx - a\,dy) = d\left[\frac{(a+b\,p)^2}{1+p^2}\right] = d\left[\frac{(a\,dx+b\,dy)^2}{dx^2+dy^2}\right].$$

Now, the equations:

$$\sqrt{a^2+b^2}\;\xi = b\,x - a\,y\,, \qquad \sqrt{a^2+b^2}\;\eta = a\,x + b\,y$$

represent a rectangular transformation, such that:

$$dx^2 + dy^2 = d\xi^2 + d\eta^2 .$$

The equation above can then be written:

$$4\sqrt{a^2+b^2}\;d\xi = d\left[\frac{(a^2+b^2)\,d\eta^2}{d\xi^2+d\eta^2}\right],$$

from which, it follows upon integration that:

$$x + c = \tfrac{1}{4}\sqrt{a^2+b^2}\;\frac{d\eta^2}{d\xi^2+d\eta^2}\,,$$

$$\frac{d\eta}{d\xi} = \sqrt{\frac{\xi+c}{\tfrac{1}{4}\sqrt{a^2+b^2}-(c+\xi)}}\,.$$

From § **12**, that will imply that the desired curves, i.e., the extremals, are cycloids. When the tangents are not prescribed at the endpoints, the expression:

$$P_1\,\delta x + Q_1\,\delta y + P_2\,\delta x' + Q_2\,\delta y'\,\Big|_0^1$$

must vanish for the allowable variations. If the points 0 and 1 are given, so:

$$\delta x_0 = \delta y_0 = \delta x_1 = \delta y_1 = 0\,,$$

then the equation:

$$P_2\,\delta x' + Q_2\,\delta y'\Big|_0^1 = 0$$

must be true for arbitrary variations $\delta x'$, $\delta y'$, since they are freely-available. It will then follow that:

$$\left.\frac{y'(x'^2+y'^2)^2}{x'\,y''-x''\,y'}\right|^{0,1} = 0\,, \qquad \left.\frac{-x'(x'^2+y'^2)^2}{x'\,y''-x''\,y'}\right|^{0,1} = 0\,.$$

The points 0 and 1 must then be cusps of the cycloid.

§ 51. – Cases in which the order of the differential equations for the extremal is decreased.

If we consider y to be a function of x and accordingly set $x = t$ then the extremals will satisfy the equation:

$$Q\,(f) = \frac{\partial f}{\partial y} - \frac{d}{dt}\frac{\partial f}{\partial y'} + \cdots + (-1)^n \frac{d^n}{dt^n}\frac{\partial f}{\partial y^{(n)}} = 0\,, \tag{20}$$

which represents a finite equation for $n = 0$, but in general a differential equation of order $2n$. Obviously, the quantity $y^{(2n)}$ occurs only in the last term, and indeed with the factor:

$$(-1)^n \frac{\partial^2 f}{\partial y^{(n)}\,\partial y^{(n)}}\ .$$

Therefore, the order of the differential equation will be reduced if and only if the function f depends upon $y^{(n)}$ linearly, such that one can set:

$$f\,[x, y, y', \ldots, y^{(n)}] = g\,[x, y, \ldots, y^{(n-1)}] + y^{(n)}\,h[x, y, \ldots, y^{(n-1)}]\ .$$

In that case, according to **Euler**, the integral J is replaced with another one whose integrand is free of $y^{(n)}$. Namely, one poses the equation:

$$\int f\,dx = G[x, y, \ldots, y^{(n-1)}] + \int H\,[x, y, \ldots, y^{(n-1)}]\,dx\,, \tag{21}$$

or what amounts to the same thing:

$$g\,dx + h\,dy^{(n-1)} = H\,dx + \left\{\frac{\partial G}{\partial x} + \frac{\partial G}{\partial y}y' + \frac{\partial G}{\partial y'}y'' + \cdots + \frac{\partial G}{\partial y^{(n-2)}}y^{(n-1)}\right\}dx + \frac{\partial G}{\partial y^{(n-1)}}dy^{(n-1)}\,,$$

so one needs only to determine G as a function of the independent arguments $x, y, \dots, y^{(n-1)}$ (which is possible by means of a quadrature) such that:

$$\frac{\partial G}{\partial y^{(n-1)}} = h[x, y, \dots, y^{(n-1)}] ,$$

and in addition to set:

$$H = g[x, y, \dots, y^{(n-1)}] - \frac{\partial G}{\partial x} - \sum_{\mathfrak{a}=0}^{n-2} \frac{\partial G}{\partial y^{(\mathfrak{a})}} y^{(\mathfrak{a}+1)} .$$

Equation (21) will then be valid, and that will imply the given transformation of the integral J. If one goes to the definite integral then that will give:

$$J_{01} = G[x, y, \dots, y^{(n-1)}]\Big|_0^1 + \int_0^1 H[x, y, \dots, y^{(n-1)}]\,dx .$$

Therefore, for prescribed values of the quantities $y,\ y',\ \dots,\ y^{(n-1)}$ at the locations 0 and 1, the integrals:

$$J, \qquad \int H[x, y, \dots, y^{(n-1)}]\,dx$$

will be simultaneous extrema.

If one further writes equation (21) in the form:

$$f = H + \frac{dG}{dx}$$

and considers Q to be the symbol of an operation that is defined by equation (20) then that will give:

$$Q(f) = Q\left(\frac{dG}{dx}\right) + Q(H) .$$

The first summand on the right-hand side vanishes identically, because if one sets:

$$\Phi = \frac{dG}{dx} = \frac{\partial G}{\partial x} + \sum_{\mathfrak{a}=0}^{n-1} \frac{\partial G}{\partial y^{(\mathfrak{a})}} y^{(\mathfrak{a}+1)}$$

then one will obviously have:

$$\frac{\partial \Phi}{\partial y^{(\mathfrak{a})}} = \frac{\partial G}{\partial y^{(\mathfrak{b}-1)}} + \frac{d}{dx}\frac{\partial G}{\partial y^{(\mathfrak{b})}}$$

for $\mathfrak{b} = 1, 2, \dots, n-1$, so:

$$\frac{d^{\mathfrak{b}}}{dx^{\mathfrak{b}}}\frac{\partial \Phi}{\partial y^{(\mathfrak{b})}} = \frac{d^{\mathfrak{b}}}{dx^{\mathfrak{b}}}\frac{\partial G}{\partial y^{(\mathfrak{b}-1)}} + \frac{d^{\mathfrak{b}+1}}{dx^{\mathfrak{b}+1}}\frac{\partial G}{\partial y^{(\mathfrak{b})}} .$$

If one sets $\mathfrak{b}$ equal to the given values in succession and adds the equations that are obtained when one multiplies them by the factor $(-1)^{\mathfrak{b}}$ to the identities:

$$\frac{\partial \Phi}{\partial y} = \frac{d}{dx}\frac{\partial G}{\partial y}, \qquad (-1)^n \frac{d^n}{dx^n}\frac{\partial \Phi}{\partial y^{(n)}} = (-1)^n \frac{d^n}{dx^n}\frac{\partial G}{\partial y^{(n-1)}}$$

then all terms on the right-hand side will drop out, and one will get the result that:

(22) $$Q(\Phi) = Q\left(\frac{dG}{dx}\right) = 0, \qquad Q(f) = Q(H).$$

Since H is free of $y^{(n)}$, when $y^{(2n)}$ does not occur in the expression $Q(f)$, it will not contain $y^{(2n-1)}$ either. In general, $y^{(2n-2)}$ will appear in the expression $Q(H)$. If that were not the case then one could apply the argument that was just presented for f to the expression H, and one would get:

$$\int H\,dx = G_1[x, y, \ldots, y^{(n-2)}] + \int H_1[x, y, \ldots, y^{(n-2)}]\,dx,$$

$$Q(f) = Q(H) = Q(H_1).$$

The expression $Q(f)$ would then include no derivatives of y of order higher than $2n - 4$. If it were also free of them then one could continue the same process. Therefore, the highest derivative of y that occurs in the expression $Q(f)$ will always have even order – say, $y^{(2m)}$. One will then have:

(23) $$J = G[x, y, \ldots, y^{(n-1)}] + \sum_{\mathfrak{a}=1}^{n-m-1} G_{\mathfrak{a}}[x, y, \ldots, y^{(n-\mathfrak{a}-1)}] + \int H_{n-m-1}[x, y, \ldots, y^{(m)}]\,dx,$$

$$Q(f) = Q(H) = \ldots = Q(H_{n-m-1}),$$

and the expressions $G, H, G_1, H_1, \ldots, G_{n-m-1}, H_{n-m-1}$ can be represented with the help of $n - m$ quadratures.

In the special case $m = 0$, one has:

$$Q(f) = Q(H_{n-1}) = \frac{\partial H_{n-1}}{\partial y}.$$

Hence, when the quantity $Q(f)$ vanishes identically, H_{n-1} will be free of y, and equation (23) will yield:

(24) $$J = G[x, y, \ldots, y^{(n-1)}] + \sum_{\mathfrak{a}=1}^{n-1} G_{\mathfrak{a}}[x, y, \ldots, y^{(n-\mathfrak{a}-1)}] + \int H_{n-1}(x)\,dx.$$

$f[x, y, \ldots, y^{(n)}]$ will then be the complete differential quotient of the right-hand side of x. With that, it has been proved that the identity:

$$Q(f) = 0 \tag{25}$$

represents the integrability condition. When it is assumed, the function f will be integrable with no restriction, and its integral will be represented explicitly in equation (24). The problem of extremizing the integral J for prescribed values of the quantities y, y', …, $y^{(n)}$ at the endpoints will obviously lose any meaning because J will already be determined by those values.

The fact that equation (25) is also a necessary condition for the integrability of the function f is evident from the argument that led to equation (22).

§ 52. – The arc-length as an independent variable.

If one introduces the arc-length as the parameter t then the equation:

$$x'^2 + y'^2 = 1 .$$

will be valid. If one differentiates that $m - 1$ times with respect to t then that will give:

$$x' x^{(m)} + y' y^{(m)} + \cdots = 0 , \tag{26}$$

and the omitted terms will include only derivatives of order less than m. Thus, if, e.g., x' is non-zero then the quantities x'', x'', …, $x^{(m)}$ can be expressed rationally in terms of the quantities y', y'', …, $y^{(m)}$ and the converse will be true when y' does not vanish. If one further sets:

$$\omega = \int_0^t (x' y'' - x'' y')\, dt = \int_0^t \frac{x' y'' - x'' y'}{x'^2 + y'^2}\, dt = \arctan \frac{y'}{x'}\bigg|_0^t$$

then:

$$\omega = x' y'' - x'' y'$$

will be curvature of the curve, which will be taken to be positive or negative according to whether the radius of curvature has the same relationship to the direction of increasing t that the $+y$-axis has to the $+x$-axis, or is opposite to it, resp. ω will then be the angle that the direction of increasing t describes, which is taken to be positive or negative according to whether that direction rotates in the positive or negative sense, resp., and when $x_0' = \cos \alpha$, $y_0' = \sin \alpha$, one will have the equations:

$$x' = \cos(\omega + \alpha) , \qquad y' = \sin(\omega + \alpha) .$$

Moreover, when one differentiates the equation for ω', that will give:

$$\omega^{(m-1)} = -y'\,x^{(m)} + x'\,y^{(m)} + \cdots, \tag{27}$$

and the omitted terms have the same character as they did in equation (26). If one determines the quantities $x^{(m)}$, $y^{(m)}$ from that equation and equation (27) then the determinant of the coefficients will be + 1. The quantities x', x'', …, $x^{(m)}$, y', y'', …, $y^{(m)}$ will then be determined uniquely in terms of the quantities ω, ω', …, $\omega^{(m-1)}$. If the former quantities have the same values for two arc elements that start from the same point then the element will have *contact of order m* or an *osculation of order m* – 1. One then calls the quantity $\omega^{(\mathfrak{a})}$, i.e., the $(\mathfrak{a} - 1)^{\text{th}}$ differential quotient of the curvature with respect to arc-length, an *osculation invariant of order* $\mathfrak{a}$. In the special case where x' does not vanish along an arc, e.g., the quantities:

$$\frac{dy}{dx}, \quad \frac{d^2y}{dx^2}, \quad \ldots,$$

will also have the same property of guaranteeing contact up to a certain order by coincidence. If they coincide up to the m^{th} derivative then one will have an osculation of order m – 1.

Equation (26) will become:

$$x'\,x^{(2n)} + y'\,y^{(2n)} + \cdots = 0 \tag{28}$$

in the case of $m = 2n$. We combine that with the equations of the extremals:

$$P = Q = 0,$$

which can be written in the forms:

$$\begin{aligned} \frac{\partial^2 F}{\partial x^{(n)}\,\partial x^{(n)}}\,x^{(2n)} + \frac{\partial^2 F}{\partial x^{(n)}\,\partial y^{(n)}}\,y^{(2n)} + \cdots &= 0, \\ \frac{\partial^2 F}{\partial y^{(n)}\,\partial x^{(n)}}\,x^{(2n)} + \frac{\partial^2 F}{\partial y^{(n)}\,\partial y^{(n)}}\,y^{(2n)} + \cdots &= 0, \end{aligned} \tag{29}$$

resp., in which only the terms that did not include $x^{(2n)}$ and $y^{(2n)}$ were omitted. If we imagine that the latter quantities are determined by one of those equations and equation (28) then the determinant of the coefficients will be one of the expressions:

$$\begin{vmatrix} x' & y' \\ \dfrac{\partial^2 F}{\partial x^{(n)}\,\partial x^{(n)}} & \dfrac{\partial^2 F}{\partial x^{(n)}\,\partial y^{(n)}} \end{vmatrix}, \qquad \begin{vmatrix} x' & y' \\ \dfrac{\partial^2 F}{\partial y^{(n)}\,\partial x^{(n)}} & \dfrac{\partial^2 F}{\partial y^{(n)}\,\partial y^{(n)}} \end{vmatrix}.$$

If one now differentiates the identity [§ **48**, (8)] with respect to $x^{(n)}$ and $y^{(n)}$ then that will give:

$$x'\frac{\partial^2 F}{\partial x^{(n)}\,\partial x^{(n)}} + y'\frac{\partial^2 F}{\partial x^{(n)}\,\partial y^{(n)}} = 0\,,$$

$$x'\frac{\partial^2 F}{\partial y^{(n)}\,\partial x^{(n)}} + y'\frac{\partial^2 F}{\partial y^{(n)}\,\partial y^{(n)}} = 0\,.$$

There will then be an everywhere-finite quantity F_1 along the curve for which the equations:

$$\frac{\partial^2 F}{\partial x^{(n)}\,\partial x^{(n)}} = y'^2 F_1\,, \qquad \frac{\partial^2 F}{\partial x^{(n)}\,\partial y^{(n)}} = -\,x'\,y'\,F_1\,, \qquad \frac{\partial^2 F}{\partial y^{(n)}\,\partial y^{(n)}} = x'^2 F_1$$

are true if (and indeed *only if*) t is not precisely the arc-length. If one has $x = t$, in particular, then one will have simply:

$$F_1 = \frac{\partial^2 F}{\partial y^{(n)}\,\partial y^{(n)}}\,.$$

With those values for the second derivatives of F, the two determinants above will become:

$$-\,y'(x'^2 + y'^2)\,F_1, \qquad +\,x'(x'^2 + y'^2)\,F_1,$$

resp., so when t means the arc-length, they will be:

$$-\,y'\,F_1, \qquad +\,x'\,F_1,$$

resp. Now since at least one of the quantities x', y' is non-zero, the same thing will be true of at least one of those determinants when the assumption is introduced that F_1 does not vanish. One will then obtain expressions for $x^{(2n)}$, $y^{(2n)}$ from two of equations (28), (29) whose numerators include the quantities:

(30) $$x,\ x',\ \ldots,\ x^{(2n-1)},\ y,\ y',\ \ldots,\ y^{(2n-1)},$$

and will be regular when that is true for the system of arguments $x, x', \ldots, x^{(n)}, y, y', \ldots, y^{(n)}$ that is included in the series (30). If F_1 is non-zero for the latter then the expressions that are obtained for $x^{(2n)}, y^{(2n)}$:

$$
\text{(31)} \qquad \begin{aligned} x^{(2n)} &= \Phi[x, x', \ldots, x^{(2n-1)}, y, y', \ldots, y^{(2n-1)}], \\ y^{(2n)} &= \Psi[x, x', \ldots, x^{(2n-1)}, y, y', \ldots, y^{(2n-1)}] \end{aligned}
$$

will also be regular for the system (30) in question. If one adds the equations:

$$
\text{(32)} \qquad \frac{dx^{(\mathfrak{a})}}{dt} = x^{(\mathfrak{a}+1)}, \qquad \frac{dy^{(\mathfrak{a})}}{dt} = y^{(\mathfrak{a}+1)} \qquad (\mathfrak{a} = 0, 1, \ldots, 2n-2)
$$

then one will have, in total, $4n$ first-order differential equations for determining the quantities (30), to which the general theorems of § **27** can be applied.

A regular segment of an extremal $\mathfrak{C}$ along which F_1 does not vanish determines a solution to the system of equations that was obtained that will have a special character when the equations:

$$
\text{(33)} \qquad \begin{aligned} \varphi_1 &= x'^2 + y'^2 - 1 = 0, \\ \varphi_{\mathfrak{a}} &= x' x^{(\mathfrak{a})} + y' y^{(\mathfrak{a})} + \cdots = 0 \quad (\mathfrak{a} = 2, 3, \ldots, 2n-1) \end{aligned}
$$

are further established, and which are coupled by the formula:

$$
\frac{d\varphi_{\mathfrak{a}}}{dt} = \varphi_{\mathfrak{a}+1} .
$$

Those relations are true for any system of integrals of the equations (31), (32) in general when they are assumed, e.g., for the location $t = 0$:

$$
\text{(34)} \qquad \varphi_{\mathfrak{a}}\Big|^{0} = 0 , \qquad \mathfrak{a} = 1, 2, \ldots, 2n-1 .
$$

Namely, since equation (28) or:

$$
\varphi_{2n} = 0
$$

will follow from the system (31), φ_{2n+1} will be constant. The same thing will follow from this for φ_{2n+2} from the penultimate of those equations, etc. Now, from § **27**, one can embed the segment of the extremal $\mathfrak{C}$ considered in a $4n$-fold manifold of other extremal segments that can be represented by equations of the form:

$$
\begin{aligned} x &= X[t, x_0, x_0', \ldots, y_0^{(2n-1)}] , \\ y &= Y[t, x_0, x_0', \ldots, y_0^{(2n-1)}] . \end{aligned}
$$

The functions X, Y will be regular when the arguments define a system of values that is associated with the arc $\mathfrak{C}$. However, t is not the arc-length on those curves, in general. That will be first achieved when one subjects the $4n$ integration constants:

$$x_0,\ x'_0,\ \ldots,\ x_0^{(2n-1)},\ y_0,\ y'_0,\ \ldots,\ y_0^{(2n-1)}$$

to the $2n - 1$ equations (34). If those equations, and therefore equations (33), are fulfilled then $2n + 1$ constants will remain arbitrary, e.g., when x'_0 does not vanish on the curve $\mathfrak{C}$, one of the systems of quantities:

$$x_0, y_0,\ y'_0,\ \ldots,\ y_0^{(2n-1)}; \qquad x_0, y_0,\ \left.\frac{dy}{dx}\right|^0,\ \ldots,\ \left.\frac{d^{2n-1}y}{dx^{2n-1}}\right|^0 .$$

When one fixes the first $n + 1$ of those quantities, one can then embed the arc of the curve $\mathfrak{C}$ in a family of extremals that go through the point 0 and have the first $n - 1$ osculation invariants in common with $\mathfrak{C}$, while the following n will remain available as arbitrary constants in the vicinity of the values that they have in common with the curve. If we denote those n constants by a, b, …, k and their values for the curve $\mathfrak{C}$ by α, β, …, κ then the curve $\mathfrak{C}$ will appear to be a member of a family:

$$x = \xi\,(t, a, b, \ldots, k)\ , \quad y = \eta\,(t, a, b, \ldots, k)\ , \tag{35}$$

and since t is the arc-length, as measured from the point 0, the quantities:

$$\xi\,\big|^0,\ \left.\frac{\partial\xi}{\partial t}\right|^0,\ \ldots,\ \left.\frac{\partial^{n-1}\xi}{\partial t^{n-1}}\right|^0,\ \eta\,\big|^0,\ \left.\frac{\partial\eta}{\partial t}\right|^0,\ \ldots,\ \left.\frac{\partial^{n-1}\eta}{\partial t^{n-1}}\right|^0$$

will be independent of a, b, …, k such that the equations:

$$\xi_{\mathfrak{a}}\,\big|^0 = \xi'_{\mathfrak{a}}\,\big|^0 = \ldots = \xi_{\mathfrak{a}}^{(n-1)}\,\big|^0 = \eta_{\mathfrak{a}}\,\big|^0 = \ldots = \eta_{\mathfrak{a}}^{(n-1)}\,\big|^0 = 0 \tag{36}$$

will be true and remain valid when one replaces a with any of the quantities b, c, …, k.

§ 53. – Differential formulas for the integral *J* when it is defined along an arc of the defined family. Fields.

Let 0, 1, 2, be three points of the curve $\mathfrak{C}$ that follow each other in the direction of increasing *t*, and let 3 be a variable point on an arbitrary curve (35) that is associated with the argument *t*, and along which the integral $\bar{J}_{03}$ is defined. One obviously has:

$$\frac{\partial \bar{J}_{03}}{\partial t} = F(\xi, \xi', \ldots, \eta^{(n)}) \Big|^{3}$$

then, or from [§ **48** (7)]:

$$\frac{\partial \bar{J}_{03}}{\partial t} = \sum_{\mathfrak{a}=1}^{n} [P_{\mathfrak{a}}\, \xi^{(\mathfrak{a})} + Q_{\mathfrak{a}}\, \eta^{(\mathfrak{a})}] \Big|^{3} .$$

One will then get:

$$\frac{\partial \bar{J}_{03}}{\partial a} = \int_0^3 dt \left(\frac{\partial F}{\partial x}\, \xi_a + \frac{\partial F}{\partial x'}\, \xi'_a + \cdots + \frac{\partial F}{\partial y^{(n)}}\, \eta_a^{(n)} \right),$$

in which the values (35) are substituted in the function symbols *P*, *Q*, *F*. If one partially integrates in the last equation using the method in § **48** then that will give:

$$\frac{\partial \bar{J}_{03}}{\partial t} = \sum_{\mathfrak{a}=1}^{n} [P_{\mathfrak{a}}\, \xi_a^{(\mathfrak{a})} + Q_{\mathfrak{a}}\, \eta_a^{(\mathfrak{a}-1)}] \Big|_0^3 + \int_0^3 (P\, \xi_a + Q\, \eta_a)\, dt ,$$

so when one recalls the equations for the extremals and the formulas (36), one will have:

(37)
$$\frac{\partial \bar{J}_{03}}{\partial t} = \sum_{\mathfrak{a}=1}^{n} [P_{\mathfrak{a}}\, \xi_a^{(\mathfrak{a})} + Q_{\mathfrak{a}}\, \eta_a^{(\mathfrak{a}-1)}] \Big|^{3} ,$$

in which one can naturally replace *a* with *b*, *c*, *k*. That formula will also remain valid in many cases when not all of the assumptions that were introduced are fulfilled, e.g., the extremals are singular at the point. However, our further arguments will employ only that equation itself, such that the assumptions need to be verified only in the indicated cases if one of to make the following theory applicable.

If the quantities *t*, *a*, *b*, …, *k* are differentiable functions of one variable *t* then it will follow from the last equation that:

$$\frac{d\bar{J}_{03}}{d\tau} = \sum_{\mathfrak{a}=1}^{n} \left\{ P_{\mathfrak{a}} \left(\xi^{(\mathfrak{a})} \frac{dt}{d\tau} + \xi_a^{(\mathfrak{a}-1)} \frac{da}{d\tau} + \cdots \right) + Q_{\mathfrak{a}} \left(\eta^{(\mathfrak{a})} \frac{dt}{d\tau} + \eta_a^{(\mathfrak{a}-1)} \frac{da}{d\tau} + \cdots \right) \right\}$$

(38)

$$= \sum_{\mathfrak{a}=1}^{n} \left(P_{\mathfrak{a}} \frac{dx^{(\mathfrak{a}-1)}}{d\tau} + Q_{\mathfrak{a}} \frac{dy^{(\mathfrak{a}-1)}}{d\tau} \right) \Bigg|^{3} .$$

Now let the points 1 and 2 be connected by a curve $\mathfrak{L}$ that contacts the curve $\mathfrak{C}$ at those points and is characterized more precisely by the following assumptions: The first n – 1 osculation invariants $\omega, \omega', \ldots, \omega^{(n-2)}$ coincide with those of the curve $\mathfrak{C}$ at the points 1 and 2 and are functions $\varphi(\tau)$, in the sense of § **17**, along the curve $\mathfrak{L}$. For each element of the curve $\mathfrak{C}$, there is a corresponding element of the curve $\mathfrak{L}$ that lies between 1 and 2, for which the coordinates of the starting point and the first n osculation invariants deviate from those of the former element by differences whose absolute values lie below a positive constant ε. Hence, if the determinant:

$$\Delta = \frac{\partial(\xi, \eta, \omega, \omega', \ldots, \omega^{(n-2)})}{\partial(t, a, b, \ldots, k)}$$

is non-zero everywhere along the curve $\mathfrak{C}$ between the points 1 and 2 then we will say that the extremals (35) define a *field* of the arc 12. If one has represented the extremal that goes through the point 0 by means of a parameter s, and if one has:

$$\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 > 0$$

for the segment of the curve $\mathfrak{C}$ in question then $\partial t / \partial s$ will be finite and non-zero. Now since:

$$\Delta = \frac{\partial(\xi, \eta, \omega, \ldots)}{\partial(s, a, b, \ldots)} \frac{\partial s}{\partial t} ,$$

the content of the assumption that was introduced will remain the same when t does not mean the arc-length.

In the following argument, t shall however preserve its meaning up to now as the arc-length.

If the extremals (35) define a field then one can add to the arguments $t, a, b, \ldots, k$ the demand that the quantities $x, y, \omega, \omega', \ldots, \omega^{(n-2)}$ can assume any prescribed system of values $\mathfrak{S}$ that differs from the one that is attained by the arc 12 by sufficiently little, e.g., so little that the differences between corresponding quantities have absolute values that are less than ε_0 . If the system $\mathfrak{S}$ goes continuously into one that is associated with the curve $\mathfrak{C}$ itself then $a, b, \ldots, k$ will assume the values $\alpha, \beta, \ldots, \kappa$. If one then assumes that the quantity ε that was introduced above is less than ε_0 and lets the point 3 move along the curve $\mathfrak{L}$ then one can always construct an extremal 03 that belongs to the family:

$$x = \xi(t, a, b, \dots, k), \qquad y = \eta(t, a, b, \dots, k)$$

that has contact of order $n - 1$ with the curve $\mathfrak{L}$. Then let τ be the arc-length of the curve $\mathfrak{L}$, as measured from the point 1 onward. From § **52**, one has:

$$(39) \qquad \begin{aligned} x' &= \frac{dx}{d\tau}, & x'' &= \frac{d^2x}{d\tau^2}, & \dots, && x^{(n-1)} &= \frac{d^{n-1}x}{d\tau^{n-1}}, \\ y' &= \frac{dy}{d\tau}, & y'' &= \frac{d^2y}{d\tau^2}, & \dots, && y^{(n-1)} &= \frac{d^{n-1}y}{d\tau^{n-1}}, \end{aligned}$$

such that for $\mathfrak{a} = 1, 2, 3, \dots, n - 1$, that will give:

$$\frac{dx^{(\mathfrak{a}-1)}}{d\tau} = \frac{d^{\mathfrak{a}}x}{d\tau^{\mathfrak{a}}}, \qquad \frac{dy^{(\mathfrak{a}-1)}}{d\tau} = \frac{d^{\mathfrak{a}}y}{d\tau^{\mathfrak{a}}}.$$

Now, since Δ does not vanish, the quantities $t, a, b, \dots, k$ will be regular functions of $x, y, \omega, \dots, \omega^{(n-2)}$ in the neighborhood of the arc 12 and the system of values that is assigned to the curve $\mathfrak{C}$, so they will also be functions $\varphi(\tau)$ in the sense of § **17**. Formula (38) can then be applied, and when one recalls the last of equations (39), that will give:

$$\begin{aligned} \frac{d\bar{J}_{03}}{d\tau} &= \sum_{\mathfrak{a}=1}^{n-1} [P_{\mathfrak{a}}\, x^{(\mathfrak{a})} + Q_{\mathfrak{a}}\, y^{(\mathfrak{a})}] + P_n \frac{dx^{(n-1)}}{d\tau} + Q_n \frac{dy^{(n-1)}}{d\tau} \\ &= \sum_{\mathfrak{a}=1}^{n-1} [P_{\mathfrak{a}}\, x^{(\mathfrak{a})} + Q_{\mathfrak{a}}\, y^{(\mathfrak{a})}] + P_n \frac{d^n x}{d\tau^n} + Q_n \frac{d^n y}{d\tau^n}. \end{aligned}$$

If one then sets:

$$F = F(x, x', \dots, y^{(n)}), \qquad \bar{F} = F\left(x, \frac{dx}{d\tau}, \dots, \frac{d^n y}{d\tau^n}\right),$$

$$p = x^{(n)}, \qquad q = y^{(n)}, \qquad \bar{p} = \frac{d^n x}{d\tau^n}, \qquad \bar{q} = \frac{d^n y}{d\tau^n}$$

and employs the identity (7) in § **48** then it will follow that:

$$\frac{d\bar{J}_{03}}{d\tau} = F + \frac{\partial F}{\partial p}(\bar{p} - p) + \frac{\partial F}{\partial q}(\bar{q} - q).$$

On the other hand, one will get:

$$\frac{dJ_{32}}{d\tau} = -\bar{F}$$

for the integral J_{32} that is defined along the curve $\mathfrak{L}$, so it will follow that:

$$\text{(40)} \qquad \frac{d\,(\bar{J}_{03}+J_{32})}{d\tau} = F - \bar{F} + \frac{\partial F}{\partial p}(\bar{p}-p) + \frac{\partial F}{\partial q}(\bar{q}-q) = \mathcal{E}\,.$$

Now the extremal 03 will go to $\mathfrak{C}$ when it has an osculation of order $n - 2$ with the curve $\mathfrak{L}$, so, e.g., when the point 3 assumes the positions 1 and 2. The initial and final values of the quantity $\bar{J}_{03} + J_{32}$ are then the following:

$$\bar{J}_{03}+J_{32}\Big|^{\tau_1} = \bar{J}_{01}+J_{12}\ ,$$

$$\bar{J}_{03}+J_{32}\Big|^{\tau_2} = \bar{J}_{01}+\bar{J}_{12} = \bar{J}_{02}\ .$$

It will then follow that the difference:

$$\bar{J}_{03}+J_{32}\Big|_{\tau_1}^{\tau_2} = \bar{J}_{12}-J_{12}$$

will have the same sign as the quantity $\mathcal{E}$ when it has a fixed sign for all curves that are compared to $\mathfrak{L}$ without ever vanishing along the entire length of such a curve. The curve $\mathfrak{C}$ will then provide a maximum or minimum of the integral J in comparison to all curves $\mathfrak{L}$ according to whether $\mathcal{E}$ is positive or negative, respectively. Since the sign of the quantity $\mathcal{E}$ is fixed, moreover, it will follow that F_1 can be assumed to be non-zero. Namely, since $\bar{p} - p$ and $\bar{q} - q$ are small quantities, one can develop:

$$\begin{aligned}\bar{F} &= F + \frac{\partial F}{\partial p}(\bar{p}-p) + \frac{\partial F}{\partial q}(\bar{q}-q) \\ &+ \frac{1}{2}\left(\frac{\partial^2 F}{\partial p^2}\right)_m(\bar{p}-p)^2 + \frac{1}{2}\left(\frac{\partial^2 F}{\partial p\,\partial q}\right)_m(\bar{p}-p)(\bar{q}-q) + \frac{1}{2}\left(\frac{\partial^2 F}{\partial q^2}\right)_m(\bar{q}-q)^2,\end{aligned}$$

in which the subscript m means that a certain system of values for p and q has been taken:

$$p_m = p + \theta(\bar{p}-p)\,, \qquad q_m = q + \theta(\bar{q}-q)$$

for which θ lies between – 1 and + 1. However, since:

$$\frac{\partial^2 F}{\partial p^2} = y'^2\,F_1\ , \qquad \frac{\partial^2 F}{\partial p\,\partial q} = -\,x'\,y'\,F_1\ , \qquad \frac{\partial^2 F}{\partial q^2} = x'^2\,F_1\ ,$$

equation (40) will then give:

$$\mathcal{E} = -\tfrac{1}{2}(F_1)_m \{y'(\bar{p}-p) - x'(\bar{q}-q)\}^2 .$$

§ 54. – $\mathcal{E}$ vanishes everywhere along a curve only when it is an extremal of a field.

When the second factor in the expression that was obtained for $\mathcal{E}$ vanishes, since the omitted terms in the equation:

$$\omega^{(n-1)} = x' y^{(n)} - y' x^{(n)} + \cdots = x' q - y' p + \cdots$$

include only derivatives of order at most $n - 1$, that equation will show that the quantity $\omega^{(n-1)}$ has the same value for the curve $\mathcal{L}$ and the extremal 03, so the curves have contact of order n. However, that cannot occur everywhere along the entire curve $\mathfrak{L}$, in any event, because if, e.g., x' is non-zero at the point 3 in question then the system of values:

$$\frac{dy}{dx}, \quad \frac{d^2 y}{dx^2}, \quad \ldots, \quad \frac{d^{\mathfrak{a}} y}{dx^{\mathfrak{a}}}; \quad \omega, \ \omega', \ \ldots, \ \omega^{(\mathfrak{a}-1)} \tag{41}$$

for the first one can be expressed uniquely in terms of the system for the second one for every value of $\mathfrak{a}$. Likewise, the second one can be expressed uniquely in terms of the first, except for the multi-valuedness of the quantity:

$$\omega = \arctan \left.\frac{dy}{dx}\right|_0^3 ,$$

and the functional determinant:

$$\frac{\partial(\omega, \omega', \ldots, \omega^{(n-2)})}{\partial\left(\dfrac{dy}{dx}, \dfrac{d^2 y}{dx^2}, \ldots, \dfrac{d^{n-1} y}{dx^{n-1}}\right)}$$

will be non-zero. Therefore, if one sets $x = t$ in a neighborhood of the location considered then the determinants:

$$\Delta = \frac{\partial(x, y, \omega, \omega', \ldots, \omega^{(n-2)})}{\partial(x, a, b, c, \ldots, k)} = \frac{\partial(y, \omega, \omega', \ldots, \omega^{(n-2)})}{\partial(a, b, \ldots, k)}$$

and

$$\frac{\partial\left(y, \dfrac{dy}{dx}, \ldots, \dfrac{d^{n-1} y}{dx^{n-1}}\right)}{\partial(a, b, \ldots, k)}$$

will differ by only a finite, non-zero factor, and the latter determinant is non-zero. Hence, when one differentiates the expression:

$$y = \eta\,(x, a, b, \ldots, k)$$

n times with respect to x, one can eliminate the constants a, b, …, k and obtain a differential equation:

$$y^{(n)} = \Phi[x, y, y', \ldots, y^{(n-1)}] \tag{42}$$

that will be true for all extremals of the field and whose right-hand side will be regular in a neighborhood of the system of values for its arguments that is obtained for the abscissa x on the curve $\mathfrak{C}$ in question. That system of values will be denoted by $\mathfrak{S}$. Equation (42) must also be satisfied by the curve $\mathfrak{L}$ when it has contact of order n with the extremal 03 everywhere. However, the extremals of the field give arbitrarily-prescribed values to the quantities y, y', …, $y^{(n-1)}$ for the values of x considered in the vicinity of the system $\mathfrak{S}$, so they represent the general integral of equation (42). Now, since the system of quantities y, y', …, $y^{(n-1)}$ also lies close to the system $\mathfrak{S}$ along the curve $\mathfrak{L}$, that curve must coincide with an arc of an extremal of the field at the location in question. Moreover, since the quantities ω, ω', …, $\omega^{(n-2)}$ vary continuously on the curve $\mathfrak{L}$, it cannot be composed of segments of different extremals of the field, and since it also contacts the curve $\mathfrak{C}$ to order n at the point 1, it cannot differ from the latter curve.

The quantity $\mathcal{E}$ will therefore be non-zero for every curve $\mathfrak{L}$ that is different from $\mathfrak{C}$ when that is true for the quantity F_1 along the curve $\mathfrak{C}$. A singularity-free arc of any extremal will then provide an extremum in comparison to the curves $\mathfrak{L}$, firstly, when a field exists and secondly, when the quantity F_1 does not vanish. That corresponds to the assumptions that were introduced for the weak extremum in § **17**. One will arrive at the analogue for the strong one only when quantities ω, ω', …, $\omega^{(n-2)}$ have neighboring values on the curves $\mathfrak{L}$ and $\mathfrak{C}$.

Now, the following consideration will show that an extremal can always be surrounded by a field piece-wise: If, e.g., x_0' is non-zero then as a result of the relation that exists between the system of quantities (41) for a, b, …, k, one can introduce quantities:

$$y_0^{(n)} = \left.\frac{d^n y}{dx^n}\right|^0, \quad y_0^{(n+1)} = \left.\frac{d^{n+1} y}{dx^{n+1}}\right|^0, \qquad \ldots, \qquad y_0^{(2n-1)} = \left.\frac{d^{2n-1} y}{dx^{2n-1}}\right|^0,$$

which are defined for $x = t$. The quantity Δ will then be non-zero in a neighborhood of the location 0 when that is true of the determinant:

$$\frac{\partial(y, y', \ldots, y^{(n-1)})}{\partial(y_0^{(n)}, y_0^{(n+1)}, \ldots, y_0^{(2n-1)})}.$$

However, one can see that it does not vanish identically from the **Taylor** development:

$$y = \sum_{\mathfrak{a}=0}^{\infty} y_0^{(\mathfrak{a})} \frac{(x-x_0)^{\mathfrak{a}}}{\mathfrak{a}!}, \quad y^{(\mathfrak{b})} = \sum_{\mathfrak{a}=0}^{\infty} y_0^{(\mathfrak{b}+\mathfrak{a})} \frac{(x-x_0)^{\mathfrak{a}}}{\mathfrak{a}!},$$

which shows that the lowest power of the argument $x - x_0$ in it has the following coefficient:

$$\begin{vmatrix} \frac{1}{n!} & \frac{1}{(n+1)!} & \cdots & \frac{1}{(2n-1)!} \\ \frac{1}{(n-1)!} & \frac{1}{n!} & \cdots & \frac{1}{(2n-2)!} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{1!} & \frac{1}{2!} & \cdots & \frac{1}{n!} \end{vmatrix}.$$

The fact that this quantity is non-vanishing is easy to show and follows indirectly from the fact that it is impossible to determine an entire rational function of degree $2n - 1$ such that it will assume given values at two locations, along with its derivatives up to order $n - 1$.

With that, it has been shown that a sufficiently-small piece of an extremal along which F_1 is non-zero will always yield an extremum of the type that was defined.

Example. Problem XII (§ 50). – If t is the arc-length then F will be the radius of curvature, and the expressions for P_2, Q_2 that were derived before will imply that:

$$p\frac{\partial F}{\partial p} + q\frac{\partial F}{\partial q} = x'' \frac{\partial F}{\partial x''} + y'' \frac{\partial F}{\partial y''} = -F.$$

If one further sets:

$$\bar{F} = \frac{(x'^2 + y'^2)^2}{x'\bar{q} - y'\bar{p}}$$

then it will follow that:

$$\bar{p}\frac{\partial F}{\partial p} + \bar{q}\frac{\partial F}{\partial q} = -\frac{F^2}{\bar{F}},$$

so

$$\mathcal{E} = -\frac{F^2}{\bar{F}} + F - \bar{F} + F = -\frac{(F-\bar{F})^2}{\bar{F}}.$$

Now, the integral J will represent the area being examined, taken positively, only when the radius of curvature F remains positive and finite. The same thing will then be true for $\bar{F}$, since $\bar{F} - F$ is a small quantity. $\mathcal{E}$ will be negative then. One further has the equations:

$$\frac{\partial^2 F}{\partial p^2} = \frac{2y'^2}{(x'y'' - x''y')^3} = 2y'^2 F^3, \quad F_1 = 2F^3,$$

such that F_1 will be positive between two successive cusps of the cycloid.

One will get a two-fold infinite family of extremal that go through a point in the following way: In the coordinate system $\bar{x}$, $\bar{y}$, a cycloid that arises by rolling a circle along the $\bar{x}$-axis and possesses a cusp at the coordinate origin 0 will be represented by the equations:

(43) $$\bar{x} = a\,(t - t \sin t), \qquad \bar{y} = a\,(1 - \cos t)\,.$$

If one now sets:

(44) $$x = \bar{x} \cos b - \bar{y} \sin b\,, \qquad y = \bar{x} \sin b + \bar{y} \cos b$$

then the point (x, y) will describe a cycloid whose base is an arbitrary line that goes through the origin 0, and that will imply that:

$$x = a\,[\sin(b - t) + t \cos b - \sin b]\,,$$

$$y = a\,[-\cos(b - t) + t \sin b + \cos b]\,,$$

$$\frac{dx}{dy} = \frac{\cos b \dfrac{d\bar{x}}{dt} - \sin b \dfrac{d\bar{y}}{dt}}{\sin b \dfrac{d\bar{x}}{dt} + \cos b \dfrac{d\bar{y}}{dt}} = \tan\left(\frac{t}{2} - b\right),$$

$$\omega = -\frac{t}{2}\,.$$

The expressions for x, y, ω lead to the following equation for Δ:

$$-\Delta = \frac{\partial(x, y, \tfrac{1}{2}t)}{\partial(t, a, b)} = \begin{vmatrix} -a\cos(b-t) + a\cos b & -a\sin(b-t) + a\sin b & \tfrac{1}{2} \\ \sin(b-t) + t\cos b - \sin b & -\cos(b-t) + t\sin b + \cos b & 0 \\ a\cos(b-t) - at\sin b - a\cos b & a\sin(b-t) + t\sin b + \cos b & -1 \end{vmatrix}$$

$$= \begin{vmatrix} \sin(b-t) + t\cos b - \sin b & -\cos(b-t) + t\sin b + \cos b \\ -\cos(b-t) + \cos b - t\sin b & -\sin(b-t) + \sin b + t\cos b \end{vmatrix}.$$

One needs to investigate that determinant only for one special extremal. For example, if one sets $b = 0$ then one will get:

$$-\Delta = t^2 - 2\,(1 - \cos t) = 4\left[\left(\frac{t}{2}\right)^2 - \sin^2 \frac{t}{2}\right],$$

which will then be a positive quantity as long as t is non-zero. However, one should notice that the values $t = 0$, $t = 2\pi$ correspond to singular points of the extremal. In order to conclude the existence of an extremum from that, one must then, first of all, verify formula (37), after making a remark that is connected with it, which is very easy to do with the help of formulas (43), (44). Secondly, one must restrict oneself to arcs that do not reach the point $t = 2\pi$. For an arc of a cycloid between two successive cusps, based upon the general theory, the calculations that were performed will then show that it actually provides a minimum of the area J in comparison to all curves that have the same endpoints and tangents at the endpoints as it has, and whose tangents and radii of curvature differ sufficiently-little from those of the cycloidal arc.

§ 55. – Relative extremum. Necessary conditions. Method of multipliers.

One poses a generalized isoperimetric problem when the integral J is to be extremized while the value of another one of the same form:

$$K = \int G[x, x', \dots, x^{(n)}, y, y', \dots, y^{(n)}]\, dt$$

is prescribed. In that way, G has the same properties that were assumed for F and implies the identity (3). In particular, let the quantities G_1, $R = R_0$, R_1, …, R_n, $S = S_0$, S_1, …, S_n be constructed from the function G in the same way that F_1, P_0, …, Q_n are constructed from F, such that the identities:

$$G = \sum_{\mathfrak{a}=1}^{n} [R_{\mathfrak{a}}\, x^{(\mathfrak{a})} + S_{\mathfrak{a}}\, y^{(\mathfrak{a})}] \;,$$

$$R = R_0 = \sum_{\mathfrak{a}=1}^{n} (-1)^{\mathfrak{a}} \frac{d^{\mathfrak{a}}}{dt^{\mathfrak{a}}} \frac{\partial G}{\partial x^{(\mathfrak{a})}} \;, \qquad S = S_0 = \sum_{\mathfrak{a}=1}^{n} (-1)^{\mathfrak{a}} \frac{d^{\mathfrak{a}}}{dt^{\mathfrak{a}}} \frac{\partial G}{\partial y^{(\mathfrak{a})}}$$

will be valid. When the same continuity properties are demanded of the desired curve as in § **48**, the desired extremum can be provided by only segments of the extremals of the integral $J + \lambda K$, where λ means a constant, so by curves that satisfy the equations:

$$P + \lambda R = 0 \;, \qquad Q + \lambda S = 0$$

and depend upon $2n + 1$ constants, in general, for arbitrary values of λ. The argument in § **32** will show that with a very easy modification. Another proof that is based upon general principles will be given § **58**.

In order to derive sufficient conditions for the extremum, we assume that an $(n + 1)$-fold family of extremals go through the point 0 that can be represented by the equations:

$$x = \xi(t, a, b, \dots, k, \lambda) \;, \qquad y = \eta(t, a, b, \dots, k, \lambda) \;.$$

As in § **53**, let 1 and 2 be two points on a certain one of those extremals, which will be denoted by $\mathfrak{C}$, and let $\mathfrak{L}$ be a curve 12 that is close to the arc 12 that belongs to the curve $\mathfrak{C}$ in the same way as in § **53** and implies the equation:

$$K_{12} = \bar{K}_{12}, \tag{45}$$

in addition, in which one integrates over $\mathfrak{L}$ on the left-hand side and over $\mathfrak{C}$ on the right. If 03 is an extremal of the family along which the integrals $\bar{J}_{03}$, $\bar{K}_{03}$ are defined and 3 is a point of the curve $\mathfrak{L}$, moreover, and if $a, b, \ldots, k, \lambda$ are functions of the quantity τ that have the same meanings that they had in § **53** then one will have:

$$\frac{d\bar{J}_{03}}{d\tau} = \sum_{\mathfrak{a}=1}^{m}\left[P_{\mathfrak{a}}\frac{dx^{(\mathfrak{a}-1)}}{d\tau} + Q_{\mathfrak{a}}\frac{dy^{(\mathfrak{a}-1)}}{d\tau}\right]\Bigg|^{3} + \int_0^3 dt\left(P\frac{dx}{d\tau} + Q\frac{dx}{d\tau}\right), \tag{46}$$

instead of equation (38), and analogously:

$$\frac{d\bar{K}_{03}}{d\tau} = \sum_{\mathfrak{a}=1}^{m}\left[R_{\mathfrak{a}}\frac{dx^{(\mathfrak{a}-1)}}{d\tau} + S_{\mathfrak{a}}\frac{dy^{(\mathfrak{a}-1)}}{d\tau}\right]\Bigg|^{3} + \int_0^3 dt\left(R\frac{dx}{d\tau} + S\frac{dx}{d\tau}\right). \tag{47}$$

One now determines the extremal 03 by the requirement that it should not only have the same geometric relationship to the curve $\mathfrak{L}$ as in § **53**, but should also give a constant value to the quantity $\bar{K}_{03} + K_{32}$, whose last summand relates to $\mathfrak{L}$. That will be possible when the functional determinant:

$$\Delta^0 = \frac{\partial(\xi, \eta, \omega, \omega', \ldots, \omega^{(n-2)}, \bar{K}_{03})}{\partial(t, a, b, \ldots, k, \lambda)}$$

is non-zero along the curve $\mathfrak{C}$, except for the point 0. One can then, in fact, let the quantities ξ, η, $\omega, \ldots, \omega^{(n-2)}$, $\bar{K}_{04}$, which belong to any location 4 on the curve $\mathfrak{C}$ that lies between 1 and 2, increase by an arbitrarily-given amount by varying the arguments $t, a, \ldots, k, \lambda$ as long as they do not exceed certain limits. For example, when the point 3 is close to the point 4 on the curve $\mathfrak{L}$, one can let the quantities $\xi, \ldots, \omega^{(n-2)}$ go to the corresponding ones at the point 3 on the curve $\mathfrak{L}$, but let $\bar{K}_{04}$ go to the neighboring value $\bar{K}_{03} = \bar{K}_{01} + K_{13}$, in which the first summand refers to $\mathfrak{C}$, while the second one refers to $\mathfrak{L}$. $\mathfrak{C}$ will then be the initial and final positions of the extremal 03 as a result of equation (45), and one will get:

(48) $$\frac{d(\bar{K}_{03}+K_{32})}{d\tau} = \frac{d\bar{K}_{03}}{d\tau} - G\left(x, \frac{dx}{d\tau}, \ldots, \frac{d^n y}{d\tau^n}\right)\Bigg|^3 = 0 .$$

If one multiplies that expression by λ and adds it to the analogous one that is defined by J then, as a result of equations (46), (47), if the relations in § **53** are true, that will give:

$$\frac{d(\bar{K}_{03}+K_{32})}{d\tau} = F + \lambda\, G + \frac{\partial(F+\lambda G)}{\partial p}(\bar{p}-p) + \frac{\partial(F+\lambda G)}{\partial q}(\bar{q}-q) - \bar{F} - \lambda\,\bar{G} = \mathcal{E}^0 .$$

The right-hand side of that equation is the expression $\mathcal{E}$ that was defined above, but is defined for the function $F + \lambda\, G$, instead of F. If it has a fixed sign then the same thing will be true for the difference $J_{12} - \bar{J}_{12}$. Furthermore, if $F_1 + \lambda\, G_1$ is non-zero along the curve $\mathfrak{C}$ then, from § **54**, $\mathcal{E}^0$ will vanish along the entire curve $\mathfrak{L}$ only when the following equations are true:

$$\frac{d^n x}{d\tau^n} = x^{(n)}, \qquad \frac{d^n y}{d\tau^n} = y^{(n)},$$

in addition to equations (39). One will then have:

$$\omega_t^{(\mathfrak{a}-1)} \frac{dt}{d\tau} + \omega_a^{(\mathfrak{a}-1)} \frac{da}{d\tau} + \cdots + \omega_\lambda^{(\mathfrak{a}-1)} \frac{d\lambda}{d\tau} = \omega_t^{(\mathfrak{a}-1)}$$

for $\mathfrak{a} = 1, 2, \ldots, n - 1$, along with two other equations that will arise when one replaces ξ with η in the last $\omega^{(\mathfrak{a}-1)}$. Furthermore, from (48), one has:

$$\frac{d\bar{K}_{03}}{d\tau} = -\frac{dK_{32}}{d\tau} = G(x, x', \ldots, x^{(n)}, y, \ldots, y^{(n)}) = \frac{\partial \bar{K}_{03}}{\partial t} .$$

When one writes that equation in the form:

$$\frac{\partial \bar{K}_{03}}{\partial t}\frac{dt}{d\tau} + \frac{\partial \bar{K}_{03}}{\partial a}\frac{da}{d\tau} + \cdots = \frac{\partial \bar{K}_{03}}{\partial t} ,$$

that equation, along with the $n + 1$ foregoing ones, will give $n + 2$ linear homogeneous equations for the quantities:

$$\frac{dt}{d\tau} - 1, \ \frac{da}{d\tau}, \ \frac{db}{d\tau}, \ \ldots, \ \frac{d\lambda}{d\tau},$$

whose determinant is Δ^0, so it will be non-zero. Those quantities will then vanish (i.e., $\mathfrak{C}$ and $\mathfrak{L}$ will coincide) when $\mathcal{E}^0$ vanishes everywhere.

Sufficient conditions for the arc 12 to provide the desired extremum will then consist of Δ^0 being non-zero and $\mathcal{E}^0$ having a fixed sign. The former quantity can also be defined with an arbitrary parameter t here.

Problem XIII. – Determine the equilibrium figure of a planar elastic spring with given endpoints and tangents at the endpoints when the potential energy (per unit of arc-length) is measured by the square of the curvature.

The given length of the spring is:

$$K = \int \sqrt{x'^2 + y'^2}\, dt \,,$$

so the energy will be:

$$J = \int \frac{\sqrt{x'^2 + y'^2}}{\rho^2}\, dt \,,$$

when ρ means the radius of curvature. Thus:

$$\rho^2 = \frac{(x'^2 + y'^2)^3}{(x'\,y'' - x''\,y')^2}\,, \qquad F + \lambda\, G = \sqrt{x'^2 + y'^2}\left(\frac{1}{\rho^2} + \lambda\right).$$

Since x and y do not occur in the integrands, from § **50**, one will have the two first integrals:

$$P_1 + \lambda\, R_1 = a\,, \qquad Q_1 + \lambda\, S_1 = b$$

for the extremals of the integral $J + \lambda\, K$. Since, one further has:

$$P_2 + \lambda\, R_2 = \frac{\partial(F + \lambda\, G)}{\partial x''} = \frac{-2y'\,(x'\,y'' - x''\,y')}{(x'^2 + y'^2)^{5/2}}\,,$$

$$Q_2 + \lambda\, S_2 = \frac{\partial(F + \lambda\, G)}{\partial y''} = \frac{2x'\,(x'\,y'' - x''\,y')}{(x'^2 + y'^2)^{5/2}}\,,$$

the identity (7), or:

$$F + \lambda\, G = (P_1 + \lambda\, R_1)\, x' + (Q_1 + \lambda\, S_1)\, y' + (P_2 + \lambda\, R_2)\, x'' + (Q_2 + \lambda\, S_2)\, y''\,,$$

will give the equation:

$$\left(\frac{1}{\rho^2} + \lambda\right)\sqrt{x'^2 + y'^2} = a\, x' + b\, y' + \frac{2\sqrt{x'^2 + y'^2}}{\rho^2}\,,$$

$$\lambda - \frac{a\,x' + b\,y'}{\sqrt{x'^2 + y'^2}} = \frac{1}{\rho^2},$$

when one recalls the expression for ρ^2, or when α and ω have the same meaning that they had in § **52** and *ds* is the element of arc-length, the equation:

(49) $$\lambda - a \cos(\omega + \alpha) - b \sin(\omega + \alpha) = \frac{1}{\rho^2} = \left(\frac{d\omega}{ds}\right)^2,$$

so

(50) $$ds = \frac{d\omega}{\sqrt{\lambda - a\cos(\omega+\alpha) - b\sin(\omega+\alpha)}} .$$

Now since:

$$x = \int \cos(\omega+\alpha)\,ds, \quad y = \int \sin(\omega+\alpha)\,ds,$$

that will give:

$$x = x_0 + \int_\alpha^{\omega+\alpha} \frac{\cos\omega\,d\omega}{\sqrt{\lambda - a\cos\omega - b\sin\omega}},$$

$$y = y_0 + \int_\alpha^{\omega+\alpha} \frac{\sin\omega\,d\omega}{\sqrt{\lambda - a\cos\omega - b\sin\omega}} .$$

Those equations represent a family of extremals that start from the point 0 with a constant direction, so when one lets λ, a, b be variable, a family with the character that is required by the general theory. The quantity ω itself appears as the parameter *t*. One will then have:

$$\Delta^0 = \frac{\partial(x, y, \omega, \bar{K}_{03})}{\partial(\omega, a, b, \lambda)} = \frac{\partial(x, y, \bar{K}_{03})}{\partial(a, b, \lambda)},$$

and from formula (50), that will give:

$$\bar{K}_{03} = \int_\alpha^{\omega+\alpha} \frac{d\omega}{\sqrt{\lambda - a\cos\omega - b\sin\omega}} = s .$$

The calculation of Δ^0 obviously poses no difficulty. If one sets:

$$\sqrt{\lambda - a\cos\omega - b\sin\omega} = \psi,$$

$$A = \int_{\alpha}^{\omega+\alpha} \frac{\cos^2\omega\, d\omega}{\psi^3}, \quad B = \int_{\alpha}^{\omega+\alpha} \frac{\sin\omega\cos\omega\, d\omega}{\psi^3}, \quad C = \int_{\alpha}^{\omega+\alpha} \frac{\sin^2\omega\, d\omega}{\psi^3},$$

$$M = \int_{\alpha}^{\omega+\alpha} \frac{\cos\omega\, d\omega}{\psi^3}, \quad N = \int_{\alpha}^{\omega+\alpha} \frac{\sin\omega\, d\omega}{\psi^3}$$

then one will have:

$$-8\,\Delta^0 = \begin{vmatrix} A & B & M \\ B & C & N \\ M & N & A+C \end{vmatrix} = \begin{vmatrix} A & B & M \\ B & C & N \\ M & N & 0 \end{vmatrix} + (A + C)\,(A\,C - B^2)\,.$$

The quantity $\mathcal{E}^0$ is negative, since:

$$\frac{\partial^2(F+\lambda\,G)}{\partial q^2} = \frac{\partial^2(F+\lambda\,G)}{\partial y''\,\partial y''} = \frac{2x'^2}{(x'^2+y'^2)^{5/2}}$$

is positive. Hence, a minimum of the potential energy, and therefore a stable equilibrium, will be present only when the equation $\Delta^0 = 0$ has the single root $t = 0$.

One will get the usual equation of the elastic curve when one introduces a new rectangular coordinate system by means of the equations:

$$c = \sqrt{a^2+b^2}\,, \qquad a = c\cos\beta, \qquad b = c\sin\beta,$$

$$\bar{x} = x\cos\beta + y\sin\beta,\ \bar{y} = -x\sin\beta + y\cos\beta\,.$$

Obviously, that will give:

$$\bar{x} - \bar{x}_0 = \int_{\alpha}^{\omega+\alpha} \frac{\cos(\omega-\beta)\,d\omega}{\sqrt{\lambda - c\cos(\omega-\beta)}},$$

$$\bar{y} - \bar{y}_0 = \int_{\alpha}^{\omega+\alpha} \frac{\sin(\omega-\beta)\,d\omega}{\sqrt{\lambda - c\cos(\omega-\beta)}}.$$

The second integral can be calculated, and $\bar{x}$ will take the form of an elliptic integral of the second kind with the argument $\bar{y}$. One can also say that the coordinate transformation takes one from the general case to the special case in which $b = 0$. In the latter, the integrals B and N are expressed by elementary functions.

Incidentally, when one differentiates equation (49) with respect to s, one will get:

$$[a \sin(\omega + \alpha) - b \cos(\omega + \alpha)] \, \frac{d\omega}{ds} = -\frac{2}{\rho^3}\frac{d\rho}{ds},$$

or, since $\rho\, d\omega = ds$:

$$a \sin(\omega + \alpha) - b \cos(\omega + \alpha) = a\frac{dy}{ds} - b\frac{dx}{ds} = -\frac{2}{\rho^2}\frac{d\rho}{ds},$$

so when one integrates:

$$\frac{2}{\rho} = a\,y - b\,x + \text{const.},$$

which is an equation that expresses a main property of elastic curves.

CHAPTER SEVEN

THE MOST GENERAL PROBLEM IN THE CALCULUS OF VARIATIONS FOR A SINGLE INDEPENDENT VARIABLE

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§ 56. – Formulation of the problem.

The problems in Chapters Two and Three can be formulated in the following way: Determine the quantities z and y as functions of x such that the equation:

$$\frac{dz}{dx} = f\left(x, y, \frac{dy}{dx}\right)$$

is satisfied, and the quantity z_1 will be an extremum for given values y_0, z_0, y_1, if the values of the unknowns for $x = x_0$ and $x = x_1$ are denoted with the indices 0 and 1, resp. Similarly, for the isoperimetric problem, arrive at an extremum for the quantity z_1 under the assumption that the equations:

$$\frac{dz}{dx} = f\left(x, y, \frac{dy}{dx}\right), \quad \frac{du}{dx} = g\left(x, y, \frac{dy}{dx}\right)$$

are satisfied, and the values of y_0, z_0, u_0, y_1 are given. The problem of Chapter Six for $n = 2$ can be expressed by saying that the equations:

$$\frac{dz}{dx} = f\left(x, y, \frac{dy}{dx}\right), \qquad \frac{dy}{dx} = u$$

are prescribed, the values y_0, z_0, u_0, u_1, y_1 are given, and z_1 is, in turn, to be extremized. One will get a problem that is related to the ones that were cited that is not, however, accessible to the methods that were used up to now when an integral whose integrand depends upon the length of the desired curve is to be extremized. For example, one might have the integral:

$$\omega = \int_{x_0}^{x_1} f\left(x, y, \frac{dy}{dx}, \sigma\right),$$

in which one has set:

$$\sigma = \int_{x_0}^{x_1} dx \sqrt{1 + \left(\frac{dy}{dx}\right)^2}.$$

Obviously, the equations:

$$\frac{d\omega}{dx} = f\left(x, y, \frac{dy}{dx}, \sigma\right), \qquad \left(\frac{d\sigma}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2$$

are given, and the values y_0, y_1, ω_0, σ_0 are prescribed, while ω_1 is to be extremized, but σ_1 is not subject to any restriction.

All of those problems are special cases of the following very general one: Let $y_0, y_1, \ldots, y_{n-1}$ be unknown functions of x that are subject to the $r + 1$ equations:

$$\psi_{\mathfrak{a}}\left(x, y_0, y_1, \ldots, y_{n-1}, \frac{dy_0}{dx}, \frac{dy_1}{dx}, \ldots, \frac{dy_{n-1}}{dx}\right) = 0 \qquad (\mathfrak{a} = 0, 1, \ldots, r).$$

Let the values of the quantities $y_0, y_1, y_2, \ldots, y_{n-1}$ be given for $x = x_0$, and likewise the values of some of them for $x = x_1$. The unknown functions shall then be determined in such a way that the value of y_0 is extremized for $x = x_1$. If one imagines that $x, y_0, \ldots, y_{n-1}$ are represented as single-valued functions of a parameter t along the desired manifold then the equations will take the form:

(1) $$\varphi_{\mathfrak{a}}(y_0, y_0, \ldots, y_n, y_0', \ldots, y_n') = 0 .$$

Let the function $\varphi_{\mathfrak{a}}$ be homogeneous of degree q in the quantities y'. In order to achieve well-defined arcs, we further assume that the quantities y are continuous functions of the argument t that are provided with continuous first and second derivatives. If the latter runs through the interval from t_0 to t_1 then let systems of values:

(2) $$y_0, \ldots, y_n, \; y_0', \ldots, \; y_n'$$

for which all functions $\varphi_{\mathfrak{a}}$ are regular be always defined by those functions. Call the set of all such systems of values $\mathfrak{M}$. If one sets:

$$\varphi_{\mathfrak{a}\mathfrak{b}} = \frac{\partial \varphi_{\mathfrak{a}}}{\partial y_{\mathfrak{b}}}, \qquad \bar{\varphi}_{\mathfrak{a}\mathfrak{b}} = \frac{\partial \varphi_{\mathfrak{a}}}{\partial y_{\mathfrak{b}}'} \qquad (\mathfrak{a} = 0, 1, \ldots, r, \quad \mathfrak{b} = 0, 1, \ldots, n)$$

then let the determinant:

(3) $$\sum \pm \bar{\varphi}_{00}\,\bar{\varphi}_{11} \cdots \bar{\varphi}_{rr} = \frac{\partial(\varphi_0, \varphi_1, \ldots, \varphi_r)}{\partial(y_0', y_1', \ldots, y_r')}$$

be non-zero for the indicated system of values (2), so equations (1) will be soluble for $y_0', \ldots, y_n'$. Finally, let the quantities $y_1, \ldots, y_r$ include ones whose values are not prescribed for $t = t_1$ (say, the quantities $y_{r+1}, y_{r+2}, \ldots, y_r$) such that in general the relation:

$$0 \leq s \leq r < n$$

will be satisfied, and in the case of $r = s$, all unknown functions except for y_0 are also given for $t = t_1$.

For the isoperimetric problem that was referred to above, one will have, e.g.:

$$r = s = 1, \qquad n = 3, \qquad z = z_0 , \qquad u = y_1 , \qquad y = y_2 , \qquad x = y_3 .$$

By contrast, for the problem of the integral ω :

$$s = 0, \qquad r = 1, \qquad n = 3, \qquad \omega = y_0 , \qquad \sigma = y_1 , \qquad y = y_2 , \qquad x = y_3 .$$

In general, each of the indices $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, … shall denote a certain series of numbers from now on, and indeed let:

$$\begin{aligned} \mathfrak{a}, \mathfrak{e} &= 0, 1, \ldots, r , \\ \mathfrak{b} &= 0, 1, \ldots, n , \\ \mathfrak{c} &= s + 1, s + 2, \ldots, r, \\ \mathfrak{d} &= r + 1, r + 2, \ldots, n, \\ \mathfrak{g}, \mathfrak{f} &= 0, 1, \ldots, s . \end{aligned}$$

Now in order to examine whether a system of functions with the given behavior will yield an extremum for the value of y_0 for $t = t_1$, we replace it with the system $y_\mathfrak{b} + \Delta y_\mathfrak{b}$, for which the equations:

$$\varphi_\mathfrak{a} (y_\mathfrak{b} + \Delta y_\mathfrak{b} , \; y'_\mathfrak{b} + \Delta y'_\mathfrak{b}) = 0 \tag{4}$$

are valid in the entire interval from t_0 to t_1, but the equations:

$$\Delta y_\mathfrak{b} \big|^{t_0} = 0 , \quad \Delta y_\mathfrak{b} \big|^{t_1} = 0 , \quad \Delta y_1 \big|^{t_1} = 0 , \quad \Delta y_2 \big|^{t_1} = \ldots = \Delta y_s \big|^{t_1} = 0 \tag{5}$$

need to be true only at the limits. Equations (4) can be written:

$$\sum_{\mathfrak{b}=0}^{n} (\varphi_{\mathfrak{a}\mathfrak{b}} \, \Delta y_\mathfrak{b} + \bar{\varphi}_{\mathfrak{a}\mathfrak{b}} \, \Delta y'_\mathfrak{b}) + [\Delta y_\mathfrak{b}, \Delta y'_\mathfrak{b}]_2 = 0 . \tag{6}$$

Therefore, if the quantities $\Delta y_\mathfrak{b}$ are given as functions of t then the quantities Δy_0, Δy_1, …, Δy_r will be determined by a system of $r + 1$ first-order differential equations, and indeed completely, since their values are given for $t = t_0$. One sees from the last relations (5) that as long as s is non-zero, the quantities $\Delta y_\mathfrak{b}$ cannot be completely arbitrary functions of t, since one would not expect those equations to be satisfied, in general. Should the desired extremum be provided by the functions $y_\mathfrak{b}$ in question, then obviously $\Delta y_0 \Big|^{t_1}$ would have to possess a fixed sign.

The further arguments shall be connected with only equations (5), (6), and will be based upon the fact the their left-hand sides are regular functions of 4 (n + 1) quantities y, y', Δy, $\Delta y'$, while the fact that $\varphi_{\mathfrak{ab}}$ and $\bar{\varphi}_{\mathfrak{ab}}$ are partial derivatives of a function $\varphi_\mathfrak{a}$ will not be used. If one drops the last assumption then the problem that was formulated above will be generalized considerably, since differential equations of constraint will be given for not only the unknown functions, but also for some of the increments that those functions will take on under the transition to neighboring manifolds of a certain type. Thus, those manifolds that should yield an extremum for y_0 in comparison to the origin $\mathfrak{M}$ will be defined. Mechanics presents such problems with non-integrable equations of constraint.

Now, since the determinant (3) does not vanish for the system of values that comes under consideration, equations (6) will imply expressions for $\Delta y_0'$, $\Delta y_1'$, …, $\Delta y_r'$ that take the form:

(7) $$\Delta y_\mathfrak{a}' = [\Delta y_\mathfrak{b}, \Delta y_\mathfrak{b}', y_\mathfrak{b} - \eta_\mathfrak{b}, y_\mathfrak{b}' - \eta_\mathfrak{b}']_1 = R_\mathfrak{a} ,$$

when $t = \tau$, $y_\mathfrak{b} = \eta_\mathfrak{b}$, $y_\mathfrak{b}' = \eta_\mathfrak{b}'$ is any location on the manifold $\mathfrak{M}$. The series $R_\mathfrak{a}$ might converge for all values of τ that lie between t_0 and t_1 when the absolute values of the arguments (regardless of whether they are real or complex) do not exceed the positive constant α. Since the arguments $y_\mathfrak{b} - \eta_\mathfrak{b}$, $y_\mathfrak{b}' - \eta_\mathfrak{b}'$ are continuous functions of t, that can be achieved for some of them by initially assuming that:

(8) $$|\, t - \tau \,| \leq \beta$$

and understanding β to mean a suitably-chosen positive constant that is independent of τ. If one adds the demands that:

(9) $$|\, \Delta y_\mathfrak{b} \,| \leq \alpha , \qquad |\, y_\mathfrak{b}' \,| \leq \alpha$$

to that inequality then the series $R_\mathfrak{a}$ will have values whose absolute values do not exceed a certain positive constant γ under the further condition that:

$$|\, \Delta y_\mathfrak{a} \,| \leq \alpha .$$

In particular, one now sets:

(10) $$\Delta y_\mathfrak{a} = z_\mathfrak{a} , \qquad \Delta y_\mathfrak{d} = \sum_{\mathfrak{m}=0}^{m} \varepsilon_\mathfrak{m} \, u_{\mathfrak{dm}} .$$

In that way, let $\varepsilon_{\mathfrak{m}}$ be real or complex constants, let $u_{\mathfrak{d}\mathfrak{m}}$ real functions of t that are continuous between t_0 and t_1 and possess continuous first derivatives and vanish for $t = t_0$ and $t = t_1$, and the absolute values remain below the fixed positive constant g, along with those of their first derivatives. If ζ is a positive constant and:

$$|\varepsilon_{\mathfrak{m}}| < \zeta < \frac{\alpha}{(m+1)\,g}$$

then the inequalities (9) will be valid. The series $f_{\mathfrak{a}}(z_0, z_1, \ldots, z_r)$ to which the expressions $R_{\mathfrak{a}}$ go under the substitution (10) are power series in the arguments $z_{\mathfrak{a}}$, $\varepsilon_{\mathfrak{m}}$ with continuous functions of t as coefficients, and they will converge when t belongs to the interval (8) under the assumption that:

$$z_{\mathfrak{a}} < \alpha .$$

In that way, the following inequalities will be satisfied:

$$(11)\qquad |f_{\mathfrak{a}}(z_0, z_1, \ldots, z_r)| < \gamma .$$

The following conclusion can be inferred from that: The quantity α is split into two positive summands, such that:

$$\alpha = \alpha_1 + \alpha_2, \quad 0 < 2\,\alpha_1 < \alpha_2 .$$

When one then assumes that:

$$(12)\qquad |z_{\mathfrak{a}}| < \alpha_1, \qquad |\bar{z}_{\mathfrak{a}}| < \alpha_1, \qquad |\bar{z}_{\mathfrak{a}} - z_{\mathfrak{a}}| < 2\,\alpha_2 ,$$

in which $z_{\mathfrak{a}}$ and $\bar{z}_{\mathfrak{a}}$ can be real or complex quantities, one can develop:

$$f_{\mathfrak{a}}(\bar{z}_0, \bar{z}_1, \ldots, \bar{z}_r) = f_{\mathfrak{a}}(z_0, z_1, \ldots, z_r) + \sum_{\mathfrak{e}=0}^{r} (\bar{z}_{\mathfrak{e}} - z_{\mathfrak{e}})\, M_{\mathfrak{a}\mathfrak{e}} ,$$

$$M_{\mathfrak{a}\mathfrak{e}} = [\bar{z}_0 - z_0, \bar{z}_1 - z_1, \ldots, \bar{z}_r - z_r]_0 ,$$

and from a fundamental theorem of the theory of functions, the inequalities:

$$\frac{1}{\mathfrak{r}_0!\,\mathfrak{r}_1!\cdots}\left|\frac{\partial^{\mathfrak{r}_0+\mathfrak{r}_1+\cdots} f_{\mathfrak{a}}}{\partial z_0^{\mathfrak{r}_0}\,\partial z_1^{\mathfrak{r}_1}\cdots}\right| < \gamma\,\alpha_2^{-\mathfrak{r}_0-\mathfrak{r}_1-\cdots}$$

will be valid.

The absolute values of the terms in the series $M_{\mathfrak{a}\mathfrak{e}}$ are then smaller than the corresponding ones in certain convergent series in increasing positive powers of $2\alpha_1 / \alpha_2$ with positive terms that depend upon only γ, α_1, α_2, but not upon τ. There will then be a quantity γ_0 that is independent of τ and has the property that under the assumption (12), one will always have:

$$|f_{\mathfrak{a}}(\bar{z}_0, \bar{z}_1, \ldots, \bar{z}_r) - f_{\mathfrak{a}}(z_0, z_1, \ldots, z_r)| < \gamma_0 \sum_{\mathfrak{e}=0}^{r} |\bar{z}_{\mathfrak{e}} - z_{\mathfrak{e}}| . \tag{13}$$

We now fix τ in any way, split the quantity α_1 into two positive summands, such that:

$$\alpha_1 = \alpha_3 + \alpha_4, \qquad \alpha_3 > 0, \qquad \alpha_4 > 0,$$

and determine a system of solutions to equations (7) or:

$$z'_{\mathfrak{a}} = f_{\mathfrak{a}}(z_0, z_1, \ldots, z_r), \tag{14}$$

for which:

$$z_{\mathfrak{a}}\big|^{\tau} = z_{\mathfrak{a}0} = [\varepsilon]_1, \qquad |z_{\mathfrak{a}0}| < \alpha_3 . \tag{15}$$

Following an idea of **Picard**, we define:

$$z_{\mathfrak{a},\mathfrak{n}+1} = z_{\mathfrak{a}0} + \int_{\tau}^{t} f_{\mathfrak{a}}(z_{0\mathfrak{n}}, z_{1\mathfrak{n}}, \ldots, z_{r\mathfrak{n}})\, dt,$$

in general, and assume that:

$$|t - \tau| < \beta_0 < \beta .$$

When:

$$|z_{\mathfrak{a}\mathfrak{n}}| < \alpha_1, \tag{16}$$

the relation (11) will next give the consequence:

$$|z_{\mathfrak{a},\mathfrak{n}+1} - z_{\mathfrak{a}0}| < \beta_0\, \gamma .$$

If one chooses β_0 to be small enough that:

$$\beta_0\, \gamma < \alpha_4 \tag{17}$$

then when one recalls the last relation in (15), it will follow that:

$$|z_{\mathfrak{a},\mathfrak{n}+1} - z_{\mathfrak{a}0}| < \alpha_4, \qquad |z_{\mathfrak{a},\mathfrak{n}+1}| < \alpha_1,$$

such that the relation (16) will be true in general. The equation:

$$z_{\mathfrak{a},\mathfrak{n}+1} - z_{\mathfrak{a}\mathfrak{n}} = \int_\tau^t dt \{ f_\mathfrak{a}(z_{0\mathfrak{n}}, z_{1\mathfrak{n}}, \dots, z_{r\mathfrak{n}}) - f_\mathfrak{a}(z_{0,\mathfrak{n}-1}, z_{1,\mathfrak{n}-1}, \dots, z_{r,\mathfrak{n}-1}) \}$$

will then imply that:

$$| z_{\mathfrak{a},\mathfrak{n}+1} - z_{\mathfrak{a}\mathfrak{n}} | < \beta_0 \gamma_0 \sum_{\mathfrak{e}=0}^{r} | z_{\mathfrak{e}\mathfrak{n}} - z_{\mathfrak{e},\mathfrak{n}-1} | ,$$

$$\sum_{\mathfrak{e}=0}^{r} | z_{\mathfrak{e},\mathfrak{n}+1} - z_{\mathfrak{e}\mathfrak{n}} | < (r+1) \beta_0 \gamma_0 \sum_{\mathfrak{e}=0}^{r} | z_{\mathfrak{e}\mathfrak{n}} - z_{\mathfrak{e},\mathfrak{n}-1} | ,$$

on the basis of the relations (13), (16). If one further assumes that:

$$(r+1)\, \beta_0\, \gamma_0 < 1 ,$$

which can obviously be achieved, just like the inequality (17), when one understands β_0 to mean a positive value that depends upon only α_4, γ, γ_0, but not τ, then the absolute values of the terms in the series:

(18) $$z_{\mathfrak{a}0} + \sum_{\mathfrak{u}=0}^{\infty} | z_{\mathfrak{a},\mathfrak{u}+1} - z_{\mathfrak{a}\mathfrak{u}} |$$

will be less than the corresponding terms in a convergent geometric progression whose ratio is $\beta_0 \gamma_0$. The series will then converge uniformly in the entire region:

$$| t - \tau | < \beta_0 , \quad | \varepsilon_\mathfrak{m} | < \zeta ,$$

in which $\varepsilon_\mathfrak{m}$ can also be complex quantities.

Now since the series $R_\mathfrak{a}$ does not include any terms that are free of $\Delta y_\mathfrak{b}$ and $\Delta y'_\mathfrak{b}$, so one can set:

$$f_\mathfrak{a}(z_0, z_1, \dots, z_r) = [z_0, z_1, \dots, z_r, \varepsilon_0, \varepsilon_1, \dots, \varepsilon_\mathfrak{m}]_1 ,$$

one will also have:

$$f_\mathfrak{a}(z_{00}, z_{10}, \dots, z_{r0}) = [\varepsilon]_1$$

then, and all quantities $z_{\mathfrak{a}\mathfrak{n}}$ with continuous functions of t as their coefficients will have the same form. From the **Weierstrass** double series theorem, the series (18) can also be converted into a single series $[\varepsilon]_1$, and it will follow from the uniform convergence with respect to the argument t that for a suitable choice of the quantity ζ_0, under the assumption that:

(19) $$| \varepsilon_\mathfrak{m} | < \zeta_0 , \qquad | t - \tau | < \beta_0 ,$$

the absolute value of the expression (18) will lie below a prescribed limit, e.g., below α_3 .

Finally, it is clear that the expressions (18) represent a system of integrals of equations (14). Namely, if one denotes them by $z_\mathfrak{a}$ then one will obviously have:

$$z_\mathfrak{a} = \lim_{\mathfrak{n}\to\infty} z_{\mathfrak{an}} = z_{\mathfrak{a}0} + \lim_{\mathfrak{n}\to\infty} \int_\tau^t f_\mathfrak{a}(z_{0,\mathfrak{n}-1}, z_{1,\mathfrak{n}-1}, \ldots, z_{r,\mathfrak{n}-1})\, dt = z_{\mathfrak{a}0} + \int_\tau^t f_\mathfrak{a}(z_0, z_1, \ldots, z_r)\, dt\,,$$

which proves the statement that was asserted. Equations (14) now show that the quantities $z'_\mathfrak{a}$ can also be represented in the form $[\varepsilon]_1$ and that they are continuous functions of t, ε_0 , …, $\varepsilon_\mathfrak{m}$. The same thing is also true for $\partial z_\mathfrak{a} : \partial\varepsilon_\mathfrak{m}$. That is because if the inequality:

$$\left| \sum_{\mathfrak{n}=h}^{h+k} (z_{\mathfrak{a},\mathfrak{n}+1} - z_{\mathfrak{an}}) \right| < \sigma$$

is satisfied, e.g., in the region (19), then the absolute values of the coefficients of $\varepsilon_0^{\mathfrak{r}_0}\,\varepsilon_1^{\mathfrak{r}_1}$, … in that sum will be less than $\sigma\zeta_0^{-\mathfrak{r}_0-\mathfrak{r}_1-\cdots}$. The sum of the corresponding coefficients in all expressions $z_{\mathfrak{a},\mathfrak{n}+1} - z_{\mathfrak{an}}$ will therefore be likewise uniformly convergent when t runs from $t - \beta_0$ to $t + \beta_0$. The derivatives:

$$\frac{\partial z_\mathfrak{a}}{\partial t}, \quad \frac{\partial z_\mathfrak{a}}{\partial \varepsilon_\mathfrak{m}}, \quad \frac{\partial}{\partial \varepsilon_\mathfrak{m}}\left(\frac{\partial z_\mathfrak{a}}{\partial t}\right)$$

will then exist and be continuous, e.g., in the real neighborhood of the location $t = \tau$, $\varepsilon_\mathfrak{m} = 0$. From a theorem by **Schwartz**, it will follow from this that the derivative:

$$\frac{\partial}{\partial t}\left(\frac{\partial z_\mathfrak{a}}{\partial \varepsilon_\mathfrak{m}}\right)$$

will also exist and be equal to the third of those quantities. Hence, if one sets:

$$z_\mathfrak{a} = \sum_{\mathfrak{m}=0}^{m} v_{\mathfrak{am}}\, \varepsilon_\mathfrak{m} + [\varepsilon]_2$$

then the $v_{\mathfrak{am}}$ will be continuous functions of t that have continuous first derivatives.

Now the assumption (15) will be fulfilled when one sets $\tau = t_0$, $z_{\mathfrak{a}0} = 0$. A system of integrals with the given properties will then exist in the region:

$$t_0 \leq t \leq t_0 + \beta_0 , \qquad\qquad | \varepsilon_{\mathfrak{m}} | < \zeta_0 ,$$

in any case, and will give values of $z_{\mathfrak{a}}$ fir $t = t_0 + \beta_0$ for which:

$$| z_{\mathfrak{a}} | < \alpha_3 , \qquad z_{\mathfrak{a}} = [\varepsilon]_1 .$$

If one sets $\tau = t_0 + \beta_0$ then the relations (15) will be true, and one can continue the system of integrals up to the value $t_0 + 2\beta_0$, regardless of its properties, but one might possibly need to reduce ζ_0. If one repeats that argument then since β_0 is independent of τ, after a finite number of steps, one will get a value for ζ_0 and a system of integrals $z_{\mathfrak{a}}$ that has the properties that were indicated above in the entire interval from t_0 to t_1 under the assumption that $| \varepsilon_{\mathfrak{m}} | < \zeta_0$. One can set $z_{\mathfrak{a}} = \Delta y_{\mathfrak{a}}$, since $\Delta y_{\mathfrak{a}}$ vanishes for $t = t_0$, just like $z_{\mathfrak{a}}$, and the systems (7), (14) will coincide under the assumption (10).

Thus, if all of the quantities $\Delta y_{\mathfrak{b}}$ have the form $[\varepsilon]_1$, and if any quantity w takes on the increase Δw when one replaces $y_{\mathfrak{b}}$ with $y_{\mathfrak{b}} + \Delta y_{\mathfrak{b}}$ then we would like to let δw denote the expression for what remains when one drops all terms that are of at least second power in the quantities ε. In particular, for the system Δy that was considered above, one will then have:

$$\delta y_{\mathfrak{b}} = \Delta y_{\mathfrak{b}} , \qquad \delta y_{\mathfrak{a}} = \sum_{\mathfrak{m}} v_{\mathfrak{am}}\, \varepsilon_{\mathfrak{m}} .$$

If Δw is a series $[\varepsilon]_1$ with continuous and differentiable functions of t as its coefficients then the following equation will be valid:

$$\frac{d\,\delta w}{dt} = \delta w' .$$

The developments that were implemented create a stable foundation upon which one can operate with the symbol δ on the quantities $y, \ y'$ that are coupled by equations (1) in the same way that one operates with the differentiation symbol on a parameter that is independent of t.

§ 57. – Necessary conditions for an extremum. Method of multipliers.

With the defined meaning for the symbol δ, equations (6) will imply that:

$$\sum_{\mathfrak{b}=0}^{n} (\varphi_{\mathfrak{ab}}\, \delta y_{\mathfrak{b}} + \bar{\varphi}_{\mathfrak{ab}}\, \delta y'_{\mathfrak{b}}) = 0 ,$$

so when one multiplies by any factors $\mu_\mathfrak{a}$ and adds:

$$\sum_{\mathfrak{a}=0}^{r} \mu_\mathfrak{a} \sum_{\mathfrak{b}=0}^{n} (\varphi_{\mathfrak{ab}}\, \delta y_\mathfrak{b} + \bar{\varphi}_{\mathfrak{ab}}\, \delta y'_\mathfrak{b}) = 0 ,$$

or from the rules for operating with the symbol δ:

$$(20) \qquad 0 = \sum_{\mathfrak{b}=0}^{n} \delta y_\mathfrak{b} \sum_{\mathfrak{a}=0}^{r} \left[\mu_\mathfrak{a}\, \varphi_{\mathfrak{ab}} - \frac{d(\mu_\mathfrak{a}\, \bar{\varphi}_{\mathfrak{ab}})}{dt} \right] + \frac{d}{dt} \sum_{\mathfrak{b}=0}^{n} \delta y_\mathfrak{b} \sum_{\mathfrak{a}=0}^{r} \mu_\mathfrak{a}\, \bar{\varphi}_{\mathfrak{ab}} .$$

If we now determine the multipliers such that:

$$(21) \qquad \sum_{\mathfrak{a}=0}^{r} \left[\mu_\mathfrak{a}\, \varphi_{\mathfrak{ae}} - \frac{d(\mu_\mathfrak{a}\, \bar{\varphi}_{\mathfrak{ae}})}{dt} \right] = 0 \qquad (\mathfrak{e} = 0, 1, \ldots, r)$$

then that will define $r + 1$ linear homogeneous differential equations in which the determinant of the coefficients of the quantities $\mu'_\mathfrak{a}$ has the value:

$$\sum \pm\, \bar{\varphi}_{00}\, \bar{\varphi}_{11} \cdots \bar{\varphi}_{rr} ,$$

so it does not vanish. One can then determine $r + 1$ systems of integrals:

$$(22) \qquad \mu_{\mathfrak{a}0}, \quad \mu_{\mathfrak{a}1}, \quad \ldots, \; \mu_{\mathfrak{a}r} ,$$

in which the second index of the system distinguishes the first individual within a system, and one assumes that the determinant:

$$D(\mu) = \sum \pm\, \mu_{00}\, \mu_{11} \cdots \mu_{rr} \Big|^{t_1}$$

is non-zero. The fact that the quantities μ are continuous functions of t over the entire interval from t_0 to t_1 follows from the fact that equations (21) yield linear forms for the quantities $\mu'_\mathfrak{a}$ in the arguments $\mu_\mathfrak{a}$ whose coefficients are finite and continuous in the indicated interval. Equation (20) now implies that:

$$0 = \sum_{\mathfrak{d}=r+1}^{n} \delta y_\mathfrak{d} \sum_{\mathfrak{a}=0}^{r} \left[\mu_{\mathfrak{ae}}\, \varphi_{\mathfrak{ad}} - \frac{d(\mu_{\mathfrak{ae}}\, \bar{\varphi}_{\mathfrak{ad}})}{dt} \right] + \frac{d}{dt} \sum_{\mathfrak{b}=0}^{n} \delta y_\mathfrak{b} \sum_{\mathfrak{a}=0}^{r} \mu_{\mathfrak{ae}}\, \bar{\varphi}_{\mathfrak{ab}} \qquad (\mathfrak{e} = 0, 1, \ldots, r) ,$$

so when one integrates that from t_0 to t_1, one will also have:

$$\sum_{\mathfrak{b}=0}^{n} \delta y_{\mathfrak{b}} \sum_{\mathfrak{a}=0}^{r} \mu_{\mathfrak{a}\mathfrak{e}}\, \bar{\varphi}_{\mathfrak{a}\mathfrak{b}} \Bigg|_{t_0}^{t_1} = \int_{t_0}^{t_1} dt \sum_{\mathfrak{d}=r+1}^{n} \delta y_{\mathfrak{d}} \sum_{\mathfrak{a}=0}^{r} \left[\mu_{\mathfrak{a}\mathfrak{e}}\, \varphi_{\mathfrak{a}\mathfrak{d}} - \frac{d\,(\mu_{\mathfrak{a}\mathfrak{e}}\, \bar{\varphi}_{\mathfrak{a}\mathfrak{d}})}{dt} \right],$$

or, since the quantities $\delta y_{\mathfrak{b}}$ vanish for $t = t_0$:

$$(23)\qquad - \sum_{\mathfrak{b}=0}^{n} \delta y_{\mathfrak{b}} \sum_{\mathfrak{a}=0}^{r} \mu_{\mathfrak{a}\mathfrak{e}}\, \bar{\varphi}_{\mathfrak{a}\mathfrak{b}} \Bigg|^{t_1} = \int_{t_0}^{t_1} dt \sum_{\mathfrak{d}=r+1}^{n} \delta y_{\mathfrak{d}} \sum_{\mathfrak{a}=0}^{r} \left[\mu_{\mathfrak{a}\mathfrak{e}}\, \varphi_{\mathfrak{a}\mathfrak{d}} - \frac{d\,(\mu_{\mathfrak{a}\mathfrak{e}}\, \bar{\varphi}_{\mathfrak{a}\mathfrak{d}})}{dt} \right].$$

If we set:

$$(24)\qquad \sum_{\mathfrak{a}=0}^{r} \mu_{\mathfrak{a}\mathfrak{g}}\, \bar{\varphi}_{\mathfrak{a}\mathfrak{e}} \Bigg|^{t_1} = 0 \qquad \begin{pmatrix} \mathfrak{c} = a+1,\ldots,r \\ \mathfrak{g} = 0,1,\ldots,s \end{pmatrix}$$

in order to remove the variations δy_{s+1} , δy_{s+2} , …, δy_r (which are subject to no restriction here) from the first $s + 1$ equations then that requirement can always be satisfied by a suitable choice of the quantities (22) without the determinant $D\,(\mu)$ vanishing, and since the quantities $\delta y_{\mathfrak{b}}$ vanish for $t = t_1$, the first $s + 1$ equations (23) will yield:

$$(25)\qquad \sum_{\mathfrak{f}=0}^{s} \delta y_{\mathfrak{f}} \sum_{\mathfrak{a}=0}^{r} \mu_{\mathfrak{a}\mathfrak{g}}\, \bar{\varphi}_{\mathfrak{a}\mathfrak{f}} \Bigg|^{t_1} = \int_{t_0}^{t_1} dt \sum_{\mathfrak{d}=r+1}^{n} \delta y_{\mathfrak{d}} \sum_{\mathfrak{a}=0}^{r} \left[\frac{d\,(\mu_{\mathfrak{a}\mathfrak{g}}\, \bar{\varphi}_{\mathfrak{a}\mathfrak{d}})}{dt} - \mu_{\mathfrak{a}\mathfrak{g}}\, \varphi_{\mathfrak{a}\mathfrak{d}} \right] (\mathfrak{g} = 0, 1, \ldots, s).$$

The determinant of the $(s + 1)^2$ quantities:

$$(26)\qquad \sum_{\mathfrak{a}=0}^{r} \mu_{\mathfrak{a}\mathfrak{g}}\, \bar{\varphi}_{\mathfrak{a}\mathfrak{f}} \Bigg|^{t_1}$$

is non-zero because that is true for the determinant of the $(r + 1)^2$ quantities:

$$(27)\qquad \sum_{\mathfrak{a}=0}^{r} \mu_{\mathfrak{a}\mathfrak{e}}\, \bar{\varphi}_{\mathfrak{a}\mathfrak{i}} \Bigg|^{t_1} \qquad (\mathfrak{e}, \mathfrak{i} = 0, 1, \ldots, r),$$

whose value is obviously:

$$\sum \pm\, \mu_{00}\, \mu_{11} \cdots \mu_{rr} \sum \pm\, \bar{\varphi}_{00}\, \bar{\varphi}_{11} \cdots \bar{\varphi}_{rr} \,\Big|^{t_1}.$$

Now, from (24), the terms in the system (27) will vanish for which $\mathfrak{e}$ is one of the numbers 0, 1, …, s, and $\mathfrak{i}$ is one of the numbers $s + 1$, $s + 2$, …, r, so when the row index $\mathfrak{e}$ is constant, all terms that belong to the first $s + 1$ rows and the last $r – s$ columns. The determinant will then be the product of the two determinants that are defined by the terms in the first $s + 1$ rows and columns and the last $r – s$ rows and columns. However, the former is the determinant of the quantities (26); it cannot have the value zero then. Therefore, from equations (25), the $s + 1$ quantities $\delta y_{\mathfrak{b}}$ can be calculated in the following form:

$$\delta y_{\mathfrak{f}}\Big|^{t_1} = \sum_{\mathfrak{g}=0}^{s} c_{\mathfrak{f}\mathfrak{g}} \int_{t_0}^{t_1} dt \sum_{\mathfrak{d}=r+1}^{n} \delta y_{\mathfrak{d}} \sum_{\mathfrak{a}=0}^{r} \left[\mu_{\mathfrak{a}\mathfrak{g}}\, \varphi_{\mathfrak{a}\mathfrak{d}} - \frac{d\,(\mu_{\mathfrak{a}\mathfrak{g}}\, \bar{\varphi}_{\mathfrak{a}\mathfrak{d}})}{dt} \right] .$$

If one sets:

$$\nu_{\mathfrak{a}\mathfrak{f}} = \sum_{\mathfrak{g}=0}^{s} c_{\mathfrak{f}\mathfrak{g}}\, \mu_{\mathfrak{a}\mathfrak{g}} \qquad (\mathfrak{a} = 0, \dots, r) ,$$

in general, then that will give:

$$\delta y_{\mathfrak{f}}\Big|^{t_1} = \int_{t_0}^{t_1} dt \sum_{\mathfrak{d}=r+1}^{n} \delta y_{\mathfrak{d}} \sum_{\mathfrak{a}=0}^{r} \left[\nu_{\mathfrak{a}\mathfrak{f}}\, \varphi_{\mathfrak{a}\mathfrak{d}} - \frac{d\,(\nu_{\mathfrak{a}\mathfrak{f}}\, \bar{\varphi}_{\mathfrak{a}\mathfrak{d}})}{dt} \right] . \tag{28}$$

In that way, the quantities:

$$\nu_{0\mathfrak{f}},\ \nu_{1\mathfrak{f}},\ \dots,\ \nu_{r\mathfrak{f}}$$

define a system of solutions to equations (21), since that will be true for any system:

$$\mu_{0\mathfrak{g}},\ \mu_{1\mathfrak{g}},\ \dots,\ \mu_{r\mathfrak{g}} .$$

If one further multiplies equations (24) by $c_{\mathfrak{f}\mathfrak{g}}$ and sums over $\mathfrak{g}$ then that will give:

$$\sum_{\mathfrak{g}=0}^{s} c_{\mathfrak{f}\mathfrak{g}} \sum_{\mathfrak{a}=0}^{r} \mu_{\mathfrak{a}\mathfrak{g}}\, \bar{\varphi}_{\mathfrak{a}\mathfrak{c}}\Big|^{t_1} = 0$$

or

$$\sum_{\mathfrak{a}} \bar{\varphi}_{\mathfrak{a}\mathfrak{c}} \sum_{\mathfrak{g}} c_{\mathfrak{f}\mathfrak{g}}\, \mu_{\mathfrak{a}\mathfrak{g}}\Big|^{t_1} = \sum_{\mathfrak{a}} \nu_{\mathfrak{a}\mathfrak{f}}\, \bar{\varphi}_{\mathfrak{a}\mathfrak{c}}\Big|^{t_1} = 0 .$$

Now since $\mathfrak{f}$ and $\mathfrak{g}$ mean the same system of numerals, equations (24) will remain true when one replaces μ with ν. For the sake of completeness, we then set:

$$\nu_{\mathfrak{a}\mathfrak{c}} = \mu_{\mathfrak{a}\mathfrak{c}} , \qquad \nu_{\mathfrak{a}\mathfrak{d}} = \mu_{\mathfrak{a}\mathfrak{d}} \qquad \begin{pmatrix} \mathfrak{c} = s+1, \dots, r \\ \mathfrak{d} = r+1, \dots, n \end{pmatrix} .$$

The $(r + 1)^2$ quantities $\nu_{\mathfrak{a}\mathfrak{e}}$ are then expressed linearly in terms of the $\mu_{\mathfrak{a}\mathfrak{e}}$ by means of the system of coefficients:

$$\begin{matrix} c_{00} & c_{01} & \cdots & c_{0s} & 0 & 0 & \cdots \\ c_{10} & c_{11} & \cdots & c_{1s} & 0 & 0 & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \\ c_{s0} & c_{s1} & \cdots & c_{ss} & 0 & 0 & \cdots \\ 0 & 0 & \cdots & 0 & 1 & 0 & \cdots \\ 0 & 0 & \cdots & 0 & 0 & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \end{matrix}$$

whose determinant has the value:

$$\sum \pm c_{00} \, c_{11} \cdots c_{ss} ,$$

so it will be non-zero from the way that the quantities c came about. The determinant:

$$\sum \pm \nu_{00} \, \nu_{11} \cdots \nu_{rr} \Big|^{t_1}$$

will also be non-zero then, and the quantities ν will have all of the properties that were assumed for the quantities μ .

Now the present extremum problem demands that $\Delta y_0 \big|^{t_1}$ must have a fixed value when all quantities Δy vanish for $t = t_0$ and all of them except for Δy_0 , Δy_{s+1} , Δy_{s+2} , …, Δy_r vanish for $t = t_1$. With the change Δ that was defined in the previous section, the quantities $\Delta y_\mathfrak{b}$ will vanish automatically for $t = t_1$. All that will remain then are the equations:

(29) $$\Delta y_1 \big|^{t_1} = \Delta y_2 \big|^{t_1} = \ldots = \Delta y_s \big|^{t_1} = 0 ,$$

under the assumption that $\Delta y_0 \big|^{t_1}$ should have a fixed sign. Now, by definition, one has:

$$\Delta y_0 = \delta y_0 + [\varepsilon]_2 , \qquad \Delta y_\mathfrak{b} = \delta y_\mathfrak{b} + [\varepsilon]_2 .$$

If one then regards the quantities u as fixed and the constants ε as freely available then the theorem in § **7** will say that when one regards the equations:

(30) $$\delta y_1 \big|^{t_1} = \delta y_2 \big|^{t_1} = \ldots = \delta y_s \big|^{t_1} = 0$$

as linear relations between the quantities ε, that will imply the equation:

(31) $$\delta y_0 \big|^{t_1} = 0 .$$

In order to be able to fulfill equations (29), we add that the previously-undetermined number of the quantities ε should not exceed the number of those equations, and set:

$$m = s ,$$

such that:

$$\delta y_{\mathfrak{b}} = \sum_{\mathfrak{g}=0}^{s} \varepsilon_{\mathfrak{g}}\, u_{\mathfrak{d}\mathfrak{g}} .$$

If we further define:

$$W_{\mathfrak{f}}\,(u_{\mathfrak{g}}) = \int_{t_0}^{t_1} dt \sum_{\mathfrak{d}=r+1}^{n} u_{\mathfrak{d}\mathfrak{g}} \sum_{\mathfrak{a}=0}^{r} \left[\nu_{\mathfrak{a}\mathfrak{f}}\, \varphi_{\mathfrak{a}\mathfrak{d}} - \frac{d\,(\nu_{\mathfrak{a}\mathfrak{f}}\, \bar{\varphi}_{\mathfrak{a}\mathfrak{d}})}{dt} \right]$$

then the formulas (28) will give:

$$\begin{aligned}
\delta y_{\mathfrak{f}} \Big|^{t_1} &= \int_{t_0}^{t_1} dt \sum_{\mathfrak{d}=r+1}^{n} \left(\sum_{\mathfrak{g}=0}^{s} \varepsilon_{\mathfrak{g}}\, u_{\mathfrak{d}\mathfrak{g}} \right) \sum_{\mathfrak{a}=0}^{r} \left[\nu_{\mathfrak{a}\mathfrak{f}}\, \varphi_{\mathfrak{a}\mathfrak{d}} - \frac{d\,(\nu_{\mathfrak{a}\mathfrak{f}}\, \bar{\varphi}_{\mathfrak{a}\mathfrak{d}})}{dt} \right] \\
&= \sum_{\mathfrak{g}=0}^{s} \varepsilon_{\mathfrak{g}} \int_{t_0}^{t_1} dt \sum_{\mathfrak{d}=r+1}^{n} u_{\mathfrak{d}\mathfrak{g}} \sum_{\mathfrak{a}=0}^{r} \left[\nu_{\mathfrak{a}\mathfrak{f}}\, \varphi_{\mathfrak{a}\mathfrak{d}} - \frac{d\,(\nu_{\mathfrak{a}\mathfrak{f}}\, \bar{\varphi}_{\mathfrak{a}\mathfrak{d}})}{dt} \right] \\
&= \sum_{\mathfrak{g}=0}^{s} \varepsilon_{\mathfrak{g}}\, W_{\mathfrak{f}}\,(u_{\mathfrak{g}}) \qquad\qquad (\mathfrak{f} = 0, 1, \ldots, s) .
\end{aligned}$$

With the given relation between equations (30), (31) it will then follow that:

$$\begin{vmatrix} W_0(u_0) & W_0(u_1) & \cdots & W_0(u_s) \\ W_1(u_0) & W_1(u_1) & \cdots & W_1(u_s) \\ \vdots & \vdots & \ddots & \vdots \\ W_s(u_0) & W_s(u_1) & \cdots & W_s(u_s) \end{vmatrix} = 0 .$$

Now the functions u are subject to no other restrictions than that they must be continuous, have continuous first derivatives, and vanish for $t = t_0$ and $t = t_1$. Therefore, if C are quantities that are independent of the functions $u_{\mathfrak{d}0}$ then one will have:

$$\sum_{\mathfrak{f}=0}^{s} C_{\mathfrak{f}}\, W_{\mathfrak{f}}(u_0) = 0 ,$$

or from the definition of the expressions W:

$$\sum_{\mathfrak{f}=0}^{s} C_{\mathfrak{f}} \int_{t_0}^{t_1} dt \sum_{\mathfrak{d}=r+1}^{n} u_{\mathfrak{d}\mathfrak{g}} \sum_{\mathfrak{a}=0}^{r} \left[\nu_{\mathfrak{a}\mathfrak{f}}\, \varphi_{\mathfrak{a}\mathfrak{d}} - \frac{d\,(\nu_{\mathfrak{a}\mathfrak{f}}\, \bar{\varphi}_{\mathfrak{a}\mathfrak{d}})}{dt} \right] = 0 ,$$

or finally when one sets:

$$\sum_{\mathfrak{f}=0}^{s} C_{\mathfrak{f}}\, \nu_{\mathfrak{af}} = \lambda_{\mathfrak{a}} ,$$

one will get:

$$\int_{t_0}^{t_1} dt \sum_{\mathfrak{d}=r+1}^{n} u_{\mathfrak{d}\mathfrak{g}} \sum_{\mathfrak{a}=0}^{r} \left[\lambda_{\mathfrak{af}}\, \varphi_{\mathfrak{ad}} - \frac{d\,(\lambda_{\mathfrak{af}}\, \bar{\varphi}_{\mathfrak{ad}})}{dt} \right] = 0 .$$

If one lets all of the quantities $u_{\mathfrak{d}0}$ vanish identically, except for one of them, then from the conclusion of § **8**, that will give the equations:

$$\sum_{\mathfrak{a}=0}^{r} \left[\lambda_{\mathfrak{af}}\, \varphi_{\mathfrak{ad}} - \frac{d\,(\lambda_{\mathfrak{af}}\, \bar{\varphi}_{\mathfrak{ad}})}{dt} \right] = 0 , \tag{32}$$

since we have assumed that the quantities y, y', y'' are continuous functions of t along the manifold $\mathfrak{M}$ in question. The quantities $\lambda_{\mathfrak{a}}$ are then composed from the quantities $\nu_{\mathfrak{af}}$ in the same way that that they are composed from the quantities $\mu_{\mathfrak{af}}$. The argument that was made above will then say that the quantities $\lambda_{\mathfrak{a}}$ are solutions of the system (21) that satisfy the equations:

$$\sum_{\mathfrak{a}=0}^{r} \lambda_{\mathfrak{af}}\, \bar{\varphi}_{\mathfrak{ac}} \Big|^{t_1} = 0 \qquad (\mathfrak{c} = s + 1, s + 2, \ldots, r) . \tag{33}$$

If one adds that system to the equations (32) that were proved then one will see that the $r + 1$ quantities $\lambda_{\mathfrak{a}}$ satisfy the $n + 1$ equations:

$$\sum_{\mathfrak{a}=0}^{r} \left[\lambda_{\mathfrak{a}}\, \varphi_{\mathfrak{ab}} - \frac{d\,(\lambda_{\mathfrak{a}}\, \bar{\varphi}_{\mathfrak{ab}})}{dt} \right] = 0 \qquad (\mathfrak{b} = 0, 1, \ldots, n) . \tag{34}$$

The simultaneous existence of the last two systems of equations then defines a necessary condition for the desired extremum for a manifold $\mathfrak{M}$ with the given continuity properties. As one easily sees, that will be fulfilled automatically when the quantities $C_{\mathfrak{f}}$ vanish identically.

If one once more assumes that $\varphi_{\mathfrak{ab}}$ and $\bar{\varphi}_{\mathfrak{ab}}$ are partial derivatives of the function $\varphi_{\mathfrak{a}}$, and sets:

$$\Omega = \sum_{\mathfrak{a}=0}^{r} \lambda_{\mathfrak{a}}\, \varphi_{\mathfrak{a}} , \qquad \Omega_{\mathfrak{b}} = \frac{\partial \Omega}{\partial y'_{\mathfrak{b}}}$$

then obviously:

$$\Omega = 0 , \tag{35}$$

and the equations that one gets will be:

(36) $$\Omega_{\mathfrak{c}}\Big|^{t_1} = 0\,, \qquad \frac{\partial\Omega}{\partial y_{\mathfrak{b}}} - \frac{d\Omega_{\mathfrak{b}}}{dt} = 0\,.$$

The last of these equations are mutually independent, because since the functions $\varphi_{\mathfrak{a}}$ (and therefore Ω, as well) are homogeneous of degree q in the arguments $y'_{\mathfrak{b}}$, one will have:

(37) $$\sum_{\mathfrak{b}} y'_{\mathfrak{b}}\,\Omega_{\mathfrak{b}} = q\,\Omega = 0\,,$$

when only the equations $\varphi_{\mathfrak{a}} = 0$ are valid, but the λ are completely arbitrary quantities. If one differentiates the last equation with respect to t then one will get:

$$\sum_{\mathfrak{b}} \left(y''_{\mathfrak{b}}\,\Omega_{\mathfrak{b}} + y'_{\mathfrak{b}}\,\frac{d\Omega_{\mathfrak{b}}}{dt} \right) = 0\,.$$

On the other hand, it follows from equation (35) that:

$$\sum_{\mathfrak{b}} \left(y''_{\mathfrak{b}}\,\Omega_{\mathfrak{b}} + y'_{\mathfrak{b}}\,\frac{\partial\Omega}{\partial y_{\mathfrak{b}}} \right) = 0\,,$$

since the quantities $\lambda'_{\mathfrak{a}}$ contain the vanishing factors $\varphi_{\mathfrak{a}}$. If one then subtracts the last equation from the one that preceded it then that will give:

(38) $$\sum_{\mathfrak{b}} y'_{\mathfrak{b}} \left(\frac{\partial\Omega}{\partial y_{\mathfrak{b}}} - \frac{d\Omega_{\mathfrak{b}}}{dt} \right) = 0\,,$$

from which the dependency of the last equations in (36) upon each other will become clear.

If the quantities $y_{\mathfrak{b}}$, $\lambda_{\mathfrak{a}}$ are determined as functions of t in such a way that the condition equations $\varphi_{\mathfrak{a}} = 0$ and the equations (36) are fulfilled then we will call the simple manifold that corresponds to the system of values $y_{\mathfrak{b}}$, $\lambda_{\mathfrak{a}}$ for different values of t an *extremal*.

Obviously, the result obtained can also be formulated in the following way: One multiplies the equations:

$$\sum_{\mathfrak{b}=0}^{n} (\varphi_{\mathfrak{a}\mathfrak{b}}\,\delta y_{\mathfrak{b}} + \bar{\varphi}_{\mathfrak{a}\mathfrak{b}}\,\delta y'_{\mathfrak{b}}) = 0\,,$$

which are either given directly or obtained by varying the equations $\varphi_{\mathfrak{a}} = 0$, by the factors $\lambda_{\mathfrak{a}}$ and adds them. One integrates the resulting equation over t from t_0 to t_1 and eliminates the quantities $\delta y'$ by partial integration using the formula:

$$\frac{d\,\delta y}{dt} = \delta y' .$$

One then sets the factors that multiply the quantities δy under the integral sign equal to zero, and likewise, the factors of those quantities $\delta y_{\mathfrak{c}}\Big|^{t_1}$ outside of the integral sign that will not be assumed to vanish from the outset. In that way, one gets equations (33) and (34) precisely. Naturally, the first one will appear only when r and s are different.

§ 58. – Examples. Most general isoperimetric problem.

A more important special case that was treated by itself in the previous chapter is the one in which one of the equations $\varphi_{\mathfrak{a}} = 0$ has the form:

$$\varphi_0 = y_0' - \psi(y_1, y_2, \dots, y_1', \dots, y_n') = 0 ,$$

and y_0 does not occur in the remaining ones. The first of equations (34) will then read simply:

$$\frac{d\lambda_0}{dt} = 0 .$$

When one sets:

$$\frac{\lambda_1}{\lambda_0} = l_1 , \qquad \frac{\lambda_2}{\lambda_0} = l_2 , \qquad \dots, \qquad \frac{\lambda_r}{\lambda_0} = l_r ,$$

the differential equations of the problem will be obtained from the formula:

$$\delta \int (\psi + l_1\,\varphi_1 + l_2\,\varphi_2 + \cdots + l_r\,\varphi_r)\,dt = 0 .$$

If one had $\lambda_0 = 0$ then one would have a manifold in the domain of the quantities $y_1, y_2, \dots, y_n$ for which the conditions for the extremum of one of the quantities:

$$y_1\Big|^{t_1} , \quad y_2\Big|^{t_1} , \quad \dots, \quad y_n\Big|^{t_1}$$

with given values for the other ones would be fulfilled. That case shall be considered to be an exception.

First example: The general isoperimetric problem when higher derivatives enter into the integrand.

For example, let $x = t$, and extremize the integral:

$$u_0 = \int f_0 (x, y, y', y'', y''')\, dx$$

for given values of the integrals:

$$u_1 = \int f_1 (x, \ldots, y''')\, dx\,, \qquad u_2 = \int f_2 (x, \ldots, y''')\, dx\,.$$

One will then have the equations:

$$u_0' = f_0 (x, y, z, w, w')\,, \qquad u_1' = f_1 (x, y, \ldots, w')\,, \qquad u_2' = f_2 (x, y, \ldots, w')\,,$$

$$y' - z = 0\,, \qquad z' - w = 0\,,$$

and the values of y, z, w, u_1, u_2 are given at the limits. From the general rule, one must construct:

$$\int dx[\lambda_0 (\delta u_0' - \delta f_0) + \lambda_1 (\delta u_1' - \delta f_1) + \lambda_2 (\delta u_2' - \delta f_2) + \mu (\delta y' - \delta z) + \nu (\delta z' - \delta w)]$$

$$= \lambda_0\, \delta u_0 + \lambda_1\, \delta u_1 + \lambda_2\, \delta u_2 + \mu\, \delta y + \nu\, \delta z$$
$$- \int dx (\lambda_0'\, \delta u_0' + \lambda_1'\, \delta u_1 + \lambda_2'\, \delta u_2' + \lambda_0\, \delta f_0 + \lambda_1\, \delta f_1 + \lambda_2\, \delta f_2 + \mu\, \delta z + \nu\, \delta w + \mu'\, \delta y + \nu'\, \delta z)\ .$$

Now since u_0, u_1, u_2 do not enter into the functions f, one will next get:

$$\lambda_0' = \lambda_1' = \lambda_2' = 0\ .$$

If one sets:

$$g = \lambda_0 f_0 + \lambda_1 f_1 + \lambda_2 f_2$$

then the integrand in the last integral will be:

$$\delta g + \mu\, \delta z + \nu\, \delta w + \mu'\, \delta y + \nu'\, \delta z = \frac{\partial g}{\partial w'}\, \delta w' + \left(\frac{\partial g}{\partial w'} + \mu'\right) \delta y + \left(\frac{\partial g}{\partial z} + \mu + \nu'\right) \delta z + \left(\frac{\partial g}{\partial w} + \nu\right) \delta w\,.$$

If one integrates the first term partially then what will remain under the integral sign will be:

$$\left(\frac{\partial g}{\partial w'} + \mu'\right) \delta y + \left(\frac{\partial g}{\partial z} + \mu + \nu'\right) \delta z + \left(\frac{\partial g}{\partial w} - \frac{d}{dt}\frac{\partial g}{\partial w'} + \nu\right) \delta w\ .$$

All expressions in parentheses are set equal to zero now, and it will follow that:

$$\nu = -\frac{\partial g}{\partial y''} + \frac{d}{dx}\frac{\partial g}{\partial y'''}, \qquad \mu = -\frac{\partial g}{\partial y'} + \frac{d}{dx}\frac{\partial g}{\partial y''} - \frac{d^2}{dx^2}\frac{\partial g}{\partial y'''},$$

$$\mu' + \frac{\partial g}{\partial y} = \frac{\partial g}{\partial y} - \frac{d}{dx}\frac{\partial g}{\partial y'} + \frac{d^2}{dx^2}\frac{\partial g}{\partial y''} - \frac{d^3}{dx^3}\frac{\partial g}{\partial y'''} = 0 .$$

One will then get the differential equation for the desired curve when one attaches constant factors λ :

$$\delta\int(\lambda_0\, f_0 + \lambda_1\, f_1 + \lambda_2\, f_2)\, dx = 0 .$$

The desired curves are then identical to the extremals of that integral in the case of an absolute extremum.

Second example: The principle of least action in its broader form that was given by **Lagrange** is expressed as follows: Let the coordinates of the masses in a system be determined as functions of the parameters y, z, … that do not include time explicitly. Let x be time, let T be the *vis viva*, and let $Y\,\delta y + Z\,\delta z + \ldots$ be the work done by the applied forces under the transition from the position (y, z, …) to the neighboring one ($y + \delta y$, $z + \delta z$, …). If one then compares the natural motion of the system in the domain of the variables y, z, … with a neighboring one that is constructed only mathematically, but has the same starting and ending position, and is so arranged that under the transition from one motion to the other one, the equation:

(39) $$\delta T = Y\,\delta y + Z\,\delta z + \ldots$$

will be valid then the time integral of the *vis viva* under the natural motion will be a minimum:

$$\delta\int T\, dx = 0 .$$

The quantity x cannot be taken to be an independent variable here since its value is not prescribed for the final position of the system. One must then regard all quantities x, y, z, … as functions of a parameter t that have the same initial and final values t_0, t_1 for all comparable motions. The number of unknowns will then be greater by two than the number of parameters y, z, … The number of given equations is two, since one must consider the defining equation of the action u :

$$\frac{du}{dt} = T\,\frac{dx}{dt},$$

along with the relation (39). One will then have:

$$r = 1, \quad s = 0 .$$

From the general rule, one must integrate equation (39), multiplied by λ, over t and add it to the equation:

$$\delta \int T\,dx = \int_{t_0}^{t_1} \delta\left(T\,\frac{dx}{dt}\right) dt = 0$$

which will give:

$$\int_{t_0}^{t_1} \{\delta T\,(x' + \lambda) + T\,\delta x' - \lambda\,(Y\,\delta y + Z\,\delta z + \cdots)\}\,dt = 0 .$$

When one considers the equations:

$$\delta x\,\Big|^{t_0} = \delta y\,\Big|^{t_0} = \delta z\,\Big|^{t_0} = \ldots = \delta y\,\Big|^{t_1} = \delta z\,\Big|^{t_1} = \ldots = 0 ,$$

the usual partial integration will give a result of the form:

$$\delta x\left\{T + (\lambda + x')\frac{\partial T}{\partial x'}\right\}\Bigg|^{t_1} + \int_{t_0}^{t_1} dt\{\xi\,\delta x + \eta\,\delta y + \zeta\,\delta z + \cdots\} = 0 ,$$

and one will then have to impose the equations:

$$\xi = \eta = \zeta = \ldots = 0 .$$

Now since T is a quadratic form in the quantities $\frac{dy}{dx}$, $\frac{dz}{dx}$, … whose coefficients depend upon only y, z, … with the assumptions that were introduced, say:

$$T = \Phi\left(\frac{dy}{dx}, \frac{dz}{dx}, \cdots\right) ,$$

one will obviously have:

(40) $$T = \frac{\Phi\,(y', z', \ldots)}{x'^2} ,$$

so:

$$\frac{\partial T}{\partial x'} = -\,\frac{2T}{x'} .$$

One then gets the equation for the quantities x that corresponds to (33):

(41)
$$T - (\lambda + x') \cdot \frac{2T}{x'}\bigg|^{t_1} = 0\,, \qquad 2\lambda + x'\big|^{t_1} = 0\,.$$

One further has the differential equation:

$$\xi = -\frac{dT}{dt} - \frac{d}{dt}\left\{(\lambda + x')\frac{\partial T}{\partial x'}\right\} = 0\,,$$

or

$$T - 2(\lambda + x')\frac{T}{x'} = \text{const.}$$

It will then follow from equation (41) that in general:

(42)
$$2\lambda + x' = 0\,.$$

The aforementioned partial integration will then give:

$$\eta = -\lambda Y + (\lambda + x')\frac{\partial T}{\partial y} - \frac{d}{dt}\left\{(\lambda + x')\frac{\partial T}{\partial y'}\right\}\,.$$

Now since it will result from equation (40) that when one sets:

$$\frac{dy}{dx} = p\,,$$

the equation:

$$x'\frac{\partial T}{\partial y'} = \frac{\partial T}{\partial p}$$

will be valid, when one recalls equation (42), that will yield:

$$\begin{aligned}
\eta &= -\lambda Y - \lambda\frac{\partial T}{\partial y} - \frac{d}{dt}\left(\frac{1}{2}\frac{\partial T}{\partial p}\right) \\
&= -\lambda\left\{Y + \frac{\partial T}{\partial y} + \frac{1}{2\lambda}\frac{d}{dt}\left(\frac{\partial T}{\partial p}\right)\right\} \\
&= -\lambda\left\{Y + \frac{\partial T}{\partial y} - \frac{d}{dx}\frac{\partial T}{\partial p}\right\}.
\end{aligned}$$

The expressions ζ, … have analogous forms, and one will get the **Lagrange** equations:

(43) $$Y + \frac{\partial T}{\partial y} = \frac{d}{dx}\frac{\partial T}{\partial p}, \qquad \ldots$$

If the parameters y, z, … are not freely available, but are subject to condition equations:

$$Y_1\,\delta y + Z_1\,\delta z + \ldots = 0\,, \qquad Y_2\,\delta y + Z_2\,\delta z + \ldots = 0\,, \qquad \ldots,$$

whose left-hand sides do not necessarily need to be variations of finite expressions, then one needs only to differentiate those equations in order to come to a problem of the type that was considered up to now. One will get:

$$Y_1\,\delta y' + Z_1\,\delta z' + \cdots + Y_1'\,\delta y + \cdots = 0\,,$$
$$Y_2\,\delta y' + Z_2\,\delta z' + \cdots + Y_2'\,\delta y + Z_2'\,\delta z + \cdots = 0\,,$$
$$\ldots,$$

and that system is equivalent to the previous one, since the variations δy, δz, … will vanish for $t = t_0$. One has only to add the values of η, ζ, … above to the summands:

$$\lambda_1\,Y_1' - \frac{d(\lambda_1\,Y_1)}{dt} + \lambda_2\,Y_2' - \frac{d(\lambda_2\,Y_2)}{dt} + \cdots = -\lambda_1'\,Y_1 - \lambda_2'\,Y_2 - \cdots,$$
$$\lambda_1\,Z_1' - \frac{d(\lambda_1\,Z_1)}{dt} + \lambda_2\,Z_2' - \frac{d(\lambda_2\,Z_2)}{dt} + \cdots = -\lambda_1'\,Z_1 - \lambda_2'\,Z_2 - \cdots$$

Equations (43) will then become:

$$Y + \frac{\partial T}{\partial y} - \frac{\lambda_1'}{\lambda}Y_1 - \frac{\lambda_2'}{\lambda}Y_2 - \cdots = \frac{d}{dt}\left(\frac{\partial T}{\partial p}\right),$$

etc.

As an application, we consider a ball of radius a that rolls without slipping on the yz-plane of the coordinate system and contacts it at the point (y, z). Let θ, φ, ψ be the **Euler** angles, which determine the positions of three mutually-perpendicular directions in the ball with respect to the coordinate system. If M is then the mass of the ball and m is its moment of inertia with respect to a diameter then:

$$2T = M\left\{\left(\frac{dy}{dx}\right)^2 + \left(\frac{dz}{dx}\right)^2\right\} + m\left\{\left(\frac{d\theta}{dx}\right)^2 + \sin^2\theta\left(\frac{d\varphi}{dx}\right)^2 + \left(\frac{d\psi}{dx} - \cos\theta\,\frac{d\varphi}{dx}\right)^2\right\},$$

and one will have the geometrically easily-recognized equations of constraint for a rolling motion:

(44)
$$\begin{aligned}\delta y &= -a\sin\varphi\sin\theta\,\delta\psi + a\cos\varphi\,\delta\theta,\\ \delta z &= \quad a\cos\varphi\sin\theta\,\delta\psi + a\sin\varphi\,\delta\theta,\end{aligned}$$

whose multipliers are λ_1, λ_2 . Let the work done by applied forces be:

$$Y\,\delta y + Z\,\delta z + \Phi\,\delta\varphi + \Psi\,\delta\psi + \Theta\,\delta\theta\,.$$

When one sets:

$$\lambda\,\mu = \lambda_1', \qquad \lambda\,\nu = \lambda_2',$$

the rule above will then give the equations of motion:

$$M\,\frac{d^2y}{dx^2} = Y + \mu\,, \qquad M\,\frac{d^2z}{dx^2} = Z + \nu\,,$$

$$m\left(\frac{d^2\theta}{dx^2} - \sin\theta\,\frac{d\varphi}{dx}\frac{d\psi}{dx}\right) = -\,\mu\,a\cos\varphi - \nu\,a\sin\varphi + \Theta\,,$$

$$m\frac{d}{dx}\left(\frac{d\psi}{dx} - \cos\theta\,\frac{d\varphi}{dx}\right) = \mu\,a\sin\varphi\sin\theta - \nu\,a\cos\varphi\sin\theta + \Psi\,,$$

$$m\frac{d}{dx}\left(\frac{d\varphi}{dx} - \cos\theta\,\frac{d\psi}{dx}\right) = \Phi\,.$$

One adds equations (44) to these, in which one can replace the symbol δ with $d : dx$.

Third example: Find the brachistochrone (§ **12**) in a resisting medium when the resistance is a given function of the velocity.

Let y, z be the coordinates, let v be the velocity, let 1 be the mass of the point, and let $f(v)$ be the absolute value of the resistance, so $-f(v)\sqrt{dy^2 + dz^2}$ will be the work it does. Let g be the constant of gravity, so the + z-axis points vertically downwards. That will then give the energy equation:

$$d\,\frac{v^2}{2} = g\,dz - f(v)\sqrt{dy^2 + dz^2}\,,$$

or when all quantities are functions of the parameter t :

(45)
$$v\,v' - g\,z' - f(v)\sqrt{y'^2 + z'^2} = 0\,.$$

Now, the integral:

$$u = \int \frac{\sqrt{dy^2 + dz^2}}{v}$$

is to be extremized. The initial values of u, v, y, z, and final values of y, z are given. If one writes the definition of u in the form:

$$u' - \frac{\sqrt{y'^2 + z'^2}}{v} = 0$$

then one will have $r = 1$, $s = 0$ (with the notation of § **56**), and since u itself does not enter into consideration, one can set $\lambda_0 = 1$, so:

$$\Omega = u' - \frac{\sqrt{y'^2 + z'^2}}{v} + \lambda\,(v\,v' - g\,z' + f(v)\sqrt{y'^2 + z'^2}\,)\,.$$

The equations of the extremals that correspond to the unknowns y, z are:

$$\frac{d}{dt}\frac{\partial\Omega}{\partial y'} = \frac{d}{dt}\frac{\partial\Omega}{\partial z'} = 0\,,$$

and immediately give:

(46)
$$\begin{aligned} \frac{y'}{\sqrt{y'^2 + z'^2}}\left(-\frac{1}{v} + \lambda\,f(v)\right) &= a, \\ -\lambda\,g + \frac{z'}{\sqrt{y'^2 + z'^2}}\left(-\frac{1}{v} + \lambda\,f(v)\right) &= b. \end{aligned}$$

It follows from this that:

$$b = -\,\lambda\,g + \frac{a\,z'}{y'}\,, \qquad \lambda = \frac{1}{g}\left(\frac{a}{p} - b\right)\,,$$

when one sets:

$$\frac{dy}{dz} = p\,.$$

Equation (33) is to be defined for the variable v since its final value is not given. When 1 is the endpoint of the desired curve and v does not vanish at it, one will get:

$$\left.\frac{\partial\Omega}{\partial v'}\right|^1 = 0\,, \qquad \lambda\,\Big|^1 = 0\,, \qquad \lambda = \frac{1}{g}\left(\frac{1}{p} - \frac{1}{p_1}\right)\,.$$

If one calculates p and $\sqrt{1+p^2}$ as functions of v from the first equation in (46) and substitutes the value obtained for $\sqrt{1+p^2}$ in the equation:

$$v\frac{dv}{dz} = g - f(v)\sqrt{1+p^2}\,,$$

which is equivalent to (45), then one will get simple expressions of the form $\Phi(v)\,dv$ for dz and $p\,dz$, which will reduce the determination of the desired curve to quadratures.

Fourth example: Let:

$$w = f(u, v)\,, \qquad u = \int F(x, y, x', y')\,dt\,, \qquad v = \int G(x, y, x', y')\,dt\,,$$

and find the extremum of the quantity w for given initial and final values of x and y. If one writes the defining equations in the form:

$$w' = f_u\,u' + f_v\,v'\,, \qquad u' - F(x, y, x', y') = 0\,, \qquad v' - G(x, y, x', y') = 0$$

then one will get a problem of the type that was considered. One has $r = 2$, $n = 4$, $s = 0$.

§ 59. – Concept and properties of the field in the problem in § 56.

In order to be able to exhibit sufficient conditions for an extremum, we confine ourselves to the case of $r = s$, i.e., we assume that the initial and final values of all quantities $y_{\mathfrak{b}}$, with the exception of the final value of y_0, are given, but that the latter is to be extremized. We now make the following convention in regard to the indices:

$$\begin{aligned} \mathfrak{a} &= 0, 1, \ldots, r\,, \\ \mathfrak{p}, \mathfrak{b} &= 0, 1, \ldots, n\,, \\ \mathfrak{d} &= r+1, r+2, \ldots, n\,, \\ \mathfrak{c} &= 1, 2, \ldots, n\,, \end{aligned}$$

such that $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{d}$ keep their meanings, but $\mathfrak{e}$, $\mathfrak{f}$, … remain arbitrary.

Now let an $(n - 1)$-fold family of extremals be defined in the region of the quantities $y_{\mathfrak{b}}$ by the equations:

$$(47) \qquad y_{\mathfrak{b}} = \theta_{\mathfrak{b}}(t, a, b, \ldots, h)\,, \qquad \lambda_{\mathfrak{a}} = \zeta_{\mathfrak{a}}(t, a, b, \ldots, h)\,,$$

whose right-hand sides include $n - 1$ arbitrary constants $a, b, \dots, h$. When t belongs to the interval from t_{00} to t_2 , let the functions $\theta_\mathfrak{b}$ and $\lambda_\mathfrak{a}$ be regular at the location ($t, a_0, b_0, \dots, h_0$), and let the functions $\varphi_\mathfrak{a}$ be regular at the location:

$$y_\mathfrak{b} = \theta_\mathfrak{b}\,(t, a_0, \dots, h_0)\,, \qquad y'_\mathfrak{b} = \theta'_\mathfrak{b}\,(t, a_0, \dots, h_0)\,.$$

Differentiating the equations $\varphi_\mathfrak{a} = 0$ with respect to a will then give:

$$\sum_\mathfrak{a} \lambda_\mathfrak{a} \sum_\mathfrak{b} \left(\varphi_{\mathfrak{ab}} \frac{\partial y_\mathfrak{b}}{\partial a} + \bar{\varphi}_{\mathfrak{ab}} \frac{\partial y'_\mathfrak{b}}{\partial a} \right) = 0\,,$$

$$\frac{d}{dt} \sum_\mathfrak{b} \frac{\partial y_\mathfrak{b}}{\partial a} \sum_\mathfrak{a} \lambda_\mathfrak{a}\, \bar{\varphi}_{\mathfrak{ab}} + \sum_\mathfrak{b} \frac{\partial y_\mathfrak{b}}{\partial a} \sum_\mathfrak{a} \left[\lambda_\mathfrak{a}\, \varphi_{\mathfrak{ab}} - \frac{d(\lambda_\mathfrak{a}\, \bar{\varphi}_{\mathfrak{ab}})}{dt} \right] = 0\,,$$

or from the equations of the extremals in the form (36):

$$(48) \qquad \frac{d}{dt} \sum_\mathfrak{b} \frac{\partial y_\mathfrak{b}}{\partial a} \sum_\mathfrak{a} \lambda_\mathfrak{a}\, \bar{\varphi}_{\mathfrak{ab}} = \frac{d}{dt} \sum_\mathfrak{b} \Omega_\mathfrak{b} \frac{\partial y_\mathfrak{b}}{\partial a} = 0\,,$$

and the equation will remain valid when one replaces a with $b, c, \dots, h$. When the manifolds (47) have the system of values:

$$y_\mathfrak{b}^0 = \theta_\mathfrak{b}\,(t_{00}, a_0, \dots, h_0)\,,$$

in common, which can correspond to different values of the constants for different values of t_0, then one will have:

$$(49) \qquad y_\mathfrak{b}^0 = \theta_\mathfrak{b}\,(t_0, a, \dots, h)\,,$$

so

$$0 = \theta'_b \frac{\partial t_0}{\partial a} + \frac{\partial \theta_b}{\partial a} \bigg|^{t=t_0}\,,$$

along with analogous equations for $b, c, \dots, h$. The identity (37) will then imply that:

$$\sum_\mathfrak{b} \Omega_\mathfrak{b} \frac{\partial y_\mathfrak{b}}{\partial a} \bigg|^{t=t_0} = 0\,,$$

so as a result of equation (48), when one integrates over t :

(50) $$\sum_{\mathfrak{b}} \Omega_{\mathfrak{b}} \frac{\partial y_{\mathfrak{b}}}{\partial a} = \sum_{\mathfrak{b}} \Omega_{\mathfrak{b}} \frac{\partial y_{\mathfrak{b}}}{\partial b} = \ldots = 0 .$$

The fact that the quantity t_0 can be represented as a regular function of $a, b, \ldots, h$ at the location $(a_0, \ldots, h_0)$ in at least one of equations (49) has been used only implicitly in this, such that not all of the quantities $\theta'_{\mathfrak{b}}(t_{00}, a_0, b_0, \ldots, h_0)$ vanish. That assumption will be established.

Now let t_1 be placed between t_{00} and t_2 . In the domain of the quantities $y_{\mathfrak{b}}$, the system of values:

$$\theta_{\mathfrak{b}}(t_{00}, a_0, \ldots) , \qquad \theta_{\mathfrak{b}}(t_1, a_0, \ldots) , \qquad \theta_{\mathfrak{b}}(t_2, a_0, \ldots)$$

at the locations 0, 1, 2 might be different. They belong to the well-defined manifold:

$$y_{\mathfrak{b}} = \theta_{\mathfrak{b}}(t, a_0, b_0, \ldots, h_0) ,$$

which will be denoted by $\mathfrak{C}$. Let another manifold $\mathfrak{L}$ that connects the locations 1 and 2 be defined by the equations:

$$Y_{\mathfrak{d}} = f_{\mathfrak{d}}(\tau) ,$$

whose right-hand sides have the properties of the functions that were denoted by $\varphi(\tau)$ in § **17**. τ assumes the values τ_1 and τ_2 at the locations 1 and 2, and the quantities $f'_{\mathfrak{d}}(\tau)$ might nowhere vanish simultaneously. The quantities $Y_{\mathfrak{a}}$ are defined as functions of $\varphi(\tau)$ along the manifold $\mathfrak{L}$ by the equations:

(51) $$\varphi_{\mathfrak{a}}(Y_{\mathfrak{b}}, P_{\mathfrak{b}}) = 0 , \qquad Y_{\mathfrak{b}}\Big|^{\tau_1} = \theta_{\mathfrak{b}}(t_1, a_0, b_0, \ldots, h_0) ,$$

in which one generally sets:

$$P_{\mathfrak{b}} = \frac{dY_{\mathfrak{b}}}{d\tau} .$$

The latter equations (51) are already included in the assumptions that were introduced previously for $\mathfrak{b} = r + 1, r + 2, \ldots, n$. Moreover, if the n equations:

$$Y_{\mathfrak{c}}\Big|^{2} = \theta_{\mathfrak{c}}(t_2, a_0, b_0, \ldots, h_0)$$

are true then $\mathfrak{L}$ will be one of the manifolds for which the segment 12 that satisfies the equations:

$$y_{\mathfrak{b}} = \theta_{\mathfrak{b}}(t, a_0, b_0, \ldots, h_0)$$

is compared in the extremum problem. When the difference:

$$Y_0 \Big|^2 - \theta_0 (t_2, a_0, b_0, \ldots, h_0)$$

possesses a fixed sign for all manifolds $\mathfrak{L}$ that deviate only slightly from $\mathfrak{C}$, the segment 12 will provide an extremum. The extremum shall be referred to as *strong* when all of the manifolds $\mathfrak{L}$ are brought into consideration for which each value of τ between τ_1 and τ_2 can be associated with a value of t that belongs to the interval from t_1 to t_2 such that the quantities:

$$| Y_{\mathfrak{b}} - \theta_{\mathfrak{b}} (t, a_0, \ldots, h_0) |$$

do not exceed a certain positive constant ε. By contrast, the extremum will be referred to as *weak* when one introduces the further restriction on $\mathfrak{L}$ that the inequalities:

$$\left| \frac{P_{\mathfrak{b}}}{P_{\mathfrak{p}}} - \frac{\theta'_{\mathfrak{b}}(t, a_0, \ldots, h_0)}{\theta'_{\mathfrak{p}}(t, a_0, \ldots, h_0)} \right| < \varepsilon \tag{52}$$

also exist for the mutually-associated values t, τ, and the two quantities $P_{\mathfrak{b}}$, $\theta'_{\mathfrak{b}}(t, a_0, \ldots, h_0)$ will have the same sign as long as they are both non-zero.

The new assumption will now be introduced that the functional determinant:

$$\Delta = \frac{\partial(\theta_1, \theta_2, \ldots, \theta_n)}{\partial(t, a, \ldots, h)} = \Delta\,(t, a, \ldots, h)$$

is non-zero for $a = a_0$, $b = b_0$, …, $h = h_0$ in the interval from t_1 to t_2 . As we would like to say, the extremals (47) will then define a *field*. If t^0 is an arbitrary location between t_1 and t_2 one sets:

$$\eta_{\mathfrak{b}} = \theta_{\mathfrak{b}} (t^0, a_0, b_0, \ldots, h_0)\,, \qquad \eta'_{\mathfrak{b}} = \theta'_{\mathfrak{b}}(t^0, a_0, b_0, \ldots, h_0)$$

then one will have the equations:

$$y_{\mathfrak{c}} - \eta_{\mathfrak{c}} = [t - t^0, a - a_0, \ldots, h - h_0]_1 \,,$$

and the determinant of the linear terms on the right will be $\Delta\,(t^0, a_0, b_0, \ldots, h_0)$. One can then solve those equations in the form:

$$t - t^0 = [y_{\mathfrak{c}} - \eta_{\mathfrak{c}}]_1\,, \qquad a - a_0 = [y_{\mathfrak{c}} - \eta_{\mathfrak{c}}]_1\,, \qquad \ldots, \qquad h - h_0 = [y_{\mathfrak{c}} - \eta_{\mathfrak{c}}]_1\,. \tag{53}$$

There will then be a positive constant ε_0 with the property that the series converges for arbitrary values of t^0 that belong to the established interval as long as one has:

$$|y_\mathfrak{c} - \eta_\mathfrak{c}| < \varepsilon ,$$

in general. If one then gives the value of ε_0 to the quantity ε that was introduced above then one can generally replace $y_\mathfrak{c}$ with $Y_\mathfrak{c}$ in equations (53), and in that way define $t, a, b, \ldots, h$ as functions of t that have the same continuity properties as the functions $f_\mathfrak{b}(\tau)$. Obviously, the values that are defined for $a, b, \ldots$ will go to $a_0, b_0, \ldots$ for $\tau = \tau_1$ and $\tau = \tau_2$. With that, the analogue of the **Weierstrass** construction has been implemented, since each of the systems of values $t, a, b, \ldots$ that were defined satisfies the equations:

$$Y_\mathfrak{c} = \theta_\mathfrak{c}(t, a, \ldots, h) , \tag{54}$$

so the quantities $a, b, \ldots, h$ will then define an extremal of the family (47) in the domain of the quantities $y_\mathfrak{b}$ that includes a location 3 that runs through the manifold $\mathfrak{L}$ and implies the equations:

$$y_\mathfrak{c} = Y_\mathfrak{c}$$

for it. Naturally, y_0 and Y_0 will be different, in general.

If one restricts oneself to the manifold $\mathfrak{L}$ that comes under consideration for a weak extremum, such that the inequalities (52) are true for a suitable association of t and t, then it will follow directly from the continuity of the functions $\theta_\mathfrak{b}$ that the quantities:

$$\left| \frac{P_\mathfrak{b}}{P_\mathfrak{p}} - \frac{\theta'_\mathfrak{b}(t, a, \ldots, h)}{\theta'_\mathfrak{p}(t, a, \ldots, h)} \right| , \tag{55}$$

in which $t, a, \ldots, h$ have the values that are defined by the **Weierstrass** construction, also remain below a limit that decreases indefinitely along with ε.

§ 60. – Sufficient conditions for an extremum.

If one regards $t, a, b, \ldots$ as functions of τ then based upon the **Weierstrass** construction, one will obviously have:

$$\frac{dy_\mathfrak{b}}{d\tau} = \frac{\partial\theta_\mathfrak{b}}{\partial a}\frac{da}{d\tau} + \cdots + \frac{\partial\theta_\mathfrak{b}}{\partial h}\frac{dh}{d\tau} + \frac{\partial\theta_\mathfrak{b}}{\partial t}\frac{dt}{d\tau} , \qquad \frac{dy_\mathfrak{c}}{d\tau} = \frac{dY_\mathfrak{c}}{d\tau} = P_\mathfrak{c} , \tag{56}$$

and equations (50) will yield:

(57)
$$\sum_{\mathfrak{b}} \Omega_{\mathfrak{b}} \frac{dy_{\mathfrak{b}}}{d\tau} = \frac{dt}{d\tau} \sum_{\mathfrak{b}} \Omega_{\mathfrak{b}}\, y'_{\mathfrak{b}} = 0 .$$

On the other hand, when one suggests the substitution of $Y_{\mathfrak{b}}$ for $y_{\mathfrak{b}}$ and $P_{\mathfrak{b}}$ for $y'_{\mathfrak{b}}$ by overbars:

$$\sum_{\mathfrak{b}} P_{\mathfrak{b}} \bar{\Omega}_{\mathfrak{b}} = 0 ,$$

in which the arguments $\lambda_{\mathfrak{a}}$ keep the values $\zeta_{\mathfrak{a}}$ that were defined above. When one recalls the last equation in (56), equation (57) can then be written:

$$\Omega_0 \frac{dy_0}{d\tau} - \bar{\Omega}_0\, P_0 + \sum_{\mathfrak{c}} P_{\mathfrak{c}} (\Omega_{\mathfrak{c}} - \bar{\Omega}_{\mathfrak{c}}) = 0$$

or

$$\Omega_0 \frac{d\,(y_0 - Y_0)}{d\tau} = \sum_{\mathfrak{b}} P_{\mathfrak{b}} (\bar{\Omega}_{\mathfrak{b}} - \Omega_{\mathfrak{b}}) = - \sum_{\mathfrak{b}} P_{\mathfrak{b}}\, \bar{\Omega}_{\mathfrak{b}} .$$

If one then sets:

$$\Omega^0 = \Omega\,(y_0, Y_1, \ldots, Y_n, P_0, P_1, \ldots, P_n) = \Omega\,(y_0, \ldots, y_n, P_0, \ldots) ,$$

$$\mathcal{E} = \Omega^0 - \sum_{\mathfrak{b}} P_{\mathfrak{b}}\, \Omega_{\mathfrak{b}} ,$$

and if the factors λ also have their values in (47) here then it will follow that:

(58)
$$\Omega_0 \frac{d\,(y_0 - Y_0)}{d\tau} = - \Omega^0 + \mathcal{E} .$$

The expressions Ω^0 and $\bar{\Omega}$, the last of which has the value zero from (51), moreover, differ in only their first arguments, so the quantity:

$$\frac{-\Omega^0}{y_0 - Y_0} = - \frac{\Omega^0 - \bar{\Omega}}{y_0 - Y_0} = \Phi$$

will remain finite for $y_0 = Y_0$ and will then be an integrable function of τ in the interval from τ_1 to τ_2 in any case. If one then writes equation (58) in the form:

(59)
$$\Omega_0 \frac{d\,(y_0 - Y_0)}{d\tau} - \Phi\,(y_0 - Y_0) = \mathcal{E}$$

and assumes that Ω_0 is non-zero for the system of values $y_{\mathfrak{b}}$ considered then one can set:

$$\Phi = \Omega_0\,\Psi, \qquad y_0 - Y_0 = z\exp\left(-\int_{\tau_1}^{\tau}\Psi\,d\tau\right).$$

The integral has the properties of the function $\varphi(\tau)$ in § **17**, and equation (59) will reduce to the following one:

$$\frac{dz}{d\tau} = \frac{\mathcal{E}}{\Omega_0}\exp\left(-\int_{\tau_1}^{\tau}\Psi\,d\tau\right).$$

Now since $y_0 - Y_0$, and therefore z, vanishes for $t = t_1$, from (51), it will follow that:

$$y_0 - Y_0 = \left(\exp\int_{\tau_1}^{\tau}\Psi\,d\tau\right)\int_{\tau_1}^{\tau}\frac{\mathcal{E}}{\Omega_0}\exp\left(-\int_{\tau_1}^{\tau}\Psi\,d\tau\right)d\tau\,.$$

That difference will be equal to zero when $\mathcal{E}$ vanishes everywhere along the entire manifold $\mathfrak{L}$. The next case in which $\mathcal{E}$ vanishes is the one in which the equations:

$$P_{\mathfrak{b}} = m\,y_{\mathfrak{b}}' = m\,\theta_{\mathfrak{b}}'(t, a, \ldots, h)$$

are valid. We then say that $\mathcal{E}$ vanishes in an *ordinary* way. If that happens everywhere along the manifold $\mathfrak{L}$ then equations (54) will give:

$$\frac{\partial\theta_{\mathfrak{c}}}{\partial a}\frac{da}{d\tau} + \cdots + \frac{\partial\theta_{\mathfrak{c}}}{\partial h}\frac{dh}{d\tau} + \frac{\partial\theta_{\mathfrak{c}}}{\partial t}\left(\frac{dt}{d\tau} - m\right) = 0\,.$$

However, since Δ is non-zero, it would follow from this that:

$$\frac{da}{d\tau} = \frac{db}{d\tau} = \ldots = \frac{dh}{d\tau} = 0\,,$$

so $\mathfrak{L}$ and $\mathfrak{C}$ must coincide. If that is not the case and an extraordinary vanishing of the quantity $\mathcal{E}$ is excluded, but its sign is constant, then since Ω_0 is non-zero, the latter will also be true of $\mathcal{E}\,\Omega_0$ and $y_0 - Y_0$, and those quantities will have a common fixed sign. With that, the extremum for the quantity y_0 is proved. One also obtains a **Jacobi** condition as a sufficient condition here, namely, the existence of a field, along with the **Weierstrass** sign condition that is expressed by means of the quantity $\mathcal{E}$.

Now in order to also derive the analogue of the **Legendre** condition, we set:

(60)
$$\sqrt{P_0^2 + P_1^2 + \cdots + P_n^2} = S\,, \qquad \sqrt{y_0'^2 + y_1'^2 + \cdots + y_n'^2} = s\,, \qquad S > 0, \quad s > 0,$$
$$P_\mathfrak{b} = S\,R_\mathfrak{b}\,, \qquad y'_\mathfrak{b} = s\,z_\mathfrak{b}\,, \qquad \sum_\mathfrak{b} R_\mathfrak{b}^2 = \sum_\mathfrak{b} z_\mathfrak{b}^2 = 1\,.$$

One can then introduce the quantity:

$$\bar{\tau} = \int_{\tau_1}^{\tau} S\,d\tau$$

as an independent variable along the manifold $\mathfrak{L}$, in which $R_\mathfrak{b}$ enters in place of $P_\mathfrak{b}$. If one imagines implementing the foregoing development with the variable $\bar{\tau}$ and denotes that by τ then one will have $S = 1$. If one further assumes that $q = 1$, such that the quantities $\Omega_\mathfrak{b}(y, y')$ in the arguments will be homogeneous of degree zero in the y', then one will get:

(61)
$$\mathcal{E} = \Omega\,(y, R) - \sum_{\mathfrak{b}=0}^{n} R_\mathfrak{b}\,\Omega_\mathfrak{b}(y, z)\,,$$

or when one recalls the identity (37):

$$\mathcal{E} = \Omega\,(y, R) - \sum_{\mathfrak{b}=0}^{n} (R_\mathfrak{b} - z_\mathfrak{b})\,\Omega_\mathfrak{b}(y, z)\,.$$

If one is dealing with a weak extremum, in particular, then from the remarks that were made about the quantities (55), the differences $R_\mathfrak{p} - z_\mathfrak{p}$ will be arbitrarily small as long as ε is assumed to be sufficiently small. One can then develop the quantity $\Omega\,(y, R)$ in powers of the differences $R - z$ and get:

$$\mathcal{E} = \tfrac{1}{2}\sum_{\mathfrak{b},\mathfrak{p}=0}^{n} (R_\mathfrak{b} - z_\mathfrak{b})(R_\mathfrak{p} - z_\mathfrak{p})\frac{\partial^2\Omega(y,z)}{\partial z_\mathfrak{b}\,\partial z_\mathfrak{p}} + [R-z]_3\,.$$

The differences $R - z$ are subject to, first of all, the equation:

(62)
$$\sum_\mathfrak{b} z_\mathfrak{b}(R_\mathfrak{b} - z_\mathfrak{b}) + [R-z]_2 = 0\,,$$

which follows from the identities (60), and furthermore, the equations that follow from the **Taylor** development of the expression:

$$\varphi_\mathfrak{a}\,(Y_0, y_1, \ldots, y_n, R_0, \ldots, R_n) - \varphi_\mathfrak{a}\,(y_0, y_1, \ldots, y_n, z_0, \ldots, z_n) = 0\,,$$

i.e., the equations:

(63) $$0 = \varphi_{\mathfrak{a}0}(Y_0 - y_0) + \sum_{\mathfrak{b}} \bar{\varphi}_{\mathfrak{ab}}(R_{\mathfrak{b}} - z_{\mathfrak{b}}) + [Y_0 - y_0, R_{\mathfrak{p}} - z_{\mathfrak{p}}]_2 .$$

If one imagines that, say, $y_0 - Y_0$ and $r + 1$ of the quantities $R - z$ are expressed as power series in terms of the remaining ones using the last equation and equation (62) then one will get a power series in just those arguments for $\mathcal{E}$ whose quadratic terms are defined in precisely the same way as when no quadratic terms are present in equations (62), (63) and no cubic terms are present in $\mathcal{E}$. The quantity will then have a fixed sign when the quadratic form:

$$Q = \sum_{\mathfrak{p},\mathfrak{q}=0}^{n} \frac{\partial^2 \Omega}{\partial y'_{\mathfrak{p}}\, \partial y'_{\mathfrak{q}}} u_{\mathfrak{p}}\, u_{\mathfrak{q}}$$

can be reduced to a definite form in $n - r - 1$ or $n - r$ of the arguments u using the $r + 2$ condition equations:

(64) $$\sum_{\mathfrak{b}} y'_{\mathfrak{b}} u_{\mathfrak{b}} = 0 , \qquad \varphi_{\mathfrak{a}0}\, v + \sum_{\mathfrak{b}} \bar{\varphi}_{\mathfrak{ab}}\, u_{\mathfrak{b}} = 0$$

according to whether the quantities $\varphi_{\mathfrak{a}0}$ all vanish identically or not, resp.

In that entire development, the quantities y, y' always referred to the extremal 03 that is obtained in the **Weierstrass** construction and the point 3 that runs along the manifold $\mathfrak{L}$. One must then take $y_{\mathfrak{b}}$ to have the expression $\theta_{\mathfrak{b}}(t, a, \ldots, h)$. However, since the quantities θ are continuously variable in their arguments, the quadratic form Q will have the desired properties in a certain neighborhood of the manifold $\mathfrak{C}$ when that is true for the elements of the latter themselves. That is, one can replace the constants $a, b, \ldots, h$ with $a_0, b_0, \ldots, h_0$ in the sufficient condition for the fixed sign of the expression $\mathcal{E}$ above and refer the quantities y, y' to the extremal $\mathfrak{C}$ being investigated.

In order to further simplify that condition, we set:

$$y'_{\mathfrak{e}} = p_{\mathfrak{e}}\, y'_n , \quad u_{\mathfrak{e}} - p_{\mathfrak{e}}\, u_n = v_{\mathfrak{e}} \qquad (\mathfrak{e} = 0, 1, \ldots, n-1)$$

and consider the fact that in the case of $q = 1$, the homogeneity of the quantities $\Omega_{\mathfrak{b}}(y, y')$ will imply the identities:

$$\sum_{\mathfrak{b}=0}^{n} y'_{\mathfrak{b}} \frac{\partial^2 \Omega}{\partial y'_{\mathfrak{b}}\, \partial y'_{\mathfrak{p}}} = 0 , \qquad \frac{\partial^2 \Omega}{\partial y'_n\, \partial y'_{\mathfrak{p}}} + \sum_{\mathfrak{e}=0}^{n-1} p_{\mathfrak{e}} \frac{\partial^2 \Omega}{\partial y'_{\mathfrak{e}}\, \partial y'_{\mathfrak{p}}} = 0 .$$

That will then easily imply that:

(65) $$Q = \sum_{\mathfrak{e},\mathfrak{f}=0}^{n-1} \frac{\partial^2 \Omega}{\partial y'_{\mathfrak{e}} \, \partial y'_{\mathfrak{f}}} v_{\mathfrak{e}} \, v_{\mathfrak{f}} \, .$$

If one further sets:

$$\varphi_{\mathfrak{a}} = y'_n \; f_{\mathfrak{a}} \, (y_n, y_0, \ldots, y_{n-1}, p_0, \ldots, p_{n-1}) \, , \qquad \frac{\partial f_{\mathfrak{a}}}{\partial p_{\mathfrak{e}}} = f_{\mathfrak{a}\mathfrak{e}} \, , \qquad v \, y'_n = w$$

in order to make $q = 1$ then the last of equations (64) will go to the following one:

(66) $$\frac{\partial f_{\mathfrak{a}}}{\partial y_0} w + \sum_{\mathfrak{e}=0}^{n-1} f_{\mathfrak{a}\mathfrak{e}} \, v_{\mathfrak{e}} = 0 \, ,$$

since obviously:

$$\bar{\varphi}_{\mathfrak{a}} = f_{\mathfrak{a}} - \sum_{\mathfrak{e}=0}^{n-1} p_{\mathfrak{e}} \; f_{\mathfrak{a}\mathfrak{e}} \, ,$$

and the first equation in (58) can be ignored. Now, the usual case in the calculus of variations is the one in which y_0 is defined by a quadrature. One can then assume that y_0 does not occur in any of the functions $\varphi_{\mathfrak{a}}$, and y'_0 occurs in only φ_0 , and indeed such that:

$$f_0 = -\frac{y'_0}{y'_n} + f \, (y_n, y_1, \ldots, y_{n-1}, p_0, \ldots, p_{n-1}) \, .$$

If one writes x for y_n then one will have to extremize the integral:

$$y_0 = \int F \, (x, y_1, \ldots, y_{n-1}, p_1, \ldots, p_{n-1}) \, dx$$

with the conditions that:

$$f_{\mathfrak{i}} \, (x, y_1, \ldots, y_{n-1}, p_0, \ldots, p_{n-1}) = 0 \qquad (\mathfrak{i} = 1, 2, \ldots, r) \, .$$

The first equation in (66) can be dropped, since, from (65), the form Q is free of v_0 . The remaining ones will become:

(67) $$\sum_{\mathfrak{g}=1}^{n-1} \frac{\partial f_{\mathfrak{i}}}{\partial p_{\mathfrak{g}}} v_{\mathfrak{g}} = 0 \, ,$$

and since λ_0 is constant, from § **58**, when one sets:

$$\lambda_{\mathfrak{i}} = l_{\mathfrak{i}} \, \lambda_0 \, , \qquad F = f + l_1 \, f_1 + \ldots + l_r \, f_r \, , \qquad \Omega = \lambda_0 \, (x' F - y'_0) \, ,$$

one will have:

$$\frac{\partial F}{\partial y_{\mathfrak{g}}} - \frac{d}{dx}\frac{\partial F}{\partial p_{\mathfrak{g}}} = 0$$

as the equations of the extremals. Now formula (59) gives:

$$x' Q = \lambda_0 \sum_{\mathfrak{g},\mathfrak{h}=1}^{n-1} \frac{\partial^2 F}{\partial p_{\mathfrak{g}}\, \partial p_{\mathfrak{h}}} v_{\mathfrak{g}}\, v_{\mathfrak{h}} .$$

Therefore, if the quantity x' is positive along a certain segment of the extremal $\mathfrak{C}$ and the form:

$$Q^0 = \sum_{\mathfrak{g},\mathfrak{h}=1}^{n-1} \frac{\partial^2 F}{\partial p_{\mathfrak{g}}\, \partial p_{\mathfrak{h}}} v_{\mathfrak{g}}\, v_{\mathfrak{h}}$$

can be reduced to a definite form in $n - 1 - r$ arguments using the conditions (67) then $\mathcal{E}$ will have a fixed sign. In that way, since $\Omega_0 = -\lambda_0$, the sign of $\mathcal{E}\,\Omega_0$ will be opposite to that of the form Q^0. The condition for a minimum or a maximum will be fulfilled according to whether the latter is positive or negative, resp. With that, we have derived the criterion for a weak extremum that is analogous to **Legendre**'s in the form that goes back to **Clebsch**.

§ 61. – Dependency on the integration constants. Existence of a field.

The relation (38) that was proved in § **57** shows that one of the differential equations for an extremal will always follows from the other ones. Therefore, if one sets:

$$y_n = x = t , \qquad y_n' = 1, \qquad y_{\mathfrak{e}}' = \frac{dy_{\mathfrak{e}}}{dx} ,$$

in particular, then $y_0, y_1, \ldots, y_{n-1}$ will be determined as functions of x by the $n + r + 1$ equations:

(68) $$\varphi_{\mathfrak{a}} = 0 , \qquad \frac{\partial \Omega}{\partial y_{\mathfrak{e}}} - \frac{d\Omega_{\mathfrak{e}}}{dx} = 0 \qquad (\mathfrak{e} = 0, 1, \ldots, n-1) ,$$

and the last equation for the extremals is a consequence of them. The system:

$$\frac{d\varphi_{\mathfrak{a}}}{dx} = 0 , \qquad \frac{\partial \Omega}{\partial y_{\mathfrak{e}}} - \frac{d\Omega_{\mathfrak{e}}}{dx} = 0$$

includes $n + r + 1$ equations that can generally be solved for the $n + r + 1$ quantities $y''_\mathfrak{e}$, $\lambda'_\mathfrak{a}$. They will then give $y_\mathfrak{e}$, $\lambda_\mathfrak{a}$ as functions of x and $2n + r + 1$ arbitrary constants, say, the initial values of $y_\mathfrak{e}$, $y'_\mathfrak{e}$, $\lambda_\mathfrak{a}$. $r + 1$ of them are determined from the remaining one by means of the equations $\varphi_\mathfrak{a} = 0$, such that $2n$ constants will remain. The latter equations are free of $\lambda_\mathfrak{a}$, so one can assume that the initial values of the quantities $\lambda_\mathfrak{a}$ enter into the remaining $2n$ constants. Equations (34) will now remain fulfilled when one leaves the functions $y_\mathfrak{e}$ completely unchanged in a system of solutions but multiplies the quantity $\lambda_\mathfrak{a}$ by a constant factor. One can then fix one of the $2n$ constants, with no loss of generality, and set:

$$y_\mathfrak{e} = \psi_\mathfrak{e}\,(x, c_1, c_2, \ldots, c_{n-1}) \tag{69}$$

accordingly.

In the usual problem in the calculus of variations, one has:

$$\varphi_0 = -\,y'_0 + f\left(y_\mathfrak{e}, \frac{y'_\mathfrak{c}}{y'_n}\right)\cdot y'_n\ ,$$

i.e., one must extremize the definite integral:

$$y_0 = \int_{x_0}^{x} f\left(x, y_1, \ldots, y_{n-1}, \frac{dy_1}{dx}, \ldots, \frac{dy_{n-1}}{dx}\right) dx\,.$$

The value of y_0 is therefore zero for $x = x_0$ from the outset, which will then decrease the number of arbitrary constants to $2\,(n-1)$.

If one directs one's attention in the general case to all extremals (69) that have the system of values:

$$x = x_0\,, \qquad y_0 = y_0^0\,, \qquad \ldots, \qquad y_{n-1} = y_{n-1}^0$$

in common then the constants c will be subject to the equations:

$$y_\mathfrak{e}^0 = y_\mathfrak{e}\,(x_0, c_1, c_2, \ldots, c_{2n-1})\,. \tag{70}$$

Those will be identities when one identifies n of the constants c with the quantities $y_\mathfrak{e}^0$. If one denotes the remaining $n - 1$ of them by $a, b, \ldots, h$ then it will be obvious that:

$$\frac{\partial(y_1, y_2, \ldots, y_{n-1})}{\partial(a, b, \ldots, h)} = \frac{\partial(x, y_1, \ldots, y_{n-1})}{\partial(x, a, \ldots, h)} = \frac{\partial(y_0^0, y_1^0, \ldots, y_{n-1}^0, y_1, y_2, \ldots, y_{n-1})}{\partial(c_1, c_2, \ldots, c_{2n-1})} = \Delta\,.$$

As long as the latter determinant is non-zero, the extremals that are defined by equations (70) will define a field. However, it has by no means been proved that this determinant cannot also vanish identically. We shall give sufficient conditions for that to not happen in the following example.

Example: Find the extremum of the integral:

$$w = \int f(x, y, y', v, v')\, dx$$

with the condition equation:

$$v' = g(x, y, y', v) \tag{71}$$

and given initial and final values of the quantities x, y, v. Let λ be the multiplier of one of them, while the other one can be set equal to 1, such that:

$$\Omega = w' - f + \lambda(v' - g) \ .$$

When the index 0 always refers to the initial location $x = x_0$, one can then regard the quantities y_0, v_0, y_0', λ_0 as constants of the extremals, and then have:

$$\Delta = \frac{\partial(y_0, v_0, y, v)}{\partial(y_0, v_0, y_0', \lambda_0)} = \frac{\partial(y, v)}{\partial(y_0', \lambda_0)} \ .$$

If one now determines the quantity μ from the equation:

$$\mu' + \mu \frac{\partial g}{\partial v} = 0$$

and sets:

$$G = \mu \frac{\partial g}{\partial y} - \frac{d}{dx}\left(\mu \frac{\partial g}{\partial y'}\right)$$

then when one differentiates equation (71) with respect to λ_0 and multiplies by μ, it will obviously follow that:

$$\frac{d}{dx}\left(\mu \frac{\partial v}{\partial \lambda_0}\right) = \frac{d}{dx}\left(\mu \frac{\partial g}{\partial y'} \frac{\partial y}{\partial \lambda_0}\right) + \left\{\mu \frac{\partial g}{\partial y} - \frac{d}{dx}\left(\mu \frac{\partial g}{\partial y'}\right)\right\} \frac{\partial y}{\partial \lambda_0} = \frac{d}{dx}\left(\mu \frac{\partial g}{\partial y'} \frac{\partial y}{\partial \lambda_0}\right) + G \frac{\partial y}{\partial \lambda_0} \ , \tag{72}$$

and that equation will remain valid when λ_0 is replaced with y_0'. Furthermore, since:

$$y = y_0 + y_0' (x - x_0) + [x - x_0]_2 \ , \tag{73}$$

that will give:

$$\frac{\partial y}{\partial \lambda_0} = \frac{\partial y}{\partial y_0'} = 0$$

for $x = x_0$. If one then integrates equation (72) over x then it will follow that:

$$\mu \frac{\partial v}{\partial \lambda_0} = \mu \frac{\partial g}{\partial y'} \frac{\partial y}{\partial \lambda_0} + \int_{x_0}^{x} G \frac{\partial y}{\partial \lambda_0}\, dx ,$$

$$\mu \frac{\partial v}{\partial y_0'} = \mu \frac{\partial g}{\partial y'} \frac{\partial y}{\partial y_0'} + \int_{x_0}^{x} G \frac{\partial y}{\partial y_0'}\, dx ,$$

and from that:

$$\mu\,\Delta = \frac{\partial y}{\partial y_0'} \int_{x_0}^{x} G \frac{\partial y}{\partial \lambda_0}\, dx - \frac{\partial y}{\partial \lambda_0} \int_{x_0}^{x} G \frac{\partial y}{\partial y_0'}\, dx .$$

If one substitutes the development (73) in this equation:

$$\frac{\partial y}{\partial \lambda_0} = L\,(x - x_0)^l + [x - x_0]_{l+1} , \qquad \frac{\partial y}{\partial y_0'} = x - x_0 + [x - x_0]_2$$

and sets:

$$G = M\,(x - x_0)^m + [x - x_0]_{m+1} ,$$

in which L and M do not vanish, then one will have $l \geq 2$, and one will get:

$$\mu\,\Delta = (x - x_0 + \ldots) \int_{x_0}^{x} \{L\,M\,(x - x_0)^{l+m} + \cdots\}\, dx - [L(x - x_0)^l + \cdots] \int_{x_0}^{x} \{M\,(x - x_0)^{m+1} + \cdots\}\, dx$$

$$= L\,M\,(x - x_0)^{l+m+2} \left(\frac{1}{l+m+1} - \frac{1}{m+2} \right) + [x - x_0]_{l+m+3} .$$

Now since the difference:

$$\frac{1}{l+m+1} - \frac{1}{m+2} = \frac{1-l}{(m+2)(l+m+1)}$$

is non-zero, Δ can vanish identically only when one of the equations:

$$\frac{\partial y}{\partial \lambda_0} = 0 , \qquad G = 0$$

is fulfilled for all values of x along the extremal in question.

CHAPTER EIGHT

THE EXTERMUM FOR A DOUBLE INTEGRAL

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§ 62. – Parametric representation of a surface.

Let $\mathfrak{S}$ be a surface that is defined by setting the rectangular coordinates x, y, z equal to functions of two parameters u, v. Let the derivatives with respect to the latter be denoted in the usual way by subscripts, such that:

$$\frac{\partial x}{\partial u} = x_u\,, \qquad \frac{\partial x}{\partial v} = x_v\,, \qquad \ldots$$

We then consider double integrals that are taken over the surface $\mathfrak{S}$:

$$J = \iint_{\mathfrak{S}} \Phi\,(x, y, z, x_u, y_u, z_u, x_v, y_v, z_v)\, du\, dv\,,$$

which are determined by the surface $\mathfrak{S}$ alone, but do not depend upon the special nature of the connection between the x, y, z, on the one hand, and u, v, on the other. One can also get the surface $\mathfrak{S}$ when one represents x, y, z as functions of the parameter p, q, so one will have:

$$J = \iint_{\mathfrak{S}} \Phi\,(x, y, z, x_p, y_p, z_p, x_q, y_q, z_q)\, dp\, dq$$

when one integrates over the region in the variables p, q that corresponds to the surface $\mathfrak{S}$, as the notation would suggest. If one regards the latter variables as functions of u and v whose functional determinant is positive then the last equation will give:

$$J = \iint_{\mathfrak{S}} \Phi\,(x, y, z, x_p, y_p, z_p, x_q, y_q, z_q)\, \frac{\partial\,(p, q)}{\partial\,(u, v)}\, du\, dv\,,$$

from which it will follow that if one replaces the surface $\mathfrak{S}$ with an arbitrary-small piece of it, one will have the identity:

$$(1) \qquad \Phi\,(x, y, z, x_u, y_u, z_u, x_v, y_v, z_v) = \Phi\,(x, y, z, x_p, y_p, z_p, x_q, y_q, z_q)\, \frac{\partial\,(p, q)}{\partial\,(u, v)}\,.$$

When the function Φ is arranged such that this relation exists for any arbitrary surface $\mathfrak{S}$, a series of useful identities can be derived from that.

In particular, let a surface patch $\mathfrak{S}_0$ be defined by the equations:

$$x = f(p, q), \quad y = g(p, q), \quad z = h(p, q),$$

whose right-hand sides are regular functions of their arguments in a certain region and whose elements yield only those systems of values:

$$x, y, z, x_p, y_p, z_p, x_q, y_q, z_q$$

for which the function Φ is regular. The same surface patch will also be represented by the equations:

$$x = f(u + \rho, v + \sigma), \quad y = g(u + \rho, v + \sigma), \quad z = h(u + \rho, v + \sigma),$$

in which ρ, σ mean regular functions of u, v that we can regard as small. They might, perhaps, be endowed with a constant factor that can be made arbitrarily small. The system of values p, q and u, v that are associated with the same point of $\mathfrak{S}_0$ are then coupled by the equations:

$$p = u + \rho, \qquad q = v + \sigma,$$

and the functional determinant:

$$\frac{\partial(p,q)}{\partial(u,v)} = \begin{vmatrix} 1+\rho_u & \sigma_u \\ \rho_v & 1+\sigma_v \end{vmatrix} = 1 + \rho u + \sigma v + [\varepsilon]_2$$

is positive. If one further sets:

$$\xi = \rho f_p(u, v) + \sigma f_q(u, v), \quad \eta = \rho g_p(u, v) + \sigma g_q(u, v), \quad \zeta = \rho h_p(u, v) + \sigma h_q(u, v)$$

then one will have the **Taylor** developments:

$$x = f(u, v) + \xi + [\varepsilon]_2, \qquad y = g(u, v) + \eta + [\varepsilon]_2, \qquad z = h(u, v) + \zeta + [\varepsilon]_2,$$

from which it will follow immediately that when one differentiates with respect to u and v, one will get:

$$x_u - f_p(u, v) = \xi_u + [\varepsilon]_2, \qquad y_u - g_p(u, v) = \eta_u + [\varepsilon]_2, \qquad z_u - h_p(u, v) = \zeta_u + [\varepsilon]_2,$$

$$x_v - f_q(u, v) = \xi_v + [\varepsilon]_2, \qquad y_v - g_q(u, v) = \eta_v + [\varepsilon]_2, \qquad z_v - h_q(u, v) = \zeta_v + [\varepsilon]_2,$$

and the assumed identity (1) will imply that:

$$\mathcal{F}(f(p, q), \ldots, f_p(p, q), \ldots)(1 + \rho u + \sigma v + [\varepsilon]_2)$$

(2)
$$
\begin{aligned}
&= \Phi(x, \ldots, x_u, \ldots) \\
&= \Phi\left(f(u+\rho, v+\sigma), \ldots, \frac{\partial f(u+\rho, v+\sigma)}{\partial u}, \ldots \right) \\
&= \Phi(f(u, v) + \xi + [\varepsilon]_2, \ldots, f_p(u, v) + \xi_u + [\varepsilon]_2, \ldots) .
\end{aligned}
$$

The last of those expressions can be written:

$$\Phi + \Phi_x\,\xi + \Phi_y\,\eta + \Phi_z\,\zeta + \Phi_{x_u}\,\xi_u + \Phi_{y_u}\,\eta_u + \Phi_{z_u}\,\zeta_u + \Phi_{x_v}\,\xi_v + \Phi_{y_v}\,\eta_v + \Phi_{z_v}\,\zeta_v + [\varepsilon]_2 ,$$

which will make the arguments of the function symbol Φ be:

(3) $\quad x = f(u, v), \quad y = g(u, v), \quad z = h(u, v), \quad x_u = f_p(u, v), \ldots, \quad x_v = f_q(u, v), \ldots$

On the other hand, one can develop:

$$\Phi(f(p, q), \ldots, f_p(p, q), \ldots) = \Phi + \rho\frac{\partial\Phi}{\partial u} + \sigma\frac{\partial\Phi}{\partial v} + [\varepsilon]_2 ,$$

in which the arguments (3) are likewise substituted in the Φ symbol on the right-hand side. If one then drops the term $[\varepsilon]_2$ from equation (2) then that will give:

$$
\begin{aligned}
&\Phi_x\,\xi + \Phi_y\,\eta + \Phi_z\,\zeta + \Phi_{x_u}\,\xi_u + \Phi_{y_u}\,\eta_u + \Phi_{z_u}\,\zeta_u + \Phi_{x_v}\,\xi_v + \Phi_{y_v}\,\eta_v + \Phi_{z_v}\,\zeta_v \\
&\qquad = \Phi(\rho_u + \sigma_v) + \rho\frac{\partial\Phi}{\partial u} + \sigma\frac{\partial\Phi}{\partial v} \\
&\qquad = \frac{\partial(\rho\Phi)}{\partial u} + \frac{\partial(\sigma\Phi)}{\partial v} .
\end{aligned}
$$

We transform the left-hand side of this equation by means of the identity:

$$\Phi_{x_u}\,\xi_u = \frac{\partial(\xi\,\Phi_{x_u})}{\partial u} - \xi\frac{\partial\,\Phi_{x_u}}{\partial u}$$

and its analogues, and set:

$$P = \Phi_x - \frac{\partial\Phi_{x_u}}{\partial u} - \frac{\partial\,\Phi_{x_v}}{\partial v} ,$$

$$Q = \Phi_y - \frac{\partial \Phi_{y_u}}{\partial u} - \frac{\partial \Phi_{y_v}}{\partial v},$$
$$R = \Phi_z - \frac{\partial \Phi_{z_u}}{\partial u} - \frac{\partial \Phi_{z_v}}{\partial v}.$$

We then get:

$$P\,\xi + Q\,\eta + R\,\zeta = \frac{\partial}{\partial u}(\rho\Phi - \xi\Phi_{x_u} - \eta\Phi_{y_u} - \zeta\Phi_{z_u}) + \frac{\partial}{\partial v}(\sigma\Phi - \xi\Phi_{x_v} - \eta\Phi_{y_v} - \zeta\Phi_{z_v}) \;.$$

If we substitute the values of ξ, η, ζ and consider the facts that the system of arguments (3) has been chosen, and that furthermore ρ, σ, are arbitrary functions, except for the factor ε that they contain, then the factors of the quantities ρ, σ, ρ_u, σ_u, ρ_v, σ_v on both sides can be set equal to each other, and that will yield the identities:

(4) $$P\,x_u + Q\,y_u + R\,z_u = 0, \qquad P\,x_v + Q\,y_v + R\,z_v = 0,$$

(5) $$\begin{aligned} \Phi &= x_u\,\Phi_{x_u} + y_u\,\Phi_{y_u} + z_u\,\Phi_{z_u} = x_v\,\Phi_{x_v} + y_v\,\Phi_{y_v} + z_v\,\Phi_{z_v}\,, \\ 0 &= x_u\,\Phi_{x_v} + y_u\,\Phi_{y_v} + z_u\,\Phi_{z_v} = x_v\,\Phi_{x_u} + y_v\,\Phi_{y_u} + z_v\,\Phi_{z_u}\,, \end{aligned}$$

in which x, y, z are regarded as arbitrary functions of u and v.

§ 63. – Variation of a double integral

The integral J will now be defined when one integrates over a well-defined surface patch $\mathfrak{S}$ on which x and y are single-valued, continuous functions of u and v that are provided with first and second derivatives that are also like that, and the functional determinants:

(6) $$\frac{\partial(y,z)}{\partial(u,v)}, \qquad \frac{\partial(z,x)}{\partial(u,v)}, \qquad \frac{\partial(x,y)}{\partial(u,v)}$$

are nowhere-vanishing. If one interprets u and v as rectangular coordinates in the plane then the points of the surface patch $\mathfrak{S}$ might correspond to a region $\mathfrak{U}$ that is bounded by a curve $\mathfrak{K}$ that returns to itself and does not intersect itself. It meets none of the lines u = const., v = const. infinitely often and consists of a finite number of pieces along which u and v are continuous functions of a parameter t that are provided with continuous first derivatives. The curve $\mathfrak{K}$ corresponds to the boundary line of the surface $\mathfrak{S}$, which will be denoted by $\mathfrak{C}$.

Furthermore, let δx, δy, δz be functions of u and v that possess the same continuity properties as x, y, z. The region $\mathfrak{U}$ then corresponds to a surface patch $\mathfrak{S}^0$ on $\mathfrak{S}$ in a single-valued invertible way that goes through the point $(x + \delta x, y + \delta y, z + \delta z)$. Any quantity ω whose value is determined by the surface patch will go to $\omega + \Delta\omega$ when that patch is replaced with $\mathfrak{S}^0$. If one regards the quantities δx, δy, δz and their first derivatives as small – i.e., one neglects all terms of dimension two and higher in the progressive **Taylor** development of those quantities – then $\Delta\omega$ will go to the variation $\delta\omega$. Now, since x obviously goes to $x + \delta x$ and x_u to $\partial (x + \delta x) / \partial u$, one will have:

$$\delta x = \Delta x , \qquad \delta x_u = \Delta x_u = \frac{\partial\,\delta x}{\partial u}, \qquad \delta x_v = \Delta x_v = \frac{\partial\,\delta x}{\partial v},$$

and similar equations will be true for y and z. The surface patch $\mathfrak{S}^0$ has all of the properties that are assumed for $\mathfrak{S}$. The assumption that relates to the quantities (6) will also be fulfilled when the absolute values of the quantities δx, δx_u, …, δz_v are sufficiently small.

The rule given for neglecting terms will be implemented especially in the **Taylor** development:

$$\Delta\Phi = \Phi_x\,\delta x + \Phi_{x_u}\delta x_u + \Phi_{x_v}\delta x_v + \cdots + [\delta x, \delta x_u, \delta x_v, \ldots]_2 ,$$

in which the terms that contain x will be dropped, along with the similar terms in which x is replaced with y and z, as shall be done frequently from now on. The definition of the δ symbol then implies that:

$$\delta\Phi = \Phi_x\,\delta x + \Phi_{x_u}\delta x_u + \Phi_{x_v}\delta x_v + \cdots$$

The integral J will then take on the increment:

$$\iint_{\mathfrak{S}} \Delta\Phi\, du\, dv = \iint_{\mathfrak{S}} du\, dv\, (\delta\Phi + [\delta x, \delta x_u, \delta x_v, \ldots]_2)$$

when $\mathfrak{S}$ goes to $\mathfrak{S}^0$. It follows from this that:

$$\delta J = \iint_{\mathfrak{S}} du\, dv\, \delta\Phi .$$

We transform the expression by means of the identity:

$$\Phi_{x_u}\delta x_u + \delta x \frac{\partial \Phi_{x_u}}{\partial u} = \frac{\partial(\Phi_{x_u}\,\delta x)}{\partial u}$$

and its analogues. We then get:

$$\delta J = \iint_{\mathfrak{S}} du\, dv\,(P\,\delta x + Q\,\delta y + R\,\delta z) + \iint_{\mathfrak{S}} du\, dv \left(\frac{\partial U}{\partial u} + \frac{\partial V}{\partial v}\right),$$

in which we have set:

$$U = \Phi_{x_u}\delta x + \Phi_{y_u}\delta y + \Phi_{z_u}\delta z, \qquad V = \Phi_{x_v}\delta x + \Phi_{y_v}\delta y + \Phi_{z_v}\delta z.$$

The last summand in the expression for δJ can be converted into a simple integral. Namely, if one chooses the direction of increase in the parameter t along the curve $\mathfrak{K}$ – i.e., the boundary of the region $\mathfrak{U}$ – such that it lies with respect to the interior-pointing normal to the region $\mathfrak{U}$ just as the + u-axis lies with respect to the + v-axis then the direction of increasing t will make the same concave angle that the indicated normal makes with the + v direction. Now, $du : dt$ is positive or negative according to whether the direction of increasing t defines an acute angle with the + u-axis or an obtuse one, resp. Thus, if one moves along a line u = const. that enters the region $\mathfrak{U}$ and leaves it again in the direction of increasing v then $du : dt$ will be positive when it enters and negative when it leaves. One now defines the integral:

$$\iint \frac{\partial V}{\partial u}\, du\, dv \ ,$$

in which one first integrates along a strip of width $|\, du\, |$ that is bounded by two lines u = const. If 0 is an entry point when one advances in the direction + v and 1 is the following exit point then the piece of the strip that lies between both points will yield the following contribution to the integral:

$$|\, du\, | \int \frac{\partial V}{\partial u}\, dv = |\, du\, |\ V \Big|_0^1 ,$$

and from the above, when dt is positive, one will have:

$$|\, du\, | = \frac{du}{dt}\, dt \Big|^0 = -\frac{du}{dt}\, dt \Big|^1 .$$

The partial integral thus-obtained can then be written:

$$-\ dt\ \frac{du}{dt} V \Big|^1 - dt \frac{du}{dt} V \Big|^0 .$$

The sum of all those partial integrals is:

$$\iint_{\mathfrak{G}} \frac{\partial V}{\partial u}\, du\, dv = -\int_0^{t_0} V \frac{du}{dt}\, dt ,$$

when the entire curve $\mathfrak{K}$ corresponds to the interval from 0 to t_0 . That argument is justified by the assumed properties of the curve $\mathfrak{K}$ and the fact that the integral on the right has a well-defined sense and value.

If one switches the v and u axes then the direction of increasing t must be inverted in order for its previous relationship to the axes to remain unchanged. If one sets:

$$t' = -t + t_0$$

then one will get:

$$\iint_{\mathfrak{S}} \frac{\partial U}{\partial u}\, du\, dv = -\int_0^{t_0} U \frac{dv}{dt'}\, dt',$$

and when one once more introduces t :

$$\iint_{\mathfrak{S}} \frac{\partial U}{\partial u}\, du\, dv = -\int_{t_0}^{0} U \frac{dv}{dt'}\, dt' = +\int_0^{t_0} U \frac{dv}{dt}\, dt \;.$$

If one then understands the differential that is defined by positive dt in the simple integrals over du, dv :

$$du = \frac{du}{dt}\, dt\,, \quad dv = \frac{dv}{dt}\, dt$$

then one can put the expression for δJ that was obtained into the following form:

$$\delta J = \iint_{\mathfrak{S}} du\, dv\, (P\,\delta x + Q\,\delta y + R\,\delta z) + \int_{\mathfrak{C}} (U\, dv - V\, du) \;.$$

The sense in which the simple integral is defined is determined on the surface in the following way: Let 0 be a point in its interior that corresponds to u_0, v_0 in the uv-plane. The elements that start from 0:

$$dv = 0, \qquad du > 0\,; \qquad du = 0, \qquad dv > 0\,,$$

which might be called (u) and (v), correspond to the directions + u, + v that are drawn from the point (u_0, v_0). If a line element moves through the concave angle that is defined by (u) and (v) from the former to the latter, which will define a certain positive sense of rotation, then the element will not fall along any of the lines u = const., v = const. along the way. The corresponding direction in the uv-plane then moves from the position + u to the direction + v through the angle that they define, so it rotates in the sense in which a direction that agrees with that of increasing t along the curve $\mathfrak{K}$ must rotate in order to go to a direction that points to the interior of the region $\mathfrak{U}$. The latter corresponds to a direction on the surface $\mathfrak{S}$ that points to its interior from a point on the boundary line $\mathfrak{C}$. Such a thing will then have the same relationship to the direction of integration that (v) has to (u). The opposite orientation of the directions (u), (v) is the same at all points of the surface $\mathfrak{S}$, moreover, since the lines u = const., v = const. do not have the same direction anywhere, due to the assumption that that was made in regard to the quantities (6). As seen from a well-defined side of the surface $\mathfrak{S}$, (v) lies in relations to (u) as South does in relation to West. As seen from that side, the direction of integration is seen to coincide with an orbit from West to East towards the South.

We shall always denote the normal that has the same relationship to (u) and (v) that the $+ z$ axis has to $+ x$ and $+ y$ by n, and its direction cosines by X, Y, Z. It will keep its direction when we introduce new parameters for u and v if the functional determinant with respect to u and v is positive.

§ 64. – Necessary conditions for an absolute and relative extremum when the boundary line is given.

Should the surface $\mathfrak{S}$ yield an extremum of the integral J in regard to all neighboring surfaces with the same boundary line $\mathfrak{C}$, then ΔJ would have to possess a fixed sign as long as the absolute values of the quantities δx, δy, δz, and their first derivatives lie below a certain limit. In order to infer conclusions from that, we assume the right angle in which:

$$u_0 \leq u \leq u_1 , \quad v_0 \leq v \leq v_1 \tag{7}$$

always belong to the interior of $\mathfrak{U}$, and let:

$$\delta x = \varepsilon\, T, \qquad T = (u - u_0)^3\, (u_1 - u)^3\, (v - v_0)^3\, (v_1 - v)^3, \qquad \delta y = \delta z = 0$$

for that right angle, while:

$$\delta x = \delta y = \delta z = 0$$

outside of $\mathfrak{U}$. The quantities δx, δy, δz, along with their first and second derivatives, will then be continuous on the entire surface $\mathfrak{S}$ and vanish along the curve $\mathfrak{C}$. The surface $\mathfrak{S}^0$ will then have all of the properties that were assumed for $\mathfrak{S}$ and must make the sign of the quantity ΔJ constant for sufficiently-small values of the constant ε. Since:

$$\Delta J = \delta J + [\varepsilon]_2 ,$$

and δJ is the only term that is linear in ε, from § **7**, that will require that:

$$\delta J = \varepsilon \iint P\, T\, du\, dv = 0 ,$$

in which one integrates over the region (7). Since the latter is arbitrarily small and can be limited at arbitrary locations, but T is positive in its interior, it will follow that:

$$P = 0\,, \qquad \Phi_x - \frac{\partial \Phi_{x_u}}{\partial u} - \frac{\partial \Phi_{x_v}}{\partial v} = 0$$

for the entire surface $\mathfrak{S}$. One likewise gets:

$$Q = 0\,, \qquad \Phi_y - \frac{\partial \Phi_{y_u}}{\partial u} - \frac{\partial \Phi_{y_v}}{\partial v} = 0\,,$$

$$R = 0\,, \qquad \Phi_z - \frac{\partial \Phi_{z_u}}{\partial u} - \frac{\partial \Phi_{z_v}}{\partial v} = 0\,,$$

and the identities (4) will show that one of those equations is always a consequence of the remaining ones. A surface that satisfies those three equations is called an *extremal* of the integral J.

Similarly, one can also derive necessary conditions for a certain relative extremum of the integral J. The assumptions that were introduced for it and its integrand might also be true for the integral:

$$K = \iint_{\mathfrak{S}} \Psi\,(x, y, z, x_u, y_u, z_u, x_v, y_v, z_v)\,du\,dv\;.$$

The quantities P, Q, R might go to $\bar{P}$, $\bar{Q}$, $\bar{R}$ when one replaces Φ with Ψ. In addition to the region (7), we also vary the region that is defined by the inequalities:

(8) $$u_2 \le u \le u_3\,, \qquad v_2 \le v \le v_3\,,$$

which is disjoint from the other region, but likewise still inside of $\mathfrak{U}$, and set:

$$T_0 = (u - u_2)^3\,(u_3 - u)^3\,(v - v_2)^3\,(v_3 - v)^3.$$

For the region (7), let:

$$\delta x = \alpha\,T, \qquad \delta y = \beta\,T, \qquad \delta z = \gamma\,T,$$

while:

$$\delta x = \alpha_0\,T_0\,, \qquad \delta y = \beta_0\,T_0\,, \qquad \delta z = \gamma_0\,T_0$$

for the region (8), in which we understand α, α_0, … to mean constants, and:

$$\delta x = \delta y = \delta z = 0$$

everywhere outside of both regions. The variations will then have the same continuity properties as above over the entire surface $\mathfrak{S}$ and will vanish on the boundary line. If one sets:

$$p = \int_{u_0}^{u_1}\int_{v_0}^{v_1} T\,P\,du\,dv\,, \qquad \bar{p} = \int_{u_0}^{u_1}\int_{v_0}^{v_1} T\,\bar{P}\,du\,dv\,,$$

$$p_0 = \int_{u_2}^{u_3}\int_{v_2}^{v_3} T_0\,P\,du\,dv\,, \qquad \bar{p}_0 = \int_{u_2}^{u_3}\int_{v_2}^{v_3} T_0\,\bar{P}\,du\,dv\,,$$

and defines the quantities $q, \bar{q}, \ldots, r, \bar{r}, \ldots$ similarly by replacing P with Q and R then one will have:

$$\delta J = \iint_{\mathfrak{S}} (P\,dx + Q\,dy + R\,dz)\,du\,dv$$
$$= \alpha\,p + \beta\,q + \gamma\,r + \alpha_0\,p_0 + \beta_0\,q_0 + \gamma_0\,r_0\,,$$

$$\delta K = \alpha\,\bar{p} + \beta\,\bar{q} + \gamma\,\bar{r} + \alpha_0\,\bar{p}_0 + \beta_0\,\bar{q}_0 + \gamma_0\,\bar{r}_0\,.$$

Furthermore, one has:

$$\Delta J = \delta J + [\alpha, \beta, \ldots, \gamma_0]_2\,, \qquad \Delta K = \delta K + [\alpha, \beta, \ldots, \gamma_0]_2\,.$$

Therefore, should the quantity ΔJ have a fixed sign under the assumption that:

$$\Delta K = 0\,,$$

then the theorem in § **7** would immediately imply that every determinant that is defined by two columns in the matrix:

$$\begin{matrix} p, & q, & r, & p_0, & q_0, & r_0, \\ \bar{p}, & \bar{q}, & \bar{r}, & \bar{p}_0, & \bar{q}_0, & \bar{r}_0 \end{matrix} \tag{9}$$

should vanish. Now, since the regions (7) and (8), in which one fixes the locations $0 = (u_0, v_0)$ and $2 = (u_2, v_2)$, can be shrunk indefinitely, the considerations in § **32** will give certain mean values, in such a way that the same equations will be true for the determinants from the matrix:

$$\begin{matrix} P\,|^0, & Q\,|^0, & R\,|^0, & P\,|^2, & Q\,|^2, & R\,|^2, \\ \bar{P}\,|^0, & \bar{Q}\,|^0, & \bar{R}\,|^0, & \bar{P}\,|^2, & \bar{Q}\,|^2, & \bar{R}\,|^2 \end{matrix}$$

that are true for the ones in the matrix (9). It follows from this that when one regards 0 as variable and 2 as fixed, the differential equations:

$$\mu\,P + \bar{\mu}\,\bar{P} = \mu\,Q + \bar{\mu}\,\bar{Q} = \mu\,R + \bar{\mu}\,\bar{R} = 0$$

will be true along the entire surface $\mathfrak{S}$ for a suitable choice of the constants μ, $\bar{\mu}$. A surface that satisfies those equations for an arbitrary system of non-zero values of μ, $\bar{\mu}$ is called an *extremal* for the given problem of a relative extremum. If one overlooks the cases in which $\mathfrak{S}$ is an extremal, in the sense of an absolute extremal, for one of the integrals J, K then one will have the differential equations:

$$P + \lambda \bar{P} = Q + \lambda \bar{Q} = R + \lambda \bar{R} = 0$$

for the necessary conditions for the relative extremum, in which λ means a finite, non-vanishing constant.

Problem XIV. – Find the surface of least area for a given boundary.

If we, with **Gauss**, set:

$$E = x_u^2 + y_u^2 + z_u^2, \qquad F = x_u\, x_v + y_u\, y_v + z_u\, z_v, \qquad G = x_v^2 + y_v^2 + z_v^2$$

then the surface integral will be:

$$J = \iint \sqrt{EG - F^2}\, du\, dv .$$

We will then have to set:

$$\Phi = \sqrt{EG - F^2} = \sqrt{(y_u\, z_v - z_u\, y_v)^2 + (z_u\, x_v - x_u\, z_v)^2 + (x_u\, y_v - y_u\, x_v)^2} .$$

Since J is independent of the choice of the variables u, v, the identities that were derived in § **62** are valid. The quantities:

$$X = \frac{y_u\, z_v - y_v\, z_u}{\sqrt{EG - F^2}}, \qquad Y = \frac{z_u\, x_v - z_v\, x_u}{\sqrt{EG - F^2}}, \qquad Z = \frac{x_u\, y_v - x_v\, y_u}{\sqrt{EG - F^2}},$$

which are defined with the positive square roots, are the direction cosines of the normal n that was defined in § **63**. One further finds that:

$$(10) \qquad \begin{aligned} \Phi_{x_u} &= y_v\, Z - z_v\, Y, & \Phi_{y_u} &= z_v\, X - x_v\, Z, & \Phi_{z_u} &= x_v\, Y - y_v\, X, \\ \Phi_{x_v} &= - y_u\, Z - z_u\, Y, & \Phi_{y_v} &= - z_u\, X - x_u\, Z, & \Phi_{z_v} &= - x_u\, Y - y_u\, X. \end{aligned}$$

The equations of the extremals are then:

$$- P = \frac{\partial (y_v\, Z - z_v\, Y)}{\partial u} - \frac{\partial (y_u\, Z - z_u\, Y)}{\partial v} = 0 , \ldots,$$

or when written out:

$$y_v\, Z_u - z_v\, Y_u - (y_u\, Z_v - z_u\, Y_v) = 0\,,$$

$$z_v\, X_u - x_v\, Z_u - (z_u\, X_v - x_u\, Z_v) = 0\,,$$

$$x_v\, Y_u - y_v\, X_u - (x_u\, Y_v - z_u\, X_v) = 0\,.$$

If one multiplies them by X, Y, Z, resp., and adds them then it will follow that:

(11) $$\begin{vmatrix} X & Y & Z \\ x_v & y_v & z_v \\ X_u & Y_u & Z_u \end{vmatrix} - \begin{vmatrix} X & Y & Z \\ x_u & y_u & z_u \\ X_v & Y_v & Z_v \end{vmatrix} = -P\,X - Q\,Y - R\,Z = 0\,.$$

One sets X, Y, Z equal to their values that were given above in the first determinant and drops the factor $(EG - F^2)^{-1/2}$. One will then get:

$$\begin{vmatrix} y_u & z_u \\ y_v & z_v \end{vmatrix} \begin{vmatrix} y_v & z_v \\ Y_u & Z_u \end{vmatrix} + \begin{vmatrix} z_u & x_u \\ z_v & x_v \end{vmatrix} \begin{vmatrix} z_v & x_v \\ Z_u & X_u \end{vmatrix} + \begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix} \begin{vmatrix} x_v & y_v \\ X_u & Y_u \end{vmatrix} = \begin{vmatrix} F & x_u\, X_u + y_u\, Y_u + z_u\, Z_u \\ G & x_v\, X_v + y_v\, Y_v + z_v\, Z_v \end{vmatrix}.$$

One again sets:

$$\begin{aligned} D &= X\, x_{uu} + Y\, y_{uu} + Z\, z_{uu} = -x_u\, X_u - y_u\, Y_u - z_u\, Z_u\,, \\ D' &= -X_v\, x_u - Y_v\, y_u - Z_v\, z_u = -x_v\, X_u - y_v\, Y_u - z_v\, Z_u\,, \\ D'' &= X\, x_{vv} + Y\, y_{vv} + Z\, z_{vv} = -x_v\, X_v - y_v\, Y_v - z_v\, Z_v\,, \end{aligned}$$

in which the equality of the second and third sum in each row follows from the identities:

$$X\, x_u + Y\, y_u + Z\, z_u = X\, x_v + Y\, y_v + Z\, z_v = 0$$

when one differentiates with respect to u and v. If one then operates on the second determinant in equation (11) in the same way that one did for the first then that will give:

$$\begin{vmatrix} F & -D \\ G & -D' \end{vmatrix} + \begin{vmatrix} F & -D'' \\ E & -D' \end{vmatrix} = 0\,, \qquad D\,G + D''E - 2\,D'F = 0\,.$$

Now, if ρ, ρ' are the radii of principal curvature, taken positively or negatively according to whether the direction n points to the concave or convex side of the associated principal section, resp., then:

(12) $$\frac{1}{\rho} + \frac{1}{\rho'} = \frac{2\,F\,D' - E\,D'' - G\,D}{E\,G - F^2}\,.$$

The extremals of our problem, namely, the minimal surfaces, will then have zero mean curvature.

Problem XV. – Find the equilibrium figure of a heavy fluid under the action of capillary forces.

Let part of the fluid be on one side of a fixed wall, and part of it be bounded by a free surface $\mathfrak{S}$ that is bounded by the curve $\mathfrak{C}$ along the wall. Let *dt* be a spatial element of the fluid, let *n* be the interior normal to its free surface, and let *ds* be an element of the latter. Let all quantities that relate to the wall be distinguished from the corresponding ones that relate to the free surface by only the superscript 0. Let the density of the fluid be 1, and let gravity have the + *z* direction. According to **Gauss**, the potential energy of all forces at work will be:

$$J = a\int ds + b\int ds^0 + c\int z\,d\tau ,$$

in which *a*, *b*, *c* mean constants, and each integral is extended over the entire region whose element appears in the integrand. One must then make *J* an extremum for a given value of the volume:

$$K = \int d\tau .$$

By means of the partial integration in § **63**, one now finds that:

$$\begin{aligned} \int z\,d\tau &= -\int \frac{z^2}{2}\cos(n, z)\,ds - \int \frac{(z^0)^2}{2}\cos(n^0, z)\,ds^0 , \\ \int d\tau &= -\int x\cos(n, x)\,ds - \int x^0\cos(n^0, x)\,ds^0 , \end{aligned}$$

and when one forms the analogous expressions for *y* and *z*, one will for *K* that:

$$3\,K = 3\int d\tau$$

$$= -\int [x\cos(n,x) + y\cos(n,y) + z\cos(n,z)]\,ds - \int [x^0\cos(n^0,x) + y^0\cos(n^0,y) + z^0\cos(n^0,z)]\,ds^0 .$$

If one further introduces coordinates u^0, v^0 and *u*, *v* on the wall and free surface such that the normal *n* that was defined in § **63** points to the interior of the fluid everywhere then one will have, with positive square roots:

$$\begin{aligned} \cos(n, x) &= X = (EG - F^2)^{-1/2}(y_u\,z_v - y_v\,z_u) , \\ \cos(n^0, x) &= X^0 = (E^0G^0 - (F^0)^2)^{-1/2}(y_u^0\,z_v^0 - y_v^0\,z_u^0) , \end{aligned}$$

and analogous equations. That then will yield:

$$J = -\ a\iint_{\mathfrak{S}} du\,dv\sqrt{EG-F^2} - c\iint_{\mathfrak{S}^0} \frac{z^2}{2}(x_u\,y_v - x_v\,y_u)\,du\,dv$$

$$+\ b\iint_{\mathfrak{S}^0} du^0\,dv^0\sqrt{E^0G^0-(F^0)^2} - c\iint_{\mathfrak{S}^0} \frac{(z^0)^2}{2}(x_u^0\,y_v^0 - x_v^0\,y_u^0)\,du^0\,dv^0\,,$$

$$K = -\iint_{\mathfrak{S}} \frac{du\,dv}{3}\begin{vmatrix} x & y & z \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix} - \iint_{\mathfrak{S}} \frac{du^0\,dv^0}{3}\begin{vmatrix} x^0 & y^0 & z^0 \\ x_u^0 & y_u^0 & z_u^0 \\ x_v^0 & y_v^0 & z_v^0 \end{vmatrix}\ .$$

If one varies the free surface in such a way that the separation line $\mathfrak{C}$ remains fixed then the third and fourth summands in the expression for J and the second summand in the expression for K will be constant, so they can then be overlooked. If we now form the expression $PX + QY + RZ$ for the integral J then, from the calculations that were performed in Problem XIV, the term that is multiplied by a will give the contribution:

$$-\ a\frac{ED'' - 2FD' + GD}{\sqrt{EG-F^2}}\ .$$

The calculation for the terms multiplied by c is easy and on the basis for formula (12), one will get:

$$P\,X + Q\,Y + R\,Z = -\sqrt{EG-F^2}\left[a\left(\frac{1}{\rho}+\frac{1}{\rho'}\right)+c\,z\right]\ .$$

Analogously, one gets for the integral K :

$$\bar{P}\,X + \bar{Q}Y + \bar{R}Z = -\sqrt{EG-F^2}\ ,$$

and the equation of the extremals will be:

$$(P+\lambda\,\bar{P})\,X + (Q+\lambda\,\bar{Q})Y + (R+\lambda\,\bar{R})Z = 0\ ,$$

so

$$a\left(\frac{1}{\rho}+\frac{1}{\rho'}\right)+c\,z = -\,\lambda\ .$$

If one sets the gravity constant equal to zero then c will vanish, and one will get the equation:

$$a\left(\frac{1}{\rho}+\frac{1}{\rho'}\right) = -\,\lambda$$

as the solution to the problem of attaining the smallest area for a given volume. The desired surfaces will then have constant mean curvature.

§ 65. – The boundary line is not given, but only subject to conditions. Method of multipliers.

If the boundary line $\mathfrak{C}$ is not given, but only restricted by certain conditions, and if they are fulfilled by a surface $\mathfrak{S}$ that yields the required extremum then the conditions will still be fulfilled when one varies $\mathfrak{S}$ in such a way that the boundary line remains unchanged, e.g., as was done in § **64**. In this case, as well, that will next imply that the desired surface $\mathfrak{S}$ must be an extremal. Assuming that, from § **63**, in the case of an absolute extremum, one now has the formula:

$$\Delta J = \int_{\mathfrak{C}} (U\,dv - V\,du) + \iint_{\mathfrak{S}} du\,dv[\delta x, \delta x_u, \ldots, \delta z_v]_2 \ .$$

If one seeks a relative extremum, and $\bar{U}$, $\bar{V}$ have the same meaning for the integral K that U, V have for J then:

$$\Delta\,(J + \lambda\,K) = \int_{\mathfrak{C}} [(U + \lambda\bar{U})\,dv - (V + \lambda\bar{V})\,du] + \iint_{\mathfrak{S}} du\,dv[\delta x, \delta x_u, \ldots, \delta z_v]_2 \ , \tag{13}$$

and that expression will go to ΔJ when ΔK vanishes. Now should an extremum be present, the quantity ΔJ would have to have a fixed sign for all variations that are compatible with the prescribed conditions. On the basis of § **7**, and under the assumption on the constants ε that:

$$\delta x = \sum_{\mathfrak{a}} \varepsilon_{\mathfrak{a}}\,\xi_{\mathfrak{a}} \ , \qquad \delta y = \sum_{\mathfrak{a}} \varepsilon_{\mathfrak{a}}\,\eta_{\mathfrak{a}} \ , \qquad \delta z = \sum_{\mathfrak{a}} \varepsilon_{\mathfrak{a}}\,\zeta_{\mathfrak{a}} \ ,$$

that requirement will imply new necessary conditions for the extremum only when the restrictions on the boundary line $\mathfrak{C}$ imply relations of the form:

$$[\varepsilon]_1 = 0 \ .$$

The most important cases are the ones in which the curve $\mathfrak{C}$ is constrained by a fixed surface, or an integral that is taken along that curve has a prescribed value. In the former case, one has an equation of the form:

$$p\ \delta x + q\ \delta y + r\ \delta z + [\delta x,\ \delta y,\ \delta z]_2 = 0 \ . \tag{14}$$

One can multiply that by an undetermined factor l (§ **13**) and integrate. That will then yield:

$$\Delta J = \int_{\mathfrak{C}} [U\,dv - V\,du + l\,(p\,\delta x + q\,\delta y + r\,\delta z)\,dt] + \int_{\mathfrak{C}} [\delta x, \delta y, \delta z]_2\,dt + \iint_{\mathfrak{S}} du\,dv [\delta x, \delta y, \ldots]_2 \;.$$

Now, since U, V are linear in the variations δx, δy, δz, one can determine the arbitrary factor l in such a way that a variation under the first integral sign that can be expressed in terms of the other ones as a result of equation (14) – say, δz – will drop out. The factors of the variations δx and δy must also vanish then, as long as they are not subject to any further restrictions. Namely, if one sets:

$$\delta x = \varepsilon\,\xi\,, \qquad \delta y = \varepsilon\,\eta$$

and let ξ, η denote arbitrary functions of t then the second and third integral in the expression for ΔJ will yield only the expression $[\varepsilon]_2$, and in order for ΔJ to have a fixed sign, the term that is linear in ε must vanish. One can then simply impose the equation:

$$(15) \qquad 0 = \int_{\mathfrak{C}} [U\,dv - V\,du + l\,(p\,\delta x + q\,\delta y + r\,\delta z)\,dt]$$

and treat it like the equation $\delta J = 0$ for an absolute extremum of the simple integral.

Furthermore, should an integral:

$$L = \int_{\mathfrak{C}} W\left(x, y, z, \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) dt$$

assume a prescribed value along the curve $\mathfrak{C}$, then the argument in § **32** will imply, with no modification. that a constant μ exists for which the equation:

$$\int_{\mathfrak{C}} (U\,dv - V\,du + \mu\,\delta W\,dt) = 0$$

is true for arbitrary variations δx, …

Example. – **Fourier**'s equations for heat conduction as solutions of an isoperimetric problem.

Let ds be the element of a given planar surface patch $\mathfrak{M}$, let dt be the arc element of its periphery, and let n be the interior normal to the latter. The integral:

$$J = \int_{\mathfrak{M}} ds \left[\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2\right]$$

shall be minimized for prescribed values of:

$$K = \int_{\mathfrak{M}} f^2 ds \;, \quad L = \int_{\mathfrak{L}} f^2 dt \;.$$

If one wishes to construct the model that has been used up to now then f must be regarded as the third rectangular space coordinate. $\mathfrak{C}$ is a space curve whose projection onto the xy-plane is $\mathfrak{L}$. Obviously, one has:

$$\delta J + \lambda\, \delta K = \delta \int_{\mathfrak{M}} ds \left[\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \lambda f^2 \right]$$

$$= 2\int_{\mathfrak{M}} ds \left(-\frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial y^2} + \lambda f \right) \delta f - 2\int_{\mathfrak{C}} dt \left(\frac{\partial f}{\partial x} \cos(n, x) + \frac{\partial f}{\partial y} \cos(n, y) \right) \delta f \;.$$

Therefore, when one next fixes $\mathfrak{C}$, that will give:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \lambda f \,.$$

If that equation is fulfilled and one varies $\mathfrak{C}$ then one will have:

$$\Delta\,(J + \lambda\, K) = -\, 2\int_{\mathfrak{L}} dt \,\frac{\partial f}{\partial n}\, \delta f + \int_{\mathfrak{M}} [\delta f , \delta f_x , \delta f_y]_2 \, ds \;.$$

Now, one has:

$$\delta L = 2\int_{\mathfrak{L}} f \,\delta f \, dt \,,$$

so that will give:

$$\int_{\mathfrak{L}} dt \,\delta f \left(\frac{\partial f}{\partial n} + \mu f \right) = 0 \,, \qquad \frac{\partial f}{\partial n} + \mu f = 0$$

as the last necessary condition for an extremum.

Problem XIV. – One finds immediately from the formulas that were given in § **64** that:

$$U\, dv - V\, du = \delta x\, [(y_v\, Z - z_v\, Y)\, dv + (y_u\, Z - z_u\, Y)\, du] + \ldots = \begin{vmatrix} \delta x & \delta y & \delta z \\ dx & dy & dz \\ X & Y & Z \end{vmatrix} .$$

Hence, if $\mathfrak{C}$ belongs to a given surface, which implies the relation:

$$p\, \delta x + q\, \delta y + r\, \delta z + [\delta x, \delta y, \delta z]_2 = 0 \,,$$

then one must set the coefficients of the variation in equation (15) equal to zero:

$$Z\,dy - Y\,dz + l\,p = 0\,, \qquad X\,dz - Z\,dx + l\,q = 0\,, \qquad Y\,dx - X\,dy + lr = 0\,.$$

Since l obviously cannot vanish, it follows from this that:

$$p\,X + q\,Y + r\,Z = 0\,,$$

i.e., the given surface must be perpendicular to the desired minimal surface.

Problem XV (§ **64**). – If one varies $\mathfrak{C}$ then the last summands in the expressions J and K will also be variable.

One next gets:

$$3(\bar{U}\,dv - \bar{V}\,du) = \delta x\,(z\,dy - y\,dz) + \delta x^0(z^0dy^0 - y^0dz^0) + \ldots$$

Now, one has:

(16) $$x = x^0, \qquad \delta x = \delta x^0, \qquad y = \delta y^0, \qquad \ldots$$

The direction of integration in which one has to take the simple integral that appears in the difference ΔK is, however, opposite to the one on the wall and the free surface, since it must go around the direction n in the positive sense each time. One will then have to set:

(17) $$dx = -\,dx^0, \qquad dy = -\,dy^0, \qquad dz = -\,dz^0,$$

such that the expression above will vanish. Similar statements are true for those parts of the expression $U\,dv - V\,du$ that originate in the second and fourth summands of the quantity J. They are:

$$\frac{c\,z^2}{2}(-\delta x\,dy + \delta y\,dz) + \frac{c\,(z^0)^2}{2}(-\delta x^0dy^0 + \delta y^0dz^0)\,,$$

and from (16), (17), that will give a sum of zero. Thus, all that remains in the expression (13) under the simple integral sign is what originates in the terms that are multiplied by a and b:

$$\delta x\,[a\,(y_v\,Z - z_v\,Y)\,dv - a\,(z_u\,Y - y_u\,Z)\,du + b(y_v^0\,Z^0 - z_v^0\,Y^0)\,dv^0 - b(z_u^0\,Y^0 - y_u^0\,Z^0)\,du^0] + \ldots$$

$$= a\begin{vmatrix} \delta x & \delta y & \delta z \\ dx & dy & dz \\ X & Y & Z \end{vmatrix} + b\begin{vmatrix} \delta x & \delta y & \delta z \\ -dx & -dy & -dz \\ X^0 & Y^0 & Z^0 \end{vmatrix}\,.$$

Now let *dt* be an element of arc-length along the curve $\mathfrak{C}$, while ν and ν^0 are those of their normals that contact the wall and the free surface. One will then have:

$$\cos(\nu, x) = Z\frac{dy}{dt} - Y\frac{dz}{dt}, \qquad \cos(\nu^0, x) = Z^0\frac{dy}{dt} - Y^0\frac{dz}{dt}, \quad \ldots,$$

and one can say that the integral:

$$\int_{\mathfrak{C}} dt\,\{a[\delta x\cos(\nu,x)+\delta y\cos(\nu,y)+\delta z\cos(\nu,z)]-b[\delta x\cos(\nu^0,x)+\delta y\cos(\nu^0,y)+\delta z\cos(\nu^0,z)]\}$$

must vanish, as long as:

$$X^0\,\delta x + Y^0\,\delta y + Z^0\,\delta z = 0\,.$$

It then follows that with a suitable choice of l :

$$a\cos(\nu, x) - b\cos(\nu^0, x) = l\,X^0,$$
$$a\cos(\nu, y) - b\cos(\nu^0, y) = l\,Y^0,$$
$$a\cos(\nu, z) - b\cos(\nu^0, z) = l\,Z^0.$$

Now, the direction ν^0 is tangential to the fixed wall, so:

$$X^0\cos(\nu^0, x) + Y^0\cos(\nu^0, y) + Z^0\cos(\nu^0, z) = 0\,.$$

The previous equations then give:

$$a\cos(\nu, \nu^0) - b = 0\,,$$

i.e., the angle of inclination of the free surface with respect to the wall is constant.

§ 66. – Definitions and properties of three quantities Φ_{11}, Φ_{12}, Φ_{13} by which certain second derivatives of the integrands can be represented.

Let:

$$e_{\mathfrak{ab}} = e_{\mathfrak{ba}} \qquad (\mathfrak{a}, \mathfrak{b} = 1, 2, 3)$$

be any symmetric system of quantities. If these equations exist:

$$\text{(18)} \qquad \begin{aligned} e_{\mathfrak{a}1}\,a + e_{\mathfrak{a}2}\,b + e_{\mathfrak{a}3}\,c &= 0, \\ e_{\mathfrak{a}1}\,\alpha + e_{\mathfrak{a}2}\,\beta + e_{\mathfrak{a}3}\,\gamma &= 0, \end{aligned} \qquad (\mathfrak{a} = 1, 2, 3)$$

and one sets:

$$b\,\gamma - c\,\beta = \xi\,, \qquad c\,\alpha - a\,\gamma = \eta\,, \qquad a\,\beta - b\,\alpha = \zeta\,,$$

and if at least one of those quantities is non-zero then it will follow that:

$$e_{\alpha 1} : e_{\alpha 2} : e_{\alpha 3} = \xi : \eta : \zeta\,,$$

or also:

$$e_{11} : e_{12} : e_{13} = \xi^2 : \xi\eta : \xi\zeta\,, \quad e_{21} : e_{22} : e_{23} = \eta\,\xi : \eta^2 : \eta\,\zeta\,, \quad e_{31} : e_{32} : e_{33} = \zeta\xi : \zeta\,\eta : \zeta^2\,.$$

That will then give a finite quantity m such that the following equations exist:

$$e_{11} = m\,\xi^2\,, \quad e_{12} = m\,\xi\,\eta\,, \quad e_{13} = m\,\xi\,\zeta\,, \quad e_{22} = m\,\eta^2\,, \quad e_{23} = m\,\eta\,\zeta\,, \quad e_{33} = m\,\zeta^2\,.$$

We shall make use of those general remarks in regard to the equations that follow from equations (5) in § **62** upon differentiating with respect to $x_u , y_u , \ldots, z_v$. We set:

$$x_u = a\,, \quad y_u = b\,, \quad z_u = c\,, \quad x_v = \alpha\,, \quad y_v = \beta\,, \quad z_v = \gamma\,,$$

to abbreviate. As a result of the assumption that was introduced in § **63** in regard to the quantities (6), not all three of ξ, η, ζ will have the value zero. If we differentiate equations (5) with respect to a and α then we will get the eight equations:

$$(19)\qquad \begin{aligned}
0 &= a\,\Phi_{aa} + b\,\Phi_{ba} + c\,\Phi_{ca}\,, \\
\Phi_\alpha &= a\,\Phi_{a\alpha} + b\,\Phi_{b\alpha} + c\,\Phi_{c\alpha}\,, \\
0 &= \alpha\,\Phi_{aa} + \beta\,\Phi_{ba} + \gamma\,\Phi_{ca}\,, \\
0 &= \alpha\,\Phi_{a\alpha} + \beta\,\Phi_{b\alpha} + \gamma\,\Phi_{c\alpha} + \Phi_a\,, \\[1ex]
\Phi_a &= \alpha\,\Phi_{\alpha a} + \beta\,\Phi_{\beta a} + \gamma\,\Phi_{\gamma a}\,, \\
0 &= \alpha\,\Phi_{\alpha\alpha} + \beta\,\Phi_{\beta\alpha} + \gamma\,\Phi_{\gamma\alpha}\,, \\
0 &= a\,\Phi_{\alpha a} + b\,\Phi_{\alpha a} + c\,\Phi_{\gamma a} + \Phi_\alpha\,, \\
0 &= a\,\Phi_{\alpha\alpha} + b\,\Phi_{\beta\alpha} + c\,\Phi_{\gamma\alpha}\,.
\end{aligned}$$

These will be followed by sixteen others that arise when one simultaneously replaces the second subscripts a and α with b and β or c and γ and makes the same substitution in the first derivatives that occur. The latter are easily eliminated. One will obviously get the equations:

$$\begin{aligned}
a\,(\Phi_{a\alpha} + \Phi_{\alpha a}) + b\,(\Phi_{b\alpha} + \Phi_{\beta a}) + c\,(\Phi_{c\alpha} + \Phi_{\gamma a}) &= 0\,, \\
\alpha\,(\Phi_{a\alpha} + \Phi_{\alpha a}) + \beta\,(\Phi_{b\alpha} + \Phi_{\beta a}) + \chi\,(\Phi_{c\alpha} + \Phi_{\gamma a}) &= 0\,,
\end{aligned}$$

from which, four similar ones will once more arise with the substitutions that were given above. The six equations that are thus obtained define a system of the form (18) when one gives the quantities $e_{\mathfrak{ab}}$ the following values:

$$\begin{array}{ccc} 2\Phi_{a\alpha}, & \Phi_{b\alpha}+\Phi_{\beta a}, & \Phi_{c\alpha}+\Phi_{\gamma a}, \\ \Phi_{a\beta}+\Phi_{\alpha b}, & 2\Phi_{b\beta}, & \Phi_{c\beta}+\Phi_{\gamma b}, \\ \Phi_{a\gamma}+\Phi_{\alpha c}, & \Phi_{b\gamma}+\Phi_{\beta c}, & 2\Phi_{c\gamma}. \end{array}$$

When we introduce the values:

$$\xi = X\sqrt{EG-F^2}, \ldots,$$

it will then follow from the general remark above that a certain quantity Φ_{12} will satisfy the following equations:

$$\Phi_{a\alpha} = \Phi_{12}\,X^2, \qquad \Phi_{b\alpha}+\Phi_{\beta\alpha} = 2\,\Phi_{12}\,X\,Y, \qquad \Phi_{c\alpha}+\Phi_{\gamma a} = 2\,\Phi_{12}\,X\,Z,$$

$$\Phi_{a\beta}+\Phi_{\alpha b} = 2\,\Phi_{12}\,Y\,X, \qquad \Phi_{b\beta} = \Phi_{12}\,Y^2, \qquad \Phi_{c\beta}+\Phi_{\gamma b} = 2\,\Phi_{12}\,Y\,Z,$$

$$\Phi_{a\gamma}+\Phi_{\alpha c} = 2\,\Phi_{12}\,Z\,X, \qquad \Phi_{b\gamma}+\Phi_{c\beta} = 2\,\Phi_{12}\,Z\,Y, \qquad \Phi_{c\gamma} = \Phi_{12}\,Z^2.$$

Equations (19), which are free of the first derivatives of the function Φ, and the ones that are derived from them, which are twelve in number, likewise decompose into two systems of the form (18), in which the quantities $e_{\mathfrak{ab}}$ are replaced with one of the following two systems of quantities:

$$\begin{array}{ccccccc} \Phi_{aa}, & \Phi_{ab}, & \Phi_{ac}, & \qquad & \Phi_{\alpha\alpha}, & \Phi_{\alpha\beta}, & \Phi_{\alpha\gamma}, \\ \Phi_{ba}, & \Phi_{bb}, & \Phi_{bc}, & & \Phi_{\beta\alpha}, & \Phi_{\beta\beta}, & \Phi_{\beta\gamma}, \\ \Phi_{ca}, & \Phi_{cb}, & \Phi_{cc}, & & \Phi_{\gamma\alpha}, & \Phi_{\gamma\beta}, & \Phi_{\gamma\gamma}. \end{array}$$

There are then quantities Φ_{11}, Φ_{22} such that the following equations are true:

$$\Phi_{aa} = \Phi_{11}\,X^2, \quad \Phi_{ab} = \Phi_{11}\,X\,Y, \quad \Phi_{ac} = \Phi_{11}\,X\,Z, \quad \Phi_{bb} = \Phi_{11}\,Y^2, \quad \Phi_{bc} = \Phi_{11}\,Y\,Z, \quad \Phi_{cc} = \Phi_{11}\,Z^2,$$

$$\Phi_{\alpha\alpha} = \Phi_2\,X^2, \quad \Phi_{\alpha\beta} = \Phi_{22}\,Y^2, \quad \Phi_{\gamma\gamma} = \Phi_{22}\,Z^2, \quad \Phi_{\beta\gamma} = \Phi_{22}\,Y\,Z, \quad \Phi_{\gamma\alpha} = \Phi_{22}\,Z\,X, \quad \Phi_{\alpha\beta} = \Phi_{22}\,X\,Y.$$

When one recalls the identity:

$$X^2+Y^2+Z^2 = 1,$$

one can obviously conclude from the latter equations and the analogous equations above that:

(20) $\Phi_{11} = \Phi_{aa} + \Phi_{bb} + \Phi_{cc}$, $\Phi_{22} = \Phi_{\alpha\alpha} + \Phi_{\beta\beta} + \Phi_{\gamma\gamma}$, $\Phi_{12} = \Phi_{a\alpha} + \Phi_{b\beta} + \Phi_{c\gamma}$.

Example. – If one sets, as in Problem XIV:

$$\Phi = \sqrt{EG - F^2} \, ,$$

then one will immediately get from equations (10) that:

$$\frac{\partial^2 \Phi}{\partial x_u^2} = \Phi_{aa} = \frac{y_v^2 + z_v^2}{\Phi} - \frac{(y_v Z - z_v Y)^2}{\Phi},$$

$$\frac{\partial^2 \Phi}{\partial x_u \, \partial x_v} = \Phi_{a\alpha} = \frac{-y_u \, y_v - z_u \, z_v}{\Phi} - \frac{(y_v Z - z_v Y)(y_u Z - z_u Y)}{\Phi},$$

$$\frac{\partial^2 \Phi}{\partial x_v^2} = \Phi_{\alpha\alpha} = \frac{y_u^2 + z_u^2}{\Phi} - \frac{(y_u Z - z_u Y)^2}{\Phi}.$$

In each of those expressions, one simultaneously permutes the symbols x, y, z and X, Y, Z cyclically. One will then get Φ_{bb}, Φ_{cc}, $\Phi_{b\beta}$, $\Phi_{c\gamma}$, $\Phi_{\beta\beta}$, $\Phi_{\gamma\gamma}$, and equations (20) yield:

$$\Phi_{11} = \frac{G}{\sqrt{EG - F^2}}, \qquad \Phi_{12} = \frac{-F}{\sqrt{EG - F^2}}, \qquad \Phi_{22} = \frac{E}{\sqrt{EG - F^2}}.$$

The quantities Φ_{11}, Φ_{12}, Φ_{22} obviously depend upon the choice of the system of variables u, v, but have certain properties that are independent of it. In particular, when the form:

$$\psi = \Phi_{11} h^2 + 2\Phi_{12} hk + \Phi_{22} k^2$$

is definite, we would like to show that the same thing is true for the corresponding form ψ^0 to which one will arrive with another system of independent variables r, s. Namely, the form ψ will be definite (e.g., when X is non-vanishing) if and only if that is true for the form:

$$\theta = \Phi_{aa} h^2 + 2\Phi_{a\alpha} hk + \Phi_{\alpha\alpha} k^2 = X^2 \psi \, .$$

If one now sets:

$$\frac{\partial x}{\partial r} = l, \qquad \frac{\partial y}{\partial r} = m, \qquad \frac{\partial z}{\partial r} = n, \qquad \frac{\partial x}{\partial s} = \lambda, \qquad \frac{\partial y}{\partial s} = \mu, \qquad \frac{\partial z}{\partial s} = \nu,$$

then one will have:

$$l = a\, u_r + \alpha\, v_r\,, \quad \lambda = a\, u_s + \alpha\, v_s\,,$$

as well as two similar equations, which do not, however, contain a and α, and the following identity will be true as a result of the assumption (1) that was introduced for the function Φ :

$$\Phi\,(x, y, z, a, b, c, \alpha, \beta, \gamma) = \Phi\,(x, y, z, l, m, n, \lambda, \mu, \nu)\,\frac{\partial(r,s)}{\partial(u,v)}\,,$$

or when written more briefly:

$$\Phi = \Phi^0 \cdot \rho\,.$$

Since Φ^0 includes the quantities a and α only in the arguments l, λ, it will follow from this that:

$$\begin{aligned}
\Phi_{aa} &= \left(\Phi_l^0\,\frac{\partial l}{\partial a} + \Phi_\lambda^0\,\frac{\partial \lambda}{\partial a}\right)\rho \;=\; \rho(\Phi_l^0\, u_r + \Phi_\lambda^0\, u_s)\,,\\
\Phi_{\alpha a} &= \rho(\Phi_{ll}^0\, u_r^2 + 2\Phi_{l\lambda}^0\, u_r\, u_s + \Phi_{\lambda\lambda}^0\, u_s^2)\,,\\
\Phi_{a\alpha} &= \rho(\Phi_{ll}^0\, u_r\, u_s + \Phi_{l\lambda}^0\,(u_r\, v_s + u_s\, v_r) + \Phi_{\lambda\lambda}^0\, u_s\, v_s)\,,\\
\Phi_{\alpha\alpha} &= \rho(\Phi_{ll}^0\, v_r^2 + 2\Phi_{l\lambda}^0\, v_r\, v_s + \Phi_{\lambda\lambda}^0\, v_s^2)\,.
\end{aligned}$$

The form θ will then be identical to the form:

$$\rho(\Phi_{ll}^0\, h_0^2 + 2\Phi_{l\lambda}^0\, h_0\, k_0 + \Phi_{\lambda\lambda}^0\, k_0^2) \;=\; \rho\,\theta^0\,,$$

and the forms θ and θ^0 are always both definite. However, the form ψ^0 differs from the latter by only a positive factor, which proves the asserted statement. The signs of ψ and ψ^0 are identical or different according to whether the functional determinant ρ is positive or negative, respectively.

§ 67. – Second variation.

As was done before in § **28**, the double sum of the terms in the **Taylor** development of a function $f\,(x + h, y + k, \ldots)$ that are quadratic in $h, k, \ldots$ will arise from the linear ones when one applies the operation:

$$h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} + \cdots$$

and considers $h, k, \ldots$ to be constant in it. The corresponding operation for the **Taylor** development of the quantity:

$$\text{(21)} \qquad \Phi\,(x + \delta x, y + \delta y, z + \delta z, x_u + \delta x_u\,, \ldots)$$

is obviously:

$$\delta x \frac{\partial}{\partial x} + \cdots + \delta x_u \frac{\partial}{\partial x_u} + \cdots ,$$

so it is identical to the operation δ itself, such that one can write the double sum of the terms in the expression (21) that are quadratic in the variations as:

$$\delta(\delta\Phi) = \delta(\Phi_x\,\delta x + \ldots + \Phi_{x_u}\delta x_u + \ldots) ,$$

in which one sets:

$$\delta(\delta x) = \delta(\delta x_u) = \ldots = 0 .$$

We shall denote the expression that is obtained by $\delta^2\Phi$ and call it the *second variation* of Φ. We let the second variation of the integral J be the double integral of it over the surface $\mathfrak{S}$ and set:

$$\delta^2 J = \iint_{\mathfrak{S}} \delta^2\Phi\, du\, dv = \iint_{\mathfrak{S}} \delta(\delta\Phi)\, du\, dv . \tag{22}$$

One will obviously get:

$$\Delta J = \delta J + \tfrac{1}{2}\delta^2 J + \iint_{\mathfrak{S}} du\, dv [\delta x, \ldots, \delta z_v]_3$$

then. Now, the δ symbol can be permuted with integration and differentiation with respect to u and v. If one applies that remark to the outer symbol δ that appears in the right-hand side of formula (22) then that will give:

$$\delta^2 J = \delta \iint \delta\Phi\, du\, dv = \delta(\delta J) ,$$

and since, from § **63**, one can write:

$$\delta J = \int_{\mathfrak{C}} (U\, dv - V\, du) + \iint_{\mathfrak{S}} du\, dv (P\,\delta x + Q\,\delta y + R\,\delta z) , \tag{23}$$

it will follow by calculation that one has the easily-verified formula:

$$\delta^2 J = \int_{\mathfrak{C}} (\delta U\, dv - \delta V\, du) + \iint_{\mathfrak{S}} du\, dv (\delta P\,\delta x + \delta Q\,\delta y + \delta R\,\delta z) .$$

If the quantities δx, δy, δz vanish along the boundary line $\mathfrak{C}$ then the same thing will be true of δU and δV, since each term in those expressions include one of those three variations as a factor, and what will remain is:

$$\delta^2 J = \iint_{\mathfrak{S}} du\, dv (\delta P\,\delta x + \delta Q\,\delta y + \delta R\,\delta z) .$$

We shall reshape that expression under the assumption that:

$$\delta x = X\,\omega\,, \qquad \delta y = Y\,\omega\,, \qquad \delta z = Z\,\omega\,. \tag{24}$$

Such a variation is called a *normal variation*, by analogy with the concept that was introduced in § **24**. For such a thing, the connecting line from the point (x, y, z) to the $(x + \delta x, y + \delta y, z + \delta z)$ will intersect the surface $\mathfrak{S}$ perpendicularly, and the distance between the points will be $\pm\,\omega$ according to whether the direction from coincides with the normal n or is opposite to it. Therefore, when δx, δy, δz vanish on the boundary line, the same thing will be true for ω. With the notation:

$$\Omega = X\,\delta P + Y\,\delta Q + Z\,\delta R\,,$$

one will have:

$$\delta^2 J = \iint_{\mathfrak{S}} \Omega\,du\,dv\,. \tag{25}$$

As one easily sees, the expression Ω is linearly homogeneous with respect to the quantity ω and its derivatives of first and second order. If one poses the partial differential equation:

$$\Omega = 0 \tag{26}$$

for ω then one will get from formulas (23) and (25) that:

$$\Delta J = \iint_{\mathfrak{S}} du\,dv\,[\omega, \omega_u, \omega_v]_3$$

when the surface considered is an extremal. Hence, if the boundary curve $\mathfrak{C}$ is defined by the vanishing of an integral of equation (25) then since $-\,\omega$ will also be a solution of that equation, one must generally expect that ΔJ can be positive, as well as negative. An extremal patch will then no longer produce an extremum for the integral J, in general, when it contains a closed curve along which a certain integral of equation (26) vanishes.

In order to make the last statement clearer, we differentiate equations (24), which will give us:

$$\delta a = \omega_u\,X + \omega\,X_u\,, \qquad \delta b = \omega_u\,Y + \omega\,Y_u\,, \qquad \delta c = \omega_u\,Z + \omega\,Z_u\,,$$
$$\delta \alpha = \omega_v\,X + \omega\,X_v\,, \qquad \delta \beta = \omega_v\,Y + \omega\,Y_v\,, \qquad \delta \gamma = \omega_v\,Z + \omega\,Z_v\,.$$

We will then introduce an abbreviated notation for three-term linear expression whose arguments are δx, δy, δz, or X, Y, Z, or the derivatives of one of those systems of quantities with respect to u or v. Chiefly two derivatives of Φ appear as coefficients. We shall denote such a trinomial by putting its first term in brackets and establish that in all coefficients of a trinomial, the first subscript of Φ will always be the same, but the second one shall run through the values of x, y, z, or a, b, c, or α, β, γ. One then has, e.g.:

$$[\Phi_{a\alpha}\,\delta x] = \Phi_{a\alpha}\,\delta x + \Phi_{a\beta}\,\delta y + \Phi_{a\gamma}\,\delta z\,,$$
$$[\Phi_{\beta a}\,X_u] = \Phi_{\beta a}\,X_u + \Phi_{\beta b}\,Y_u + \Phi_{\beta c}\,Z_u\,.$$

In the same way, one can also represent trinomials whose coefficients arise from the ones considered up to now upon differentiating with respect to u or v, e.g.:

$$\left[\frac{\partial \Phi_{a\alpha}}{\partial u} X\right] = \frac{\partial \Phi_{a\alpha}}{\partial u} X + \frac{\partial \Phi_{a\beta}}{\partial u} Y + \frac{\partial \Phi_{a\gamma}}{\partial u} Z\,.$$

Obviously, the second index of Φ inside of the bracket can always be only a or x or α, while the first one can be each of the nine arguments that appear in Φ.

With that notation, since:

$$P = \Phi_x - \frac{\partial \Phi_a}{\partial u} - \frac{\partial \Phi_\alpha}{\partial u}\,,$$

one will have the equation:

$$\begin{aligned}\delta P = {} & [\Phi_{xx}\,\delta x] + [\Phi_{xa}\,\delta a] + [\Phi_{x\alpha}\,\delta\alpha] \\ & - \frac{\partial}{\partial u}\{[\Phi_{ax}\,\delta x] + [\Phi_{aa}\,\delta a] + [\Phi_{a\alpha}\,\delta\alpha]\} \\ & - \frac{\partial}{\partial v}\{[\Phi_{\alpha x}\,\delta x] + [\Phi_{\alpha a}\,\delta a] + [\Phi_{\alpha\alpha}\,\delta\alpha]\}\,.\end{aligned}$$

One will get the quantities δQ and δR from this when one simultaneously replaces the symbols x, a, α in the first index with y, b, β or z, c, γ, but leaves all of the other ones unchanged. We now focus on the terms that yield the trinomial:

$$[\Phi_{xa}\,\delta a] - \frac{\partial}{\partial u}[\Phi_{ax}\,\delta x]\,. \tag{27}$$

Obviously, one has:

$$\frac{\partial}{\partial u}[\Phi_{ax}\,\delta x] = \left[\frac{\partial \Phi_{ax}}{\partial u}\delta x\right] + [\Phi_{ax}\,\delta a]\,.$$

If we drop the terms that include the factor ω after the substitution (24) then the sum of (27) and its analogues for δQ and δR will give the contribution:

$$X\,\{[\Phi_{xa}\,\delta a] - [\Phi_{ax}\,\delta a]\} + Y\,\{[\Phi_{ya}\,\delta a] - [\Phi_{bx}\,\delta a]\} + Z\,\{[\Phi_{za}\,\delta a] - [\Phi_{cx}\,\delta a]\}$$

to the sum Ω, or when one once more drops the terms with the factor ω :

$$\omega_u \{X([\Phi_{xa}\, X] - [\Phi_{ax}\, X]\} + Y([\Phi_{ya}\, X] - [\Phi_{bx}\, X]) + Z([\Phi_{za}\, X] - [\Phi_{cx}\, X])\},$$

and that expression will vanish, since, e.g., X^2 and $X\,Y$ are endowed with the factors:

$$\Phi_{xa} - \Phi_{ax}, \qquad \Phi_{xb} - \Phi_{ay} + \Phi_{ya} - \Phi_{bx},$$

whose values are zero. The term (27), and likewise the term:

$$[\Phi_{x\alpha}\,\delta\alpha] - \frac{\partial}{\partial v}[\Phi_{\alpha x}\,\delta x],$$

along with their analogues in δQ and δR, will then give a contribution to Ω that includes ω, multiplied by a quantity that is independent of ω. The same thing will be true for the terms $[\Phi_{xx}\,\delta x]$, $[\Phi_{yx}\,\delta x]$, $[\Phi_{zx}\,\delta x]$. One can then set:

$$\begin{aligned}\Omega - \omega(\ldots) = -X&\left\{\frac{\partial}{\partial u}([\Phi_{aa}\,\delta a] + [\Phi_{a\alpha}\,\delta\alpha]) + \frac{\partial}{\partial v}([\Phi_{\alpha a}\,\delta a] + [\Phi_{\alpha\alpha}\,\delta\alpha])\right\}\\ -Y&\left\{\frac{\partial}{\partial u}([\Phi_{ba}\,\delta a] + [\Phi_{b\alpha}\,\delta\alpha]) + \frac{\partial}{\partial v}([\Phi_{\beta a}\,\delta a] + [\Phi_{\beta\alpha}\,\delta\alpha])\right\}\\ -Z&\left\{\frac{\partial}{\partial u}([\Phi_{ca}\,\delta a] + [\Phi_{c\alpha}\,\delta\alpha]) + \frac{\partial}{\partial v}([\Phi_{\gamma a}\,\delta a] + [\Phi_{\gamma\alpha}\,\delta\alpha])\right\}.\end{aligned}$$

If one employs equations (24) then the factor of $-X$ will be:

$$\begin{aligned}&\frac{\partial}{\partial u}\{\omega[\Phi_{aa}\,X_u] + \omega_u[\Phi_{a\alpha}\,X]) + \omega[\Phi_{a\alpha}\,X_v] + \omega_v[\Phi_{\alpha\alpha}\,X])\}\\ +&\frac{\partial}{\partial v}\{\omega[\Phi_{\alpha a}\,X_u] + \omega_u[\Phi_{\alpha\alpha}\,X]) + \omega[\Phi_{\alpha\alpha}\,X_v] + \omega_v[\Phi_{\alpha\alpha}\,X])\}.\end{aligned}$$

However, the definition of the quantity Φ_{11} (§ **66**) gives:

$$\begin{aligned}[\Phi_{aa}\,X_u] &= \Phi_{11}(X^2 X_u + X\,Y\,Y_u + X\,Z\,Z_u) = 0,\\ [\Phi_{aa}\,X] &= \Phi_{11}(X^3 + X\,Y^2 + X\,Z^2) = \Phi_{11}\,X.\end{aligned}$$

One likewise gets:

$$[\Phi_{\alpha\alpha}\,X_v] = 0, \qquad [\Phi_{\alpha\alpha}\,X] = \Phi_{22}\,X.$$

The factor of $-X$ is then:

$$\frac{\partial}{\partial u}\{\Phi_{11}\,\omega_u\,X + \omega\,[\Phi_{a\alpha}\,X_v] + \omega_v\,[\Phi_{a\alpha}\,X]\}$$

(28)

$$+ \frac{\partial}{\partial v}\{\Phi_{22}\,\omega_v\,X + \omega\,[\Phi_{a\alpha}\,X_u] + \omega_u\,[\Phi_{a\alpha}\,X]\}\ .$$

Furthermore, from § **66**, one has the identity:

$$[\Phi_{a\alpha}\,X] + [\Phi_{\alpha\alpha}\,X] = 2\,\Phi_{12}\,X\ .$$

The expression (28) can then be written as follows:

$$X\,(\Phi_{11}\,\omega_{uu} + 2\,\Phi_{12}\,\omega_{uv} + \Phi_{22}\,\omega_{vv})$$

$$+\ \omega_u\left\{[\Phi_{a\alpha}\,X_v] + \frac{\partial}{\partial v}[\Phi_{\alpha\alpha}\,X] + \frac{\partial(\Phi_{11}X)}{\partial u}\right\} + \omega_v\left\{[\Phi_{\alpha\alpha}\,X_u] + \frac{\partial}{\partial u}[\Phi_{a\alpha}\,X] + \frac{\partial(\Phi_2 X)}{\partial v}\right\} + \omega(\cdots)\ .$$

A further shortening yields the identity:

$$[\Phi_{a\alpha}\,X_v] + [\Phi_{aa}\,X_v] = 2\,\Phi_{a\alpha}\,X_v + (\Phi_{a\beta} + \Phi_{\alpha b})\,Y_v + (\Phi_{a\gamma} + \Phi_{\alpha c})\,Z_v$$

$$= 2\,\Phi_{12}\,(X^2 X_v + X\,Y\,Y_v + X\,Z\,Z_v) = 0\ ,$$

and the analogous one:

$$[\Phi_{a\alpha}\,X_u] + [\Phi_{aa}\,X_u] = 0\ .$$

The factor of $-X\,\omega_u$ will then be simply:

$$\left[\frac{\partial\Phi_{\alpha\alpha}}{\partial v}X\right] + \frac{\partial(\Phi_{11}\,X)}{\partial u}\,,$$

and in the total sum Ω, ω_u will appear with the factor:

$$-X\,\frac{\partial(\Phi_{11}\,X)}{\partial u} - Y\,\frac{\partial(\Phi_{11}\,Y)}{\partial u} - Z\,\frac{\partial(\Phi_{11}\,Z)}{\partial u} - X\left[\frac{\partial\Phi_{\alpha a}}{\partial v}X\right] - Y\left[\frac{\partial\Phi_{\beta a}}{\partial v}X\right] - Z\left[\frac{\partial\Phi_{\gamma a}}{\partial v}X\right]$$

$$= -\frac{\partial\Phi_{11}}{\partial u} - \left\{\frac{\partial\Phi_{\alpha a}}{\partial v}X^2 + \frac{\partial\Phi_{\beta a}}{\partial v}Y^2 + \frac{\partial\Phi_{\gamma a}}{\partial v}Z^2 + \frac{\partial(\Phi_{\alpha b} + \Phi_{\beta a})}{\partial v}X\,Y + \frac{\partial(\Phi_{\alpha c} + \Phi_{\gamma a})}{\partial v}X\,Z + \frac{\partial(\Phi_{\beta c} + \Phi_{\gamma b})}{\partial v}Y\,Z\right\}\ .$$

From the definitions of the quantities Φ_{11} , Φ_{12} , that quantity can be written:

$$-\ \frac{\partial\Phi_{11}}{\partial u} - \left(\frac{\partial(\Phi_{12}X^2)}{\partial v} + 2\,\frac{\partial(\Phi_{12}XY)}{\partial v}\,X\,Y + \cdots\right)$$

$$= -\frac{\partial\Phi_{11}}{\partial u} - \frac{\partial\Phi_{12}}{\partial v}(X^2 + Y^2 + Z^2)^2 - 2\Phi_{12}(X^2 + Y^2 + Z^2)(X\,X_v + Y\,Y_v + Z\,Z_v)$$

$$= -\frac{\partial\Phi_{11}}{\partial u} - \frac{\partial\Phi_{12}}{\partial v}.$$

Analogously, one will get the following expression for the factor of ω_v :

$$-\frac{\partial\Phi_{21}}{\partial u} - \frac{\partial\Phi_{22}}{\partial v},$$

and one will finally get:

$$\Omega = \omega(\dots) - \omega_u\left(\frac{\partial\Phi_{11}}{\partial u} + \frac{\partial\Phi_{12}}{\partial v}\right) - \omega_v\left(\frac{\partial\Phi_{12}}{\partial u} + \frac{\partial\Phi_{22}}{\partial v}\right) - \Phi_{11}\,\omega_{uu} - 2\Phi_{12}\,\omega_{uv} - \Phi_{22}\,\omega_{vv}$$

(29)

$$= \Phi_0\,\omega - \frac{\partial}{\partial u}(\Phi_{11}\,\omega_u + \Phi_{12}\,\omega_v) - \frac{\partial}{\partial v}(\Phi_{12}\,\omega_u + \Phi_{22}\,\omega_v)\ .$$

The explicit expression for Φ_0 is easily inferred from the calculations that one must perform. One will get it when one replaces δx, δy, δz, δa, δb, δc, $\delta\alpha$, $\delta\beta$, $\delta\gamma$ with X, Y, Z, X_u , Y_u , Z_u , X_v , Y_v , Z_v in the original expression:

$$\Omega = X\left(\delta\Phi_x - \frac{\partial\,\delta\Phi_a}{\partial u} - \frac{\partial\,\delta\Phi_\alpha}{\partial v}\right) + Y\left(\delta\Phi_y - \frac{\partial\,\delta\Phi_b}{\partial u} - \frac{\partial\,\delta\Phi_\beta}{\partial v}\right) + Z\left(\delta\Phi_z - \frac{\partial\,\delta\Phi_c}{\partial u} - \frac{\partial\,\delta\Phi_\gamma}{\partial v}\right)$$

before performing the differentiations with respect to u and v .

Example. – In order to calculate Φ_0 for Problem XIV, we first observe that the identity:

$$x_u\,X + y_u\,Y + z_u\,Z = 0$$

will give:

$$x_u\,\delta X + y_u\,\delta Y + z_u\,\delta Z + X\,(X\,\omega_u + \omega\,X_u) + \dots = 0$$

for the present normal variation, or:

$$x_u\,\delta X + y_u\,\delta Y + z_u\,\delta Z = -\,\omega_u\,.$$

One likewise gets the equation:

$$x_v\,\delta X + y_v\,\delta Y + z_v\,\delta Z = -\,\omega_v\,.$$

It will follow from this and the identity:

$$X\,\delta X + Y\,\delta Y + Z\,\delta Z = 0$$

that the quantities δX, δY, δZ, and their derivatives with respect to u, v will not contain the factor ω in any of their terms but will be linearly homogeneous in the derivatives of those quantities. The coefficient of ω in the expression Ω will then be the same as in the extended one:

(30) $$\Omega + P\,\delta X + Q\,\delta Y + R\,\delta Z = \delta(P\,X + Q\,Y + R\,Z)\,.$$

One now has the following identity (§ **64**) for the assumed form of Φ:

$$P\,X + Q\,Y + R\,Z = \begin{vmatrix} X & Y & Z \\ x_u & y_u & z_u \\ X_v & Y_v & Z_v \end{vmatrix} - \begin{vmatrix} X & Y & Z \\ x_v & y_v & z_v \\ X_u & Y_u & Z_u \end{vmatrix}\,.$$

The right-hand side of equation (30) is then composed of six determinants that arise from the one that was just written down when one puts a δ symbol in front of each term in one row. However, from the given behavior of the variations δX, δX_u, δX_v, …, those determinants will yield only the following two terms with a factor of ω:

$$\begin{vmatrix} X & Y & Z \\ \delta x_u & \delta y_u & \delta z_u \\ X_v & Y_v & Z_v \end{vmatrix} - \begin{vmatrix} X & Y & Z \\ \delta x_v & \delta y_v & \delta z_v \\ X_u & Y_u & Z_u \end{vmatrix} = \begin{vmatrix} X & \cdots \\ \omega X_u + X\omega_u & \cdots \\ X_v & \cdots \end{vmatrix} - \begin{vmatrix} X & \cdots \\ \omega X_v + X\omega_v & \cdots \\ X_u & \cdots \end{vmatrix}$$

$$= 2\omega \begin{vmatrix} X & Y & Z \\ X_u & Y_u & Z_u \\ X_v & Y_v & Z_v \end{vmatrix}\,,$$

and one will ultimately get:

$$\Phi_0 = 2 \sum \pm X\,Y_u\,Z_v\,,$$

$$\Omega = 2\omega \sum \pm X\,Y_u\,Z_v - \frac{\partial}{\partial u}\left(\frac{G\,\omega_u - F\,\omega_v}{\sqrt{EG-F^2}}\right) - \frac{\partial}{\partial v}\left(\frac{-F\,\omega_u + E\,\omega_v}{\sqrt{EG-F^2}}\right)\,.$$

Let the surface for which the quantity Ω is defined be an extremal in particular – i.e., a minimal surface – which was not assumed up to now. On such a thing, the variables u, v can be chosen such that:

$$F = 0\,, \qquad E = G\,, \qquad \frac{X \pm Y\,i}{1-Z} = u \pm v\,i\,,$$

$$X = \frac{2u}{1+u^2+v^2}, \qquad Y = \frac{2v}{1+u^2+v^2}, \qquad Z = \frac{u^2+v^2-1}{1+u^2+v^2} .$$

According to **Bonnet** and **Weierstrass**, that follows easily from the vanishing of the mean curvature. In those variables, the expression (29) will assume the following form:

$$\Omega = \frac{-8\omega}{(1+u^2+v^2)^2} - \frac{\partial^2\omega}{\partial u^2} - \frac{\partial^2\omega}{\partial v^2} .$$

Therefore, when a closed line $\omega = 0$ lies inside a piece of a minimal surface and ω satisfies the equation:

$$\frac{\partial^2\omega}{\partial u^2} + \frac{\partial^2\omega}{\partial v^2} + \frac{8\omega}{(1+u^2+v^2)^2} = 0 ,$$

the piece of the minimal surface will no longer imply a minimum for the surface, in general.

§ 68. – Conversion of the second variation that will exhibit the signs.

Since the function Φ is regular for any system of values $x, y, \ldots, z_v$ that is erected on the system $\mathfrak{S}$, one can develop that quantity in a **Taylor** series:

$$\Phi\,(x + \delta x, y + \delta y, \ldots, z_v + \delta z_v) ,$$

in which the terms that have dimensions one and two in the variations are written out, but the terms of higher dimension are summarized by means of the **Lagrange** remainder formula in an expression:

$$\rho\,(\delta x, \ldots, \delta x_u, \ldots, \delta z_v)$$

that is a cubic form in the nine variations. Their coefficients are certain derivatives of Φ that are defined for a system of values:

$$x + \theta\,\delta x , \qquad \ldots, \qquad x_u + \theta\,\delta x_u , \qquad \ldots, \qquad z_v + \theta\,\delta z_v ,$$

in which θ lies between the limits 0 and 1. Now since the derivatives of the function Φ are finite and continuous on the surface $\mathfrak{S}$, the absolute values of the coefficients of the form ρ will lie below a positive limit that is independent of the choice of surface element considered. The same thing will be true for the coefficients of the cubic form in the arguments ω, ω_u, ω_v, in which ρ goes over to the normal variation:

(31) $$\delta x = \omega\,X , \qquad \delta y = \omega\,Y , \qquad \delta z = \omega\,Z .$$

Now since the quantities:

$$\frac{\omega\,\omega_u}{\omega^2+\omega_u^2+\omega_v^2}, \qquad \frac{\omega\,\omega_v}{\omega^2+\omega_u^2+\omega_v^2}, \qquad \frac{\omega_u\;\omega_v}{\omega^2+\omega_u^2+\omega_v^2}$$

belong to the interval from – 1 to + 1 when ω, ω_u, ω_v do not vanish simultaneously, one can also regard the quantity:

$$\frac{\rho}{\omega^2+\omega_u^2+\omega_v^2}$$

as a linear form in the arguments ω, ω_u, ω_v whose coefficients lie between finite limits that are independent of ω. Under the assumption that:

$$(32) \qquad |\,\omega\,| < \varepsilon, \qquad |\,\omega_u\,| < \varepsilon, \qquad |\,\omega_v\,| < \varepsilon,$$

the absolute value of that expression will then become infinitely small with ε.

Now let $\varphi(\omega, \omega_u, \omega_v)$ be a quadratic form that has finite and continuous coefficients and is definite on the entire surface $\mathfrak{S}$. One will then have the inequality:

$$\left|\frac{\varphi(\omega,\omega_u,\omega_v)}{\omega^2+\omega_u^2+\omega_v^2}\right| > \gamma$$

for arbitrary values of ω, ω_u, ω_v that do not vanish simultaneously, in which a positive quantity that is independent of u, v, ω is on the right-hand side. It follows from this that with the assumption (32), the quantity:

$$\frac{\varphi(\omega,\omega_u,\omega_v)+\rho}{\omega^2+\omega_u^2+\omega_v^2}$$

will have the sign of the form φ as long as ε is assumed to be sufficiently small. The numerator of that expression will never have a different sign from that of the form φ then, even when the quantities ω, ω_u, ω_v can vanish simultaneously.

We shall employ that general consideration in order to determine the sign of the quantity ΔJ under the normal variation (31). Namely, from § **67**, one will have:

$$\Delta J = \iint_{\mathfrak{S}} du\,dv\{\tfrac{1}{2}\delta^2\Phi + [\delta x, \ldots, \delta z_v]_3\}$$
$$= \iint_{\mathfrak{S}} du\,dv\{\tfrac{1}{2}\omega\,\Omega + \rho\}\,.$$

Now when ω vanishes on the boundary of the surface $\mathfrak{S}$, a partial integration by means of the expression (29) will give:

$$(33) \qquad \delta^2 J = \iint_{\mathfrak{S}} \Omega\, \omega\, du\, dv = \iint_{\mathfrak{S}} du\, dv \{\Phi_0\, \omega^2 + \psi(\omega_u, \omega_v)\} ,$$

in which one sets:

$$\psi(h, k) = \Phi_{11}\, h^2 + 2\Phi_{12}\, hk + \Phi_{22}\, k^2 .$$

It will then follow that:

$$2\, \Delta J = \iint_{\mathfrak{S}} du\, dv \{\Phi_0\, \omega^2 + \psi(\omega_u, \omega_v) + 2\rho\} .$$

Moreover, since the assumed behavior of the quantity ω implies the equation:

$$\iint_{\mathfrak{S}} du\, dv \left(\frac{\partial(\alpha\, \omega^2)}{\partial u} + \frac{\partial(\beta\, \omega^2)}{\partial v} \right) = 0 ,$$

when α and β are arbitrary functions of u and v that are continuous on the surface $\mathfrak{S}$ and provided with continuous first derivatives. When one adds the last two equations, one will get:

$$(34) \qquad 2\, \Delta J = \iint_{\mathfrak{S}} du\, dv [\theta(\omega, \omega_u, \omega_v) + 2\rho] = \delta^2 J + 2 \iint_{\mathfrak{S}} \rho\, du\, dv ,$$

with the notation:

$$\theta(h, k, l) = (\Phi_0 + \alpha_u + \beta_v)\, h^2 + 2\alpha\, hk + 2\beta\, hl + \psi(k, l) .$$

If one then succeeds in determining the functions α and β in such a way that the form θ is definite on the entire surface $\mathfrak{S}$ then ΔJ will have a fixed sign. With that, we have derived a criterion for the occurrence of an extremum that we would like to call the ***Brunacci*** *criterion*. It can also be applied to isoperimetric problems, because if one has a normal variation for which the quantity ΔK (with the notation of § **64**) vanishes then:

$$\Delta J = \Delta\, (J + \lambda\, K) , \qquad \delta(J + \lambda\, K) = 0 ,$$

and the formula (34) will remain true when one replaces Φ with $\Phi + \lambda\, \Psi$ on the right-hand side.

The extremum, which is ensured by a fixed sign on the quantity ΔJ under the assumptions that were introduced, has a special character and is related to the weak extremum of § **17**. One compares the surface $\mathfrak{S}$ with all of the ones that arise from it by a sufficiently-small normal variation, so by a displacement of each point in the direction normal to it. In that way, not only will the magnitude of the displacement remain below a certain limit, but also the absolute values of its derivatives with respect to u and v. Moreover, the latter quantities need to have only those properties that an integral of an entire rational function of ω, ω_u, ω_v over the surface $\mathfrak{S}$ would have if it could be transformed by the usual rules of integral calculus, and in particular, by partial integration. The extremum that is thus defined suffices for many applications, and in particular, the mechanical ones. Furthermore, it is not difficult to show that any surface whose points and tangent planes deviate from those of $\mathfrak{S}$ sufficiently little can arise by a normal variation of the type considered.

Now, in order to make the **Brunacci** condition more suitable for application, we start from the fact that if it is possible to fulfill the condition then the form $\psi(k, l)$ must obviously be definite. One must then have:

$$\Phi_{11}\,\Phi_{22} - \Phi_{12}^2 > 0\,,$$

and the form $\theta(h, k, l)$ will likewise be definite when its determinant has the same sign as the form ψ. In order to arrive at that, one can pose the equation:

$$(\Phi_{11}\,\Phi_{22} - \Phi_{12}^2)(\Phi_0 + \alpha_u + \beta_v) - \Phi_{11}\,\beta^2 + 2\Phi_{12}\,\alpha\,\beta - \Phi_{22}\,\alpha^2 = \gamma(\Phi_{11}\,\Phi_{22} - \Phi_{12}^2)\,,$$

in which γ means a constant whose sign agrees with that of the form ψ, since the left-hand side is the determinant of the form θ. If one sets:

$$\alpha = \frac{\sigma}{w}\,, \qquad \beta = \frac{\tau}{w}$$

here then that will give:

$$(\Phi_{11}\,\Phi_{22} - \Phi_{12}^2)[\Phi_0\,w^2 + w(\sigma_u + \tau_v)]$$
$$\sigma[-(\Phi_{11}\,\Phi_{12} - \Phi_{12}^2)\,w_u + \Phi_{12}\,\tau - \Phi_{22}\,\sigma] + \tau[-(\Phi_{11}\,\Phi_{12} - \Phi_{12}^2)\,w_v - \Phi_{11}\,\tau - \Phi_{12}\,\sigma]$$
$$= \gamma\,w^2(\Phi_{11}\,\Phi_{22} - \Phi_{12}^2)\,.$$

That equation will be fulfilled when:

$$\sigma = -\Phi_{11}\,w_u - \Phi_{12}\,w_v\,, \qquad \tau = -\Phi_{21}\,w_u - \Phi_{22}\,w_v\,, \qquad \Phi_0\,w + \sigma_u + \tau_v = \gamma\,w\,,$$

or also, when w is an integral of the equation:

$$(35) \qquad (\Phi_0 - \gamma)\,w - \frac{\partial}{\partial u}(\Phi_{11}\,w_u + \Phi_{12}\,w_v) - \frac{\partial}{\partial v}(\Phi_{21}\,w_u + \Phi_{22}\,w_v) = 0\,,$$

and σ, τ are defined by the foregoing equations. If one thus finds an integral of equation (35) that is non-vanishing and continuous, along with its first derivatives, on the entire surface $\mathfrak{S}$ then the form $\theta(h, k, l)$ will become positive-definite with the determination of the functions α, β above, and the **Brunacci** condition for the extremum that was defined above will be fulfilled.

A more detailed discussion would be superfluous when Φ_0 has the sign of the form ψ on the entire surface $\mathfrak{S}$. The form $\Phi_0\,h^2 + \psi(k, l)$ would already be definite then, such that one would simply set $\alpha = \beta = 0$. If the quantity Φ_0 also assumes values with a different sign then the relationship of equation (35) to the equation $\Omega = 0$ must be observed, which is what it will go to when one sets $\gamma = 0$, $w = \omega$. If the equation $\Omega = 0$ has a nowhere-vanishing integral on the surface $\mathfrak{S}$ then one will easily find that:

$$w^2 \,\theta(h, k, l) = \psi(w\,k - w_u\,h,\; w\,l - w_v\,k)\,.$$

As a result of equation (34), the quantity $\delta^2 J$ will then have the sign of the form ψ and will vanish only when:

$$w\,\omega_u - w_u\,\omega = w\,\omega_v - w_v\,\omega = 0$$

on the entire surface $\mathfrak{S}$, i.e., w and ω differ by only a constant factor. Since ω vanishes on the boundary, that will be possible only when $\omega = 0$ everywhere.

An integral of the equation $\Omega = 0$ with the given behavior can be defined under a condition that is easy to exhibit. Let the surface patch $\mathfrak{S}$ be an individual member of a family of extremal patches that are represented by the equations:

$$x = \xi(u, v, a)\,, \qquad y = \eta(u, v, a)\,, \qquad z = \zeta(u, v, a)\,. \tag{36}$$

Let the functions ξ, η, ζ be regular when the system of values (u, v, a) belongs to a certain region $(\mathfrak{A})$, inside of which the system of values that belongs to the surface $\mathfrak{S}$ also lies. Furthermore, let the functional determinant:

$$\Delta = \frac{\partial(\xi, \eta, \zeta)}{\partial(u, v, a)}$$

be non-zero within the region $(\mathfrak{A})$. We then say that the extremal patches considered define a *field*. Let two of them that belong to the parameters a and $a + \delta a$, and the first of which we identity with $\mathfrak{S}$, be represented by the system of equations (36) and:

$$\bar{x} = \xi(\bar{u}, \bar{v}, a + \delta a)\,, \quad \bar{y} = \eta(\bar{u}, \bar{v}, a + \delta a)\,, \quad \bar{z} = \zeta(\bar{u}, \bar{v}, a + \delta a)\,,$$

respectively, and define a connection between the arguments u, v and $\bar{u}$, $\bar{v}$ such that the point $(\bar{x}, \bar{y}, \bar{z})$ lies on the normal to the surface $\mathfrak{S}$ that is erected at the point (x, y, z). In order for that to be true, it is necessary and sufficient that the three equations:

$$\begin{aligned} (\bar{x} - x)\,Y - (\bar{y} - y)\,X &= 0\,, \\ (\bar{y} - y)\,Z - (\bar{z} - z)\,Y &= 0\,, \\ (\bar{z} - z)\,X - (\bar{x} - x)\,Z &= 0 \end{aligned} \tag{37}$$

should exist. From the assumed properties of ξ, η, ζ, one can further develop:

(38) $$-(\bar{x}-x)+\xi_a\,\delta a+\xi_u\,(\bar{u}-u)+\xi_v\,(\bar{v}-v)+[\delta a,\bar{u}-u,\bar{v}-v]_2 = 0\,,$$

along with analogous equations for y and z. If one combines those equations (e.g., when Z is non-zero) with the second and third equation in (37) then, as one easily sees, the functional determinant on the left-hand side, with the arguments $\bar{u}$, $\bar{v}$, $\bar{x}$, $\bar{y}$, $\bar{z}$, will have the non-zero value $Z\sqrt{EG-F^2}$, and one will get developments for $\bar{u}-u$, $\bar{v}-v$, $\bar{x}-x$, $\bar{y}-y$, $\bar{z}-z$ that take the form of $[\delta a]_1$. Then let:

$$\bar{x}-x = \nu X\,, \quad \bar{y}-y = \nu Y\,, \quad \bar{z}-z = \nu Z\,,$$

such that ν is the distance from the point (x, y, z) to $(\bar{x},\bar{y},\bar{z})$, be taken to be positive or negative according to the direction n that points from the first of those points to the second one, or conversely. If one then multiplies equation (38) and its analogues by X, Y, Z, resp., and adds them, then that will give:

$$\nu = \frac{\delta a}{\sqrt{EG-F^2}}\frac{\partial(\xi,\eta,\zeta)}{\partial(u,v,a)}+[\delta a]_2 = \frac{\Delta\,\delta a}{\sqrt{EG-F^2}}+[\delta a]_2\,,$$

and it will follow from this that:

(39) $$\left.\frac{\partial\bar{x}}{\partial\,\delta a}\right|^{\delta a=0}\delta a = \omega X\,, \qquad \left.\frac{\partial\bar{y}}{\partial\,\delta a}\right|^{\delta a=0}\delta a = \omega Y\,, \qquad \left.\frac{\partial\bar{z}}{\partial\,\delta a}\right|^{\delta a=0}\delta a = \omega Z\,,$$

in which one sets:

(40) $$\omega = \frac{\Delta\,\delta a}{\sqrt{EG-F^2}}\,.$$

Now since the point $(\bar{x},\bar{y},\bar{z})$ describes an extremal, the equations will exist:

$$P = Q = R = 0$$

when one replaces x, y, z with $\bar{x}$, $\bar{y}$, $\bar{z}$. One can then differentiate them with respect to δa and thus obtain, e.g., the equation:

(41) $$\frac{\partial P}{\partial x}\frac{\partial\bar{x}}{\partial\,\delta a}+\frac{\partial P}{\partial x_u}\frac{\partial\bar{x}_u}{\partial\,\delta a}+\frac{\partial P}{\partial x_v}\frac{\partial\bar{x}_v}{\partial\,\delta a}+\cdots = 0\,,$$

whereby those quantities will have the value zero in the derivatives with respect to δa. Now since the symbol $\partial / \partial\,\delta a$ is subject to the same rules of operation as the variation symbol δ, and in particular, the way that the symbols can be permuted with the symbols for differentiation with respect to u and v, on the basis of equations (39), the result obtained (41) can be expressed in the follow way: The equation:

$$\delta P = 0$$

will be true when the assumption that:

$$\delta x = \omega X, \qquad \delta y = \omega Y, \qquad \delta z = \omega Z \tag{42}$$

is introduced, with the notation in (40). The same argument can be carried out for the expressions Q and R, and the equation will follow that:

$$X\,\delta P + Y\,\delta Q + Z\,\delta R = 0,$$

whose left-hand side will go to the value Ω under the assumption (42). With that, it is shown that the quantity (40) is an integral of the differential equation $\Omega = 0$, and is obviously one that is non-zero on the entire surface $\mathfrak{S}$.

Therefore, when one can conclude the existence of integral of equation (35) that is non-vanishing on the surface $\mathfrak{S}$ and is equipped with continuous first derivatives, or:

$$\Omega - \gamma\,\omega = 0 \tag{43}$$

for sufficiently-small values of $|\gamma|$, from the existence of an integral of the equation $\Omega = 0$ with the given behavior, the extremum that was defined above can be ensured under the following conditions:

1. The extremal patch $\mathfrak{S}$ can be surrounded by a field.

2. The form $\psi(k, l)$ is everywhere-definite on the surface $\mathfrak{S}$.

The suggested conclusion in regard to equation (43) can be inferred in full rigor when:

$$\Omega = \Phi_0\,\omega - \frac{\partial^2\omega}{\partial u^2} - \frac{\partial^2\omega}{\partial v^2}, \qquad \psi(k, l) = k^2 + l^2$$

for a suitable choice of the parameters u, v and Φ_0 is regular and negative on the surface $\mathfrak{S}$. From § **67**, that is true for minimal surfaces. One can then assume that the quantity $\Phi_0 - \gamma$ is also negative on the surface $\mathfrak{S}$ and apply the following theorem of **Schwarz**: In the region $\mathfrak{A}$ (§ **65**) that corresponds to the surface $\mathfrak{S}$, let the quantity p be regular and positive. Let φ be a function of u and v that is continuous, along with its first derivatives on the boundary of a region $\mathfrak{U}$ but is not everywhere-vanishing in its interior. If one then sets:

$$J_0 = \iint_{\mathfrak{A}} p\,\varphi^2\,du\,dv\;, \qquad J_1 = \iint_{\mathfrak{A}} du\,dv\,(\varphi_u^2 + \varphi_v^2)$$

then the quotient $J_0 : J_1$ will have a well-defined finite maximum c. When c is a proper fraction, there will exist a continuous integral of the equation:

$$\frac{\partial^2 \omega}{\partial u^2} + \frac{\partial^2 \omega}{\partial v^2} + p\,\omega = 0$$

that has continuous first derivatives and is everywhere non-zero in the region $\mathfrak{A}$.

It next follows from that theorem that when one sets $p = +\,1$, the ratio:

$$\iint_{\mathfrak{A}} \varphi^2\,du\,dv \;:\; \iint_{\mathfrak{A}} (\varphi_u^2 + \varphi_v^2)\,du\,dv$$

will have a well-defined finite maximum m, so for any choice of the function φ, it can be written in the form $\mu\,m$ when μ satisfies the inequality:

$$0 < \mu \leq 1\;.$$

If one then sets $p = -\,\Phi_0$ and $p = -\,\Phi_0 + \gamma$ and denotes the associated values of J_0 and c by the subscripts 0 and γ, resp., then since J_1 is independent of p, one will obviously have:

$$\frac{J_{0\gamma}}{J_1} = \frac{J_{00}}{J_1} + \gamma\,\mu\,m\,, \tag{44}$$

so when $\gamma > 0$:

$$\frac{J_{0\gamma}}{J_1} \geq \frac{J_{00}}{J_1}\,, \qquad c_\gamma \geq c_0\;.$$

It follows from this that:

$$\lim_{\gamma \to 0} c_\gamma = c_0\,, \tag{45}$$

since if that were not the case then there would be a positive constant γ^0 such that no matter how small γ^1 might be chosen, there would always exist values of γ for which the following inequalities would be true:

$$c_\gamma - c_0 > \gamma^0\,, \qquad \gamma < \gamma^1\;. \tag{46}$$

However, for those functions φ that give the ratio $J_{0\gamma} : J_1$ its greatest value, as a result of the relation (44), one would have:

$$\frac{J_{00}}{J_1} = c_\gamma - \gamma\,\mu\, m\,, \qquad c_\gamma - \gamma\,\mu\, m \leq c_0\,, \qquad c_\gamma - c_0 \leq \gamma\,\mu\, m\,,$$

which would contradict the first inequality (46), since γ^1 can be arbitrarily small. One can derive an analogous contradiction for $\gamma < 0$ when one switches c_0 and c_γ. The relation (45) is then proved. If c_0 is a proper fraction then the same thing will be true for sufficiently-small values γ of c_γ, and equation (43) will have an integral that is non-vanishing in the region $\mathfrak{A}$ or on the surface $\mathfrak{S}$.

Now, when one sets $\omega = \varepsilon\,\varphi$, one will have:

$$\delta^2 J = (J_1 - J_{00})\ \varepsilon^2$$

for the minimal surfaces. When $c_0 \geq 1$, that quantity can be negative or vanishing without ω vanishing identically. From the above, that is impossible, so $\delta^2 J$ will be positive when the equation $\Omega = 0$ possess a non-vanishing integral on the surface $\mathfrak{S}$, and in particular, when the surface $\mathfrak{S}$ can be surrounded by a field. With the latter assumption, c_0, and therefore c_γ as well, will be a proper fraction so the minimum of the surface area in the sense that was defined will be guaranteed.

§ 69. – The quantity $\mathcal{E}$.

If the extremal patch $\mathfrak{S}$ is surrounded by a field, and it corresponds to the value $a = a_0$ then, as is easy to see, one can include it in a region $\mathfrak{G}$ such that a certain extremal of the field will go through each point of the latter, so the quantity a can be regarded as a single-valued function of position.

We now compare $\mathfrak{S}$ with a surface patch $\mathfrak{T}$ that exists completely in the region $\mathfrak{G}$ and has the boundary line $\mathfrak{C}$ in common with $\mathfrak{S}$, but otherwise no common point, such that $a - a_0$ will have a fixed sign for the entire surface $\mathfrak{T}$ – say, positive – and its maximum will be attained for the value $a_1 - a_0$. Every extremal of the field for which a lies between a_0 and a_1 cuts the surface $\mathfrak{T}$ along a closed line $\mathfrak{C}_a$ that surrounds the extremal patch $\mathfrak{T}_a$, but divides the surface $\mathfrak{T}$ into two parts $\mathfrak{T}_a^0$ and $\mathfrak{T}_a$, the latter of which is bounded by the line $\mathfrak{C}$. Obviously, $\mathfrak{T}_{a_0}$ reduces to the line $\mathfrak{C}$, while $\mathfrak{T}_{a_1}$ is identical to the entire surface $\mathfrak{T}$. Therefore, when one makes the domain of integration for the symbol J unambiguous, if one defines the variable quantity:

$$W(a) = J(\mathfrak{S}_a) + J(\mathfrak{T}_a)$$

then one will have:

$$W(a_0) = J(\mathfrak{S}_{a_0}) = J(\mathfrak{S}), \quad W(a_1) = J(\mathfrak{T}_{a_1}) = J(\mathfrak{T}), \quad W(a_1) - W(a_0) = J(\mathfrak{T}) - J(\mathfrak{S}).$$

If the latter difference has a fixed sign then the surface $\mathfrak{S}$ will yield an extremum to the integral J in comparison to all surfaces $\mathfrak{T}$. Obviously that will occur when the differential $dW(a)$ is well-defined and has a fixed sign, and the function $W(a)$ possesses continuity properties such that one can infer the usual conclusions about the increase or decrease in the function from the sign of the differential. One refrains from making any precise convention about the properties of the surface $\mathfrak{T}$ that would endow the quantity $W(a)$ with the given behavior. One easily sees that this will occur in any case when $\mathfrak{T}$ is composed of a finite number of regular surface patches.

Now one obviously has:

$$dW(a) = J(\mathfrak{S}_{a+da}) - J(\mathfrak{S}_a) + J(\mathfrak{T}_{a+da}) - J(\mathfrak{T}_a). \tag{47}$$

We can regard $\mathfrak{S}_{a+da}$ as a variation of $\mathfrak{S}_a$, in the sense of § **63**, although we shall not prove that, in general, so:

$$J(\mathfrak{S}_a) = \delta J = \int_{\mathfrak{C}_a} (U\,dv - V\,du)$$

$$= \int_{\mathfrak{C}_a} dt \left\{ (\Phi_{x_u}\delta x + \Phi_{y_u}\delta y + \Phi_{z_u}\delta z)\frac{dv}{dt} - (\Phi_{x_v}\delta x + \Phi_{y_v}\delta y + \Phi_{z_v}\delta z)\frac{du}{dt} \right\},$$

in which the integration has the same sense that it had in § **63**. Moreover, the difference $J(\mathfrak{T}_{a+da}) - J(\mathfrak{T}_a)$ can be regarded as the integral J, when it is extended over the strip in the surface $\mathfrak{T}$ between the extremals of the field that belong to a and $a + da$. Thus, if one takes t to be the arc-length of the common boundary of the surfaces $\mathfrak{S}_a$ and $\mathfrak{T}_a$, and if s denotes the width of the strip then since $\sqrt{EG - F^2}\,du\,dv$ is the surface element, one can set:

$$dJ(\mathfrak{T}_a) = \int_{\mathfrak{C}_a} dt \frac{\Phi^0}{\sqrt{E^0G^0 - (F^0)^2}},$$

in which the index 0 suggests that the quantity in question refers to the element of the surface $\mathfrak{T}$. In order to determine σ, as before, we let X, Y, Z, X^0, Y^0, Z^0 denote the direction cosines of the normals to the surfaces $\mathfrak{S}_a$ and $\mathfrak{T}$, and let σ' denote the direction of a tangent to the latter that is perpendicular to the curve $\mathfrak{C}_a$ and points to the interior of the surface $\mathfrak{T}_a^0$, and in that way take the normal n^0 to the surface $\mathfrak{T}$ that lies with respect to the directions of increasing t and σ' in the same way that the + z axis lies with respect to the + x and + y axes. One will then have:

$$\sigma = \delta x \cos(\sigma' x) + \delta y \cos(\sigma' y) + \delta z \cos(\sigma' z),$$
$$0 = X^0 \cos(\sigma' x) + Y^0 \cos(\sigma' y) + Z^0 \cos(\sigma' z),$$

and when the symbol d means an advance along $\mathfrak{C}_a$ in the direction of integration:

$$0 = \cos(\sigma' x)\, dx + \cos(\sigma' y)\, dy + \cos(\sigma' z)\, dz .$$

If one solves the last two equations for the quantities $\cos(\sigma' x)$, … and uses the established orientations for the directions n^0 and σ' then one will get:

$$\cos(\sigma' x) = Y^0 \frac{dz}{dt} - Z^0 \frac{dy}{dt}, \ldots,$$

so

$$\sigma\, dt = \begin{vmatrix} \delta x & \delta y & \delta z \\ X^0 & Y^0 & Z^0 \\ dx & dy & dz \end{vmatrix} . \tag{48}$$

With those values, the differential form above will give:

$$- dW(a)$$
$$= \int_{\mathfrak{C}_a} \left\{ (\Phi_{x_v}\, \delta x + \Phi_{y_v}\, \delta y + \Phi_{z_v}\, \delta z)\, du - (\Phi_{x_u}\, \delta x + \Phi_{y_u}\, \delta y + \Phi_{z_u}\, \delta z)\, dv - \frac{\Phi^0}{\sqrt{E^0 G^0 - (F^0)^2}} \begin{vmatrix} \delta x & \delta y & \delta z \\ X^0 & Y^0 & Z^0 \\ dx & dy & dz \end{vmatrix} \right\},$$

or

$$- dW(a) = \int_{\mathfrak{C}_a} \mathcal{E}\, dt ,$$

in which we have set:

$$\mathcal{E} = (\Phi_{x_v}\, \delta x + \Phi_{y_v}\, \delta y + \Phi_{z_v}\, \delta z) \frac{du}{dt} - (\Phi_{x_u}\, \delta x + \Phi_{y_u}\, \delta y + \Phi_{z_u}\, \delta z) \frac{dv}{dt} \tag{49}$$

$$- \frac{\Phi^0}{\sqrt{E^0 G^0 - (F^0)^2}} \begin{vmatrix} \delta x & \delta y & \delta z \\ X^0 & Y^0 & Z^0 \\ \frac{dx}{dt} & \frac{dy}{dt} & \frac{dz}{dt} \end{vmatrix} .$$

The sign of that quantity is fixed in many cases, e.g., in Problem XIV. From (10), one has:

$$\mathcal{E} = \begin{vmatrix} \delta x & \delta y & \delta z \\ X & Y & Z \\ x_u & y_u & z_u \end{vmatrix} \frac{du}{dt} - \begin{vmatrix} \delta x & \delta y & \delta z \\ X & Y & Z \\ -x_u & -y_u & -z_u \end{vmatrix} \frac{dv}{dt} - \begin{vmatrix} \delta x & \delta y & \delta z \\ X^0 & Y^0 & Z^0 \\ \frac{dx}{dt} & \frac{dy}{dt} & \frac{dz}{dt} \end{vmatrix} ,$$

$$\mathcal{E}\, dt = \begin{vmatrix} \delta x & \delta y & \delta z \\ X & Y & Z \\ dx & dy & dz \end{vmatrix} - \begin{vmatrix} \delta x & \delta y & \delta z \\ X^0 & Y^0 & Z^0 \\ dx & dy & dz \end{vmatrix}$$

here. From (48), the second determinant is positive. Now, since one obviously has:

$$X^0\, \delta x + Y^0\, \delta y + Z^0\, \delta z = X^0\, dx + Y^0\, dy + Z^0\, dz = 0 ,$$

with the established orientations of the directions that correspond to the symbols d, δ, n^0, that will give:

$$X^0 = \frac{dy\,\delta z - dz\,\delta y}{\rho}, \qquad Y^0 = \frac{dz\,\delta x - dx\,\delta z}{\rho}, \qquad Z^0 = \frac{dx\,\delta y - dy\,\delta x}{\rho},$$

in which ρ means the positive square root of the sum of the squared numerator. It follows directly from this that:

$$\mathcal{E}\, dt = \rho(X\,X^0 + Y\,Y^0 + Z\,Z^0 - 1) = \rho\,(\cos\omega - 1) ,$$

when ω means the angle between the normals to the surfaces $\mathfrak{T}$ and $\mathfrak{S}_a$ that are determined by X, …, X^0, … Therefore, the quantity $\mathcal{E}\, dt$ will always be negative here when the two surfaces do not contact each other, and nowhere-positive. Thus, the difference $J(\mathfrak{S}) - J(\mathfrak{T})$ will also be negative when, say, the surfaces $\mathfrak{S}_a$ and $\mathfrak{T}_a^0$ do not contact each other everywhere. If we overlook that case then the minimum property of the surface $\mathfrak{S}$ will be proved by the assumed properties of the quantity $W(a)$.

As the formulas (49), (48) show, and as one easily verifies by calculation, the quantity $\mathcal{E}$ will keep its value when one introduces a new system of rectangular coordinates without changing the orientation of the axes, just as when one introduces new parameters *r*, *s* for *u* and *v* for which the normal *n* keeps its direction, i.e., the inequality:

$$\frac{\partial(r,s)}{\partial(u,v)} > 0 \tag{50}$$

is satisfied. The latter convention makes it possible for the directions of integration along the curves $\mathfrak{C}_a$ to remain the same. In particular, if we focus on any element of one of those curves, along with the two elements of the surfaces $\mathfrak{S}_a$ and $\mathfrak{T}$ that go through it and determine the quantity $\mathcal{E}$ then the coordinate system can be arranged so that the + *z*-axis defines acute angles with the directions *n* and n^0, so the inequalities:

$$Z^0 > 0\,, \qquad Z > 0$$

will be satisfied. Now since *Z* differs from the determinant $\frac{\partial(x,y)}{\partial(u,v)}$ only by a positive factor, the coordinates *x* and *y* can be introduced as independent variables that satisfy the condition (50) in place of *r* and *s* in each case in the neighborhood of the elements considered. Hence, one also has the inequality:

$$dx\,\delta y - dy\,\delta x > 0\,, \tag{51}$$

since the normal n^0, by definition, has the same relationship to the direction of integration and the direction that points to the interior of the surface $\mathfrak{T}_a^0$ that the + *z*-axis has to the + *x* and + *y* axes, resp.

If one now applies the special assumption that $r = x$, $s = y$ to the integral:

$$J = \iint \Phi(x, y, z, x_r, y_r, z_r, x_s, y_s, z_s)\, dr\, ds$$

and sets:

$$z_x = p\,, \qquad z_x = p\,, \qquad \Phi\,(x, y, z, 1, 0, p, 0, 1, q) = f\,(x, y, z, p, q)$$

then one will have:

$$J = \iint f\, dx\, dy\ .$$

Moreover, equations (5) in § **62** imply that:

$$\Phi^0_{x_r} = f - p\, f_p\,, \qquad \Phi^0_{x_s} = -\,p\, f_q\,,$$

$$\Phi^0_{y_r} = -q\,f_p\,, \qquad \Phi^0_{y_s} = f - q\,f_q\,.$$

With the notation:

$$p^0 = -\frac{X^0}{Z^0}, \quad q^0 = -\frac{Y^0}{Z^0},$$

one will get:

$$dz = p^0 dx + q^0 dy\,,$$

$$dx^2 + dy^2 + dz^2 = [1 + (p^0)^2]\,dx^2 + 2\,p^0 q^0\,dx\,dy + [1 + (q^0)^2]\,dy^2,$$

$$\delta z = p^0 \delta x + q^0 \delta y$$

for the surface $\mathfrak{T}$, so:

$$E^0 G^0 - (F^0)^2 = 1 + (p^0)^2 + (q^0)^2,$$

and since Z^0 is positive, one will have:

$$X^0 = \frac{-p^0}{\sqrt{1 + (p^0)^2 + (q^0)^2}}, \quad Y^0 = \frac{-q^0}{\sqrt{1 + (p^0)^2 + (q^0)^2}}, \quad Z^0 = \frac{1}{\sqrt{1 + (p^0)^2 + (q^0)^2}},$$

with the positive square root. With the help of those equations, one will get the following expression for $\mathcal{E}$:

$$\begin{aligned}\mathcal{E}\,dt = {} & \{-p\,f_q\,\delta x + (f - q\,f_q)\,\delta y + f_q\,(p^0 \delta x + q^0 \delta y)\}\,dx \\ & - \{(f - p\,f_p)\,\delta x - q\,f_p\,\delta y + f_p\,(p^0 \delta x + q^0 \delta y)\}\,dx - \frac{f\,(x, y, z, p^0, q^0)}{1 + (p^0)^2 + (q^0)^2}\begin{vmatrix} \delta x & \delta y & \delta z \\ -p^0 & -q^0 & 1 \\ dx & dy & dz \end{vmatrix}.\end{aligned}$$

The last determinant can be simplified by multiplying the first two columns by $-p^0$, $-q^0$, resp., and adding them to the third, since the equations:

(52) $$\delta z = p^0\,\delta x + q^0\,\delta y\,, \quad dz = p^0\,dx + q^0\,dy$$

are valid. The value of the determinant will then be simply:

$$(dx\,\delta y - dy\,\delta x)[1 + (p^0)^2 + (q^0)^2]\,,$$

and the entire last term will then be:

$$- f(x, y, z, p^0, q^0)(dx\ \delta y - dy\ \delta x) .$$

If one then considers the relation:

$$(p - p^0)\, dx + (q - q^0)\, dy = 0 ,$$

which follows from the second equation in (52), then that will give:

$$\mathcal{E}\, dt = \{- f(x, y, z, p^0, q^0) + f(x, y, z, p, q) + (p^0 - p) f_p + (q^0 - q) f_q\}(dx\,\delta y - dy\,\delta x) .$$

One can then make a **Taylor** development of the form:

$$f(x, y, z, p^0, q^0)$$

$$= f(x, y, z, p, q) + (p^0 - p) f_p + (q^0 - q) f_q + \tfrac{1}{2}[\bar{f}_{pp}(p^0 - p)^2 + 2\bar{f}_{pq}(p^0 - p)(q^0 - q) + \bar{f}_{qq}(q^0 - q)^2] ,$$

in which the overbar suggests that p, q are set equal to certain values p_m, q_m, the first of which lies between p and p^0, while the second one lies between q and q^0. It will then follow from the inequality (51) that the sign of $-\mathcal{E}$ coincides with that of the form:

$$\bar{f}_{pp}\, h^2 + 2\bar{f}_{pq}\, hk + \bar{f}_{qq}\, k^2 .$$

However, due to the relation (50), in which x and y can be taken to be r and s, resp., from § **66**, the forms:

$$\psi = \Phi_{11} h^2 + 2\Phi_{12}\, hk + \Phi_{22}\, k^2 , \qquad f_{pp}\, h^2 + 2 f_{pq}\, hk + f_{qq}\, k^2$$

will always be simultaneously definite and have the same sign. In particular, if the form ψ is definite and of fixed sign for all surface elements (which was true, e.g., in Problem XIV in § **66**) then $\mathcal{E}$ will likewise have an absolutely-fixed sign, and indeed, the sign of the form $-\psi$. Furthermore, if the form ψ is definite only on all elements of the extremal patch $\mathfrak{S}$ under examination and the direction of n^0 deviates from that of n sufficiently little then the differences $p^0 - p$ and $q^0 - q$ will be arbitrarily small, and likewise the quantities $p_m - p$ and $q_m - q$, such that the form ψ will have the same sign for the direction that is defined by the direction cosines:

$$\frac{-p_m}{\sqrt{1 + p_m^2 + q_m^2}} , \qquad \frac{-q_m}{\sqrt{1 + p_m^2 + q_m^2}} , \qquad \frac{1}{\sqrt{1 + p_m^2 + q_m^2}} ,$$

and the surface element that is perpendicular to it that it has for an element of the surface $\mathfrak{S}$ itself. In that case, $\mathcal{E}$ will take on a fixed sign when any tangential plane to the surface $\mathfrak{T}$ deviates from one on the surface $\mathfrak{S}$ sufficiently little. In that fact, one finds the analogues of the relations between the **Weierstrass** and **Legendre** sign conditions that correspond to cases a) and b) in § **16**.

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