# Contributions to the theory of the calculus of variations: The connection between the Weierstrass and Jacobi-Hamilton methods and a theory of integration of Cauchy 

By ADOLF KNESER in Breslau

Translated by D. H. Delphenich

In my textbook on the calculus of variations, I referred to the following problem as the most general problem of that discipline as long as one seeks functions of one variable: A number of differential equations:

$$
\begin{equation*}
\varphi_{\rho}\left(y_{0}, y_{1}, \ldots, y_{n}, d y_{0}, d y_{1}, \ldots, d y_{n}\right)=0 \quad(\rho=0,1, \ldots, r) \tag{1}
\end{equation*}
$$

exist between $n+1$ variables $y_{0}, y_{1}, \ldots, y_{n}$ that are homogeneous of degree one in the differentials. Let $\mathfrak{C}$ be a region of those variables in a bounded simple manifold to which the differentials that occur in equations (1) refer, and the initial and final values of the quantities $y$ will be subject to conventions such that the final value of at least one of them (say, $y_{0}$ ) is initially subject to no restrictions. The problem is then to determine $\mathfrak{C}$ in such a way that the final value of $y_{0}$ will be an extremum. That implies that the differential equations:

$$
\begin{equation*}
\frac{\partial \Omega}{\partial y_{v}}-d \frac{\partial \Omega}{\partial d y_{v}}=0 \quad(n=0,1, \ldots, n) \tag{2}
\end{equation*}
$$

will be true along the manifold $\mathfrak{C}$, in which one sets:

$$
\Omega=\sum_{\rho=0}^{r} \lambda_{\rho} \varphi_{\rho}\left(y_{0}, y_{1}, \ldots, y_{n}, d y_{0}, d y_{1}, \ldots, d y_{n}\right)
$$

and the $\lambda_{\rho}$ are multipliers that are determined by the combination of equations (1) and (2).
That very general problem was recently referred to as the Mayer problem, as opposed to the Lagrangian problem, which emerges from it by the following specialization: The quantity $y_{0}$ does not occur at all, and the differential $d y_{0}$ occurs in only one of equations (1) - say, the first one which one imagines to have been solved for $d y_{0}$. One might then have, say:

$$
d y_{0}-\psi\left(y_{0}, y_{1}, \ldots, y_{n}, d y_{0}, d y_{1}, \ldots, d y_{n}\right)=0
$$

and $\psi$ is once more homogeneous of degree one in the differentials that occur. The remaining equations (1) will be:

$$
\begin{equation*}
\varphi_{\rho}\left(y_{1}, \ldots, y_{n}, d y_{1}, \ldots, d y_{n}\right)=0 \quad(\rho=1,2, \ldots, r) \tag{3}
\end{equation*}
$$

and the problem is now to extremize the integral:

$$
y_{0}=\int \psi\left(y_{1}, y_{2}, \ldots, y_{n}, \frac{d y_{1}}{d t}, \frac{d y_{2}}{d t}, \ldots, \frac{d y_{n}}{d t}\right) d t
$$

in which one integrates along the manifold $\mathfrak{C}$ and $t$ means a parameter that varies along it, under the condition equations (3).

Both terms lack any historical justification. Namely, the Mayer problem was already treated by Euler, and quite thoroughly and successfully. Indeed, the examples that he gave are the only ones that have been treated seriously up to the present day. Furthermore, Lagrange had also mentioned Mayer problems expressly in his ground-breaking treatise, and he later treated an example that Euler examined with variable endpoints in the calculus of functions that has found no successors up to now, in any event. Now since, on the other hand, Mayer's most important works are dedicated to precisely the problem that is now called the Lagrange problem, one can permute both names quite well in the terminology that was introduced.

Meanwhile, let us not be pedantic. The fact that those names were established for the classes of problems that were referred to undoubtedly corresponds to an existing demand for them, and we would then like to accept the names that have been used many times already and refer to them as the subject of our investigations.

We next point out that when the Lagrange problem is regarded as the Mayer problem, the concept of extremals will need to be modified a bit. For example, if one understands that in the simplest Lagrange problem, which is given by the equation:

$$
d y_{0}-\psi\left(y_{1}, y_{2}, d y_{1}, d y_{2}\right)=0
$$

extremals mean curves in the plane in which $y_{1}$ and $y_{2}$ are coordinates. In that case, one has to set:

$$
\Omega=\lambda_{0}\left(d y_{0}-\psi\left(y_{1}, y_{2}, d y_{1}, d y_{2}\right)\right)
$$

and equations (2) yield:

$$
\begin{gathered}
d \lambda_{0}=0, \quad \lambda_{0}=\text { const., } \\
\frac{\partial\left(\lambda_{0} \psi\right)}{\partial y_{v}}-d \frac{\partial\left(\lambda_{0} \psi\right)}{\partial y_{v}}=0, \quad(v=1,2)
\end{gathered}
$$

or since $\lambda_{0}$ is constant:

$$
\begin{equation*}
\frac{\partial \psi}{\partial y_{v}}-d \frac{\partial \psi}{\partial y_{v}}=0 \quad(n=1,2) \tag{4}
\end{equation*}
$$

Those two essentially-equivalent equations characterize the extremals. By contrast, if one regards our problem as a Mayer problem then, from the definition that I gave in my textbook, the extremals will be curves in the space of three coordinates $y_{0}, y_{1}, y_{2}$ that obey the relation:

$$
y_{0}=\int \psi\left(y_{1}, y_{2}, d y_{1}, d y_{2}\right),
$$

along with equations (4). Similarly, for every Lagrange problem in the $n$-fold manifold of the variables $y_{1}, y_{2}, \ldots, y_{n}$, the extremals are initially simple manifolds in the domain of those variables, but they will also be such things in the domain of the $n+1$ variables $y_{0}, y_{1}, \ldots, y_{n}$ when one regards the problem as a Mayer problem. Now certain symmetries emerge in the latter conception of things that would otherwise remain hidden, and problems will be closely related that previously seemed distinct. For that reason, in what follows, we would always like to consider the Lagrange problem to be a Mayer problem in a space whose number of dimensions is greater by one than it was in the former problem. The procedures will first make it possible to adapt certain theories that were developed only for the Lagrange problem up to now to the Mayer problem.

Known considerations in regard to the isoperimetric problem offer examples of the benefits that one can enjoy with spaces of differing numbers of dimensions, in which, say, the integral:

$$
u=\int F(x, y, d x, d y)
$$

is to be extremized, while the value of the integral:

$$
z=\int G(x, y, d x, d y)
$$

is prescribed. $F$ and $G$ are homogeneous of degree one in the differentials. Here, one next seeks a curve, namely, the extremal in the $(x, y)$-plane. However, that problem has already been treated many times, and one seeks, say, a curve in the space of the variables $x, y, z$ that fulfills the relation:

$$
d z-G(x, y, d x, d y)=0
$$

connects two given points, and yields an extremum for $u$. The isoperimetric condition is then defined by the fact that the $z$-coordinate of the endpoint is prescribed.

If one regards the problem as a four-dimensional Mayer problem then one must set:

$$
\Omega=\lambda(d u-F)+\mu(d z-G)
$$

in equation (2), in which one replaces $\lambda_{0}, \lambda_{1}, y_{0}, y_{1}, y_{2}, y_{3}$ with $\lambda, \mu, u, x, y, z$, and one will find that:

$$
\begin{gathered}
d \lambda=d \mu=0 \\
\frac{\partial(\lambda F+\mu G)}{\partial x}-d \frac{\partial(\lambda F+\mu G)}{\partial d x}=0 \\
\frac{\partial(\lambda F+\mu G)}{\partial y}-d \frac{\partial(\lambda F+\mu G)}{\partial d y}=0
\end{gathered}
$$

The long-known fact that the roles of the quantities $u$ and $z$ can be switched without changing the extremals in the $(x, y)$-plane is now obvious from that. Indeed, they are characterized by the essentially-equivalent equations (5) and (6), which include $F$ and $G$ symmetrically, but $u$ and $z$ not at all. One will then obtain the same extremals as before in the new problem of extremizing $z$ for a prescribed value of $u$.

However, a deep relationship between both extremal problems that Mayer discovered can also be made clear, namely, that the conjugate points of both problems are the same. One calls two points 0 and 1 on a two-dimensional extremal of the first isoperimetric problem (i.e., when one seeks the extremum of $u$ ) conjugate when they can be connected by a neighboring extremal that gives the same value to the integral $z$, when extended from 0 to 1 , as the first extremal, on which those points lie. If one regards that curve as the projection of a spatial extremal whose point $1^{\prime}$ has the projection 1 then the points 0 and $1^{\prime}$ will be conjugate because they are connected by two neighboring extremals. Now the extremal fulfills the necessary condition for the extremum of the quantity $u$. If one goes from the extremal arc $01^{\prime}$ to a neighboring one with the same initial and final point then the value of $u$ will remain the same to first order. One will then get the same system of values $x, y, z, u$ at the endpoints of both two-or-three-dimensional extremal arcs. If one regards the problem as four-dimensional, such that the extremals are defined by equations (5), (6), and the relations:

$$
d u-F(x, y, d x, d y)=0, \quad d z-G(x, y, d x, d y)=0
$$

then the conjugate points will be simply the ones that can be connected by two neighboring extremals in the space of the four quantities $x, y, z, u$. No distinction should be made between $z$ and $u$ in that definition. It is equally true for both of the extremum problems that were distinguished above, and Mayer's theorem is immediately obvious.

If one next addresses the simplest class of problems, for which the integral:

$$
u=\int F(x, y, d x, d y)
$$

is to be extremized, in which $F$ is homogeneous of degree one in the differentials, then one will need a field for the Weierstrass derivation of sufficient conditions for the extremum, i.e., a family of extremals along which one defines Hamilton's principal function from the integral $u$ for suitably-chosen starting points. One restricts the field according to the Jacobi condition such that a certain region of the plane will be simply covered, and $u$ can be regarded as a function of $x$ and $y$. Finally, and above all, one chooses the family of curves and the starting point of the integration such that the equation:

$$
\begin{equation*}
\delta u=\frac{\partial F}{\partial d x} \delta x+\frac{\partial F}{\partial d y} \delta y \tag{7}
\end{equation*}
$$

will be true, in which $\delta$ means an advance along an arbitrary direction in the $(x, y)$-plane and $d$ means an advance along an extremal of the field. The essence of the Weierstrass method can now be characterized in fewer words.

Let the points 1 and 2 lie inside the field along the same extremal. One will then have:

$$
u_{12}=\left.u\right|_{1} ^{2}
$$

for the value of the integral $u$ when it is defined along the extremal arc 12 . Furthermore, if $\mathfrak{B}$ is an arbitrary curve in the field that runs from 1 to 2 , and along which the differential sign $\delta$ will apply, and one sets:

$$
d U=F(x, y, d x, d y)
$$

then:

$$
U_{12}=\left.U\right|_{1} ^{2}
$$

will be the integral $U$ when it is extended along this curve from 1 to 2 , and equation (7) will yield the equation:

$$
\delta(U-u)=F(x, y, d x, d y)-\delta x \frac{\partial F}{\partial d x}-\delta y \frac{\partial F}{\partial d y}
$$

in which $F$, with no arguments, is understood to mean the same thing as before, and the right-hand side means the Weierstrass quantity:

$$
E=E(x, y, d x, d y, \delta x, \delta y)
$$

One finds directly by integration that:

$$
\int_{1}^{2} \delta(U-u)=U_{12}-u_{12}=\int_{1}^{2} E,
$$

in which one integrates over the differential $\delta$ on the right, i.e., along the curve $\mathfrak{B}$. Now the sign of the quantity $E$ is generally easy to examine. If it is fixed then the same will be true of the sign of the difference $U_{12}-u_{12}$, and that will guarantee the extremum property of the extremal arc 12 compared to the curves $\mathfrak{B}$ that run through the interior of the field and give a fixed sign to the quantity $E$.

In that line of reasoning, Weierstrass had always employed a field whose extremals went through a fixed point and defined the integral $u$ from there on. In my textbook, I referred to the fact that the Weierstrass theory can also be developed in a field that is defined by an arbitrary
simply-infinite family of extremals. One will then have to define the integral $u$ only according to equation (7) along a curve that intersects the extremals transversally, when $\delta$ means the advance along it, i.e., such that the equation:

$$
\frac{\partial F}{\partial d x} \delta x+\frac{\partial F}{\partial d y} \delta y=0
$$

is true. Obviously, such transversals can constitute a family. For less-simple problems (e.g., when higher differential quotients occur in the isoperimetric and general Mayer problems), I have applied fields with the special nature that Weierstrass employed throughout, except that for the isoperimetric problem with a variable boundary, a more general field will appear that still has a special character, however.

In fact, the simplest problem of the type that was considered more closely above stand alone in those questions. For example, already in the problem of the shortest line in three-dimensional space, one cannot employ any arbitrary family of extremals that have the necessary number of dimensions in order to construct a field, which would be a two-fold infinite family of lines in the present case. That is because, above all, one seeks the extremum of a quantity $u$ that is defined by the equation:

$$
d u=F(x, y, z, d x, d y, d z)
$$

in which $F$ is again homogeneous of degree one in the differentials, so by analogy with equation (7), one poses the relation:

$$
\delta u=\frac{\partial F}{\partial d x} \delta x+\frac{\partial F}{\partial d y} \delta y+\frac{\partial F}{\partial d z} \delta z
$$

in a field, and the extremals of the field will be intersected transversally by every surface $u=$ const., i.e., according to the relation:

$$
\frac{\partial F}{\partial d x} \delta x+\frac{\partial F}{\partial d y} \delta y+\frac{\partial F}{\partial d z} \delta z=0
$$

In the case of the shortest line, one sets:

$$
F=\sqrt{d x^{2}+d y^{2}+d z^{2}} .
$$

Transversal position is a right-angle intersection. The family of extremals, i.e., the family of lines with which one would like to form the field, must then be intersected by surfaces orthogonally, which is known to be impossible for any two-dimensional family of lines.

That suggests the problem of characterizing those manifolds of extremals with which one can form fields and defining the general concept of a field for which the Weierstrass method can always be applied, and it is quite wonderful that this is the case for even the most general Mayer problem. For the Lagrange problem, the theory of the field was essentially established by the work of Bolza
$\left.{ }^{( }{ }^{1}\right)$ and Radon ( ${ }^{2}$ ). The main problem of the present treatise is to carry out corresponding developments for the Mayer problem.

For that, it proves to be preferable to refer the concept of an extremal in the way that was suggested above to a manifold whose number of dimensions is raised by one compared to the ordinary picture, and the concepts of transversals and the field are likewise modified. From a closer investigation, one sees that those modifications are already advantageous even for the simplest problems, say, the ones that are characterized by the equation:

$$
d u=F(x, y, d x, d y) .
$$

If one would like to apply the formulas that were cited above and valid for the Mayer problem then one must set:

$$
\Omega=\lambda(d u-F(x, y, d x, d y)) .
$$

One of equations (2) that relates to $u$ will yield:

$$
d \lambda=0, \quad \lambda=\text { const. }
$$

along the extremal. Equation (7) can then be written:

$$
\begin{equation*}
\frac{\partial \Omega}{\partial d u} \delta u+\frac{\partial \Omega}{\partial d x} \delta x+\frac{\partial \Omega}{\partial d y} \delta y=0, \tag{8}
\end{equation*}
$$

which then defines the modified transversality in the space of three variables $x, y, u$, and at the same time, the field. That can be regarded as a simply-infinite family of three-dimensional extremals, i.e., space curves. If $\delta$ gives the advance on the surface that they define, and $d$ denotes the advance along one of the curves then equation (8) will be valid. If the equation of that surface is represented in the form:

$$
\begin{equation*}
W(x, y, z)=0 \tag{9}
\end{equation*}
$$

then one will have the relation:

$$
\frac{\partial W}{\partial x} \delta x+\frac{\partial W}{\partial y} \delta y+\frac{\partial W}{\partial u} \delta u=0,
$$

and together with equation (8), that will yield:

$$
\frac{\partial W}{\partial x}: \frac{\partial W}{\partial y}: \frac{\partial W}{\partial y}=\frac{\partial \Omega}{\partial d x}: \frac{\partial \Omega}{\partial d y}: \frac{\partial \Omega}{\partial d u},
$$

[^0]since two of the differentials $\delta$ are arbitrary. One can eliminate the ratio $d y: d x$ from that proportion, which is free of $\lambda$ and $d u$ and represents two equations, so one can obtain a first-order partial differential equation for $W$ that does not include that quantity and is homogeneous in the derivatives. If one then imagines that equation (9) has been solved for $u$ then one can introduce the derivatives:
$$
\frac{\partial u}{\partial x}=-\frac{\partial W}{\partial x}: \frac{\partial W}{\partial u}, \quad \frac{\partial u}{\partial y}=-\frac{\partial W}{\partial y}: \frac{\partial W}{\partial u}
$$
into the partial differential equation and then obtain a new equation for $u$ as a function of $x$ and $y$ that is naturally identical to the Jacobi-Hamilton equation. It then characterizes the field in the space of quantities $x, y, u$ as a surface that is composed of spatial extremals.

That result can be adapted to an arbitrary Mayer problem. It will again give the problem (1). We then define a field to be an $(n-1)$-fold manifold of extremals in the space of $(n+1)$ variables $y_{0}, y_{1}, \ldots, y_{n}$ along which the relation:

$$
\sum_{v=0}^{n} \delta y_{v} \frac{\partial \Omega}{\partial d y_{v}}=0
$$

is true in the following sense: The differential $d$ refers to the advance along the extremals of the family. As many of the differentials $\delta$ are arbitrary as equations (1) will admit when one writes $\delta$ for $d$. The remaining ones are determined from those equations. One further has a Jacobi condition that has the character of an inequality.

One now assumes the following proportions:

$$
\frac{\partial W}{\partial y_{\mu}}: \frac{\partial W}{\partial y_{0}}=\frac{\partial \Omega}{\partial d y_{v}}: \frac{\partial \Omega}{\partial d y_{0}} \quad(\mu=1,2, \ldots, n)
$$

Only the $r$ ratios of the $r+1$ multipliers and the $n$ ratios of the $n+1$ differentials $d y_{v}$ occur in them. If one combines them with equations (1) then one will have a system of $n+r+1$ equations from which one can imagine eliminating the indicated $n+r$ ratios:

$$
\lambda_{\rho}: \lambda_{0}, \quad d y_{v}: d y_{0}
$$

One will then obtain a first-order partial differential equation that is free of $W$ and includes only the ratios:

$$
\frac{\partial W}{\partial y_{\mu}}: \frac{\partial W}{\partial y_{0}}
$$

If one imagines solving the equation:

$$
W=0
$$

for one of the quantities $y_{v}$ (say $y_{0}$ ), which then appears as a function of $y_{1}, y_{2}, \ldots, y_{n}$, then one will have:

$$
\frac{\partial y_{0}}{\partial y_{\mu}}=-\frac{\partial W}{\partial y_{\mu}}: \frac{\partial W}{\partial y_{0}}
$$

and one can introduce those quantities in place of the derivatives of $W$ into the partial differential equation that is obtained for $W$, which will then imply a partial differential equation for $y_{0}$ that is essentially the Jacobi-Hamilton equation. $y_{0}$ can be replaced with any of the quantities $y_{v}$ in that entire representation.

The main result of the foregoing investigation is now the following one:
An ( $n-1$ )-fold infinite family of extremals of the Mayer problem in the space of $n+1$ variables $y_{v}$ define a field in which the Weierstrass theory will unfold if and only if they combine into an $n$ fold manifold for which the Jacobi-Hamilton differential equation is satisfied.

In the context of the general concept of a field, we shall discuss the special types of fields for which all of the extremals have a fixed location in common or, more generally, intersect fixed manifolds.

## § 1. - The shortest line in the plane as a Mayer problem.

If $u$ is the arc-length in the plane then one will have the equation:

$$
\begin{equation*}
d u-\sqrt{d x^{2}+d y^{2}}=0 \tag{10}
\end{equation*}
$$

If one seeks the extremum of $u$ then will have to set:

$$
\Omega=\lambda\left(d u-\sqrt{d x^{2}+d y^{2}}\right),
$$

as a general rule, and find the equations for the extremals in the space of quantities $x, y, u$ :

$$
d \lambda=0, \quad d \frac{\lambda d x}{\sqrt{d x^{2}+d y^{2}}}=d \frac{\lambda d y}{\sqrt{d x^{2}+d y^{2}}}=0, \quad \frac{d y}{d x}=\text { const. }
$$

That must be combined with equation (10). That will show that the extremals are lines in space that are inclined at an angle of $45^{\circ}$ with respect to the $x y$-plane. One will get different parallels that lie vertically above each other (when one considers the $x y$-plane to be horizontal) depending upon the starting point from which the arc-length along a line in the plane is measured.

Three-dimensional transversality will be defined by the equation:

$$
\begin{equation*}
\delta u \frac{\partial \Omega}{\partial d u}+\delta x \frac{\partial \Omega}{\partial d x}+\delta y \frac{\partial \Omega}{\partial d y}=0 \tag{11}
\end{equation*}
$$

or

$$
\delta u=\delta x \frac{d x}{d u}+\delta y \frac{d y}{d u} .
$$

A field can be defined by any family of lines along which one measures the arc-length from an orthogonal trajectory. The corresponding spatial extremals will then be lines that are inclined by $45^{\circ}$ with respect to the horizontal plane and whose horizontal projections onto lines $\mathfrak{K}$ in the horizontal plane will be perpendicular to the lines of the family. If that line of intersection is a circle then the spatial extremals will go through a point that lies vertically above the center of the circle and defines a cone. If the curve of intersection $\mathfrak{K}$ is not a circle then at least the projections of two neighboring lines can be normals to the curvature circle of the curve $\mathfrak{K}$, and the corresponding neighboring plane of the family that are inclined by $45^{\circ}$ with respect to the horizontal will intersect. The lines of a spatial field will then define a developable surface whose generators are inclined by $45^{\circ}$ with respect to the $x y$-plane.

If one intersects that surface with a family of horizontal planes $u=$ const. and projects the lines of intersection onto the $x y$-plane then one will get the transversals, i.e., here they would be the orthogonal trajectories to the planar family of lines that one ordinarily considers.

In order to establish that geometrically-obvious fact analytically, we write equation (11) as:

$$
\begin{equation*}
-\delta u+\delta x \frac{d x}{d u}+\delta y \frac{d y}{d u}=0 \tag{12}
\end{equation*}
$$

and see from this that the direction that corresponds to the differential $\delta$ will be perpendicular to another one whose direction cosines have the ratios:

$$
-1: \frac{d x}{d u}: \frac{d y}{d u} .
$$

Since the sum of the squares of those quantities equals 2 , the direction cosine that relates to the $u$ axis will be $-1: \sqrt{2}$. The direction that we speak of will then define an angle with the vertical of $135^{\circ}$, and the directions that are denoted by $\delta$ will lie in a plane that is inclined by $45^{\circ}$ with respect to the horizontal. Now since one can also set $\delta=d$, in particular, in equation (12), those directions will define a surface element that includes a line element of the spatial extremal.

If one further regards the element $\delta$ as something that belongs to a surface $\mathfrak{F}$ and sets:

$$
\delta u=p \delta x+q \delta y
$$

accordingly, then equation (12) will immediately yield:

$$
\begin{equation*}
p^{2}+q^{2}=1, \tag{13}
\end{equation*}
$$

i.e., a first-order partial differential equation for the surface $\mathfrak{F}$. Since one can write that equation as:

$$
\frac{-1}{\sqrt{1+p^{2}+q^{2}}}= \pm \frac{1}{\sqrt{2}}
$$

the surfaces $\mathfrak{F}$ can be defined to be the ones whose normals are inclined by $45^{\circ}$ with respect to the vertical. That already implies that the surfaces must be developable because their spherical images by parallel normals will degenerate into two curves on the cone.

If one further defines equation (13) by the general method of characteristics then one will get the simultaneous system:

$$
\frac{d x}{p}=\frac{d y}{q}=\frac{d p}{0}=\frac{d q}{0}=\frac{d u}{1},
$$

so

$$
p=\text { const., } \quad q=\text { const. }, \quad \frac{d x}{d u}=p=\text { const., } \quad \frac{d y}{d u}=q=\text { const., }
$$

and from (10):

$$
\left(\frac{d x}{d u}\right)^{2}=\left(\frac{d y}{d u}\right)^{2}=1, \quad p \frac{d x}{d u}+q \frac{d y}{d u}=1
$$

Those equations make it clear that the characteristics are identical with the spatial extremals. The surface element that is associated with each point of the characteristic is the same as the one that was employed above in order to interpret equation (11).

It is also interesting to verify the general fact of Cauchy's theory of integration that when two neighboring characteristics belong to the surface element that fulfills the partial differential equation at any of their locations, that relationship will present itself over their entire extent and bound a strip on an integral surface, which will be a "characteristic strip," in Lie's terminology.

A family of spatial extremals depends upon a parameter $a$ that is constant along each extremal, while the points along that extremal will be distinguished by the values of the parameter $t$. If one sets:

$$
\begin{equation*}
\delta=d a \frac{\partial}{\partial a}+d t \frac{\partial}{\partial t} \tag{14}
\end{equation*}
$$

then the terms in the expression:

$$
-\delta u+\frac{d x}{d u} \delta x+\frac{d y}{d u} \delta y
$$

that involve $d t$ will drop out, and one will have the equation:

$$
-\delta u+\frac{d x}{d u} \delta x+\frac{d y}{d u} \delta y=\left[-\frac{\partial u}{\partial a}+\frac{d x}{d u} \frac{\partial x}{\partial a}+\frac{d y}{d u} \frac{\partial y}{\partial a}\right] d a
$$

so when one differentiates with respect to $t$ and imagines that $d x: d u$ and $d y: d u$ are constant along each extremal, one will get:

$$
\begin{equation*}
\frac{d}{d t}\left(-\delta u+\frac{d x}{d u} \delta x+\frac{d y}{d u} \delta y\right)=\left[-\frac{\partial^{2} u}{\partial a \partial t}+\frac{d x}{d u} \frac{\partial^{2} x}{\partial a \partial t}+\frac{d y}{d u} \frac{\partial^{2} y}{\partial a \partial t}\right] d a \tag{16}
\end{equation*}
$$

However, it follows from the general equation:

$$
d u=\sqrt{d x^{2}+d y^{2}}
$$

that:

$$
\begin{gathered}
\frac{\partial u}{\partial t}=\sqrt{\left(\frac{\partial x}{\partial t}\right)^{2}+\left(\frac{\partial y}{\partial t}\right)^{2}}, \quad \frac{d x}{d u}=\frac{\partial x}{\partial t}: \frac{\partial u}{\partial t}, \\
\frac{\partial^{2} u}{\partial a \partial t}=\frac{\frac{\partial x}{\partial u} \frac{\partial^{2} x}{\partial a \partial t}+\frac{\partial y}{\partial u} \frac{\partial^{2} y}{\partial a \partial t}}{\frac{\partial u}{\partial t}}=\frac{d x}{d u} \frac{\partial^{2} x}{\partial a \partial t}+\frac{d y}{d u} \frac{\partial^{2} y}{\partial a \partial t} .
\end{gathered}
$$

Equation (16) will then imply that:

$$
\begin{equation*}
\frac{d}{d t}\left(-\delta u+\frac{d x}{d u} \delta x+\frac{d y}{d u} \delta y\right)=0 \tag{17}
\end{equation*}
$$

Hence, when the equation:

$$
-\delta u+\frac{d x}{d u} \delta x+\frac{d y}{d u} \delta y=0
$$

is true at any location along an extremal of the family, along with the relation (14), it will be true everywhere. However, for independent differentials $d t$ and $d a$, that will mean that the connecting lines between a point on an extremal and all neighboring points of the neighboring curves of the family will lie in a surface element that fulfills the partial differential (13). With that, the cited theorem of Cauchy is confirmed.

Naturally, it is geometrically obvious that two neighboring lines that can be connected by a surface element at any location will then intersect when they are extremals that belong to that developable surface that we have defined to be a field.

Finally, we remark that equation (17) will also be obtained when $t$ is a function of $a$ and that quantity runs through a finite interval because the terms that are multiplied by $d t$ will also drop out under that assumption. If the equation:

$$
\begin{equation*}
-\delta u+\frac{d x}{d u} \delta x+\frac{d y}{d u} \delta y=0 \tag{18}
\end{equation*}
$$

is true for the indicated values of $t$ as a function of $a$ then it will be true along all of the extremals that correspond to the region of the quantity $a$ under consideration. One can interpret that fact geometrically as follows: If one intersects an arbitrary family of lines in the plane with an arbitrary curve and assumes that the value of $u$ (so the initial value of the length as measured along the extremal) is such that equation (18) will be verified then that will be true in general, and the lines at a distance of $u$ as measured along them will define a field.

## § 2. - A problem that is closely related to the shortest line in the plane.

The spatial extremals will remain the same as before when one seeks the extremum of the quantity $y$ in the equations:

$$
d u=\sqrt{d x^{2}+d y^{2}}, \quad d y=\sqrt{d u^{2}-d x^{2}} .
$$

The problem makes good sense: Perhaps a rope of given length and given starting point reaches up a vertical pole as high as possible. The interpretation in the $u x$-plane is somewhat different and more convenient for theoretical explanation. One seeks the shortest line of non-Euclidian length in it when one regards $\sqrt{d u^{2}-d x^{2}}$ as the differential of arc-length. We refer to that problem as (B) and distinguish it from the problem that was treated in the previous section by calling the latter (A).

Both of them have the spatial extremals in common, so they are spatial fields. However, the planar fields prove to be different. One obtains them for problem (B) when one intersects the developable surface that defines the spatial field by planes $y=$ const. and projects the line of intersection onto the $u x$-plane. As was mentioned before, for problem (A), one must intersect with planes $u=$ const. and project onto the $x y$-plane. On the developable surface, one has the equation:

$$
-\delta u+\frac{d x \delta x+d y \delta y}{\sqrt{d x^{2}+d y^{2}}}=0
$$

so along the curves $u=$ const., one will have:

$$
\delta x d x+\delta y d y=0
$$

whereas along the curves $y=$ const., one will have:

$$
\begin{equation*}
d x \delta x-d u \delta u=0 \tag{19}
\end{equation*}
$$

The last equation will give the transversality condition for the problem (B) in the $u x$-plane. A nonEuclidian orthogonality exists between the directions that are denoted by $d$ and $\delta$. They will be harmonically separated by the directions of the lines $u= \pm x$.

Among the fields, the ones for which the developable surface degenerates into a cone that must have a vertical axis and an opening angle of $90^{\circ}$ at the vertex deserve special attention. Its intersection with the planes $u=$ const. are circles whose projections onto the $x y$-plane appear to be orthogonal trajectories of a pencil of lines and transversals in the sense of problem (A). By contrast, if one intersection the cone with planes $y=$ const. then one will obtain hyperbolas in the $u x$-plane whose asymptotes run parallel to the lines $u= \pm x$. They intersect the generators of the cone (so in turn, a pencil of lines) transversally according to equation (18) under the assumption that $d y=0$, and they are the geometric loci of the points that have a constant non-Euclidian distance:

$$
y=\sqrt{d u^{2}-d x^{2}}
$$

from their centers. One will then have $y=$ const. along the intersection with the cone.
Incidentally, the plane sections of a non-degenerate developable surface have points of regression along the edge of regression. However, the edge of regression projects onto the enveloping line of the projections of the generators. One will then get an intuitive proof of the fact that the orthogonal trajectories of a family of lines will exhibit points of regression on the enveloping line, and indeed regardless of whether that orthogonality is understood in the ordinary sense or the sense of equation (19).

The projections of the edges of regression onto the $x y$ and $u x$-planes are the envelopes of the plane fields in problems (A) and (B), so they will be the loci of extremal focal points of the curves in which the developable surface is cut by the $x y$ and $u x$-planes, resp. The focal points then yield the same system of values $(x, y, u)$ for either problem when the quantity $u$ is defined by integrating the relevant curve for problem (A), and the quantity $y$ for problem (B), as one postulates in the ordinary theory. With that, the reciprocity law that Mayer exhibited for the isoperimetric problems is adapted to the simplest problem of the calculus of variations and the ones that are connected with it. The special nature of the problems brings with it only the fact that no pairs of conjugate points will appear.

## § 3. - Fields for the general Mayer problem.

Now let the $n+1$ variables $y_{0}, y_{1}, \ldots, y_{n}$ be generally constrained by the differential equations:

$$
\varphi_{\rho}\left(y_{0}, y_{1}, \ldots, y_{n}, d y_{0}, d y_{1}, \ldots, d y_{n}\right)=0 \quad(\rho=0,1, \ldots, r)
$$

in which the differentials appear homogeneously of degree one. One sets:

$$
\begin{equation*}
\Omega=\sum_{\rho=0}^{r} \lambda_{\rho} \varphi_{\rho}, \quad \frac{\partial \Omega}{\partial y_{v}}=\sum_{\rho=0}^{r} \lambda_{\rho} \frac{\partial \varphi_{\rho}}{\partial y_{v}}, \quad \frac{\partial \Omega}{\partial d y_{v}}=\sum_{\rho=0}^{r} \lambda_{\rho} \frac{\partial \varphi_{\rho}}{\partial d y_{v}}, \tag{21}
\end{equation*}
$$

and define an extremal to be any simple manifold in the domain of the quantities $y_{v}$ for which the equations:

$$
\begin{equation*}
\frac{\partial \Omega}{\partial y_{v}}-d \frac{\partial \Omega}{\partial d y_{v}}=0 \quad(v=0,1, \ldots, n) \tag{22}
\end{equation*}
$$

are true, along with equations (20), for a suitable choice of the multiplier $\lambda$ as a function of position.
An $(n-1)$-fold family of extremals will be represented with associated multipliers by the equations:

$$
y_{v}=\theta_{v}\left(t, a_{1}, a_{1}, \ldots, a_{n-1}\right), \quad \lambda_{v}=\zeta_{v}\left(t, a_{1}, a_{1}, \ldots, a_{n-1}\right),
$$

in which only $t$ varies along the individual extremals. If one imagines that those values, as well as $d y_{v}=\frac{\partial \theta_{v}}{\partial t} d t$, are substituted in equations (20) and differentiated with respect to one of the quantities $a$ then that will yield:

$$
\sum_{v=0}^{n} \frac{\partial \varphi_{\rho}}{\partial y_{v}} \frac{\partial \theta_{v}}{\partial a}+d t \sum_{v=0}^{n} \frac{\partial \varphi_{\rho}}{\partial d y_{v}} \frac{\partial^{2} \theta_{v}}{\partial t \partial a}=0
$$

so from (21), one will also have:

$$
\begin{equation*}
\sum_{v=0}^{n} \frac{\partial \Omega}{\partial y_{v}} \frac{\partial \theta_{v}}{\partial a}+d t \sum_{v=0}^{n} \frac{\partial \Omega}{\partial d y_{v}} \frac{\partial^{2} \theta_{v}}{\partial t \partial a}=0 \tag{23}
\end{equation*}
$$

If one sets:

$$
\delta^{\prime}=d a_{1} \frac{\partial}{\partial a_{1}}+d a_{2} \frac{\partial}{\partial a_{2}}+\cdots+d a_{n-1} \frac{\partial}{\partial a_{n-1}}
$$

then that symbol can be switched with the differentiation with respect to $t$ (so $\delta$ ), and the equations (23) that correspond to the different parameters $a$ will yield:

$$
\sum_{v=0}^{n} \frac{\partial \Omega}{\partial y_{v}} \delta^{\prime} y_{v}+d t \sum_{v=0}^{n} \frac{\partial \Omega}{\partial d y_{v}} d \delta^{\prime} y_{v}=0
$$

when one once more replaces $\theta_{v}$ with $y_{v}$, or also:

$$
\sum_{v=0}^{n}\left(\frac{\partial \Omega}{\partial y_{v}}-d \frac{\partial \Omega}{\partial d y_{v}}\right) \delta^{\prime} y_{v}+d \sum_{v=0}^{n} \frac{\partial \Omega}{\partial d y_{v}} \delta^{\prime} y_{v}=0
$$

from which it will follow that:

$$
\begin{equation*}
d \sum_{v=0}^{n} \frac{\partial \Omega}{\partial d y_{v}} \delta^{\prime} y_{v}=0 \tag{24}
\end{equation*}
$$

when one recalls equations (22).
Furthermore, due to the homogeneity of the quantities $\varphi_{\rho}$ and $\Omega$, one has the identity:

$$
\begin{equation*}
\sum_{v=0}^{n} \frac{\partial \Omega}{\partial d y_{v}} d y_{v}=0, \tag{25}
\end{equation*}
$$

so one will have:

$$
\begin{equation*}
d \sum_{v=0}^{n} \frac{\partial \Omega}{\partial d y_{v}} d y_{v}=0 \tag{26}
\end{equation*}
$$

a fortiori. If one sets:

$$
\delta=d t \frac{\partial}{\partial t}+\delta^{\prime}
$$

and adds equations (24) and (26) then it will follow that:

$$
\begin{equation*}
d \sum_{v=0}^{n} \frac{\partial \Omega}{\partial d y_{v}} \delta y_{v}=0 \tag{27}
\end{equation*}
$$

and since $d$ means the advance along an extremal, as always, one can express that result by saying that the quantity:

$$
\sum_{v=0}^{n} \frac{\partial \Omega}{\partial d y_{v}} \delta y_{v}
$$

is constant along any extremal. In particular, it will vanish everywhere when that is true at one location.

Now one defines an $(n-1)$-fold family of points along the extremals considered by the equation:

$$
\begin{equation*}
t=\tau=\tau\left(a_{1}, a_{2}, \ldots, a_{n-1}\right) \tag{28}
\end{equation*}
$$

Equation (24) then shows that the equation:

$$
d \sum_{v=0}^{n} \frac{\partial \Omega}{\partial d y_{v}} \delta^{\prime} y_{v}=0
$$

will be true for all values of the parameters $a$ and $t$ that come under consideration when it is true under the assumption that (28) is true. If one adds equation (25) to it, in which the $d t$ is once more independent of the parameters $a$, then one will obtain the equation:

$$
\begin{equation*}
\sum_{v=0}^{n} \delta y_{v} \frac{\partial \Omega}{\partial d y_{v}}=0 \tag{29}
\end{equation*}
$$

in general, under the same assumption, i.e., for independent values of $t, a_{1}, \ldots, a_{n-1}$.
One can also formulate that result by saying that only equation (29) enters into the assumption that one initially assumes for the manifold (28). Since equation (25) generally implies that:

$$
\sum_{v=0}^{n} \frac{\partial \theta_{v}}{\partial t} \frac{\partial \Omega}{\partial d y_{v}}=0
$$

it will follow that:

$$
\sum_{v=0}^{n} d \tau \frac{\partial y_{v}}{\partial t} \frac{\partial \Omega}{\partial d y_{v}}=0
$$

Thus, when equation (29) is true for the manifold (28), on which one sets:

$$
\delta=d t \frac{\partial}{\partial t}+\delta^{\prime}
$$

it will follow that for any location on any extremal of the family, one will have:

$$
\sum_{v=0}^{n} \delta^{\prime} y_{v} \frac{\partial \Omega}{\partial d y_{v}}=0
$$

which is the relation from which we have derived equation (29) above for independent $t, a_{\mu}$. That is, if equation (29) is true in the region $t=\tau\left(a_{1}, a_{2}, \ldots, a_{n-1}\right)$ then it will be true in general for independent $t, a_{1}, \ldots, a_{n-1}$. We then call the system of extremals:

$$
y_{v}=\theta_{v}\left(t, a_{1}, a_{2}, \ldots, a_{n-1}\right)
$$

a field. For that concept, it is essential that the number of parameters $a$ is exactly $n-1$. Obviously, no use of that number is made in the chain of inferences.

We shall explain the result obtained in more generality and specialize it with some examples.
The fact that the quantities $\partial \Omega / \partial y_{v}$ are differential quotients is not employed, except in equation (25). Accordingly, let $P_{v}, Q_{v}$ be arbitrary functions of the $n+1$ functions $y_{v}$ and their differentials. In the latter, let them be homogeneous, and indeed the quantities $P_{v}$ have degree one, while the quantities $Q_{\nu}$ have degree zero. One further has the identity:

$$
\sum_{v=0}^{n} Q_{v} d y_{v}=0 .
$$

A family of simple manifolds that fulfill the equations:

$$
P_{v}-d Q_{v}=0 \quad(v=0,1, \ldots, n)
$$

will be represented by equations:

$$
\begin{equation*}
y_{v}=\theta_{v}\left(t, a_{1}, a_{2}, \ldots\right), \tag{30}
\end{equation*}
$$

and let it be so arranged that when one sets:

$$
\delta^{\prime}=d a_{1} \frac{\partial}{\partial a_{1}}+d a_{2} \frac{\partial}{\partial a_{2}}+\cdots, \quad \delta=\delta^{\prime}+d t \frac{\partial}{\partial t}
$$

the equation:

$$
\sum_{v=0}^{n}\left(P_{v} \delta^{\prime} y_{v}+Q_{v} d \delta^{\prime} y_{v}\right)=0
$$

will be true. Therefore, if the equation:

$$
\sum_{v=0}^{n} Q_{v} \delta y_{v}=0
$$

is true relative to the family (30) under the assumption that $t=\tau\left(t, a_{1}, a_{2}, \ldots\right)$ then it will be true in general for independent $t, a_{1}, a_{2}, \ldots$

As an example, we can further take the simplest problem in the calculus of variation, for which:

$$
d u-\sqrt{d x^{2}+d y^{2}}=0
$$

in the notation of § 1. The extremals in xyu-space are lines that are inclined by $45^{\circ}$ with respect to the horizontal $x y$-plane. A field is a developable surface that is defined by a family of those extremals. From § 1, the equation:

$$
-\delta u+\frac{d x}{d u} \delta x+\frac{d y}{d u} \delta y=0
$$

that characterizes the field expresses the idea that the direction $\delta$ lies in a plane that is inclined $45^{\circ}$ with respect to the horizontal. That plane, which also includes the directions, is the tangent plane to the surface that is defined by the lines. Our theorem then has the following geometric content: A family of lines that are inclined $45^{\circ}$ with respect to the horizontal define a ruled surface. it will be developable if and only a curve can be drawn on it, along which all tangent planes are inclined $45^{\circ}$ with respect to the horizontal.

In the $x y$-plane, the extremals are lines along which the length $u$ is measured from any starting point. If we intersect a family of those lines with a curve $\mathfrak{K}$, along which $A$ and $B$ are two
neighboring points, then $u$ will vary under the transition from $A$ to $B$ just as much as the projection of the element $A B$ onto the line in the family that goes through $B$ amounts to. Any curve with which we intersect the family of lines will then have just that property. In particular, the curves $u$ = const. will intersect the lines orthogonally. (One should confer § 1.)

## § 4. - The method of Jacobi and Hamilton.

One can imagine eliminating the $n$ quantities $t, a_{1}, \ldots, a_{n}$ from the $n+1$ equations:

$$
\begin{equation*}
y_{v}=\theta_{v}\left(t, a_{1}, \ldots, a_{n-1}\right) \quad(n=0,1, \ldots, n) \tag{31}
\end{equation*}
$$

and thus obtaining an equation:

$$
\Phi\left(y_{0}, y_{1}, \ldots, y_{n}\right)=0
$$

Thus, if $\delta$ once more means the advance along the $n$-fold manifold that is defined by (31) then it will follow that:

$$
\begin{equation*}
\sum_{v=0}^{n} \frac{\partial \Phi}{\partial y_{v}} \delta y_{v}=0 \tag{32}
\end{equation*}
$$

Now it is obviously proper to assume that at least one system of $n$ of the quantities $\theta_{\nu}$ possesses a non-zero functional determinant with respect to the variables $t, a_{1}, \ldots, a_{n-1}$. Otherwise, the number of parameters $a$ could be reduced, which would contradict the concept of a field that was established in § 3. Hence, $n$ of the $n+1$ differentials $\delta y_{v}$ are certainly independent. When one recalls the equation that characterizes the field:

$$
\begin{equation*}
\sum_{v=0}^{n} \frac{\partial \Omega}{\partial d y_{v}} \delta y_{v}=0 \tag{33}
\end{equation*}
$$

it will then follow that the coefficients of the $\delta y_{v}$ in the two equations (32) and (33) must be proportional:

$$
\begin{equation*}
\frac{\partial \Phi}{\partial y_{0}}: \frac{\partial \Phi}{\partial y_{1}}: \cdots: \frac{\partial \Phi}{\partial y_{n}}=\frac{\partial \Omega}{\partial d y_{0}}: \frac{\partial \Omega}{\partial d y_{1}}: \cdots: \frac{\partial \Omega}{\partial d y_{n}} . \tag{34}
\end{equation*}
$$

Now only the $n$ ratios of the the differentials $\delta y_{v}$ occur in the ratios on the right-hand side, and likewise only the $r$ ratios of the $r+1$ multipliers. One can imagine eliminating those $n+r$ ratios from the $n+r+1$ equations, namely, the proportion (34), that represent the $n$ equations and the $r$ +1 equations $\varphi_{\rho}=0$, in which the differentials $d y_{\nu}$ likewise appear only by way of their ratios. As a result of that elimination, one will get a partial differential equation in the unknown $\Phi$ in which that unknown itself does not occur, and its derivatives $\partial \Phi / \partial y_{\nu}$ occur only by way of their ratios.

If one then imagines determining one of the quantities $y$ (say, $y_{0}$ ) as a function of the remaining ones by way of the equation $\Phi=0$ and sets:

$$
y_{0}=\psi\left(y_{1}, \ldots, y_{n}\right), \quad \frac{\partial \psi}{\partial y_{\mu}}=-\frac{\partial \Phi}{\partial y_{\mu}}: \frac{\partial \Phi}{\partial y_{0}} \quad(\mu=1,2, \ldots, n)
$$

then that will yield a first-order partial differential equation for $\psi$, say:

$$
\begin{equation*}
F\left(y_{1}, y_{2}, \ldots, y_{n}, \psi, \frac{\partial \psi}{\partial y_{1}}, \frac{\partial \psi}{\partial y_{2}}, \ldots \frac{\partial \psi}{\partial y_{n}}\right)=0 \tag{35}
\end{equation*}
$$

which is the Jacobi-Hamilton differential equation.
With that, we have shown that any field, when regarded in the sense of our definition as a manifold in the domain of the $n+1$ variables $y_{v}$, will fulfill the Jacobi-Hamilton differential equation.

Conversely, if any solution of the Jacobi-Hamilton equation is given in the form:

$$
\Phi\left(y_{0}, y_{1}, \ldots y_{n}\right)=0
$$

then any equation:

$$
\begin{equation*}
\Phi\left(y_{0}, y_{1}, \ldots y_{n}\right)=C=\text { const. } \tag{36}
\end{equation*}
$$

will also give a solution, and one can start from $n+r$ of the $n+r+1$ equations (34) and $\varphi_{\rho}=0$ and determine from them the $n$ ratios of the differentials $d y_{v}$ and the $r$ ratios of the quantities $\lambda_{\rho}$, one of which can be chosen arbitrarily. The last of those $n+r+1$ equations will be fulfilled in any event on the basis of the assumption (35). Now since the equation:

$$
\sum_{v=0}^{n} \frac{\partial \Phi}{\partial y_{v}} \delta y_{v}=0
$$

is valid for the advance along the manifold (36), it will follow from the proportion (34) that:

$$
\begin{equation*}
\sum_{v=0}^{n} \frac{\partial \Phi}{\partial d y_{v}} \delta y_{v}=0 \tag{37}
\end{equation*}
$$

in which $\Omega$ is naturally defined with the multipliers that were just determined such that one will still not know whether it is the quantity that was previously denoted in that way.

In any event, any point of the manifold (36) is associated with a certain direction of advance on that manifold in that way by the ratios of the $d y_{v}$, and there will be a certain simple manifold that starts from any point and for which equation (34) is true. The simple manifold (36) will then be divided into $\infty^{n-1}$ simply-extended ones then.

The proportion (34) further says that there exists a quantity $\mu$ that fulfills the equations:

$$
\frac{\partial \Phi}{\partial y_{v}}=\mu \frac{\partial \Omega}{\partial y_{v}} \quad(v=0,1, \ldots, n) .
$$

If one sets:

$$
\mu \Omega=\Omega^{\prime}, \quad \frac{\partial \Omega^{\prime}}{\partial d y_{v}}=\Omega_{v}
$$

then it will generally follow that:

$$
\frac{\partial \Omega_{v}}{\partial y_{\mu}}=\frac{\partial \Omega_{\mu}}{\partial y_{v}} \quad(\mu, v=0,1, \ldots, n)
$$

Those equations are based upon the intuition that one can determine the ratios of the differentials $d y_{v}$ and the quantities $\lambda_{\rho}$ that occur in $\Omega$ as functions of the quantities $y_{v}$ in the way that was given above. They can be regarded as independent variables due to the arbitrary constant $C$ that appears in equation (36).

Now let $t$ be a parameter that varies along one of the simple manifolds, say the value of a function of position $t\left(y_{0}, y_{1}, \ldots, y_{n}\right)$, such that the quantities $y_{v}^{\prime}=d y_{v}: d t$ can be considered to be functions of the arguments $y_{\mu}$. If one generally introduces $y_{v}^{\prime}$ in place of $d y_{v}$ in $\Omega^{\prime}$, which will take that expression to $\Omega^{\prime \prime}$, then the derivatives with respect to the differentials will be equal to ones that are taken with respect to the quantities $y_{v}^{\prime}$ :

$$
\frac{\partial \Omega^{\prime}}{\partial d y_{v}}=\frac{\partial \Omega^{\prime \prime}}{\partial d y_{v}^{\prime}}=\Omega_{v}
$$

Furthermore, due to the homogeneity of the expression $\Omega^{\prime \prime}$, one has the equations:

$$
\begin{equation*}
\frac{\partial \Omega^{\prime \prime}}{\partial d y_{\mu}}=\sum_{v=0}^{n} \frac{\partial^{2} \Omega}{\partial y_{v}^{\prime} \partial y_{\mu}} y_{v}^{\prime}, \quad \sum_{v=0}^{n} y_{v}^{\prime} \frac{\partial^{2} \Omega^{\prime \prime}}{\partial y_{v}^{\prime} \partial y_{\mu}^{\prime}}=0 . \tag{38}
\end{equation*}
$$

Now, one obviously has:

$$
\begin{array}{r}
\frac{\partial \Omega_{0}}{\partial y_{0}}=\frac{\partial^{2} \Omega}{\partial y_{0}^{\prime} \partial y_{0}}+\frac{\partial^{2} \Omega}{\partial y_{0}^{\prime} \partial y_{0}^{\prime}} \frac{\partial y_{0}^{\prime}}{\partial y_{0}}+\frac{\partial^{2} \Omega}{\partial y_{0}^{\prime} \partial y_{1}^{\prime}} \frac{\partial y_{1}^{\prime}}{\partial y_{0}}+\cdots, \\
\frac{\partial \Omega_{0}}{\partial y_{1}}=\frac{\partial \Omega_{1}}{\partial y_{0}}=\frac{\partial^{2} \Omega^{\prime \prime}}{\partial y_{1}^{\prime} \partial y_{0}}+\frac{\partial^{2} \Omega}{\partial y_{1}^{\prime} \partial y_{0}^{\prime}} \frac{\partial y_{0}^{\prime}}{\partial y_{0}}+\frac{\partial^{2} \Omega}{\partial y_{1}^{\prime} \partial y_{1}^{\prime}} \frac{\partial y_{1}^{\prime}}{\partial y_{0}}+\cdots,
\end{array}
$$

$$
\begin{array}{r}
\frac{\partial \Omega_{1}}{\partial y_{1}}=\frac{\partial^{2} \Omega}{\partial y_{1}^{\prime} \partial y_{1}}+\frac{\partial^{2} \Omega}{\partial y_{1}^{\prime} \partial y_{0}^{\prime}} \frac{\partial y_{0}^{\prime}}{\partial y_{1}}+\frac{\partial^{2} \Omega}{\partial y_{1}^{\prime} \partial y_{1}^{\prime}} \frac{\partial y_{1}^{\prime}}{\partial y_{1}}+\cdots, \\
\frac{\partial \Omega_{1}}{\partial y_{2}}=\frac{\partial \Omega_{2}}{\partial y_{1}}=\frac{\partial^{2} \Omega^{\prime \prime}}{\partial y_{2}^{\prime} \partial y_{1}}+\frac{\partial^{2} \Omega}{\partial y_{2}^{\prime} \partial y_{0}^{\prime}} \frac{\partial y_{0}^{\prime}}{\partial y_{1}}+\frac{\partial^{2} \Omega}{\partial y_{2}^{\prime} \partial y_{1}^{\prime}} \frac{\partial y_{1}^{\prime}}{\partial y_{1}}+\cdots,
\end{array}
$$

etc. When one starts from the first of equations (38), it will follow from this that:

$$
\frac{\partial \Omega^{\prime \prime}}{\partial y_{0}}=\sum_{v=0}^{n} \frac{\partial^{2} \Omega^{\prime \prime}}{\partial y_{v}^{\prime} \partial y_{0}} y_{v}^{\prime}=\sum_{v=0}^{n} \frac{\partial \Omega_{0}}{\partial y_{v}} y_{v}^{\prime}=\frac{d \Omega_{0}}{d t}
$$

because the quantities $\partial y_{v}^{\prime}: \partial y_{0}$ on the right-hand side vanish due to the second of equations (38). One likewise finds in general that:

$$
\frac{\partial \Omega^{\prime \prime}}{\partial y_{v}}-\frac{d}{d t} \frac{\partial \Omega^{\prime \prime}}{\partial y_{v}^{\prime}}=0
$$

or also:

$$
\frac{\partial \Omega^{\prime}}{\partial y_{v}}-\frac{d}{d t} \frac{\partial \Omega^{\prime}}{\partial y_{v}}=0 .
$$

Now since $\Omega^{\prime}$ is an expression of the same form as $\Omega$, only it is formed with other multipliers, that will show that the simple manifolds that are constructed are extremals of the original variational problem. The families that are constructed from them, which correspond to a well-defined value of the constant in the equation:

$$
\begin{equation*}
\Phi\left(y_{0}, y_{1}, \ldots, y_{n}\right)=C, \tag{39}
\end{equation*}
$$

are fields in the sense of our definition, due to the relation (37), or:

$$
\sum_{v=0}^{n} \frac{\partial \Omega^{\prime}}{\partial d y_{v}} \delta y_{v}=0 .
$$

Any solution of the Jacobi-Hamilton equation of the form (39) then leads to a family of fields that each correspond to a value of the constant $C$.

As an example, we consider the shortest line in space. We set:

$$
n=3, \quad r=0, \quad \Omega=\lambda\left(d u-\sqrt{d x^{2}+d y^{2}+d z^{2}}\right),
$$

and find that along the extremals:

$$
d \lambda=0, \quad d\left(\frac{d x}{d u}\right)=d\left(\frac{d y}{d u}\right)=d\left(\frac{d z}{d u}\right)=0 .
$$

The extremals are then lines with assigned lengths. Due to the proportion:

$$
\begin{equation*}
\frac{\partial \Phi}{\partial u}: \frac{\partial \Phi}{\partial x}: \frac{\partial \Phi}{\partial y}: \frac{\partial \Phi}{\partial z}=-1: \frac{d x}{d u}: \frac{d y}{d u}: \frac{d z}{d u} \tag{40}
\end{equation*}
$$

so

$$
\left(\frac{\partial \Phi}{\partial u}\right)^{2}=\left(\frac{\partial \Phi}{\partial x}\right)^{2}+\left(\frac{\partial \Phi}{\partial y}\right)^{2}+\left(\frac{\partial \Phi}{\partial z}\right)^{2}
$$

or when one imagines calculating $u=\psi(x, y, z)$ from the equation:

$$
\begin{equation*}
\Phi(x, y, z, u)=C \tag{41}
\end{equation*}
$$

the Jacobi-Hamilton differential equation will then imply that:

$$
\begin{equation*}
\left(\frac{\partial \psi}{\partial x}\right)^{2}+\left(\frac{\partial \psi}{\partial y}\right)^{2}+\left(\frac{\partial \psi}{\partial z}\right)^{2}=1 \tag{42}
\end{equation*}
$$

and the proportion (40) will yield:

$$
\begin{equation*}
\frac{\partial \psi}{\partial x}=\frac{d x}{d u}, \quad \frac{\partial \psi}{\partial y}=\frac{d y}{d u}, \quad \frac{\partial \psi}{\partial z}=\frac{d z}{d u} . \tag{43}
\end{equation*}
$$

The lines of the field will then be intersected normally by the surfaces $u=$ const. or $\psi=$ const. If one places the starting point from which the length $u$ is measured in a suitable way then one will get the family of fields that correspond to the different values of the constant $C$ in equation (41).

The fact that any solution of equation (42) will produce such a family of fields is clear: One needs only to measure out lengths along the normals to any surface $y=$ const. that are measured from any other such surface in order to obtain a family of three-fold manifolds in the space of four variables $x, y, z, u$ that fulfill the characteristic equation of the field. Indeed, equations (43) imply that:

$$
\delta u=\frac{\partial \psi}{\partial x} \delta x+\frac{\partial \psi}{\partial y} \delta y+\frac{\partial \psi}{\partial z} \delta z=\frac{d x}{d u} \delta x+\frac{d y}{d u} \delta y+\frac{d z}{d u} \delta z
$$

or

$$
\frac{\partial \Omega}{\partial d u} \delta u+\frac{\partial \Omega}{\partial d x} \delta x+\frac{\partial \Omega}{\partial d y} \delta y+\frac{\partial \Omega}{\partial d z} \delta z=0
$$

## § 5. - Focal points and the Weierstrass theory.

In the Mayer extremum problem, one compares a piece $\mathfrak{C}$ of an extremal with a piece of another simple manifold $\mathfrak{M}$ in the domain of the $n+1$ quantities along which those quantities have the values $Y_{v}$ and might be functions of a parameter $\tau$ that fulfills the $r+1$ equations:

$$
\varphi_{\rho}\left(Y_{0}, Y_{1}, \ldots, Y_{n}, \frac{d Y_{0}}{d \tau}, \frac{d Y_{1}}{d \tau}, \ldots, \frac{d Y_{n}}{d \tau}\right)=0
$$

and the Weierstrass theory will show that when one considers $y_{0}$ and $Y_{0}$ at the endpoint of the extremal arc $\mathfrak{C}$ and at those of the manifold $\mathfrak{M}$, the difference $y_{0}-Y_{0}$ will possess a fixed sign for all manifolds $\mathfrak{M}$ that are characterized in a certain way, which would then make $y_{0}$ take the form of an extremum.

The extremals $\mathfrak{C}$ belong to a field whose extremals might be denoted by the symbols $y_{v}$ and $d y_{v}$. Let:

$$
y_{v}=\theta_{v}\left(t, a_{1}, a_{2}, \ldots, a_{n-1}\right) \quad(n=0,1, \ldots, n)
$$

in it, as before. The Weierstrass construction, whose possibility we initially assume, but will later verify under certain assumptions, consists of looking for an extremal of the field for any system of values $Y_{v}$ that is constructed on $\mathfrak{M}$ and a location that is associated with it for which the $n$ equations:

$$
y_{\mu}=Y_{\mu} \quad(\mu=1,2, \ldots, n)
$$

are valid, such that $y_{\mu}$ takes the form of a function of $\tau$, and the equations:

$$
\begin{equation*}
\frac{d y_{\mu}}{d \tau}=\frac{d Y_{\mu}}{d \tau} \tag{44}
\end{equation*}
$$

can be added to them, while $y_{0}-Y_{0}$ generally remains non-zero. Furthermore, since the system of values $y_{\nu}$ belongs to the field, any change in $\tau$ will imply an advance along the field, and its characteristic will then yield:

$$
\sum_{v=0}^{n} \frac{\partial \Omega}{\partial d y_{v}} \frac{d y_{v}}{d \tau}=0
$$

or when we set:

$$
\frac{\partial \Omega(y, d y)}{\partial d y_{v}}=\Omega_{v}
$$

and employ equations (44):

$$
\begin{equation*}
\sum_{v=0}^{n} \Omega_{v} \frac{d Y_{v}}{d \tau}+\Omega_{0}\left(\frac{d y_{0}}{d \tau}-\frac{d Y_{0}}{d \tau}\right)=0 . \tag{45}
\end{equation*}
$$

If one further introduces the quantities:

$$
y_{v}^{\prime}=\frac{\partial \theta_{v}}{\partial t},
$$

such that one can set:

$$
\Omega_{v}=\frac{\partial \Omega\left(y, y^{\prime}\right)}{\partial y_{v}^{\prime}},
$$

and if one recalls the identity:

$$
\sum_{v=0}^{n} \Omega_{v} d y_{v}=0, \quad \sum_{v=0}^{n} \Omega_{v} y_{v}^{\prime}=0
$$

then one can regard the expression:

$$
\sum_{v=0}^{n} \Omega_{v} \frac{d Y_{v}}{d \tau}=\sum_{v=0}^{n} \Omega_{v}\left(\frac{d Y_{v}}{d \tau}-y_{v}^{\prime}\right)
$$

as the sum of the linear terms in the Taylor development of the quantity:

$$
\Omega\left(y, \frac{d Y}{d \tau}\right)=\Omega\left(y, \frac{d Y}{d \tau}\right)-\Omega\left(y, y^{\prime}\right),
$$

in which naturally each symbol $y, Y$ should appear with the $n+1$ indices $0,1, \ldots$ that they are assigned; one then develops the quantities $\frac{d Y_{v}}{d \tau}-y_{v}^{\prime}$. It will then follow that the quantity

$$
-\Omega\left(y, \frac{d Y}{d \tau}\right)+\sum_{v=0}^{n} \Omega_{v} \frac{d Y_{v}}{d \tau}=E\left(y, y^{\prime}, \frac{d Y}{d \tau}\right)
$$

with the notation that was chosen by Weierstrass, can be represented as the remainder term in the aforementioned Taylor development and a quadratic form in the quantities $y^{\prime}-\frac{d Y}{d \tau}$, which will make it possible to discuss the sign of that quantity.

Now, equation (45) yields:

$$
\begin{equation*}
\Omega_{0} \frac{d\left(y_{0}-Y_{0}\right)}{d \tau}=E\left(y, y^{\prime}, \frac{d Y}{d \tau}\right)+\Omega\left(y, \frac{d Y}{d \tau}\right)-\Omega\left(Y, \frac{d Y}{d \tau}\right) \tag{46}
\end{equation*}
$$

since the last term has the value zero. Now since all of the quantities $y_{r}$, except for $y_{0}$, coincide with the $Y$ with the same index, one can set:

$$
\Omega\left(y, \frac{d Y}{d \tau}\right)-\Omega\left(Y, \frac{d Y}{d \tau}\right)=-\left(y_{0}-Y_{0}\right) \Psi,
$$

and the quantity $\Psi$ will be finite when the difference $\left|y_{0}-Y_{0}\right|$ does not exceed a certain limit. Therefore, if one restricts the extremal arc $\mathfrak{C}$ by the assumption that $\Omega_{0}$ should not vanish along it then one can write equation (46) as:

$$
\frac{d}{d \tau}\left\{\left(y_{0}-Y_{0}\right) e^{\int \Psi \Omega_{0}^{-1} d \tau}\right\}=\frac{E e^{\int \Psi \Omega_{0}^{-1} d \tau}}{\Omega_{0}}
$$

That equation will give information about the sign of the quantity $y_{0}-Y_{0}$ at the end of the manifold $\mathfrak{M}$ when $E$ has a fixed sign along it and does not vanish everywhere, and furthermore when $y_{0}-$ $Y_{0}$ vanishes at the starting point of $\mathfrak{M}$ or has the sign of the product $E \Omega_{0}$ and its absolute value remains below a certain limit along the manifold $\mathfrak{M}$. The equation shows that the signs of the quantities $y_{0}-Y_{0}$ and $E \Omega_{0}$ are also the same at the endpoint of $\mathfrak{M}$. With that, the extremum is verified.

However, the entire line of reasoning that was followed rests upon the possibility of making the Weierstrass construction. It will be possible when a condition is fulfilled that we can properly refer to as the Jacobi condition.

One sets:

$$
\Delta_{0}=\frac{\partial\left(\theta_{1}, \theta_{2}, \ldots, \theta_{n}\right)}{\partial\left(t, a_{1}, \ldots, a_{n}\right)}=\Delta_{0}\left(t, a_{1}, a_{2}, \ldots, a_{n-1}\right)
$$

and assumes that along the extremal $\mathfrak{C}$, one has:

$$
a_{1}=a_{1}^{0}, \quad a_{2}=a_{2}^{0}, \quad \ldots, \quad a_{n-1}=a_{n-1}^{0} .
$$

Hence, when the quantity $\Delta_{0}\left(t, a_{1}^{0}, a_{2}^{0}, \ldots, a_{n-1}^{0}\right)$ is non-zero along the manifold $\mathfrak{C}$ - and that is the Jacobi condition - one can surround the simple manifold $\mathfrak{C}^{\prime}$, which is defined by the equations:

$$
y_{\mu}=\theta_{\mu}\left(t, a_{1}^{0}, a_{2}^{0}, \ldots, a_{n-1}^{0}\right) \quad(\mu=1,2, \ldots, n)
$$

and represents a projection of $\mathfrak{C}$, with a region $\mathfrak{G}$ with the property that every location in it lies on precisely one of the manifolds $y_{\mu}=\theta_{\mu}\left(t, a_{1}, a_{2}, \ldots, a_{n-1}\right)$, so the values of $t, a_{1}, a_{2}, \ldots, a_{n-1}$ that are associated with that equation will be single-valued functions of position. They will be continuous, along with their derivatives, when we assume the same thing of the functions $\theta_{\mu}$, which is
reasonable. Now the previously-considered manifold $\mathfrak{M}$ gives the system of values $Y_{1}, Y_{2}, \ldots, Y_{n}$, which lies in the interior of the region $\mathfrak{G}$. The values $t, a_{1}, a_{2}, \ldots, a_{n-1}$ that were just defined then take the form of functions of $\tau$ that are continuous, along with their derivatives, and the Weierstrass construction is guaranteed.

Along with the determinant $\Delta_{0}$, we consider all determinants:

$$
\Delta_{v}=\frac{\partial\left(\theta_{0}, \theta_{1}, \ldots, \theta_{v-1}, \theta_{v+1}, \ldots, \theta_{n}\right)}{\partial\left(t, a_{1}, a_{2}, \ldots, a_{n-1}\right)}
$$

that have a simple relationship to it. Namely, from the characteristic equation of the field:

$$
\sum_{v=0}^{n} \Omega_{v} \delta y_{v}=0
$$

when we write one of the symbols:

$$
d t \frac{\partial}{\partial t}, \quad d a_{1} \frac{\partial}{\partial a_{1}}, \ldots, \quad d a_{n-1} \frac{\partial}{\partial a_{n-1}}
$$

as $\delta$, that will imply the equations:

$$
\begin{array}{ll}
\sum_{\mu=1}^{n} \Omega_{\mu} \frac{\partial \theta_{\mu}}{\partial t}=-\Omega_{0} \frac{\partial \theta_{0}}{\partial t}, \\
\sum_{\mu=1}^{n} \Omega_{\mu} \frac{\partial \theta_{\mu}}{\partial a_{\rho}}=-\Omega_{0} \frac{\partial \theta_{0}}{\partial a_{\rho}}, & (\rho=1,2, \ldots, n-1) .
\end{array}
$$

If one then adds the first row in the determinant:

$$
\Omega_{1} \Delta_{0}=\left|\begin{array}{cccc}
\Omega_{1} \frac{\partial \theta_{1}}{\partial t} & \Omega_{1} \frac{\partial \theta_{1}}{\partial a_{1}} & \cdots & \Omega_{1} \frac{\partial \theta_{1}}{\partial a_{n-1}} \\
\frac{\partial \theta_{2}}{\partial t} & \frac{\partial \theta_{2}}{\partial a_{1}} & \cdots & \frac{\partial \theta_{2}}{\partial a_{n-1}} \\
\vdots & \vdots & \cdots & \vdots \\
\frac{\partial \theta_{n}}{\partial t} & \frac{\partial \theta_{n}}{\partial a_{1}} & \cdots & \frac{\partial \theta_{n}}{\partial a_{n-1}}
\end{array}\right|
$$

to the second row when multiplied by $\Omega_{2}$, adds it to third row multiplied by $\Omega_{3}$, etc., then that will give:

$$
\Omega_{1} \Delta_{0}=-\Omega_{0} \Delta_{1}
$$

Similarly, one would generally obtain:

$$
\Omega_{\mu} \Delta_{0}= \pm \Omega_{0} \Delta_{\mu} \quad(\mu=1,2, \ldots, n)
$$

If we then introduce the new assumption for the manifold $\mathfrak{C}$ that all of the quantities $\Omega_{v}$ are nonzero on it then all of the determinants $\Delta_{v}$ will vanish only simultaneously.

That fact leads to the essential content of the Mayer reciprocity law. Now one of the quantities $y_{v}$ is to be preferred in the equations $\varphi_{\rho}=0$, as well as in the concept of a field. Hence, if one poses the problem of extremizing the quantity $y_{1}$ in the same sense that applied to $y_{0}$ before then one can employ precisely the same field as one has up to now in order to establish the extremum. However, $y_{0}$ is preferred in the quantity $\Delta_{0} ; \Delta_{1}$ enters in its place. Now since both of them vanish only at the same time, the Jacobi condition for the extremum of the quantity $y_{1}$ will be simultaneously fulfilled along with the one for the extremum of $y_{0}$. If the former is provided by the manifold $\mathfrak{C}$ then the same thing will be true for the latter extremum.

One understands a focal point for a field to mean a point where $\Delta_{0}=0$. One sees that $\Delta_{1}$ also vanishes in it, so it will also be a focal point for the problem of extremizing the quantity $y_{1}$. Naturally, any of the quantities $y_{n}$ can enter in place of $y_{1}$ in all of those considerations. That is Mayer's theorem in its full generality.

## § 6. - Special types of fields.

As was the case in § 2 for the simplest problem in the calculus of variations, one can also distinguish special types of fields in the general Mayer problem. In order to do that, as discussed in § $\mathbf{6 1}$ of my textbook, it is necessary to determine the arbitrary constants that the extremals depend upon.

We first observe the identity:

$$
\begin{equation*}
\sum_{v=1}^{n} d y_{v}\left(\frac{\partial \Omega}{\partial y_{v}}-d \frac{\partial \Omega}{\partial d y_{v}}\right)=0 \tag{47}
\end{equation*}
$$

which follows from the homogeneity of the expression $\Omega$. It shows that one can drop one of the $n$ +1 equations:

$$
\frac{\partial \Omega}{\partial y_{v}}-d \frac{\partial \Omega}{\partial d y_{v}}=0
$$

say the one that relates to $v=n$ when $d y_{n}$ is non-zero. We make that assumption, introduce $y_{n}$ as the independent variable accordingly, and set:

$$
x=y_{n}, \quad \frac{\varphi_{\rho}}{d x}=\bar{\varphi}_{\rho}, \quad \frac{\Omega}{d x}=\bar{\Omega},
$$

$$
\frac{d y_{\sigma}}{d x}=y_{\sigma}^{\prime}, \quad \frac{d^{2} y_{\sigma}}{d x^{2}}=y_{\sigma}^{\prime \prime}, \quad(\sigma=0,1, \ldots, n-1)
$$

The barred expressions will then include the differentials $d y_{v}$ only by way of the $y^{\prime}$, and the extremals will be determined by the $n+r+1$ equations:

$$
\begin{array}{cc}
\bar{\varphi}_{\rho}=0 & (\rho=0,1, \ldots, r), \\
\frac{\partial \bar{\Omega}}{\partial y_{\sigma}}-\frac{d}{d x} \frac{\partial \bar{\Omega}}{\partial y_{\sigma}^{\prime}}=0 & (\sigma=0,1, \ldots, n-1) . \tag{49}
\end{array}
$$

If we replace equations (48) with the more general ones:

$$
\frac{d \bar{\varphi}_{\rho}}{d x}=0
$$

then those equations, along with equations (49), will give us a system of $n+r+1$ equations from which the $n+r+1$ quantities $y_{\sigma}^{\prime \prime}, \lambda_{\rho}^{\prime}$ can be determined, so $y_{\sigma}$ and $\lambda_{\rho}$ will prove to be functions of $x$ and $2 n+r+1$ arbitrary constants, say, the initial values of the quantities $y_{\sigma}, y_{\sigma}^{\prime}, \lambda_{\rho}$. However, since the $r+1$ equations (49) must be true, that will reduce the number of those constants by $r+$ 1 to $2 n$. Furthermore, since it is clear from the form of equations (49) that the multipliers $\lambda_{\rho}$ can be multiplied by an arbitrary constant factor without altering the values of the quantities $y_{\sigma}$, only $2 n-1$ constants are essential in order to specify the extremal as a simple manifold in the domain of the quantities $x, y_{\sigma}$, say the initial values of the $n$ quantities $y_{\sigma}$ and the $n-1$ quantities $\lambda_{0}: \lambda_{r}$, $\lambda_{1}: \lambda_{r}, \ldots, \lambda_{r-1}: \lambda_{r}, y_{r+1}^{\prime}, y_{r+2}^{\prime}, \ldots, y_{n-1}^{\prime}$. We call the latter the second group of constants.

If one would now like to select a field from the totality of extremals that are defined in that way then precisely $n-1$ of those $2 n-1$ constants would have to remain free parameters. We then classify the possibilities that exist in such a way that the initial values of the quantities $y_{\sigma}$ are either fixed or constrained to a $k$-dimensional manifold, where one can have that $k=1,2, \ldots, n-1$. $n-$ $1-k$ of the constants in the second group are still free then, but the remaining one are established by the demand that the family of extremals to be represented must define a field. In order for that to be the case, from § 3 , it will suffice for the characteristic equation to be fulfilled for a certain domain of the constants that remain free. That equation will read:

$$
\begin{equation*}
\Omega_{n} \delta x+\sum_{\sigma=0}^{n-1} \Omega_{\sigma} \delta y_{\sigma}=0 \tag{50}
\end{equation*}
$$

with the current notations, so as a result of the identity (47), one can set:

$$
\Omega_{n}=-\sum_{\sigma=0}^{n-1} y_{\sigma} \Omega_{\sigma}, \quad \Omega_{\sigma}=\frac{\partial \bar{\Omega}}{\partial y_{\sigma}^{\prime}}=\frac{\partial \Omega}{\partial d y_{\sigma}} .
$$

Now let the initial values of the $n$ quantities $y_{\sigma}$ be established by any initial value $x_{0}$. If the symbol $\delta$ then refers to the change in the remaining $n-1$ integration constants then one will obviously have:

$$
\delta x=\delta y_{\sigma}=0
$$

at the location $x=x_{0}$. Equation (50) will then be fulfilled for an $n-1$-dimensional region. From $\S$ 3, the extremals that go through a fixed initial location will then define a field as long as one further condition is fulfilled. If the equations:

$$
x=t, \quad y_{\sigma}=\theta_{\sigma}\left(t, a_{1}, a_{2}, \ldots, a_{n-1}\right),
$$

in which $a_{1}, a_{2}, \ldots$ mean the integration constants that remain free, represent the family of extremals thus-defined then they must generally be soluble for $x, a_{1}, \ldots, a_{n-1}$. The functional determinant:

$$
\frac{\partial\left(\theta_{0}, \theta_{1}, \ldots, \theta_{n-1}\right)}{\partial\left(x, a_{1}, \ldots, a_{n-1}\right)}
$$

cannot vanish identically. Up to now, that condition, which we would like to refer to as the independence condition from now on, does not prove to be fulfilled in general. We must then show that it is fulfilled in each individual case, which is usually easy.

More generally, let the initial values of $y_{\sigma}$ not be fixed, but only given as functions of $k$ parameters $t_{1}, t_{2}, \ldots, t_{k} . n-1-k$ of the integration constants in the second group shall remain freely available, when that is possible. One must then set:

$$
\begin{equation*}
\delta x=0, \quad \delta y_{\sigma}=\sum_{\tau=1}^{k} a_{\sigma \tau} d t_{\tau} \quad(\sigma=0,1, \ldots, n-1) \tag{51}
\end{equation*}
$$

for $x=x_{0}$, which will make the $a$ functions of the parameters $t_{1}, t_{2}, \ldots, t_{k}$. The characteristic equation (50) will then be fulfilled when the $k$ equations:

$$
\begin{equation*}
\sum_{\sigma=0}^{n-1} a_{\sigma \tau} \Omega_{\sigma}=0 \quad(\tau=1,2, \ldots, k) \tag{52}
\end{equation*}
$$

are true for $x=x_{0}$. In that way, $k$ of the integration constants in the second group will be functions of the $n-1-k$ remaining ones, which will remain free and might be denoted by $t_{k+1}, \ldots, t_{n-1}$, as well as establishing the quantities $t_{1}, t_{2}, \ldots, t_{k}$, and the characteristic equation will be valid for an $n$ - 1 -dimensional region in which the quantities $t_{1}, t_{2}, \ldots, t_{n-1}$ vary freely. From $\S \mathbf{3}$, it will then be true in general for the extremal system thus-obtained, and that system will be a field when a
further independence condition is fulfilled, which we shall leave unspecified, as we did above for the general theory.

The fixed initial values of all of the quantities $y_{\sigma}$ then characterize $n$ types of fields in total, corresponding to the values $k=1,2, \ldots, n-1$ and the special case that was considered before. They correspond to the various types of integrals of the first-order partial differential equations that the Cauchy method of integration implies.

## § 7. - Examples for § 6.

We next consider the shortest line in space, in which $y_{0}, y_{1}, y_{2}$ mean the rectangular coordinates, while $x$ means the length. One then has:

$$
\Omega=d x-\sqrt{d y_{0}^{2}+d y_{1}^{2}+d y_{2}^{2}},
$$

since the multiplier is constant, as was pointed out before in $\S \mathbf{4}$, so one can set it equal to 1 . With the notations that were introduced, one further has:

$$
\begin{align*}
& \bar{\Omega}=1-\sqrt{y_{0}^{\prime} y_{0}^{\prime}+y_{1}^{\prime} y_{1}^{\prime}+y_{2}^{\prime} y_{2}^{\prime}}=0, \quad n=3,  \tag{53}\\
& \Omega_{0}=\frac{d y_{0}}{d x}=y_{0}^{\prime}, \quad \Omega_{1}=y_{1}^{\prime}, \quad \Omega_{2}=y_{2}^{\prime} .
\end{align*}
$$

If one next sets the initial values $y_{0}^{0}, y_{1}^{0}, y_{2}^{0}$ equal to constants then one will get a pencil of lines through a fixed point as the field.

If those values are functions of $t_{1}$ for which:

$$
\delta y_{0}=a_{01} d t_{1}, \quad \delta y_{1}=a_{11} d t_{1}, \quad \delta y_{2}=a_{21} d t_{1}
$$

then equation (52), with the values (53), will give:

$$
a_{01} y_{0}^{\prime}+a_{11} y_{1}^{\prime}+a_{21} y_{2}^{\prime}=0 .
$$

One will then get the family of lines that radiate perpendicular to a curve as a field of the second kind.

Finally, if the starting point $\left(y_{0}^{0}, y_{1}^{0}, y_{2}^{0}\right)$ lies on a surface along which:

$$
\delta y_{0}=a_{01} d t_{1}+a_{02} d t_{2}, \quad \delta y_{1}=a_{11} d t_{1}+a_{12} d t_{2}, \quad \delta y_{2}=a_{21} d t_{1}+a_{22} d t_{2}
$$

then equations (52) will yield:

$$
a_{01} y_{0}^{\prime}+a_{11} y_{1}^{\prime}+a_{21} y_{2}^{\prime}=0, \quad a_{02} y_{0}^{\prime}+a_{12} y_{1}^{\prime}+a_{22} y_{2}^{\prime}=0,
$$

i.e., one will get the lines that are perpendicular to the surface that runs through the starting point $\left(y_{0}^{0}, y_{1}^{0}, y_{2}^{0}\right)$.

One will then, in fact, have three types of fields that correspond to the value $n=3$. The independence condition is fulfilled since the system of lines fill up spatial regions in all three cases. If one would like to regard the fields as three-fold-extended manifolds in four-dimensional space, corresponding to the general theory, then one would only have to define, say, the distance to each point on each line as measured from the position of the starting point as the fourth coordinate. The first of equations (51) will then be fulfilled at the starting point, as well. If one increases that length by a constant then one will obtain the families of fields that were imagined at the end of § $\mathbf{5}$.

As a second example, we consider the isoperimetric problem in the plane, in which $y_{1}, y_{2}$ are rectangular coordinates, and seek the extremum of the integral:

$$
y_{0}=\int F\left(y_{1}, y_{2}, d y_{1}, d y_{2}\right)
$$

for a given value of the integral:

$$
y_{3}=x=\int G\left(y_{1}, y_{2}, d y_{1}, d y_{2}\right) .
$$

In those expressions, $F$ and $G$ are homogeneous of degree one in the differentials. We must obviously set:

$$
n=3, \quad \Omega=\lambda_{0}\left(d y_{0}-F\left(y_{1}, y_{2}, d y_{1}, d y_{2}\right)\right)+\lambda_{1}\left(d y_{3}-G\left(y_{1}, y_{2}, d y_{1}, d y_{2}\right)\right),
$$

and we will find that:

$$
\Omega_{1}=-\frac{\partial H}{\partial y_{1}^{\prime}}, \quad \Omega_{2}=-\frac{\partial H}{\partial y_{2}^{\prime}},
$$

when:

$$
H=\lambda_{0} F\left(y_{1}, y_{2}, y_{1}^{\prime}, y_{2}^{\prime}\right)+\lambda_{1} G\left(y_{1}, y_{2}, y_{1}^{\prime}, y_{2}^{\prime}\right) .
$$

The equations:

$$
\frac{\partial H}{\partial y_{1}}-\frac{d}{d x} \frac{\partial H}{\partial y_{1}^{\prime}}=0, \quad \frac{\partial H}{\partial y_{2}}-\frac{d}{d x} \frac{\partial H}{\partial y_{2}^{\prime}}=0
$$

are true for planar extremals.
If one considers the set of all of them that go through the fixed point $\left(y_{1}^{0}, y_{2}^{0}\right)$ then it will generally be consistent with the sense of the extremum problem to calculate the integrals $y_{0}$ and $y_{3}$ from that point onward, such that:

$$
\begin{equation*}
y_{0}^{0}=x_{0}=0 . \tag{54}
\end{equation*}
$$

$\infty^{2}$ or $\infty^{n-1}$ planar extremals go through the fixed point. One will get the field in four-dimensional space when one imagines that the values of $x$ and $y_{0}$ along the curves amount to further coordinates.

We will get a second type of special field when we constrain the point $\left(y_{1}^{0}, y_{2}^{0}\right)$ to a planar curve $\mathfrak{C}$ along which:

$$
\begin{equation*}
\delta y_{1}=a_{11} d t_{1}, \quad \delta y_{2}=a_{21} d t_{1} \tag{55}
\end{equation*}
$$

Equations (54) might once more remain valid, such that $a_{01}=0$. Equation (52) will then give:

$$
a_{11} \frac{\partial H}{\partial y_{1}^{\prime}}+a_{21} \frac{\partial H}{\partial y_{2}^{\prime}}=0
$$

or also, from (55):

$$
\frac{\partial H}{\partial y_{1}^{\prime}} \delta y_{1}+\frac{\partial H}{\partial y_{2}^{\prime}} \delta y_{2}=0
$$

The field will be defined by the planar extremals that lie transversal to the curve $\mathfrak{C}$. The integrals $y_{0}$ and $y_{3}$ will be defined from the curve $\mathfrak{C}$ onward. The manifold of the extremals is once more two-fold, as it must be. I have employed that type of field in § $\mathbf{3 9}$ of my textbook in order to verify the extremum for the isoperimetric problem with a variable limit.


[^0]:    $\left({ }^{1}\right)$ Rendiconti del circolo matematico di Palermo 31 (1911).
    $\left(^{2}\right)$ Sitzungsberichte der Wiener Akademie (1911).

