LINE GEOMETRY

AND ITS APPLICATIONS

BY

G. KOENIGS

PROFESSOR SUPPLIANT AT THE COLLÈGE DE FRANCE

Translated by D. H. Delphenich

COORDINATES – LINEAR SYSTEMS – INFINITESIMAL PROPERTIES OF FIRST ORDER

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INTRODUCTION

In any branch of geometry that touches upon the most essential points of all of the other branches, and whose development has been crucial to the evolution of science during the entire first part of this century, it would seem quite difficult to give the name of the inventor with any certainty. It is to Plücker that one generally attributes the glory of that invention, and meanwhile neither the idea of a congruence of lines nor likewise that of a complex has received its first proper acknowledgment. The entire world recognizes that the properties of congruences go back to the early research in geometrical optics; however, as far as complexes are concerned, one seems much too disposed to forget that Malus was the first to conceive of them in his *Traité d'Optique*, and that one will arrive at a proposition of paramount importance in this subject that was ultimately mentioned by Chasles in an interesting Mémoire de Transon on the grouping of lines in a complex into congruences of normals to a surface. It is very remarkable that the proposition of Malus strongly touches upon another line of thinking that we will have occasion to speak of, and which has been developed in a magisterial fashion by Sophus Lie.

One will find the exact citations that corroborate the current beliefs later on in the historical part. Nevertheless, it is to Plücker that goes the immortal credit for having foreseen the role of the line in geometry and having, if not practiced, then at least indicated, a method for grouping the great principles of projective geometry that *Chasles* called *homography and duality* under more advanced laws.

However, Plücker was not given the honor of reaping the fruits of his discovery. The task of making them prosper and ripen fell on no less than the great talents of a universally esteemed geometer whose was just as celebrated in analysis. Klein has recalled the ideas of Plücker by appealing to the methods of modern algebra. The symmetry and elegance of his results, notably as far as quadratic complexes are concerned, makes him justly deserving of the admiration of geometers.

We will have occasion, in the course of this study, to mention other names that are very justly worth of being cited; however, the work of Sophus Lie on that branch of geometry deserves especial mention. That illustrious geometer has established the closest links between the geometry of Plücker and the theory of differential equations; he has, in a sense, transported a doctrine that might appear, on first glance, to be almost exclusively algebraic to the transcendental domain.

I will stop with the names that I just cited in this introduction, in order to not duplicate the historical notice that accompanies this memoir. It is certain that line geometry owes much to Cayley, Sylvester, Möbius, Chasles, and Battaglini, but the three names of Plücker, Klein, and Sophus Lie characterize, in a sense, three phases of the doctrine of the straight line, and this is why I have placed them at the forefront of the present study ([1]).

([1]) This work is a partial reproduction of a course that I taught in 1887-1888 at the Collège de France.

CHAPTER I.

THE COORDINATES OF THE STRAIGHT LINE. GENERALITIES.

Dualistic and projective character of line geometry. – Double definition of Plückerian coordinates. – The fundamental quadratic form $\omega(z)$. – The polar form. – Summary of purely dualistic notions. – Linear transformation. – Pencils, sprays, and planar systems. – Ruled systems and their classification.

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1. In the beginning of his research, Plücker himself insisted on the double dualistic and projective character of the space of lines.

One may consider only the points that comprise a figure in geometry. By transforming it homographically, one obtains an analogous figure that is defined immediately by its points. This is what one expresses by saying that *the homographic transform of a point-like figure is another point-like figure.*

On the contrary, if one considers the planes that converge to generate that figure then one will have a *planar figure; its homographic transform will be another planar figure.* One can summarize these remarks by saying that the *point-like space and the planar space are transformed into spaces of* the same type, *respectively, under any* homographic *transformation.*

Now, perform a dualistic transformation – for example, a transformation by reciprocal polars. Any point-like figure will be changed into a planar figure and any planar figure, into a point-like figure.

One may summarize that double remark by saying that *the point-like space and the planar space are transformed into spaces of* the opposite type, *respectively, under any* dualistic *transformation.*

However, one must remember that duality was placed next to homography by Chasles in the first quarter of this century, and that the subsequent progress has served only to accentuate the importance, and at the same time, the similitude of these two fundamental transformations. One then understands that there will then be some interest in finding a conception – i.e., a mode of definition – of the figures that remain unaltered as a result of one or the other of these transformations.

If we consider not just the points that comprise a figure, nor even the planes that generate it, but in fact the lines that enter into its construction then we will obtain a new mode of definition that we will characterize by saying that the figure is *ruled.* The ruled figure must then be placed alongside the point-like and planar figures. However, the advantage of this mode of definition appears immediately if one observes that a line has a line for its transform *under duality, as well as homography*, since it results immediately that the transform of a ruled figure, whether by duality or by homography, is another ruled figure, which is what one expresses by saying that the *ruled space* is transformed into a space of the same type *by homography*, as well as *by duality*.

The theory of ruled figures is therefore, in a sense, the supreme expression of the grand evolution of geometry that was inaugurated by Poncelet, Gergonne, and Chasles, and which, if one does not stop, tends, on the contrary, to penetrate almost into transcendental geometry.

Any theorem concerning a figure that is point-like, planar, or ruled, resp., may be called *point-like, planar, or ruled*, resp. It is clear that any theorem that is not ruled gives rise to a conjugate proposition, namely, the one that one deduces by reciprocal polars. From this, one derives the name *geometry in a double role* that serves to recall the habit that some geometers have of associating any non-ruled theorem with its conjugate theorem. This double meaning disappears when one uses lines, so one of them will suffice for both propositions. One immediately sees an example of this in the geometry of the spray (gerbe) and in that of the planar system.

In order to make the dominant idea of this paragraph as clear as possible, consider a curve in space. One might first regard it as a set of points that depend upon one parameter, namely, the points of the curve. One might also regard it as a set of planes that depend upon the same parameter, namely, the osculating planes. Finally, one might regard it as a set of lines that always depend upon that same parameter, namely, the tangents to the curve. The knowledge of any arbitrary one of these sets will suffice to define all of the other ones by means of differential operations that are simple to execute. Nevertheless, a detailed study of geometric transformations has shown that there is good reason to distinguish one from the other, and to direct one's attention, depending on the case, to either one of them or the other, although they are, in fact, inseparable. Thus, provisionally represent the set of points of a curve, the set of osculating planes, and the set of its tangents, by E_p, E_π, E_d, respectively. If one performs a homographic transformation then each of these sets will be transformed into another such set E'_p, E'_π, E'_d, resp. On the contrary, performing a dualistic transformation on E_p will change it into a system E'_π, and E_π will change into the system E'_p that is attached to E'_π; however, by comparison, the system E_d will change into the system E'_d. Therefore, the advantage of defining the sets attached to a curve by means of the set E'_d of tangents is that the definition preserves its character under duality, as well as under homography. Any theorem concerning a system E_p, for example, will have a corresponding theorem in a system E_π; however, if one translates the theorem in such a way that the system E_d (which is attached to E_p) figures only in its statement then the proposition will be found to coincide with the conjugate proposition.

One may similarly define a surface, not only by its points or its tangent planes, but by its tangent lines; one is then led to new properties that show the advantage of the method.

2. A line itself possesses two modes of generation: It is the locus of a point and it is also the locus of a plane that turns around it. Plücker used the term *ray* to describe the line in question as a locus of points and *axis* when it is regarded as the locus of planes. The word "axis" is employed in both senses, and, on the other hand, the distinction is so inessential that we shall find no advantage in elaborating upon these locutions. To be honest, it makes no difference whether a line is regarded as a locus of points or planes; it is naturally the one or the other, and it is not up to line geometry to establish such a distinction in every case. Even more, it might be vain, since it remains indifferent to any dualistic transformation. The distinction that was established by Plücker thus points to the imperfection in his method, which never liberated him from the encumbrance of considering point-like spaces and planar spaces. One finds no such thing in the work of

Klein. All of the elements that one encounters there are *dualistic in themselves* – i.e., they transform into the same type of elements under duality – and this must be our preoccupation from the outset when we define the coordinates. It will seem at first that we have discarded that rule, but we shall not hesitate to come back to it.

3. Consider a point-like space that is referred to point-like homogeneous coordinates. Let x_1, x_2, x_3, x_4 be the coordinates of a point x, and let:

$$\xi_1 x_1 + \xi_2 x_2 + \xi_3 x_3 + \xi_4 x_4 = 0 \tag{1}$$

be the equation of a plane. The quantities ξ_1, ξ_2, ξ_3, ξ_4 will be the homogeneous coordinates of that plane, and equation (1) expresses the notion that the point x and the plane ξ are *united*; i.e., the point is in the plane.

Take two planes ξ, η; these planes intersect along a line D, and if one sets:

$$\rho\, p_{ik} = \xi_i\, \eta_k - \eta_i\, \xi_k , \tag{2}$$

where ρ is a coefficient of proportionality, then the planes that are guided by the line D and by the summits of the tetrahedron of reference will have the equations (in the current coordinates X_i):

$$\left\{\begin{array}{l} \quad * \quad + p_{12}X_2 + p_{13}X_3 + p_{14}X_4 = 0, \\ p_{21}X_1 + \quad * \quad + p_{23}X_3 + p_{24}X_4 = 0, \\ p_{31}X_1 + p_{32}X_2 + \quad * \quad + p_{34}X_4 = 0, \\ p_{41}X_1 + p_{42}X_2 + p_{43}X_3 + \quad * \quad = 0. \end{array}\right. \tag{3}$$

If one develops the zero determinant:

$$0 = \Delta = \begin{vmatrix} \xi_1 & \xi_2 & \xi_3 & \xi_4 \\ \eta_1 & \eta_2 & \eta_3 & \eta_4 \\ \xi_1 & \xi_2 & \xi_3 & \xi_4 \\ \eta_1 & \eta_2 & \eta_3 & \eta_4 \end{vmatrix}$$

then one will find:

$$\Delta = 2(p_{12}\, p_{34} + p_{13}\, p_{42} + p_{14}\, p_{23}) = 0. \tag{4}$$

Conversely, take six quantities p_{12}, p_{13}, p_{14}, p_{34}, p_{42}, p_{23} that are linked by equation (4) and form equations (3) by agreeing that $p_{ki} = - p_{ik}$. One verifies by a simple calculation that, *by virtue of* (4), the four planes (3) intersect along a common line D. One further verifies quite easily that if one makes two planes ξ, η pass through this line then the binomial (ξ_i, $\eta_k - \eta_i\, \xi_k$) will be proportional to p_{ik} . Therefore, six quantities:

$$p_{12}, p_{13}, p_{14}, p_{34}, p_{42}, p_{23} ,$$

which are linked by the equation:

$$p_{12}\,p_{34} + p_{13}\,p_{42} + p_{14}\,p_{23} = 0, \tag{5}$$

will completely define a line by means of equations (3), where it is intended that $p_{ki} = -p_{ik}$. However, we must hesitate to adopt these six quantities p for coordinates of the line due to the absence of any dualistic character in the definition of these quantities. Indeed, we have obtained them by means of equations (2) and (3), by regarding the line D as the intersection of two or more planes.

In order to eliminate the difficulty, it will suffice to appeal to the correlated definition.

Take two points x, y on the line: Any point of that line will be represented by the coordinates:

$$z_i = l\,x_i + m\,y_i\,,$$

where l, m are two parameters. We seek the trace of that line on the plane $z_\alpha = 0$; upon setting:

$$\sigma q_{ik} = x_i\,y_k \mp y_i\,x_k\,, \tag{6}$$

where σ is a factor of proportionality, we will find that the line cuts the plane $z_\alpha = 0$ at a point with the coordinates:

$$q_{\alpha 1}, \quad q_{\alpha 2}, \quad q_{\alpha 3}, \quad q_{\alpha 4} \qquad\qquad \text{(one sees that } q_{\alpha\alpha} = 0);$$

one will thus have the four points:

$$\left\{\begin{array}{cccc} (0, & q_{12}, & q_{13}, & q_{14}), \\ (q_{21}, & 0, & q_{23}, & q_{24}), \\ (q_{31}, & q_{32}, & 0, & q_{34}), \\ (q_{41}, & q_{42}, & q_{43}, & 0); \end{array}\right. \tag{7}$$

upon developing the zero determinant that is analogous to Δ:

$$\begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \end{vmatrix},$$

one will confirm that the following expression is zero:

$$q_{12}\,q_{34} + q_{13}\,q_{42} + q_{14}\,q_{23} = 0. \tag{8}$$

Conversely, take six quantities q_{12}, q_{13}, q_{14}, q_{34}, q_{42}, q_{23} that are linked by equation (8); a simple calculation proves that, thanks to the one condition (8), the four points (7), where one assumes that $p_{ki} = -p_{ik}$, will bee on the same line D.

4. We are therefore now in the presence of a new system of coordinates q for the line, where that line is now considered to be a locus of points. From what we said above on the character of duality that must be preserved in our exposition, we do not, however, have the right to choose the system of coordinates q like the system of coordinates p. However, it is fortunate that we nevertheless do not have to choose, since *these coordinates are found to be identical*.

Indeed, start with the line D, as represented by equations (3), and express the fact that the line contains the points x and y; we will have:

$$p_{12}\, x_2 + p_{13}\, x_3 + p_{14}\, x_4 = 0,$$
$$p_{12}\, y_2 + p_{13}\, y_3 + p_{14}\, y_4 = 0;$$

from this, one will conclude:

$$\frac{p_{12}}{x_3 y_4 - x_4 y_3} = \frac{p_{13}}{x_4 y_2 - x_2 y_4} = \frac{p_{14}}{x_2 y_3 - x_3 y_2};$$

i.e.:

$$\frac{p_{12}}{q_{34}} = \frac{p_{13}}{q_{42}} = \frac{p_{14}}{q_{23}}.$$

One will likewise have:

$$p_{21}\, x_2 + p_{31}\, x_3 + p_{24}\, x_4 = 0,$$
$$p_{21}\, y_2 + p_{31}\, y_3 + p_{24}\, y_4 = 0,$$

so:

$$\frac{p_{21}}{x_3 y_4 - x_4 y_3} = \frac{p_{23}}{x_4 y_1 - y_4 x_1} = \frac{p_{24}}{x_1 y_3 - y_1 x_3};$$

i.e.:

$$\frac{p_{12}}{q_{34}} = \frac{p_{23}}{q_{14}} = \frac{p_{42}}{q_{13}}.$$

From the third of equations (3), one will likewise deduce that these equal ratios are further equal to p_{34} / q_{12} , so one will finally have:

$$\frac{p_{12}}{q_{34}} = \frac{p_{13}}{q_{42}} = \frac{p_{14}}{q_{23}} = \frac{p_{34}}{q_{12}} = \frac{p_{42}}{q_{13}} = \frac{p_{23}}{q_{14}}, \tag{9}$$

and upon combining formulas (2) and (6), and changing the coefficients of proportionality slightly, we can write:

(10)
$$\begin{cases} r_{12} = \rho(\xi_1\eta_2 - \eta_1\xi_2) = \sigma(x_3y_4 - y_3x_4), \\ r_{13} = \rho(\xi_1\eta_3 - \eta_1\xi_3) = \sigma(x_4y_2 - y_4x_2), \\ r_{14} = \rho(\xi_1\eta_4 - \eta_1\xi_4) = \sigma(x_2y_3 - y_2x_3), \\ r_{34} = \rho(\xi_3\eta_4 - \eta_3\xi_4) = \sigma(x_1y_2 - y_1x_2), \\ r_{24} = \rho(\xi_4\eta_2 - \eta_4\xi_2) = \sigma(x_1y_3 - y_1x_3), \\ r_{23} = \rho(\xi_2\eta_3 - \eta_2\xi_3) = \sigma(x_1y_4 - y_1x_4), \end{cases}$$

and it is these quantities r_{ik}, which are susceptible to a double meaning, that we will adopt for the coordinates of the straight line; these coordinates verify the quadratic relation:

(11) $$\omega(r) = 2(r_{12}\, r_{34} + r_{13}\, r_{42} + r_{14}\, r_{23}) = 0.$$

This quadratic form $\omega(r)$ will play an essential role. We shall establish that fact by a proposition that is of the greatest importance.

5. We seek the condition that the two lines r, r' should meet; for this, if we start with equations (3) then the line r will be the intersection of the two planes:

(12)
$$\begin{cases} r_{12}X_2 + r_{13}X_3 + r_{14}X_4 = 0, \\ -r_{12}X_1 + r_{43}X_3 + r_{24}X_4 = 0, \end{cases}$$

and the line r' will be the intersection of the two planes:

(12′)
$$\begin{cases} r'_{12}X_2 + r'_{13}X_3 + r'_{14}X_4 = 0, \\ -r'_{12}X_1 + r'_{43}X_3 + r'_{24}X_4 = 0, \end{cases}$$

Eliminate X_1 and X_2 from (12) and (12′). We will then find that:

$$\begin{aligned} (r_{13}r'_{12} - r'_{13}r_{12})X_3 + (r_{14}r'_{12} - r'_{14}r_{12})X_3 &= 0, \\ (r_{23}r'_{12} - r'_{23}r_{12})X_3 + (r_{24}r'_{12} - r'_{24}r_{12})X_3 &= 0. \end{aligned}$$

The necessary and sufficient condition that they should meet is then:

(13) $$(r_{13}r'_{12} - r'_{13}r_{12})(r_{24}r'_{12} - r'_{24}r_{12}) - (r_{14}r'_{12} - r'_{14}r_{12})(r_{23}r'_{12} - r'_{23}r_{12}) = 0,$$

which may be written:

$$r'^2_{12}(r_{13}r_{24} - r_{14}r_{23}) + r^2_{12}(r'_{13}r'_{24} - r'_{14}r'_{23}) + r'_{12}r_{12}(-r_{13}r'_{24} - r'_{13}r_{24} + r_{14}r'_{23} + r'_{14}r_{23}) = 0.$$

However, one has:

$$r_{12}\, r_{34} + r_{13}\, r_{42} + r_{14}\, r_{23} = 0,$$

and $r_{24} = -\, r_{42}$, moreover. One will then have:

$$r_{13}\, r_{24} - r_{14}\, r_{23} = r_{12}\, r_{34}\,,$$
$$r'_{12}\, r'_{24} - r'_{14}\, r'_{23} = r'_{12}\, r'_{34}\,,$$

and equation (13) will become:

$$r_{12}r'_{12}(r_{12}r'_{34} + r'_{12}r_{34} + r_{13}r'_{42} + r'_{13}r_{42} + r_{14}r'_{23} + r'_{14}r_{23}) = 0.$$

Our calculations supposed that r_{12} and r'_{12} were not zero, which is a hypothesis of no importance. The desired condition will then be written:

(14) $$r_{12}r'_{34} + r'_{12}r_{34} + r_{13}r'_{42} + r'_{13}r_{42} + r_{14}r'_{23} + r'_{14}r_{23} = 0.$$

However, if one refers to the expression for $\omega(r)$:

$$\omega(r) = 2(r_{12}\, r_{34} + r_{13}\, r_{42} + r_{14}\, r_{23})$$

then the left-hand side of equation (14) may be written:

$$\frac{1}{2}\left[\frac{\partial\omega(r)}{\partial r_{12}}r'_{12} + \frac{\partial\omega(r)}{\partial r_{13}}r'_{13} + \frac{\partial\omega(r)}{\partial r_{14}}r'_{14} + \frac{\partial\omega(r)}{\partial r_{23}}r'_{23} + \frac{\partial\omega(r)}{\partial r_{34}}r'_{34} + \frac{\partial\omega(r)}{\partial r_{42}}r'_{42}\right].$$

One generally represents that expression by the symbol:

$$\omega(r, r') = \frac{1}{2}\left(\frac{\partial\omega}{\partial r_{12}}r'_{12} + \cdots + \frac{\partial\omega}{\partial r_{23}}r'_{23}\right);$$

the condition that they meet will then be expressed by the equation:

(15) $$\omega(r, r') = 0.$$

Therefore, *if one constructs the* POLAR FORM $\omega(r, r')$ *relative to two lines r, r′ then the vanishing of that form will express the intersection of the lines r and r′*.

This fact has the greatest importance: Thanks to it, we may henceforth free ourselves of all considerations of point-like space or planar space that have served us up to now as intermediaries for arriving at this quadratic form ω and the remarkable properties of its polar form. *All that we need to retain here is the fact that if one chooses six arbitrary quantities* r_{12}, r_{13}, r_{14}, r_{34}, r_{42}, r_{23} *that are coupled by the equation:*

$$\omega(r) = 2(r_{12}\, r_{34} + r_{13}\, r_{42} + r_{14}\, r_{23}) = 0$$

then one will find that they define a line (it is less important for the moment how the construction of the line might result from that definition) *and that, moreover, the intersection of the two lines r, r′ is expressed by the equation* $\omega(r, r') = 0$.

It is assuredly quite worthy of interest that this simple notion of the form $\omega(r)$ suffices to enlighten all of line geometry, *and with no further assumptions*.

6. Our primary concern shall be to give a broader picture of this form ω. If we express the parameters r_{ik} as linear functions of the six new parameters x_i :

$$(16) \qquad r_{ik} = A_{ik,\,1}\, x_1 + \dots + A_{ik,\,6}\, x_6$$

then nothing will prevent us from taking $x_1, x_2, \dots, x_6$ to be the new variables, since the determinant of the linear substitution (16) is non-zero. These new variables will be linked by a homogeneous quadratic relation $\xi(x) = 0$, where the form $\xi(x)$ *is the transform of the form* $\omega(r)$.

As for $\omega(r, r')$, its transform will be, after a well-known property of quadratic forms, the polar form $\xi(x, x')$. Here is, in addition, the proof of this fact: Let $(r_{12}, r_{13}, \dots, r_{23})$, $(r'_{12}, r'_{13}, \dots, r'_{23})$ be two systems of values of r and let $(x_1, x_2, \dots, x_6)$, $(x'_1, x'_2, \dots, x'_6)$ be the corresponding systems of values for x. The system $(x_1 + \lambda x'_1)$, $(x_2 + \lambda x'_2)$, …, $(x_6 + \lambda x'_6)$, where λ is arbitrary, will correspond to the system $(r_{12} + \lambda r'_{12})$, $(r_{13} + \lambda r'_{13})$, …, $(r_{23} + \lambda r'_{23})$, and one will have, in turn:

$$\omega(r + \lambda r') = \xi(x + \lambda x'),$$

so:

$$(17) \qquad \omega(r) + 2\,\omega(r, r')\, \lambda + \omega(r)\, \lambda^2 = \xi(x) + 2\xi(x, x')\lambda + \xi(x)\lambda^2,$$

and, upon identifying the coefficients of λ^2, λ, 1, one will find, apart from two obvious relations, the relation that we needed to find, namely:

$$\omega(r, r') = \xi(x, x'),$$

where, from formula (17) itself, one will have:

$$2\, \xi(x, x') = \frac{\partial \xi}{\partial x_1} x'_1 + \frac{\partial \xi}{\partial x_2} x'_2 + \cdots + \frac{\partial \xi}{\partial x_6} x'_6 .$$

The particular form that we found for the quadratic form $\omega(r)$ is not essential; a linear transformation of the parameters permits us to convert that form into an arbitrary quadratic form in six variables (which is arbitrary if one does not shrink from a linear transformation with imaginary coefficients) whose discriminant is non-zero. One may then state the following theorem:

To any system of six variables $x_1, x_2, x_3, x_4, x_5, x_6$ *that are linked by one quadratic relation* $\xi(x) = 0$ *whose discriminant is non-zero, one may associate a well-defined line in* *space, the correspondence having such a character that the equation* $\xi(x, x') = 0$ *expresses the notion that the lines x, x′ meet.*

Now that we have given all due weight to the fundamental quadratic form $\xi(x)$, we may penetrate further into the theory by employing only the dualistic and projective elements from now on.

7. The first of the elements that we shall appeal to is the plane pencil of lines – i.e., the set of lines that issue from a point in a plane. We call that point and the plane the *supports* of the pencil of lines.

A pencil is defined completely by two of its lines a and b; *all of the other ones will have coordinates of the form:*

(18) $$x_i = \lambda\, a_i + \mu\, b_i\,,$$

where λ, μ are parameters. Indeed, one first has:

$$\xi(x) = \xi(\lambda\, a + \mu\, b) = \xi(a)\,\lambda^2 + 2\xi(a, b)\,\lambda\mu + \xi(b)\,\mu^2,$$

and since $\xi(a) = 0$, $\xi(b) = 0$ and $\xi(a, b) = 0$, due to the intersection of the lines a and b, it will then result that:

$$\xi(\lambda\, a + \mu\, b) = 0.$$

Therefore, $\xi(\lambda\, a_i + \mu\, b_i)$ are the coordinate of a line x; that line is part of the pencil (a, b). Indeed, if one lets d be a line that cuts a and b then one will have:

$$\xi(a, b) = 0, \qquad \xi(b, d) = 0,$$

and consequently:

$$\xi(\lambda\, a + \mu\, b, d) = \xi(a, d)\,\lambda + \xi(b, d)\,\mu = 0.$$

The lines represented by the formula (18) thus cut any line d that cuts a and b; this will happen only if these lines x are in the plane (a, b), as one sees by taking d to be an arbitrary line of that plane. Likewise, by taking d to be an arbitrary line that issues from the point (a, b), one sees that all of the lines (18) must pass through the point (a, b). All of the lines (18) then define part of the pencil (a, b). I add that, conversely, any line of the pencil (a, b) is representable by formulas (18). In effect, take an *arbitrary* line d that cuts an arbitrary line z of the pencil (a, b). There is only one line of this pencil that cuts d (one does not suppose that d is cut by all of the lines of the pencil), and that unique line is the line z. Now, one may determine λ, μ in such a way that x cuts d; it suffices to verify the equation:

$$\xi(x, d) = \xi(a, d)\,\lambda + \xi(a, d)\,\mu = 0.$$

There is then a line (18) that cuts d, and since all of the lines (18) define part of the pencil, that line (18) that cuts d and belongs to the pencil may only be the line z that one took arbitrarily in the pencil (a, b); therefore, any line of the pencil (a, b) is identical to one and only one line of the system (18).

In summation, if one refers to formulas (18) then to any value of $\lambda : \mu$ there will correspond a line of the pencil (a, b), and conversely. Formulas (18) will thus realize the representation of the plane pencil (a, b).

However, there is more, since $\lambda : \mu$ and the lines of the pencil correspond uniquely – i.e., since any line of the pencil corresponds only one value of $\lambda : \mu$, conforming to the principle of correspondence in its simplest form, it will then result that if one takes four lines α, β, γ, δ of the pencil, and lets ρ, σ, τ, υ be the corresponding values of $\lambda : \mu$ then the anharmonic ratio (α, β, γ, δ) of the four lines will be equal to the anharmonic ratio (ρ, σ, τ, υ) of the corresponding ratios:

$$(\alpha, \beta, \gamma, \delta) = (\rho, \sigma, \tau, \upsilon). \tag{19}$$

For example, the lines $(\lambda a_i + \mu b_i)$ and $(\lambda a_i - \mu b_i)$ form a harmonic pencil with the lines a and b.

8. Two lines that intersect define a plane pencil; three lines that intersect form a triangle or a trihedron. If they form a triangle then any line that cuts them generates the system of lines of a plane (planar system). If they form a trihedron then any line that cuts them passes through their common point, and the set of these lines generates what one calls a *spray of lines*; i.e., the set of lines that issue from a fixed point. The geometry of sprays and the geometry of planar systems are reciprocal. It should then not surprise us that line geometry has the same language for both of them and is incapable of establishing a distinction between them. On the contrary, one must see that as a sign of the perfection of line geometry that I have already alluded to.

Thus, let a, b, c be three lines that intersect: We say that all of the lines that cut a, b, c form what we call a *hyper-pencil*, because it is repugnant to opt for one or the other of the expressions *spray* or *plane pencil* if they both have the same status. The word "hyper-pencil" should shock no one, and is indeed appropriate to those systems that enjoy, as one just saw, a representation that is analogous to that of the pencil. Let a, b, c be three lines of a hyper-pencil: I say that the set of lines of the system is represented by the formulas:

$$x_i = \lambda\, a_i + \mu\, b_i + \nu\, c_i\,, \tag{20}$$

where λ, μ, ν are arbitrary parameters. The proof is analogous to the one that we made use of for the case of a pencil. I will thus not repeat the details here. One first confirms that:

$$\xi(\lambda a + \mu b + \nu c) = \xi(a)\,\lambda^2 + \xi(b)\,\mu^2 + \xi(c)\,\nu^2 + 2\xi(a, b)\,\lambda\mu + 2\xi(b, c)\,\mu\nu + 2\xi(a, c)\,\lambda\nu$$

is identically zero, which proves that the quantities x_i are the coordinates of line. Consequently, one will confirm that this line x cuts a, b, c, and, as a result, belongs to the system; finally, one will prove that any line z of the system is a line (20), by showing that two arbitrary lines d, e that cut z are always cut by a line (20) that is unique, in general.

9. We shall make incessant application of these notions, after we have nonetheless made known some general notions on ruled systems.

A line depends on four absolute parameters, so the lines in space that are subject to *one* condition preserve *three* parameters, and the set of them constitutes a COMPLEX. *Two* conditions leave only *two* parameters, and the line then generates a CONGRUENCE. *Three* conditions to *one* parameter, and the line then generates a RULED SERIES; a ruled series does not always form a surface, because the tangents to a plane curve, for example, might not be regarded as forming a ruled surface, properly speaking. There is, moreover, another time when it is inconvenient to speak of ruled *surfaces*: The hyperboloid, as a surface, serves to support, in fact, *two ruled series*, and it would be a real inconvenience for us to not separate these two series by confusing them with the same name of *surface* or *hyperboloid*. Finally, four conditions determine a line or, to say it best, A SET OF LINES. There is a great inconvenience to saying, as one often does when one speaks of an arbitrary geometric element that depends upon *n* parameters, that *n* conditions define *one* element. That locution is vicious and amounts to denying the theory of binary forms. In reality, the *n* conditions – which are four, here – define *a set* of lines that is generally finite, and these elements enjoy the interesting property that one will lose the notion if one contents oneself to saying that four conditions define *one* line. To give the simplest example of this that is drawn from another school of ideas, one knows that two plane cubics define *a set* of nine points that enjoy special properties and that two curves of order *m* and *n* intersect at *mn* points, the set of which presents some general properties that are sufficiently removed from the properties of a unique point that the properties of a curve of order *mn* will be the same as those of a simple straight line.

For these reasons, we thus consider five types of ruled systems. First, the *ruled space*, or the set of all lines in the space, and then, the *complexes*, or systems with *triple* indeterminacy. Next, one has the *congruences* – or *doubly* indeterminate systems – the *ruled series* with *simple* indeterminacy, and finally the *sets* of lines with *zero* indeterminacy.

10. The condition for a line to belong to a plane pencil is equivalent to three conditions, since the lines of such a pencil constitute a ruled series; likewise, the condition of belonging to a hyper-pencil is equivalent to two conditions.

Consider, from now on, a complex of lines; it is these lines that belong to a plane pencil that constitutes a set of indeterminacy zero. The number of lines of that set is what one calls the *degree* of the complex.

On the contrary, the lines of a complex that belong to a hyper-pencil form a ruled series. If the hyper-pencil is a spray then one will have all of the lines of the complex issuing from a point. Their set obviously forms a cone, which we shall call the *cone of the complex.* Any point of space is therefore the summit of a cone of the complex. If, on the contrary, the hyper-pencil is a planar system then one will have the lines of the complex situated in one plane and enveloping a curve: the *curve of the complex*. Any plane thus contains its enveloping curve of the complex. However, note that this curve is defined by its tangents and might nevertheless degenerate into one or more points; we will soon have some examples.

THEOREM. – *The degree of any cone of a complex and the class of any planar curve that envelops the complex are equal to each other and to the degree of the complex.*

Take a point O. In order to get the degree of the cone of a complex with summit O, one must cut this cone with a plane Π that goes through O and count the number of lines of intersection. The set of generators thus obtained is nothing but the set of lines of the complex that are contained in the pencil (O, Π). The number of these lines is then precisely equal to the degree of the complex.

One can apply the same reasoning to the enveloping curve relative to a plane Π. In order to get the class of that curve, one counts the tangents that one may pass through a point O of Π. However, the set of these tangents is nothing but the set of lines of the complex that are contained in the pencil (O, Π). Therefore, the class of the curve is precisely equal to the degree of the complex.

I will return to these general questions later on. For the moment, the theorem will suffice for me.

11. Now take a congruence. Passing through a point is equivalent to two conditions for a line; similarly, being in a plane is equivalent to two conditions. Therefore, the lines of a congruence that issue from a point form a set of indeterminacy zero. The number of lines of that set is the *degree* of the congruence. Likewise, the lines of a congruence that are in a plane form a set whose number is called the *class* of the congruence.

12. If a ruled series is given then one will call the number of lines in the series that cut an arbitrary line the *degree* of the series. If the lines form a ruled surface then this *degree* will properly be that of the surface. If they envelop a planar curve then its degree will properly be the *class* of that curve.

Finally, one may call the number of lines that comprise a finite set of lines its *degree.*

I shall begin by studying the complexes of first degree, as well as their common systems.

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CHAPTER II.

LINEAR COMPLEXES OF LINES.

Pole and polar plane. – Pencils of complexes. – Conjugate lines. – Distribution of poles and polar planes on a line of the complex. – Normal correlation of a complex. – Properties of conjugate lines. – Reciprocal polars with respect to a linear complex. – Analytic representation. – Special complexes. – Klein invariant. – Conjugate lines.

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13. A complex is called *linear* when it is of the first degree – i.e., when, among the lines of an arbitrary pencil, there is only one of them that belongs to the complex. The cone of the complex will then reduce to a plane, and the enveloping curve in a plane will reduce to a point (i.e., a locus of class 1). This gives the double theorem:

The lines of a linear complex that issue from a point P generate a plane that one calls the POLAR PLANE *of the point.*

The lines of a linear complex are traced in a plane that passes through a fixed point of that plane – viz., the focus or POLE *of that plane.*

There thus exists an infinitude of pencils in space whose lines all belong to the complex: They are the pencils that are defined by a point and its polar plane, or, what amounts to the same thing, by a plane and its pole. We call these pencils the *pencils of the linear complex* ([2]).

14. One may deduce the properties of a linear complex from a unique proposition whose proof is quite simple.

Consider the set of a plane Π *and a point O in that plane: The pole O′ of the plane* Π *is in the polar plane* Π' *of the point O.*

In other words, if a point O and a plane Π are UNITED (*see* no. 3) then their corresponding polars in the complex will be a UNITED plane Π' and a point O', resp. Indeed, the line OO' will belong to the complex, since it will pass through the point O' and will be in the polar plane Π to that point. However, it must then be contained in the plane Π', which will be the polar of the point O of that line. The plane Π' will thus contain the point O'. Q. E. D.

Suppose that the line d does not belong to the complex, that O is a point of that line, and that Π is a plane through that point. From the preceding theorem, the pole of Π will be in the polar of O. However, since O is an arbitrary point of the line d, and Π is an

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([2]) One may compare this with what we shall call the *pencil of the complex* later on in the general case of an arbitrary complex.

arbitrary plane through that line, one may conclude that *the poles of all the planes through a line will be situated in the polar planes to all of the planes of that line.*

It then results immediately that:

1. *The polars to all of the points of a line d are the planes that cut it along the same line d′.*

2. *That line d′ is the locus of the poles of the plane through the line d.*

The pole of a plane through d is then the point where it pierces d', and the pole of a plane through d' is the point where it pierces d. The lines d and d' are then in a reciprocal situation with respect to each other; one calls them *conjugate lines.*

The following remarks are used frequently:

Any line x that cuts two conjugate lines d, d′ belongs to a complex.

Consider the plane $\Pi(d, x)$ through d and x, so the pole of that point is its intersection with d' – i.e., at precisely the point $P(d', x)$ of intersection of x and d'. The line x of the plane $\Pi(d, x)$ is thus found to pass through the pole $P(d', x)$ of that plane; it is therefore implicit that it belongs to the complex.

Any line of the complex that cuts a line d also cuts its conjugate d′.

Indeed, consider the plane $\Pi(d, x)$ that one passes through d and x, by hypothesis. The line x in that plane that belongs to the complex must pass through a pole of that plane. Now, that pole is the trace of the plane on the line d'; the line x will then cut d' at that point.

Two pairs of conjugate lines form four lines that are carried by the same quadric.

Indeed, let a, a' and b, b' be two pairs of conjugate lines, and consider the quadric that is generated by a line x that leans against a, a', b. The generators x of that quadric belong to the complex since they cut a and a', and, since they cut b they must also cut b'. Therefore, a, a', b, b' are four generators of the second system.

In general, suppose that the generators x of a system of a quadric belong to a linear complex. Consider a generator y of the second system and let y' be its conjugate; that conjugate will necessarily be another generator of the same system as y. Indeed, all of the generators x cut y, and, since they belong to the complex, they must cut y'. We thus arrive at the result that if a quadric is generated by the lines x that belong to a linear complex then the generators of the second system will be found to be associated pair-wise as pairs of conjugate lines. We give the name of *quadrics of the complex* to these quadrics.

In concluding this section, we make the following remarks:

We supposed to begin with that the line d did not belong to the complex. If it does belong to the complex then it will be its own conjugate, because it is the locus of poles of its planes and the envelope of the polar planes of its points.

If a line d does not belong to the complex then it will be impossible that it cuts its conjugate d', since if P is the point of intersection then any plane through d will have its pole at the point P, and the line d that passes through P and is traced in this plane will belong to the complex.

15. Consider four planes Π_1, Π_2, Π_3, Π_4 through a line that do not belong to a complex, and let P_1, P_2, P_3, P_4 be the poles of these planes. One obtains these poles by cutting the pencil of the four planes by the line d' that is conjugate to d. The *anharmonic ratio of the four poles is therefore equal to that of the four planes.*

It is interesting to prove that this proposition further extends to the case of planes passing through a line that belongs to the complex.

Indeed, let d be a line of the complex and let a, a' be two conjugate lines that do not cut d. Consider the quadric that is generated by a line x that leans on a, a', and d. That quadric will be a quadric of the complex since x cuts the conjugate lines a and a'. Pass a plane Π through the line d; that plane cuts the quadric, not only along d, but along a generator x that must cut d at the point of contact P of the plane Π with the quadric. However, two lines of the complex pass through P that are contained in the plane Π, namely, d and x. Therefore P is the pole of the point Π. From this, one infers the consequence: The pole of a plane through d is justly the point of d where that plane is tangent to the quadric.

However, one knows the beautiful theorem of Chasles on the distribution of the tangent planes along a rectilinear generator of a quadric. The anharmonic ratio of four planes through that generator is equal to that of the four points of contact of these planes with the surface.

It then results from this theorem, when combined with the preceding remark, that *if one passes four planes through a line d of a complex then the anharmonic ratio of the poles of these planes will be equal to that of the planes themselves.* From the theorem, any linear complex defines a homographic correspondence on each of its lines between the points and planes of that line ([3]). Such a correspondence is frequently found in ruled figures, and I believe it would be useful to attribute a special name to it, which is an *anharmonic correlation*, or simply, a *correlation*.

We might then say that any linear complex defines a correlation on each of its lines, namely, the one that relates a point of the line to its polar plane, and to distinguish that correlation from all of the other ones that one might imagine on that line, I will give it the name of the *normal correlation of the complex* ([4]).

16. In the preceding section, we have seen that a linear complex is provided by means of a transformation under which a point transforms to a plane, a plane, to a point,

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([3]) I will henceforth call any plane through a line the *plane of the line*.

([4]) One will verify an extension of that notion to the case of an arbitrary complex later on.

and a line to another line. We shall extend this remark and thus obtain a result that is important in several aspects.

First of all, recall the theorem that was proved in no. **14**:

I. *If a point O and a plane* Π *are united then their corresponding elements will be a plane* Π' *and a point* O' *that will also be united.*

Here are some other theorems in which lines figure:

II. *If a line d passes through a point O then its conjugate d' will be traced in the plane* Π' *that is polar to O, and conversely.*

This theorem is an immediate consequence of the definition of the conjugate lines.

III. *If two lines a and b intersect then their conjugates a', b' will also intersect.*

Indeed, since a and b pass through the same point O then, by virtue of the preceding theorem, their conjugates a', b' will be in the same plane Π that is polar to O.

It results immediately from this that the lines of a pencil then correspond to the lines of a pencil; any of the planes and lines through a point O correspond to points and lines that are traced in the plane Π' that is polar to O.

We have already said that one gives the name of *spray* to the set of planes and lines that issue from a point and that of *planar system* to the set of points and lines in a plane. One may thus say that a spray corresponds to a planar system, and conversely.

Consider, in general, a figure $\mathcal{F}$ that is composed of points, lines, and planes, so by taking the corresponding elements to all of the ones in the figure $\mathcal{F}$, one will generate a figure $\mathcal{F}'$, which we say will be the reciprocal of $\mathcal{F}$. To the points of a straight line in $\mathcal{F}$ there will correspond planes in $\mathcal{F}'$ that pass through a line, and conversely, to the lines that issue from a point, the lines in a plane, and conversely, and to the planes through a point, the points of plane, and conversely, etc.

To a polyhedron $\mathcal{P}$ there will correspond a polyhedron $\mathcal{P}'$ in which:

1. The edges will be conjugate to the edges of $\mathcal{P}$.
2. The summits will be poles of the planes of the faces of $\mathcal{P}$.
3. The planes of the faces will be the polars to the summits of $\mathcal{P}$.

To a non-developable surface S in the figure $\mathcal{F}$, when considered as a locus of points O, there will be a surface S' in $\mathcal{F}'$ that is defined to be the envelope of the plane Π' that is polar to O, and the point of contact O' of Π' with the surface S' will be pole of the plane Π that is tangent at O to the surface S, in such a way that the surface S' will also be the locus of poles of the tangent planes to S. One may further remark that the pencil of tangents to the surface S at the point O will have the pencil of tangents to the surface S' at

O' for its reciprocal. One may thus further define the surface S' to be the envelope of the lines that are conjugate to the tangents of the surface S.

Now, let C again be a curve that one might define to be either the locus of a point O, the envelope of the tangents d at that point, or the envelope of the osculating planes Π at O. The locus of the pole O' of the plane Π will be a curve C': Consider three osculating planes Π, Π_1, Π_2 to the curve C that are infinitely close, and let O', O'_1, O'_2 be their poles, which are three points of C', so the plane of these three points will be the osculating plane to the curve C' at O', and its pole will be the point of intersection of the three planes Π, Π_1, Π_2; i.e., the point O.

One may thus again define the curve C' to be the envelope of the polar planes of the points of the curve C.

Finally, take two neighboring points O, O_1 on the curve C. The line d – or OO_1 – has for its polar, the intersection d' of the planes Π', Π'_1 that are polar to the points O and O_1, and which are both osculating planes that are close to the curve C'. The line d' is then tangent to the curve C'. From this, one has the following theorem, which implies a third definition of the curve C':

The polars d' to the tangents d of a skew curve C envelop a skew curve C'.

It is in this case that one must recall the distinctions that were made at the end of no. 1. It is clear that if one considers the systems E_p, E_Π, E_d of the curve then they will transform into the systems E'_Π, E'_p, E'_d of the reciprocal curves. I will henceforth call the set E_d of tangents to a curve – which may be skew or planar, or even reduce to a point – *developable*, as in the case of the cone.

To summarize the material in this paragraph, we say that a linear complex permits us to realize a *dualistic transformation of space.*

This process did not escape Chasles in his beautiful memoir *Sur la dualité et l'homographie*, which terminated his *Aperçu historique.* The mechanical form in which that illustrious geometer presented that transformation is of the greatest importance, and because of that we shall return to it in detail later on. We also verify in the applications how that same transformation has found a very fertile place in graphical statics, thanks to the ingenious research of Maxwell.

17. Before going further, it is convenient to express the results that we just obtained analytically.

Let $\omega(x)$ be the fundamental quadratic form, and let:

$$f(x_1, x_2, \ldots, x_6) = f(x) = 0 \tag{1}$$

be the homogeneous algebraic equation in x_1, x_2, …, x_6, which, when combined with:

$$\omega(x) = 0, \tag{2}$$

represents the linear complex considered.

I would like to prove that $f(x)$ is linear in $x_1, x_2, \ldots, x_6$. Indeed, let a, b be two lines that intersect. One has seen in no. 7 that the x_i of the straight line of the pencil (a, b) are of the form $a_i\,\lambda + b_i\,\mu$, where $\lambda : \mu$ is arbitrary. Upon expressing the idea that the line x of that pencil belongs to the complex, one will have:

$$f(a_1\lambda + b_1\mu,\ a_2\lambda + b_2\mu,\ \ldots,\ a_6\lambda + b_6\mu) = 0.$$

That equation must be of first degree in $\lambda : \mu$ because only one line of the complex belongs to a given pencil; one must have, as a consequence:

$$f(x) = A_1\,x_1 + A_2\,x_2 + \ldots + A_6\,x_6\,.$$

Conversely, any linear equation in x obviously represents a *linear* complex.

18. The condition for a line x to cut a line z is expressed by the linear equation:

$$0 = 2\,\omega(z, x) = \frac{\partial\omega}{\partial z_1}x_1 + \frac{\partial\omega}{\partial z_2}x_2 + \cdots + \frac{\partial\omega}{\partial z_6}x_6\,.$$

The set of lines that cut a fixed line z thus forms a linear complex. However, one easily recognizes that this is not the most general linear complex. Indeed, identify an arbitrary linear function of x with $\omega(z, x)$; we will have:

$$\text{(3)}\qquad \frac{\dfrac{\partial\omega}{\partial z_1}}{A_1} = \frac{\dfrac{\partial\omega}{\partial z_2}}{A_2} = \frac{\dfrac{\partial\omega}{\partial z_3}}{A_3} = \ldots = \frac{\dfrac{\partial\omega}{\partial z_6}}{A_6}\,.$$

One will infer from these linear equations in $z_1, z_2, \ldots, z_6$, the values of those quantities – or rather, their ratios – and, by substituting these values into $\omega(z)$, that form will be become a homogeneous, quadratic form in $A_1, A_2, \ldots, A_6$:

$$\text{(4)}\qquad \omega(z) = \Omega(A);$$

this form $\Omega(A)$ will be the *adjoint form* to the form $\omega(z)$.

Therefore, if the z_i are the coordinates of a line z then one must have that $\Omega(A)$ is zero.

If $\Omega(A)$ is zero then, from (4), the values of z_i that one deduces from equations (3) will be the coordinates of a line, and from equations (3), that line will be cut by all of the lines of the linear complex:

$$\sum A_i\,x_i = 0.$$

One gives the name of *special complex* to such a complex, and the line z will be the called the *directrix* or *axis*; however, the word "axis" has been employed with many meanings in that same theory of lines, so the word "directrix" seems preferable.

When the expression $\Omega(A)$ is non-zero, the linear complex will possess no directrix; however, the consideration of the form $\Omega(A)$ does not become less interesting. Klein called it the *invariant of the complex.* The name of "invariant" is justified by the following remark:

If one performs a linear transformation on the variables x_i then the coefficients A_i of a linear form on x_i will be found to transform, as one knows, by the reciprocal transformation, and the form $\Omega(A)$ will be what one calls a *contravariant* of the form $\omega(z)$; this signifies that $\Omega(A)$ will be reproduced, but multiplied by a power (viz., the second) of the determinant of the direct substitution.

If, for example, one has reduced the form $\omega(x)$ to the Plücker type:

$$\omega(x) = 2(x_1\, x_4 + x_2\, x_5 + x_3\, x_6)$$

then the form $\Omega(A)$ will be the following one:

$$\Omega(A) = 2(A_1\, A_4 + A_2\, A_5 + A_3\, A_6).$$

On the contrary, if one has reduced, as we verify that Klein did, the form $\omega(x)$ to a sum of squares, namely:

$$\omega(x) = K_1 x_1^2 + K_2 x_2^2 + \cdots + K_6 x_6^2,$$

then one will have:

$$\Omega(A) = \frac{A_1^2}{K_1} + \frac{A_2^2}{K_2} + \cdots + \frac{A_6^2}{K_6}.$$

At the beginning of this chapter, we were not preoccupied with the case of the special complex. It is clear that in this case the lines of the space will all have the same conjugate – namely, the directrix – and that all of the properties that relate to the transformation by reciprocal polars will be found to be invalid.

19. Therefore, suppose that one is dealing with a non-special complex, and let z be an arbitrary line; we seek its conjugate u. To that effect, I observe that of the three equations:

$$\sum A_i\, x_i = 0, \qquad \omega(z, x) = 0, \qquad \omega(u, x) = 0,$$

one of them must be a consequence of the other two, because any line of the complex that cuts a line will cut its conjugate and any line that cuts two conjugate lines will belong to the complex.

In order to arrive at this result easily, I observe that one has identically ([5]):

([5]) Indeed, if one sets $Z_i = \partial\omega / \partial z_i$ then from the definition of the adjoint form, one will find that:

$$\omega(z) = \Omega(Z);$$

it will then result that:

$$\sum A_i x_i = \sum \frac{\partial\Omega}{\partial A_i}\frac{\partial\omega}{\partial x_i},$$

and the three equations that we have to consider may be written:

$$\sum \frac{\partial\Omega}{\partial A_i}\frac{\partial\omega}{\partial x_i} = 0, \qquad \sum z_i \frac{\partial\omega}{\partial x_i} = 0, \qquad \sum u_i \frac{\partial\omega}{\partial x_i} = 0.$$

From the remark that was made before that they must reduce to two, one can find two quantities λ, μ such that:

(5) $$\frac{\partial\Omega}{\partial A_i} = \lambda z_i + \mu u_i .$$

If one expresses the idea that $u_1, u_2, \ldots$ are the coordinates of a line then one will find that:

$$\omega\left(\frac{\partial\Omega}{\partial A} - \lambda z\right) = 0$$

or:

$$d\omega = d\Omega = \sum \frac{d\Omega}{dZ_i} dZ_i .$$

However, one will also have:

$$d\omega = \sum \frac{d\omega}{dz_i} dz_i = \sum Z_i dz_i ,$$

and, since ω is homogeneous:

$$2\omega = \sum \frac{d\omega}{dz_i} dz_i = \sum Z_i dz_i$$

Hence:

$$2\, d\omega = \sum Z_i\, dz_i + \sum z_i\, dZ_i ,$$

and finally, by subtraction:

$$d\omega = \sum z_i\, dZ_i .$$

Upon identifying $d\omega = \sum \frac{d\Omega}{dZ_i} dZ_i$, one will then have:

$$z_i = \frac{\partial\Omega(Z)}{\partial Z_i} = \frac{\partial\Omega\left(\frac{\partial\Omega}{\partial z}\right)}{\partial\frac{\partial\omega}{\partial Z_i}} ,$$

and, as a result, we will have precisely:

$$\sum A_i x_i = \sum A_i \frac{\partial\Omega\left(\frac{\partial\Omega}{\partial z}\right)}{\partial\frac{\partial\omega}{\partial Z_i}} = \sum \frac{\partial\Omega(A)}{\partial A_i}\frac{\partial\omega}{\partial x_i} .$$

$$\omega\left(\frac{\partial\Omega}{\partial A}\right) - 2\lambda\omega\left(\frac{\partial\Omega}{\partial A}, z\right) = 0,$$

upon remembering that $\omega(z) = 0$. One has, moreover:

$$\omega\left(\frac{\partial\Omega}{\partial A}\right) = \Omega(A), \qquad 2\omega\left(\frac{\partial\Omega}{\partial A}, z\right) = \sum \frac{\partial\omega\left(\frac{\partial\Omega}{\partial A}\right)}{\partial\frac{\partial\omega}{\partial A_i}} z_i = \sum A_i z_i \,;$$

one must then have:

(6) $$\Omega(A) - \lambda \sum A_i z_i = 0.$$

Equation (6) gives us λ, and equations (5) furnish the coordinates of the line u that is conjugate to z.

This calculation supposes that $\sum A_i z_i$ is not zero; i.e., that z does not belong to the complex.

The symmetric form of equations (15) indeed exhibits the reciprocity between the lines z and u.

It is simple to find an algebraic proof of the various properties of conjugate lines that were already established geometrically when one starts with formulas (5); we leave this to the reader.

CHAPTER III.

SYSTEMS OF LINEAR COMPLEXES.

Correspondence between the points and planes of a line. Inverse pairs. – Homographic correlations on a line. – Anharmonic ratio and angle of two correlations. – Involution of two correlations. – Singular correlations. – Pairs of conjugate lines common to two linear complexes. – System with two terms. – Linear congruence. – Singular linear congruence. – Case of decomposition. – Invariant of a congruence. – Anharmonic ratio of two linear complexes. – Linear complexes in involution. – Linear systems of linear complexes. – Complementary systems. – Systems with three terms. – Lines common to three complexes. – Semi-quadrics. – Complementary semi-quadrics. – Case of degeneracy. – System with four terms. – Lines common to four complexes. – Invariants of systems of linear complexes. – General form of these invariants.

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20. I will preface the study of linear systems of complexes of first degree with several remarks of a general nature concerning the correspondences that might exist between the points of a line x and the planes through that line.

Let u be a parameter that fixes the position of a point M on the line x, in such a way that to a point M there corresponds only one value of u and conversely. Likewise, let t be a parameter that uniformly corresponds to the positions of a plane π through x. For example, u is the distance from M to a fixed point of x, t is the tangent of the angle between the plane π and a fixed plane through x.

A relation between u and t:

$$f(u, t) = 0$$

defines a correspondence between the points of x and the planes of x according to a certain law. If f is of degree m in u and of degree μ in t then one may say that this correspondence is of *class* m and *degree* μ. If $m = \mu = 1$ then one recovers the *homographic correlations* that were introduced in no. 15.

Two correspondences of degree m and m' and of class μ and μ' will have, *in general:*

$$\mu\, m' + \mu'\, m$$

common *pairs* if one calls the system that consists of a point M and the corresponding plane π a *pair of a correspondence.*

For example, two homographic correlations will have two pairs in common, in general.

This is why if one is dealing with two Chasles correlations relative to a line common to two ruled surfaces then the two pairs will be the two pairs of agreement of the two surfaces.

21. Consider two pairs (M, π), (M', π') along a line. We call the pairs that one obtains by exchanging the points their *inverse pairs*; the inverse pairs will then be:

$$(M, \pi'), (M', \pi).$$

22. Consider two homographic correlations H, H' on a line x, and let (F, Φ), (F', Φ') be their common pairs. If a plane π turns around x then the homologues O and O' of π under these two correlations will correspond homographically, and F, F' will be the double points of that homography. From a well-known property of homographies, the anharmonic ratio:

$$(O', O, F, F') = k$$

will be constant.

Likewise, if a point O moves along the line then its corresponding planes π and π' will describe two homographic pencils whose double planes will be Φ, Φ'; here again, the anharmonic ratio:

$$(\pi, \pi', \Phi, \Phi') = k$$

will be constant.

I add that $k_1 = k$.

Indeed, let π be homologous to O in H, and let π' be homologous to O in H', so we will have:

$$(\pi, \pi', \Phi, \Phi') = k_1 .$$

Let O' be homologous to π' in H; the plane π' will have the point O' for its homologue in H and the point O in H', so one will have:

$$(O, O', F, F') = k.$$

Now, under homographic correspondences the anharmonic ratio of four elements will equal that of its four correspondents. Therefore, since O, O', F, F' corresponds to π, π', Φ, Φ' in H, one will indeed have:

$$k_1 = k.$$

This ratio k will be called the *anharmonic ratio* of the two correlations.

Ever since Laguerre saw fit to define angles by an anharmonic ratio, one has often attached an angle to an anharmonic ratio by setting:

$$V = \frac{1}{2\sqrt{-1}} \log k .$$

In the case of two planes, for example, if k denotes the anharmonic ratio that they define by the two isotropic planes through their common line then, according to Laguerre, V will be found to be precisely equal to the angle between these two planes.

In order to indicate an immediate application of that notion of angle between two correlations, suppose that they are the two Chasles correlations of two ruled surfaces along the common line x. Suppose, moreover, that by means of a homographic transformation one has arranged that the planes Φ and Φ' of the pairs of agreement are two isotropic planes. V will then be the angle of two tangent planes at the same point O

on two surfaces, and since that angle is constant, one sees that the transformed surfaces *will intersect along their common line with a constant angle.*

23. A particularly important case of the angle between two homographic correlations is the one in which the angle is a right angle, which amounts to the same thing as the case where one has:

$$k = -1.$$

We will then say that the two correlations are *in involution.*

One sees that in this case the pairs of points O, O' that correspond to the same plane π will correspond involutively; the same will be true for the planes that correspond to the same point.

Here, the notion of inverse pairs that I have already spoken of intervenes.

Let (M, π) be a pair for a homographic correlation Π, let H' be a homographic correlation in involution with the first one, and let (M, π') be a pair of H', in which the point M is common with the first pair. Let M' be the homologue of π' under the correlation H, in such a way (M, π), (M', π') will be two pairs of H. It is clear that since (M, π') is a pair of H', (M', π) must be another one. Indeed, M and M' correspond to the same plane π' in H and H', respectively. Therefore, due to the characteristic symmetry of the involution, the points M and M' must be homologous to the same plane H and H', respectively, and since π' is homologous to M' in H, it must be homologous to M in H'. Therefore, the pairs (M', π), (M', π') that are inverse to the pairs (M, π), (M', π') will belong to H'.

The proof itself proves that conversely: If a correlation H' admits two pairs that are inverse to a pair that belongs to a correlation H, then the homographic correlations H and H' will be in involution.

24. It will be useful to put the preceding results into an analytical form.

The equation that relates to a homographic correlation will have the form:

$$aut + bu + ct + e = 0.$$

That equation will depend upon three parameter $a : b : c : e$.

In a paper that dates to 1882, I indicated a mode of representing homographic correlations by means of a plane in space, by considering a, b, c, e to be the coefficients of the equation of a plane. I will not refer to this representation, which says nothing essential to this discussion ([6]).

Observe that if:

$$a'ut + b'u + c't + e' = 0$$

([6]) At almost the same time, Stephanos published a representation of binary homographies in the *Mathematische Annalen* that presented several ideas that were common to the ones that I alluded to here.

is the equation of another homographic correlation H' then the homography that relates to homologous planes at the same point may be written:

$$(ac' - ca')\, tt' + (ae' - b'c)\, t + (bc' - a'e)\, t' + (be' - b'e) = 0.$$

The condition of involution will then be:

$$ae' - b'c - bc' + a'e = 0.$$

If one sets:

$$\theta(a, b, c, d) = bc - ae,$$

for the moment, then this condition can be written:

$$\frac{\partial\theta}{\partial a}a' + \frac{\partial\theta}{\partial b}b' + \frac{\partial\theta}{\partial c}c' + \frac{\partial\theta}{\partial e}e' = 0.$$

This expresses the notion that the elements (a, b, c, e), (a', b', c', e') are conjugate with respect to the quadratic form $\theta(a, b, c, e)$.

The correlations for which one has:

$$bc - ae = 0$$

will be called *singular*.

Singular correlations present a peculiarity that is quite remarkable. Their equation may be written:

$$(at + b)\,(au + c) = 0,$$

or further:

$$(t - t_0)\,(u - u_0) = 0,$$

upon setting:

$$t_0 = -\frac{b}{a}, \qquad u_0 = -\frac{c}{a}.$$

Under a singular correlation, a given point O will correspond to all planes and a given plane π, to all points. Such a correlation will then be characterized and defined by a pair (O, π), and the pairs of the correlation will be divided into two classes: One of them is obtained by associating O with an arbitrary plane on the line, and the other one, by associating the plane π with an arbitrary point of the same line. The pair (O, π) will belong to both of these classes at once; we shall then call it the *singular pair* of the singular correlation.

25. What does the analytic condition of involution actually signify when one of the two homographic correlations is singular?

One has:

$$ae' - b'c - bc' + a'e = 0,$$

and if the correlation H' is singular then one can set:

$$a' = t, \qquad b' = -t_0, \qquad c' = -u_0, \qquad e' = u_0\, t_0\,,$$

where u_0, t_0 are the parameters of the singular pair. The condition of involution will become:

$$au_0\, t_0 + bu_0 + ct_0 + e = 0;$$

it expresses the idea that that the singular pair belongs to H.

Therefore, we shall continue to say that a homographic correlation H is in involution with another one H' – where H' is singular – when the singular pair of H' belongs to H.

Likewise, two singular correlations will be said to be *in involution* if their singular pairs have either the point or the plane in common.

Consider all of the homographic correlations that admit two given pairs (u_0, t_0), (u_1, t_1); their equation can be given the form:

$$\frac{u-u_0}{u-u_1} = \lambda\,\frac{t-t_0}{t-t_1}\,,$$

where λ is arbitrary. Upon developing, this will become:

$$(1-\lambda)\, ut - (u_1 - \lambda\, u_0)\, t - (t_0 - \lambda\, t_1)\, u + t_0\, u_1 - \lambda\, t_1\, u_0 = 0.$$

The condition of involution with another correlation (a', b', c', e') will then be written:

$$e'(1-\lambda) - c'(\lambda\, t_1 - t_0) - b'(\lambda\, u_0 - u_1) + a'(t_0\, u_1 - \lambda\, t_1\, u_0) = 0,$$

or further:

$$(e' + c'\, t_0 + b'\, u_1 + a'\, u_1\, t_0) - \lambda(e' + c'\, t_1 + b'\, u_0 + a'\, u_0\, t_1) = 0.$$

Therefore, consider two correlations that admit the common pairs (u_0, t_0), (u_1, t_1); these correlations will correspond to two values $\lambda = a$, $\lambda = b$ of λ, and the condition that the correlation (a', b', c', e') be in involution with each of them will give:

$$(e' + c'\, t_0 + b'\, u_1 + a'\, u_1\, t_0) - \alpha(e' + c'\, t_1 + b'\, u_0 + a'\, u_0\, t_1) = 0,$$
$$(e' + c'\, t_0 + b'\, u_1 + a'\, u_1\, t_0) - \beta(e' + c'\, t_1 + b'\, u_0 + a'\, u_0\, t_1) = 0;$$

i.e.:

$$e' + c'\, t_0 + b'\, u_1 + a'\, u_1\, t_0 = 0,$$
$$e' + c'\, t_1 + b'\, u_0 + a'\, u_0\, t_1 = 0.$$

These equations express the idea that the inverse pairs (u_1, t_0), (u_0, t_1) belong to the correlation (a', b', c', e'). One thus has the theorem:

If two correlations H, H_1 have two pairs in common then any correlation H' that is in involution with H and H_1 will contain pairs that are inverse to the first two, and *conversely, any correlation that contains these inverse pairs will obviously be in involution with H and* H_1 (no. 23).

As one sees, this theorem defines the correlations that are in involution with two given correlations, since these two correlations will generally have two pairs in common.

26. One fact dominates the theory of systems of linear complexes, and it is the following one:

Two linear complexes will generally have a pair of conjugate lines in common.

One may give a geometric proof of this theorem:

Let A and B be two complexes, let Δ be a line that does not belong to either of them, and let Δ', Δ'' be the conjugates to Δ in the two complexes. First, exclude the case where Δ', Δ'' are in the same plane. Δ, Δ', Δ'' ωιλλ then define a quadric Q, which will be the locus of the lines X that cut Δ, Δ', Δ''. The lines X will belong to the two complexes, since they will cut the pairs of conjugate lines (Δ, Δ'), (Δ, Δ'') (no. 14). Consider a generator Y of Q that is from the same system as Δ, Δ', Δ''. We know (no. 14) that the conjugates Y', Y'' of Y in the two complexes A and B will also be generators of Q of the same system as Δ, Δ', Δ''. Moreover, the principle of correspondence proves to us that Y', Y'' will correspond homographically, because Y' and Y'' will be in one-to-one correspondence. From that, one seeks the lines Y that are traced on Q by the system Δ, Δ', Δ'', which have the same conjugates in the two complexes.

One must express the idea that Y', Y'' are coincident; there are generally two positions of coincidence Y_1', Y_2'. Take Y_1', and let Y_1 be its conjugate in A. By hypothesis, Y_1' will also be the conjugate to Y_1 in B. Therefore, Y_1, as well as Y_1', will have the same conjugate under the two complexes, and since only Y_1', Y_2' enjoy that property among the systems of generators Δ, Δ', Δ'', one must have that Y_1 will coincide with the second line Y_2'. The lines Y_1', Y_2' will then be conjugate to each other in the two complexes.

I will not discuss this geometric proof. The analytical proof that I will give leads to a discussion that is much more reliable (sûre), and which will provide us with some useful formulas.

Take the two linear complexes:

$$A = \sum a_i\, x_i = 0,$$
$$B = \sum b_i\, x_i = 0.$$

The six equations:

$$\frac{\partial \Omega(a)}{\partial a_i} = \rho\, z_i + \rho' z_i' \tag{1}$$

express the idea that we learned in the final number of the preceding chapter that the lines z, z' are conjugate in the complex A. Denote the coefficients of the two special complexes whose axes are z, z' by $c_1, \ldots, c_6$, $c'_1, \ldots, c'_6$. As we saw, one has:

$$z_i = \frac{\partial\Omega(c)}{\partial c_i}, \qquad z'_i = \frac{\partial\Omega(c')}{\partial c'_i},$$

and equation (1) may be written:

$$\frac{\partial\Omega(a)}{\partial a_i} = \rho\,\frac{\partial\Omega(c)}{\partial c_i} + \rho'\,\frac{\partial\Omega(c')}{\partial c'_i},$$

or also:

$$\frac{\partial\Omega(a-\rho c-\rho' c')}{\partial(a_i-\rho c_i-\rho' c'_i)} = 0 \qquad (i = 1, 2, \ldots, 6).$$

Since the discriminant of the form Ω is not zero, these six equations will demand that one must have:

(2) $$a_i - \rho\, c_i - \rho' c'_i = 0 \qquad (i = 1, 2, \ldots, 6).$$

Likewise, six equations such that:

(3) $$b_i - \rho\, c_i - \sigma' c'_i = 0$$

will express the idea that the lines z, z' will be conjugate in the complex B.

Now, observe that $\rho\sigma' - \rho'\sigma$ will not have to be zero, since otherwise, from equations (2), (3), the complexes A, B would not be distinct, because one deduces from these equations that:

$$\rho\, b_i - \sigma\, a_i = \quad (\rho\sigma' - \sigma\rho')\; c'_i,$$
$$\rho' b_i - \sigma' a_i = -\,(\rho\sigma' - \sigma\rho')\; c_i.$$

Since $\rho\sigma' - \sigma\rho'$ is not zero, one has, after dividing by that binomial:

(4) $$c_i = \alpha\, a_i + \beta\, a_i,$$
(5) $$c'_i = \alpha'\, a_i + \beta'\, a_i,$$

which are equations that are equivalent to (2) *and* (3).

In order to solve the problem, all that remains then is to calculate $\alpha : \beta$ and $\alpha' : \beta'$. One will arrive at this upon expressing the last condition that remains for us to write, namely, that the complex $C = \sum c_i\, x_i = 0$ must be special, and a similar statement must be true for $C' = \sum c'_i x_i = 0$. One must then write:

$$\Omega(\alpha\, a + \beta\, b) = \Omega(c) = 0,$$

$$\Omega(\alpha' a + \beta' b) = \Omega(c') = 0,$$

or, upon developing this:

(6) $$\Omega(a)\,\alpha^2 + 2\Omega(a, b)\,\alpha\beta + \Omega(b)\,\beta^2 = 0,$$

and a similar equation must be true for $\alpha' : \beta'$.

This equation will give us two values of $\alpha : \beta$, and, after substituting one of these values in (4) and the other one in (5), we will indeed have two special complexes C, C' whose axes z, z' will be conjugate under the two complexes, since equations (4), (5) are equivalent to equations (2), (3), which express precisely the fact that z and z' are conjugate under A and B.

The theorem is thus established.

The imaginary character of the roots of (6) is not an obstacle; however, the proof breaks down if equation (6) in $\alpha : \beta$ has equal roots; i.e., if the expression:

(7) $$\Phi(a \mid b) = \Omega(a)\,\Omega(b) - [\Omega(a, b)]^2$$

is zero. We shall return later on to the hypothesis that Φ is zero, which is exceptional.

27. We immediately deduce a consequence of the result that we just obtained. Consider the linear complexes that are defined by the equation:

$$\lambda A + \mu B = \sum (\lambda\, a_i + \mu\, b_i)\, x_i = 0,$$

where $\lambda : \mu$ is an arbitrary parameter. We say of these complexes that they define a *pencil*, or better yet, *a system of two terms.*

The lines z, z', which are conjugate to both A and B at once, are conjugate with respect to any complex of the system with two terms (A, B).

From the last lines of no. 19, in order to prove this proposition, it will suffice to prove that one might find – λ, μ being arbitrary – two quantities τ, τ' such that:

$$\frac{\partial\Omega(\lambda a + \mu b)}{\partial(\lambda a_i + \mu b_i)} = \tau z_i + \tau' z_i',$$

or furthermore, what amounts to the same thing [*see* the way that one passes from (1) to (2)]:

$$\tau = \lambda\rho + \mu\sigma, \qquad \tau' = \lambda\rho' + \mu\sigma'.$$

Therefore: *Any of the complexes of a system of two terms* (A, B) *will have a pair of conjugate lines in common.*

Now, among the complexes of the system (A, B), there will be two of them that are special, because if one expresses the idea that the complex $\lambda A + \mu B = 0$ is special then one will be led to write:

$$\Omega(\lambda a + \mu b) = 0,$$

an equation that is nothing but equation (6), when λ replaces α and μ replaces β. At the same time, one recognizes by this means that the lines z, z' will be precisely the directrices of these special complexes. Therefore:

In any system of linear complexes (A, B) *with two terms there will bre two complexes that are special; the directrices of these special complexes will be the two lines that are conjugate to each other in all of the complexes of the system.*

28. One calls the set of lines that are common to two linear complexes a *linear congruence.*

It is clear that the congruence that is common to two complexes of a system of two terms is composed of lines that belong to all of the complexes of the system. Indeed, the equations $A = 0$, $B = 0$ imply that:

$$\lambda A + \mu B = 0.$$

In particular, the lines of that congruence belong to the special complexes, and, as a result:

The congruence that is common to two linear complexes A, B will be composed of lines that simultaneously cut the lines z, z' that are conjugate to each other in the two complexes.

For that reason, one gives the name of *directrices of the congruence* to the lines z, z'.

In order to find the line of the congruence that issues from a point P, one takes the intersection of two planes that pass through P and the two directrices. That line will be unique. It will meanwhile be indeterminate if the point P is taken on one of the two directrices.

In order to trace the line of the congruence that is situated in a plane Π, it will suffice to join the traces of the two directrices on that plane. There is only one solution; however, the problem will be indeterminate if the plane Π passes through one of the two directrices.

In summation, the linear congruence that is common to two complexes of first order will be of first order and first class, a result that one can, moreover, state *a priori.* Furthermore, two linear complexes will have an infinitude of plane pencils in common that one will generate by associating a plane Π that goes through a directrix of the common congruence with the point P where that plane cuts the other directrix.

29. We now arrive at the singular case that we left aside, in which the expression $\Phi(a \mid b)$ is zero. Since the two roots of (6) will be equal, the previous reasoning will break down.

We first exclude the case where all of the linear complexes included in the system with two terms $\lambda A + \mu B = 0$ are special; i.e., we exclude the case where:

$$\Omega(\lambda a + \mu b) = \Omega(a)\, \lambda^2 + 2\Omega(a, b)\, \lambda\mu + \Omega(b)\, \mu^2$$

is identically zero, which would demand that:

$$\Omega(a) = 0, \qquad \Omega(a \mid b) = 0, \qquad \Omega(b) = 0.$$

Equation (6) will then possess a double root, which I shall denote by $\alpha : \beta$, and there will be only one special complex in the system with two terms, namely:

$$\alpha A + \beta B = 0.$$

I further represent the coordinates of the directrix of the complex by z_i, and finally, I consider an arbitrary complex:

$$\lambda_0 A + \mu_0 B = 0$$

of the system with two terms (A, B).

One first has, by hypothesis:

$$\Omega(\alpha a + \beta b) = 0.$$

In addition, form:

$$\Omega(\alpha a + \beta b \mid \lambda_0 a + \mu_0 b).$$

λ_0, μ_0 figure linearly in that expression, and likewise, α, β. One must then write:

$$\begin{aligned} &\Omega(\alpha a + \beta b \mid \lambda_0 a + \mu_0 b) \\ &\qquad = \Omega(\alpha a + \beta b \mid a)\, \lambda_0 + \Omega(\alpha a + \beta b \mid b)\, \mu_0 , \\ &\qquad = \Omega(a \mid a)\, \alpha \lambda_0 + \Omega(b \mid a)\, \beta \lambda_0 + \Omega(a \mid b)\, \alpha \mu_0 + \Omega(b \mid b)\, \beta\mu_0 . \end{aligned}$$

However, $\Omega(a \mid a) = \Omega(a)$, $\Omega(b \mid a) = \Omega(a \mid b)$, $\Omega(b \mid b) = \Omega(b)$. One may then write:

$$= [\Omega(a)\, \alpha + \Omega(a \mid b)\, \beta]\, \lambda_0 + [\Omega(a \mid b)\, \alpha + \Omega(b)\, \beta]\, \mu_0 ;$$

i.e., this equals 0, since, because $\alpha : \beta$ is a double root of (6), one will have:

$$\begin{aligned} \Omega(a)\, \alpha + \Omega(a \mid b)\, \beta &= 0, \\ \Omega(a \mid b)\, \alpha + \Omega(b)\, \beta &= 0. \end{aligned}$$

We will thus have:

$$\Omega(\alpha a + \beta b \mid \lambda_0 a + \mu_0 b) = 0,$$

no matter what λ_0 and μ_0 are. This can be written:

$$\sum \frac{\partial \Omega(\alpha a + \beta b)}{\partial(\alpha a_i + \beta b_i)} \; (\lambda_0\, b_i + \mu_0\, b_i) = 0,$$

or again, taking our notations into account:

$$(8) \qquad \sum (\lambda_0\, b_i + \mu_0\, b_i)\, z_i = 0.$$

From this, we will get the theorem:

When $\Phi(a \mid b) = 0$, *or, more precisely, when equation* (6) *has a double root and is not an identity, the system with two terms* (A, B) *will contain a unique special complex, and the directrix of that special complex will be a line that is common to any complex of the system.*

It is clear that any line that is common to two complexes of the system will belong to all of the other complexes of the system, as in the general case. The linear congruence that is common to all of these complexes is therefore composed of lines that cut all of the directrices z of the unique special complex that belongs to the system. However, this condition is insufficient to define the congruence.

It is easy to complete this definition. Indeed, let Δ be a line of that congruence, let P be the point where it cuts the line z, and let Π be the plane through Δ and z. Consider the plane pencil (P, Π). Two lines of that pencil – namely, the line z and the line Δ – will belong to an arbitrary complex of the system. Therefore, the point P will admit the plane Π as its polar plane in all of the complexes of the system (A, B). The pencil (P, Π) will belong to all of these complexes (no. 13). From this, it will result immediately that *any complex of the system* (A, B) *will determine the same normal correlation* (no. 15) *on the line that is common to them.* Therefore, the linear congruence will admit the following definition here:

In order for a line Δ *to belong to the congruence, it is necessary and sufficient that:*

1. *It must cut the fixed line* (*directrix* z).
2. *The plane* (z, Δ) *through* z *and* Δ *and the point* (z, Δ), *which is the intersection of* z *and* Δ, *must be two corresponding elements of a homographic correlation that is on the line* z, *a priori.*

One may interject a remark here:

Consider a general linear congruence that admits the two directrices z, z'. Consider an arbitrary quadric Q through z and z'. The congruence will be composed of lines that cut the quadric Q at two points, one of which will be situated along z, and the other of which will be on z'. Since z' will approach z infinitely closely, the congruence will be nothing but the set of lines that meet the quadric at two infinitely close points, one of which will be situated on z; i.e., the set of tangents to the quadric at the various points of its generator z. The correlation that figures in the definition of the congruence wills therefore be nothing but the Chasles correlation that establishes the correspondence between the points of z and the tangent planes at these points.

The reader will easily recognize that the singular congruences that we just defined will be of first order and first class, moreover.

30. What remains is the case where equation (6) is an identity. The complexes of the system with two terms (A, B) will all be special. We seek the locus of their directrices.

Let one of these complexes be:

$$\lambda A + \mu B = 0,$$

and let y be its directrix, so one will have:

$$y_i = \frac{\partial \Omega(\lambda a + \mu b)}{\partial(\lambda a_i + \mu b_i)},$$

and since the right-hand side is linear and homogeneous in λ, μ, one may write:

$$y_i = \lambda \frac{\partial \Omega(a)}{\partial a_i} + \mu \frac{\partial \Omega(b)}{\partial b_i}.$$

One then recognizes that the directrices of the complexes of the system (A, B) (all special) will define a plane pencil. We then state the theorem:

When all of the complexes of a system with two terms are special, their directrices will define a plane pencil.

What is the congruence that is common to these complexes? The answer is simple: Any line of the congruence must cut all of the lines of the pencil of the directrices. Such a line must then either be in the plane of the pencil or pass through the center of the pencil. In a word, the congruence is found to decompose here into two hyper-pencils, one of which is the set of lines in the plane of the directrices, and the other of which is the spray of lines that issue from the point of intersection of the directrices.

Therefore, here is an example where the congruence that is common to two complexes decomposes into two of them: The one, which forms a planar system, is of order zero and class 1; the other, a spray, is of degree 1 and class zero. The sums of the classes and that of the degrees are equal to 0 + 1 = 1 + 0 = 1; i.e., to the product of the degrees of the complex.

We will have to recognize the generality of this fact for arbitrary complexes later on. It is interesting to note that it is present in the linear congruences.

Our linear congruence degenerates here into an infinitude of directrices that define a pencil (A, α). Let x be a line of that pencil. In the case of a singular congruence, a correlation on the directrix x will serve to define the congruence. It is clear that here that correlation will be, in turn, singular, because if one lets (O, Π) be any pair of that correlation – i.e., one such that any line of the pencil (O, Π) belongs to the congruence, O

is on x, and Π is a plane of x – then one must either have that O is at A, and Π is then arbitrary, or that Π coincides with the plane α, and O is arbitrary.

From the definition of the pairs (O, Π) of the correlation, one may then conclude that it will be singular, and that (A, α) will be its singular pair.

31. We have seen that the expression $\Omega(a)$ is an invariant of the complex $\sum a_i x_i = 0$. Likewise, the expression:

$$\Phi(a \mid b) = \Omega(a)\, \Omega(b) - [\Omega(a \mid b)]^2$$

is an invariant of the congruence that is common to the two complexes A and B. That invariant is of the type that one gives the name of *combinant* to. If one performs a linear transformation of the variables x_i then it will be reproduced, only multiplied by the fourth power of the determinant of the substitution, and, because of that, it will be an invariant. However, if one replaces the two equations:

$$A = 0, \qquad B = 0$$

with these:

$$\lambda A + \mu B = 0, \qquad \lambda' A + \mu' B = 0$$

then $\Phi(a \mid b)$ will be reproduced, but multiplied by $(\lambda\mu' - \mu\lambda')^2$. Indeed, one will have:

$$\begin{aligned}
&\Phi(\lambda A + \mu B \mid \lambda' A + \mu' B) \\
&\qquad = \Omega(\lambda A + \mu B)\, \Omega(\lambda' A + \mu' B) - [\Omega(\lambda A + \mu B \mid \lambda' A + \mu' B)]^2 \\
&\qquad = [\Omega(a)\, \lambda^2 + 2\, \Omega(a \mid b)\, \lambda\mu + \Omega(b)\, \mu^2]\, [\Omega(a)\, \lambda'^2 + 2\, \Omega(a \mid b)\, \lambda'\mu' + \Omega(b)\, \mu'^2] \\
&\qquad\quad - [\Omega(a)\, \lambda\lambda' + \Omega(a \mid b)(\lambda\mu' + \mu\lambda') + \Omega(b)\, \mu\mu'] \\
&\qquad = [\Omega(a)\, \Omega(a) - \Omega(a \mid b)]^2\, (\lambda\mu' + \mu\lambda')^2.
\end{aligned}$$

The properties of the invariant Φ will thus correspond to those of the linear congruence taken by itself, independently of the choice of coordinates, as well as the choice of the two linear complexes A, B by means of which one defines it; from this, one understands why the name "combinant" is given to that invariant.

32. If two linear complexes A, B are given then one can separate the properties of the two collectively into two groups: One of them belongs to the their common congruence and remains the same if one replaces A, B with two other complexes from the system of two terms (A, B): To these properties, we attach the invariant $\Phi(a \mid b)$, whose vanishing expresses the idea that the congruence is singular.

However, aside from these properties, it is the other group of properties that belongs to the two complexes A and B exclusively. That is why if one is given two spheres then their common circle will belong to all of the spheres of the pencil, whereas their angle of intersection will belong to these two spheres, in particular.

These are the properties of that order that we shall envision for the two complexes A, B.

Consider the complex:

$$A + k\,B = 0;$$

when k varies, this complex will run through all of the systems of two terms (A, B). Let Δ be a line of the congruence that is common to the complexes of that system, let Π be a plane through Δ, and let P_k denote the pole of the plane Π in the complex $A + k\,B = 0$. When k varies the point P_k will describe the line Δ. I say that P_k will correspond to the values of k in a unique fashion. Indeed, if k is given then P_k will be perfectly determined. In the second place, if one is given P_k as the pole of the plane Π in a complex $A + k\,B$ of the pencil then it will suffice to find the value of k in order to write that a line z through P_k in the plane Π will belong to the complex, which will give:

$$A(z) + k\,B(z) = 0,$$

which is an equation in k that is of the first degree.

One sees that one excludes the case where the complexes of the system (A, B) determine the same normal correlation on Δ. In this case, and only in this case, $A(z)$ and $B(z)$ will be zero for any position of the point P_k along the line Δ. Moreover, the congruence will then be singular, and Δ will be its directrix.

Since P_k and k correspond uniquely, it will then result, from the principle of correspondence, that the anharmonic ratio of the four values of k will be equal to that of the corresponding P_k . On thus has this theorem:

Let there be four complexes of the system that are obtained by taking:

$$k = \alpha, \beta, \gamma, \delta,$$

and let Δ be a line of the common congruence. The poles in the four complexes of a plane through Δ will define an anharmonic ratio that is equal to the anharmonic ratio of the quantities $\alpha, \beta, \gamma, \delta$.

This anharmonic ratio is then constant from two standpoints: First, it remains constant when the plane turns around Δ, and second, , it remains constant when Δ is displaced in its congruence.

The same reasoning leads to the following theorem, which is the transform of the preceding one under reciprocal polars.

Let four complexes of the system be given, which are obtained by taking:

$$k = \alpha, \beta, \gamma, \delta,$$

and let Δ be a line of the common congruence; the polar planes of an arbitrary point that is taken on Δ in the four complexes will define a pencil whose anharmonic ratio will be equal to that of the quantities $\alpha, \beta, \gamma, \delta$.

These two theorems still persist when the two directrices of the congruence coincide.

They will likewise preserve their *raison d'être* if all of the complexes of the system are special; in that case, the anharmonic ratio will be equal to that of the four directrices of the complex, which will define a plane pencil.

33. Now, argue under the hypothesis that the two directrices of the congruence are distinct. If two complexes:

$$A + \rho B = 0, \quad A + \rho' B = 0$$

are given then add to them the special complexes of the system:

$$A + k B = 0, \quad A + k' B = 0,$$

in such a way that k, k' will be roots of the equation:

$$\Omega(b)\, k^2 + 2\Omega(a \mid b)\, k + \Omega(a) = 0. \tag{8}$$

Let Δ be a line of the congruence that consequently intersects the directrices z, z' at two points F, F'. If one draws an arbitrary plane Π through Δ then F and F' will be the poles of that plane in the two *special* complexes (k) and (k'); they will remain fixed when the plane turns. By contrast, the poles P_ρ, $P_{\rho'}$ of that plane Π in the complexes (ρ), (ρ') will vary, but, with F, F' they will define an anharmonic ratio:

$$(P_\rho, P_{\rho'}, F, F') = (\rho, \rho', k, k'),$$

which will be constant. They will thus describe a homography on Δ whose double points will be F, F' and whose anharmonic ratio will be (ρ, ρ', k, k').

One likewise verifies that if one takes an arbitrary point P on Δ, and if Π_ρ, $\Pi_{\rho'}$ are the polar planes to P in the complexes (ρ), (ρ'), while Φ, Φ' are the planes through Δ and z, then Δ and z' will be the planes polar to P in the *special* complexes (k), (k'); they will be fixed. The anharmonic ratio of the four planes Π_ρ, $\Pi_{\rho'}$, Φ, Φ' will be equal to:

$$(\Pi_\rho, \Pi_{\rho'}, \Phi, \Phi') = (\rho, \rho', k, k');$$

it will be constant and will have the same value as the first one.

When P is displaced along Δ, only the planes Π_ρ, $\Pi_{\rho'}$ will vary, and they will then describe two homographic pencils around Δ, where Φ, Φ' will be the double planes, and (ρ, ρ', k, k') will be the constant anharmonic ratio.

This anharmonic pencil is easy to calculate; let ε denote it. We have:

$$\varepsilon = \frac{\rho - k}{\rho' - k} : \frac{\rho - k'}{\rho' - k'} = \frac{(\rho - k)}{(\rho' - k)} \frac{(\rho' - k')}{(\rho - k')}$$

$$= \frac{2(\rho\rho' + kk') - (k + k')(\rho + \rho') - (k' - k)(\rho' - \rho)}{2(\rho\rho' + kk') - (k + k')(\rho + \rho') + (k' - k)(\rho' - \rho)},$$

so:

$$\frac{\varepsilon + 1}{\varepsilon - 1} = \frac{2(\rho\rho' + kk') - (k + k')(\rho + \rho')}{(k' - k)(\rho' - \rho)},$$

and since k, k' are roots of (8), this will become:

$$\frac{\varepsilon + 1}{\varepsilon - 1} = \frac{\Omega(b)\rho\rho' + \Omega(a \mid b)\overline{\rho + \rho'} + \Omega(a)}{(\rho' - \rho)\sqrt{-\Phi(a \mid b)}}.$$

In particular, set $\rho' = \infty$, and then set $\rho = 0$, so the two complexes considered will then be A and B, and we will have:

$$\frac{\varepsilon + 1}{\varepsilon - 1} = \frac{\Omega(a \mid b)}{\sqrt{-\Phi(a \mid b)}}.$$

It will suffices to refer to what we said above on the subject of homographic correlations on a line in order to see that this constant anharmonic ratio ε will be equal to that of the two normal correlations of the complex along any of their *common* lines. The angle of these two normal correlations will be what we, along with Klein, shall call *the angle between the two complexes.*

If one sets:

$$V = \frac{1}{2i} \log \varepsilon$$

then one will easily find:

(9) $$\cos V = \frac{1}{2}\left(\frac{1}{\sqrt{\varepsilon}} + \sqrt{\varepsilon}\right) = \frac{\Omega(a \mid b)}{\sqrt{\Omega(a)\,\Omega(b)}}.$$

Although it is not necessary for us to insist upon this fact, one sees that if $V = \pi / 2$ or $\varepsilon = -1$ then the normal correlations will be in involution, and the two complexes will also be said to be *in involution* or *orthogonal*.

The condition of involution for the two complexes A and B is, moreover, the following one:

$$\Omega(a \mid b) = 0.$$

34. Let us examine some particular cases.

The notion of involution that we just gave breaks down if one of the complexes A, B is special. However, we may continue to say that two complexes are in involution whenever the simultaneous invariant $\Omega(a \mid b)$ becomes zero, or likewise if a and b are both special.

In addition, suppose that B is special, and let z be its directrix. The equation:

$$\Omega(a \mid b) = 0$$

can be written:

$$\sum \frac{\partial \Omega}{\partial b_i} a_i = 0,$$

because, since:

$$z_i = \frac{\partial \Omega}{\partial b_i},$$

one will have, upon summing:

$$\sum a_i\, z_i = 0.$$

Therefore, *a special complex will be in involution with any complex that contains its directrix, and conversely.*

More particularly, if A itself becomes special then, by an application of this theorem, one will see that *two special complexes will be in involution under the necessary and sufficient condition that their directrices agree.*

One may present the notion of complexes in involution from another viewpoint.

Suppose one has a complex:

$$\sum a_i\, x_i = 0.$$

The condition for two lines y, y' to be conjugate in the complex is written, as one knows, in the form:

$$\frac{1}{2}\frac{\partial \Omega(a)}{\partial a_i} = \rho\, y_i + \rho' y_i' \qquad (i = 1, 2, \ldots, 6),$$

where ρ, ρ' are two parameters.

Suppose that the line y describes the complex:

$$\sum b_i\, y_i = 0.$$

The equation:

$$\frac{1}{2}\sum \frac{\partial \Omega(a)}{\partial a_i} b_i = \Omega(a \mid b) = \rho \sum b_i y_i + \rho' \sum b_i y_i'$$

will give:

$$\Omega(a \mid b) = \rho' \sum b_i y_i'.$$

If we seek, moreover, the condition for the line y' to describe the complex B, as well, then we will find:

$$\Omega(a \mid b) = 0.$$

However, if y belongs to a complex B at the same time as y then this would signify that B is its own reciprocal polar with respect to the complex A.

One will thus arrive at the following theorem:

If two linear complexes are in involution then each of them will be its own reciprocal polar with respect to the other one.

If one of the complexes is special then one will find the property of the lines of a complex that they must coincide with their conjugates with respect to that complex.

35. The consideration of complexes that are in involution plays the most important role in the geometry of the straight line. It is closely linked to the theory of *linear systems of complexes of the first degree.*

We have given the name of *system with two terms* to the set of complexes that are contained in the equation:

$$\lambda A + \mu B = 0;$$

likewise, let A, B, C be three linear complexes that are not contained in the same system with two terms. We shall give the name of *system of three terms* to the set of complexes that are represented by the equation:

$$\lambda A + \mu B + \nu C = 0.$$

Furthermore, consider four linear complexes A, B, C, D that are not contained in the same system with three terms, so the linear complexes that are represented by the equation:

$$\lambda A + \mu B + \nu C + \rho D = 0$$

will define a set of four terms.

Finally, upon taking five linear complexes A, B, C, D, E that are not contained in the same system of four terms, the equation:

$$\lambda A + \mu B + \nu C + \rho D + \sigma E = 0$$

will represent a system of five terms.

There is good reason to observe that the equation of a linear complex:

$$\sum a_i x_i = 0$$

will contain six coefficients and, as a result, five parameters. If one takes six complexes:

$$\begin{gathered} A = \sum a_i x_i = 0, \\ B = \sum b_i x_i = 0, \\ C = \sum c_i x_i = 0, \\ D = \sum d_i x_i = 0, \\ E = \sum e_i x_i = 0, \\ F = \sum f_i x_i = 0, \end{gathered}$$

and one forms the expression:

$$\lambda A + \mu B + \nu C + \rho D + \sigma E + \tau F = \sum u_i x_i$$

then one will have:

(10) $$u_i = a_i\,\lambda + b_i\,\mu + c_i\,\nu + d_i\,\rho + e_i\,\sigma + f_i\,\tau \qquad (i = 1, 2, \ldots, 6).$$

If the determinant:

$$\Delta = \| a_i\ \ b_i\ \ c_i\ \ d_i\ \ e_i\ \ f_i \|$$

is not assumed to be zero then one might not find values of $\lambda, \mu, \nu, \rho, \sigma, \tau$ other than zero that annul all of the u_i. There might not exist a linear relation of the form:

$$\lambda A + \mu B + \nu C + \rho D + \sigma E + \tau F = 0$$

then, and the complexes A, B, C, D, E, F would not belong to the same system of five terms.

On the contrary, if Δ were zero then such a linear relation would be meaningful, and the six complexes would belong to the same system of five terms, or even a smaller number of terms.

If Δ is not zero – i.e., if A, B, C, D, E, F do not belong to the same system of five terms or to a system of less than five terms – then equations (10) can be solved with respect to $\lambda, \mu, \nu, \rho, \sigma, \tau$, and as a result, any linear complex:

$$\sum u_i\,x_i = 0$$

can be represented by an equation such as:

$$\lambda A + \mu B + \nu C + \rho D + \sigma E + \tau F = 0.$$

One thus has the theorem:

If the six linear complexes A, B, C, D, E, F do not belong to the same system with five terms or to the same system with a number of terms that is less than five then the equation of any linear complex can assume the form:

$$\lambda A + \mu B + \nu C + \rho D + \sigma E + \tau F = 0.$$

In other words, a system of six terms comprises all possible linear complexes. Later on, we shall have to make use of this theorem in the context of the transformation of coordinates. For the moment, we do not rule out the linear systems, which are the object of our present study.

36. Consider the system with p terms:

(11) $$\lambda_1 A_1 + \lambda_2 A_2 + \ldots + \lambda_p A_p = 0,$$

where:

$$A_\mu = a_{\mu 1}\,x_1 + a_{\mu 2}\,x_2 + \ldots + a_{\mu 6}\,x_6 .$$

Let $\sum u_i\, x_i = 0$ be a complex, and if we express the notion that this complex u is in involution with the complex (11) then we will have:

$$\Omega(u \mid \lambda_1\, a_1 + \lambda_2\, a_2 + \ldots + \lambda_p\, a_p) = 0;$$

i.e.:

$$\Omega(u \mid a_1)\, \lambda_1 + \ldots + \Omega(u \mid a_p)\, \lambda_p = 0.$$

One then sees that if one writes:

$$\Omega(u \mid a_1) = 0, \qquad \Omega(u \mid a_2) = 0, \qquad \Omega(u \mid a_p) = 0 \tag{12}$$

then the complex (u) will first of all be in involution with the complexes (a_1), (a_2), …, (a_p), and, moreover, AS A CONSEQUENCE, it will be in involution with all of the complexes of the system with p terms (11).

The equations (12) may be written:

$$\sum \frac{\partial\Omega(a_1)}{\partial a_{1i}} u_i = 0, \qquad \sum \frac{\partial\Omega(a_2)}{\partial a_{2i}} u_i = 0, \ldots \quad \sum \frac{\partial\Omega(a_p)}{\partial a_{pi}} u_i = 0.$$

We have p equations between the u_i; they are all distinct, because, if it were otherwise the one would be able to find quantities ρ_1, ρ_2, …, ρ_p that would not all zero and would verify the relations:

$$\rho_1 \frac{\partial\Omega(a_1)}{\partial a_{1i}} + \cdots + \rho_p \frac{\partial\Omega(a_p)}{\partial a_{pi}} = 0 \qquad (i = 1, 2, \ldots, 6).$$

These relations may be written:

$$\frac{\partial\Omega(\rho_1 a_1 + \cdots + \rho_p a_p)}{\partial(\rho_1 a_{1i} + \cdots + \rho_p a_{pi})} = 0 \qquad (i = 1, 2, \ldots, 6),$$

and, since Ω has a non-zero discriminant this will demand that one must have:

$$\rho_1\, a_{1i} + \ldots + \rho_p\, a_{pi} = 0 \qquad (i = 1, 2, \ldots, 6).$$

What will then result is the identity:

$$\rho_1\, A_1 + \ldots + \rho_p\, A_p = 0,$$

and the complexes A_1, …, A_p will belong to a system with (p – 1) terms or a lower number of them. That would be contrary to our hypothesis that the system (11) is a system of p terms.

Equations (12) are thus distinct and, as a consequence, they permit one to deduce p of the u_i as functions of the $6 - p$ other ones. The general values of u_i that verify equations (12) will then have the form:

$$u_i = g_{1,i}\,\mu_1 + g_{2,i}\,\mu_2 + \ldots + g_{6-p,i}\,\mu_{6-p} = 0 \qquad (i = 1, 2, \ldots, 6),$$

and the g will be constant coefficients such that for any value of μ other than zero the u_i can be annulled all at once. From this, if one sets:

$$G_1 = \sum g_{1,i}\, x_i\,, \qquad G_2 = \sum g_{2,i}\, x_i\,, \ldots \qquad G_{6-p} = \sum g_{6-p,i}\, x_i$$

then the $(6 - p)$ complexes G can verify an identity such as:

$$\rho_1\, G_1 + \rho_2\, G_2 + \rho_{6-p}\, G_{6-p} = 0,$$

which signifies that these complexes do not all belong to the same system of $6 - p - 1 = 5 - p$ terms or to a system with a lower number of terms. The set of complexes in involution with all of the systems with p terms, a set that is represented by the equation:

$$\sum u_i\, x_i = \mu_1\, G_1 + \ldots + \mu_{6-p}\, G_{6-p} = 0,$$

will thus define a system with $6 - p$ terms.

From this, one will get the theorem:

The linear complexes that are not in involution with those of a system with p terms define a system $6 - p$ *terms in their own right.*

To abbreviate, we shall say *complementary systems* to describe two systems with p and $(6 - p)$ terms whose complexes are in involution.

For example, take a system with five terms. The complementary system will includes only one complex. One will thus have the theorem:

The complexes of a system with five terms are orthogonal to a fixed linear complex.

37. Let Σ, Σ_0 be two complementary systems with p and $(6 - p)$ terms, and suppose that p is equal to at least 2. Then, among the complexes of the system Σ there is a special one, as one sees upon writing:

$$\Omega(a_1\lambda_1 + \ldots + a_p\lambda_p) = 0;$$

i.e.:

$$\Omega(a_1)\,\lambda_1^2 + \Omega(a_2)\,\lambda_2^2 + \ldots + \Omega(a_p)\,\lambda_p^2 + 2\Omega(a_1 \mid a_2)\,\lambda_1\,\lambda_2 + 2\Omega(a_1 \mid a_3)\,\lambda_1\,\lambda_3 + \ldots = 0.$$

Now, the directrices of these special complexes must belong to each of the complexes of the complementary system Σ_0 ; indeed, these special complexes will be in involution with all of the ones of the complementary system Σ_0 . Therefore, from a theorem of no. 34, their directrices will belong to these complexes. Conversely, any line that is common to all of the complexes of the system Σ_0 will be the directrix of a special complex in involution with all of the complexes of the system Σ_0 ; this special complex will thus belong to the system Σ. We thus state the theorem:

The directrices of the special complexes that are contained in a system Σ *are nothing but the lines that are common to the complexes of the complementary system* Σ_0 .

We add the remark:

The lines that are common to the complexes of a system Σ *will cut all of the lines that are common to the complexes of a complementary system* Σ_0 .

Indeed, these lines will be the directrices of special complexes that are in involution.

38. The introduction of the notion of involution greatly simplifies the problem of the search for lines that are common to several linear complexes. We have already treated the case of two complexes; what remain then are the cases of three and four complexes.

Let *A*, *B*, *C* be three complexes that do not belong to the same system with two terms. We then propose to look for their common lines.

In order to do this, consider the system of three terms:

$$\Sigma = \lambda A + \mu B + \nu C = 0;$$

the complementary system Σ_0 will likewise be a system of three terms. We seek the special complexes that are contained in the first system Σ. We write:

$$\Omega(a\lambda + b\mu + c\nu) = 0,$$

when we are given that:

$$A = \sum a_i\, x_i\,, \qquad B = \sum b_i\, x_i\,, \qquad C = \sum c_i\, x_i\,.$$

Upon developing this, we will get:

$$(13)\qquad \begin{cases} \Omega(\lambda a + \mu b + \nu c) = \Omega(a)\lambda^2 + \Omega(b)\mu^2 + \Omega(c)\nu^2 \\ \qquad\qquad + 2\Omega(a\,|\,b)\lambda\mu + 2\Omega(a\,|\,c)\lambda\nu + 2\Omega(b\,|\,c)\mu\nu = 0. \end{cases}$$

I set:

$$(14)\qquad \Psi(a\,|\,b\,|\,c) = \begin{vmatrix} \Omega(a) & \Omega(a\,|\,b) & \Omega(a\,|\,c) \\ \Omega(b\,|\,a) & \Omega(b) & \Omega(b\,|\,c) \\ \Omega(c\,|\,a) & \Omega(c\,|\,b) & \Omega(c) \end{vmatrix},$$

in such a way that Ψ will be the discriminant of the quadratic form (13). This discriminant will be a simultaneous invariant of the complexes *A*, *B*, *C*. However, there is more: It will also be a *combinant*, like the function $\Phi(a|\,b)$. Indeed, if one replaces A, B, C with combinations such as:

$$A_1 = p\,A + q\,B + r\,C,$$
$$B_1 = p'\,A + q'\,B + r'\,C,$$
$$C_1 = p''\,A + q''\,B + r''\,C,$$

where the determinant:

$$\begin{vmatrix} p & q & r \\ p' & q' & r' \\ p'' & q'' & r'' \end{vmatrix}$$

is not zero, then the function Ψ will be reproduced, but multiplied by the square of that determinant. We observe, in passing, that if one replaces A, B, C with expressions such as A_1, B_1, C_1, for which the determinant $\Sigma \pm p\,q'\,r''$ is not zero, then this will amount to performing a linear transformation of the form (13):

$$\lambda = p\lambda_1 + p'\mu_1 + p''\nu_1\,,$$
$$\mu = q\lambda_1 + q'\mu_1 + q''\nu_1\,,$$
$$\nu = r\lambda_1 + r'\mu_1 + r''\nu_1$$

on the variables λ, μ, ν. One can profit from this remark to reduce the form (13). Therefore, if the invariant Ψ is not zero then the form (13) will be reducible to a sum of three squares or, what amounts to the same thing, a form of the type:

$$\lambda\mu - \nu^2.$$

If Ψ is zero, but not all of its minors, then the form (13) will be the product of two factors, and one may suppose that the form is of the type:

$$\lambda\mu.$$

If Ψ is zero, as well as all of its minors, then the form will be a perfect square, and one may suppose that the square is:

$$\nu^2.$$

Finally, it might be the case that the form (13) is identically zero.

We thus have the four cases that might present themselves in the intersection of three complexes of first degree. We examine them successively.

39. In the first case, one must have:

$$\Omega(a) = 0, \qquad \Omega(b) = 0, \qquad \Omega(a \mid c) = 0, \quad \Omega(b \mid c) = 0, \quad 2\Omega(a \mid b) = -\,\Omega(c) = 1.$$

The first two equations show that the two complexes A and B must be special, and, since $2\Omega(a \mid b) = 1$, one sees that their directrices cannot intersect, since $\Omega(a \mid b) = 0$ would be the condition for them to meet. The third and fourth equations show that these directrices must belong to the complex C.

Now, one verifies the following equation in the most general fashion:

$$\lambda\mu - \nu^2 = 0,$$

by taking:

$$\lambda = t^2, \qquad \mu = 1, \qquad \nu = t,$$

where t is a parameter, in such a way that all of the special complexes of the system will be represented by the following equation:

$$\sum (a_i\, t^2 + c_i\, t + b_i)\, x_i = 0.$$

The coordinates of the directrix z of one of these complexes will be:

$$z_i = \frac{\partial\Omega(at^2 + ct + b)}{\partial(a_i t^2 + c_t t + b_i)},$$

or further:

$$z_i = \frac{\partial\Omega(a)}{\partial a_i} t^2 + \frac{\partial\Omega(c)}{\partial c_i} t + \frac{\partial\Omega(b)}{\partial b_i}.$$

The locus of these directrices z will then be a ruled series, and in fact, a ruled series of second order, because if one seeks the number of lines of the series that cut the fixed line y_i then one will be led to the equation of the second order in t:

$$\begin{aligned} 0 = \omega(y \mid z) &= \sum \frac{\partial\omega}{\partial y_i} z_i \\ &= \left[\sum \frac{\partial\omega}{\partial y_i}\frac{\partial\Omega(a)}{\partial a_i}\right] t^2 + \left[\sum \frac{\partial\omega}{\partial y_i}\frac{\partial\Omega(c)}{\partial c_i}\right] t + \left[\sum \frac{\partial\omega}{\partial y_i}\frac{\partial\Omega(b)}{\partial b_i}\right] = 0. \end{aligned}$$

A ruled series can be composed of the generators of a ruled surface, of those of a cone, or even of the tangents to a planar curve. In the last two cases, the ruled series will be contained in a hyper-pencil.

Now, this is not the case here, since if this were true then the directrices of the special complexes A and B would have to intersect, since they belong to the same hyper-pencil. The expression $\Omega(a \mid b)$ would then be zero, which is not true.

One must then conclude that the directrices of our special complexes form a ruled surface, which is of second degree, since the ruled series of directrices is of second order.

One thus has this theorem:

The directrices of the special complexes of a system with three terms will generally constitute the rectilinear generators of a family of quadrics.

To abbreviate, we say that they form a *semi-quadric*. The generators of the second system of that quadric constitute what we will call the *complementary semi-quadric to the first one.*

It is now easy to obtain the lines that are common to the three complexes A, B, C. The set of these lines belongs to all of the complexes of the system of three terms:

$$\lambda A + \mu B + \nu C = 0,$$

and may be defined by taking three arbitrary complexes A_1, B_1, C_1 of this system, provided that these three complexes do not belong to the same system of two terms. Now, this is precisely the case for the three special complexes A_1, B_1, C_1 of the system; indeed, their directrices cannot intersect, since they are the directrices of the same semi-quadric. The system of two terms:

$$\rho A_i + \sigma B_i = 0$$

will then includes no other special complex besides A_1 and B_1, and, as a result, A_1, B_1, C_1 will not belong to the same system with three terms (this might no longer be true if the directrices of A_i and B_i intersect).

The lines that are common to the complexes of the system with three terms are therefore defined by the condition that they cut three arbitrary generators of the semi-quadric $\mathcal{Q}$ that is the locus of the directrices of the special linear complex of the system. These lines will thus constitute the complementary quadric $\mathcal{Q}_0$. We then assert this theorem:

The lines that are common to three complexes A, B, C do not likewise comprise a system of two terms, and as a result the lines that are common to all of the complexes of the system of three terms:

$$\Sigma = \lambda A + \mu B + \nu C = 0$$

will define a semi-quadric $\mathcal{Q}_0$ *that is complementary to the semi-quadric* $\mathcal{Q}$ *that is the locus of the directrices of the special complexes that are contained in the system of three terms.*

One should not neglect to observe that the system Σ_0 with three terms that is complementary to the system Σ will admit the semi-quadric $\mathcal{Q}_0$ as the locus of directrices of its special complexes, and that the lines of the semi-quadric $\mathcal{Q}$ will, on the contrary, be common to all of the complexes of the system Σ_0 . This will result from the corollary at the end of no. 37.

40. Our reasoning assumed only that Ψ was not zero; now, assume that $\Psi = 0$. We know that the form (13) can be reduced to $\lambda\mu$. This would give us:

$$\Omega(a) = 0, \qquad \Omega(b) = 0, \qquad \Omega(c) = 0, \qquad \Omega(b \mid c) = 0, \quad \Omega(c \mid a) = 0;$$

however, $\Omega(a \mid b)$ is not zero.

From the first three equations, the complexes A, B, C must be special. The last two show us that, in addition, the directrix Δ_C of the complex C must cut the directrices Δ_A , Δ_B of the other two complexes; the latter two will not intersect, since $\Omega(a \mid b)$ is non-zero.

Let F be the point of intersection of Δ_C and Δ_A , and let F' be that of Δ_C and Δ_B . Let Φ be the plane of Δ_C and Δ_A , while Φ' is that of Δ_C and Δ_B .

The special complexes of the system of three terms will decompose into two families with regard to the equation:

$$\lambda\,\mu = 0,$$

namely:

$$\mu\,B + \nu\,C = 0 \qquad\qquad \text{and} \qquad \lambda\,A + \nu\,C = 0.$$

Each of these families will constitute a system of two terms, and, from what we know about these systems, since the complexes that comprise them are all special, their directrices will define a plane pencil.

The family:

$$\lambda\,A + \nu\,C = 0$$

will then be composed of the special complexes whose generators generate the plane pencil (F, Φ), while the pencil (F', Φ') will correspond to the second family.

One will observe that the two planar pencils (F, Φ), (F', Φ') will have a common line Δ_C , which will be the directrix of the complex C.

It is now easy to obtain the lines that are common to the complexes of three terms. Any of these lines will be defined by the condition that it must cut all of the lines of the pencils (F, Φ), (F', Φ'). If it does not pass through F then it will be in the plane Φ, and if it does not pass through F' then it will be in the plane Φ'. These lines will then be those of the two planar pencils (F, Φ'), (F', Φ). One sees that the pairs (F, Φ'), (F', Φ) will be the inverses (no. 21) of the pairs (F, Φ), (F', Φ').

The lines of the pencils (F, Φ), (F', Φ') will then constitute a degeneracy in the lines of the semi-quadric, while the inverse pencils (F, Φ'), (F', Φ) will constitute the degenerate complementary semi-quadric.

It is quite appropriate to remark that for these geometric entities a new conception of quadrics leads to a mode of degeneracy that one does not encounter when one defines them by their points or their planes. Moreover, one knows that the latter two definitions will each lead to their own degeneracies: viz., cones or planes for the point-like quadrics and conics or points for the tangential quadrics.

A point-like quadric cannot become a conic or a point, nor can a tangential quadric become a cone or a plane. From the new viewpoint that that we now take, the quadric can degenerate into four inverse plane pencils (F, Φ), (F', Φ'), (F, Φ'), (F, Φ'), the first two of which represent the generators of one system and the last two of which represent the generators of the other.

41. Now assume that the invariant Ψ is zero, along with its first-order minors. The form will then be a perfect square, which one may assume is ν^2. In this case, one will have:

$$\Omega(a) = 0, \qquad \Omega(b) = 0, \qquad \Omega(a \mid b) = 0, \quad \Omega(a \mid c) = 0, \quad \Omega(b \mid c) = 0,$$

but $\Omega(c)$ will not be zero.

The complexes A, B will be special, and due to the fact that $\Omega(a \mid b) = 0$, their directrices will intersect. Let F be that point of intersection, and let Φ be their common plane. Since $\Omega(a \mid c) = 0$, $\Omega(b \mid c) = 0$, the lines that are the directrices of A and B will belong to the non-special complex C, and as a result, F will be the pole of the plane Φ in this complex. One immediately deduces that the lines that are common to the complexes A, B, C will be nothing but the lines of the plane pencil (F, Φ). Moreover, since the special complexes of the system will define the system of two terms:

$$\lambda A + \mu B = 0,$$

it is clear that the directrices of these special complexes will likewise be the lines of the plane pencil (F, Φ).

If one envisions the complementary system Σ_0 of the system Σ considered, namely:

$$\Sigma = \lambda A + \mu B + \nu C$$

then one will see that the complexes of the system Σ_0 have the lines of the pencil (F, Φ) in common, since the lines of the system Σ and the special complexes will be further represented by:

$$\lambda A + \mu B = 0.$$

This is nothing but the preceding, except that the pencils (F, Φ) and (F', Φ') will coincide.

42. If we are to exhaust the systems with three terms then what finally remains for us to do is to treat the case in which the form (13) is identically zero. All of the complexes of the system:

$$\Sigma = \lambda A + \mu B + \nu C = 0$$

will then be special. Let z be the directrix of one of these complexes, so one will have:

$$z_i = \frac{\partial\Omega(a)}{\partial a_i}\lambda + \frac{\partial\Omega(b)}{\partial b_i}\mu + \frac{\partial\Omega(c)}{\partial c_i}\nu,$$

and, as a result, these directrices will define a hyper-pencil.

The system Σ will therefore be composed of special complexes whose directrices form a hyper-pencil. The lines themselves of that hyper-pencil will be, moreover, the only ones that are common to all of the complexes of the system.

This case exhibits the remarkable aspect that the system Σ will coincide with its complentary system. The reader will easily prove that this is the only case in which this situation prevails.

It is assuredly quite remarkable that the line complexes of such a system of three terms must have a congruence of lines in common (of class 1 and degree zero or class zero and degree 1), since otherwise there would exist a linear relation between any three of these complexes. This fact shows the degree of circumspection that one must treat these questions with, and the fact that one must explain things with care in perhaps a bit more detail than we have seen fit to invest in this part of our exposition.

43. We now arrive at the lines that are common to four linear complexes, and to the systems of four terms. Let Σ be such a system that is represented by the equation:

$$\Sigma = \lambda A + \mu B + \nu C + \rho D = 0.$$

Since the complementary system Σ_0 has two terms, we may utilize what we know about the systems with two terms and complementary systems. The systems with two terms contain two special complexes, which might coincide in certain cases. They may also be composed of special complexes whose directrices form a plane pencil. Let us see what the corresponding complementary systems with four terms would be.

First, in the general case, we see that the complexes of the system Σ with four terms will have two common lines Δ, Δ', which are directrices of the linear congruence that is common to the complexes of the system with two terms. Thus:

Four linear complexes that are included in the same system of three terms will have two common lines Δ, Δ', *in general.*

The congruence whose directrices are Δ, Δ' is the locus of the directrices of the special complex of the system.

Δ and Δ' might coincide accidentally: The congruence of the directrices of the special complexes would then be singular.

Finally, there is the case where all of the complexes of the complementary system of two terms Σ_0 are special. Let (F, Φ) be the plane pencil that is formed by the directrices of the special complex. The complexes of the system with four terms will then have (from a theorem that was established in no. 37) all of the lines of the pencil (F, Φ) in common, and no other ones.

The complexes of the system with four terms will then be defined by the property that they admit a given plane pencil of lines (F, Φ).

As in the preceding case, one may introduce the form in λ, μ, ν, ρ:

$$\Omega(a\,\lambda + b\,\mu + c\,\nu + d\,\rho) = 0$$

and its discriminant:

$$\begin{vmatrix} \Omega(a) & \Omega(a \mid b) & \Omega(a \mid c) & \Omega(a \mid d) \\ \Omega(b \mid a) & \Omega(b) & \Omega(b \mid c) & \Omega(b \mid d) \\ \Omega(c \mid a) & \Omega(c \mid b) & \Omega(c) & \Omega(c \mid d) \\ \Omega(d \mid a) & \Omega(d \mid b) & \Omega(d \mid c) & \Omega(d) \end{vmatrix},$$

which is a combinant. If this discriminant is non-zero then one will be dealing with the general case. If it is zero then the lines Δ, Δ' will coincide.

If its minors of first order are all zero then one will be dealing with the case where the complexes have a plane pencil of lines in common.

I leave to the reader the task of proving these results, which are analogous to the ones that we already encountered for the systems of three terms. One verifies that the form in λ, μ, ν, ρ might not be identically zero, nor likewise be a perfect square.

44. We complete this study of linear systems of complexes with a remark that concerns systems with five terms.

Let A, B, C, D, E be five complexes that are not contained in the same system with four terms. If one solves the five equations:

$$A = 0, \quad B = 0, \quad C = 0, \quad D = 0, \quad E = 0$$

then the corresponding values of $x_1, \ldots, x_6$ will not generally verify the equation:

$$\omega(x) = 0;$$

in a word, the five linear complexes will not have a common line, in general.

The complementary system will reduce to a unique linear complex, as we have remarked in no. 37, and from the results that were obtained in the same place, since the common lines to the complexes of a systems will be the directrices of the special complex of the conjugate system, five linear complexes that are not included in the same system of four terms might have only one common line – namely, the directrix of the complementary complex – *which must again be special.*

Conversely, the linear complexes that contain a given line z will define a system with five terms, namely, the complementary system to the system with one term that is composed of the linear complex whose directrix is z.

45. In the course of this exposition, we have introduced successively the invariants $\Omega(a)$, $\Phi(a \mid b)$, $\Psi(a \mid b \mid c)$, and we have indicated another one that relates to systems with four terms; the systems with five terms also have an invariant. One might represent these combinants in a uniform fashion as follows: Let:

$$\omega(x) = \sum \omega_{ik}\, x_i\, x_k$$

be the fundamental form, so one will have, up to a constant factor, this expression for $\Omega(a)$:

$$\begin{vmatrix} \omega_{11} & \omega_{12} & \cdots & \omega_{16} & a_1 \\ \omega_{21} & \omega_{22} & \cdots & \omega_{26} & a_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \omega_{61} & \omega_{62} & \cdots & \omega_{66} & a_6 \\ a_1 & a_2 & \cdots & a_6 & 0 \end{vmatrix}.$$

One will likewise have for $\Phi(a \mid b)$:

$$\begin{vmatrix} \omega_{11} & \cdots & \omega_{16} & a_1 & b_1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \omega_{61} & \cdots & \omega_{66} & a_6 & b_6 \\ a_1 & \cdots & a_6 & 0 & 0 \\ b_1 & \cdots & b_6 & 0 & 0 \end{vmatrix},$$

and for $\Psi(a \mid b \mid c)$:

$$\begin{vmatrix} \omega_{11} & \cdots & \omega_{16} & a_1 & b_1 & c_1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \omega_{61} & \cdots & \omega_{66} & a_6 & b_6 & c_6 \\ a_1 & \cdots & a_6 & 0 & 0 & 0 \\ b_1 & \cdots & b_6 & 0 & 0 & 0 \\ c_1 & \cdots & c_6 & 0 & 0 & 0 \end{vmatrix}.$$

The invariant of a system with four terms will be obtained by bordering this, on the right and at the bottom, with the line $d_1 \ldots d_6$ 0 0 0 0; upon adding another border on the right and at the bottom that consists of the line $e_1 \ldots e_6$ 0 0 0 0 0, one will get the invariant for the system with five terms. This invariant will be annulled if the complexes of the system have a common line; i.e., if the complementary complex is special.

46. In concluding this chapter, we finally remark that it results from the preceding discussion that any complex P that contains lines that are common to p other than A, B, …, D, and which is *not contained* in a system with $(p - 1)$ terms or one with a number of terms less than $(p - 1)$, will belong to the system with p terms:

$$P = \lambda A + \mu B + \ldots + \rho D = 0.$$

The reader will easily verify this remark, while I shall content myself by merely stating it here.

CHAPTER IV.

PRINCIPLES OF INFINITESIMAL GEOMETRY IN LINE COORDINATES.

Skew surfaces. – Chasles correlation. – Quadrics of agreement and congruence of tangents. – Osculating hyperboloid. – Contact of a skew surface with a linear complex. – Contact of various orders. Ruled series with envelope. – Osculating pencil. – Case of the cone and the plane curve. – Theorem on curves whose tangents belong to a linear complex. – Lie contact elements. – Plane pencils that depend upon one parameter. – Bands. – General theorem on plane pencils with envelopes. – Infinitesimal properties of first order for line complexes. – Tangent complexes. – Normal correlation. – Its properties. – Plane pencils of a complex. – Klein invariant. – Singular lines. – Surface of singularities. – Pasch's theorem. – Singular complexes. – Cayley-Klein theorem. – Congruences. – Focal surfaces. – Focal pairs. – Developables. – Tangent linear complexes. – Special cases. – Invariant. – Congruences of asymptotic tangents. – Case of degeneracy.

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47. In this chapter, we shall develop the *first principles* of infinitesimal geometry in line coordinates.

Suppose that a line x depends upon one parameter t; it generates a ruled series that consitutes a skew surface, a developable, a cone, or the set of tangents to a plane curve.

We first examine the case where the series constitutes a skew surface. One knows that the distribution of tangent planes at each point of a rectilinear generator is constructed by means of a homographic correlation that we have already spoken of, and which we called the *Chasles correlation.*

The set of tangents to the surface at all of the points of the generator x constitutes a singular linear congruence; all of the linear complexes that contain that congruence will define the same normal correlation on x, namely, the Chasles correlation.

These complexes form a system with two terms that is easy to represent.

One may regard the congruence of the tangents as the set of lines that are subject to cutting the neighboring lines x and $x + x'\,dt$, where $x' = dx / dt$. They are therefore defined by the following two equations, where y is the current line:

$$2\,\omega(x \mid y) = \sum \frac{\partial \omega}{\partial x_i} y_i = 0,$$

$$2\,\omega(x + dx \mid y) = \sum \left[\frac{\partial \omega}{\partial x_i} + \frac{\partial \omega(x')}{\partial x_i'} dt \right] y_i = 0;$$

i.e.:

(1) $$\sum \frac{\partial \omega(x)}{\partial x_i} y_i = 0, \qquad \sum \frac{\partial \omega(x')}{\partial x_i'} y_i = 0.$$

The system with two terms that is considered will thus have the equation:

(2) $$\sum\left[\lambda\frac{\partial\omega(x)}{\partial x_i}+\mu\frac{\partial\omega(x')}{\partial x_i'}\right]y_i\,,$$

or further:

(3) $$\omega(\lambda\, x + \mu\, x' \mid y) = 0.$$

One sees that it contains only one special complex, since, from the form of equation (3), the special complexes of the system will have those lines whose coordinates are described by the formula:

$$\lambda\, x_i + \mu x_i'$$

for its directrices. Now, these expressions will be the coordinates of a line only if:

$$\omega(\lambda\, x + \mu\, x') = 0;$$

i.e., if:

$$\omega(x)\,\lambda^2 + 2\,\omega(x \mid x')\,\lambda\mu + \omega(x')\,\mu^2 = 0,$$

or finally if:

$$\omega(x')\,\mu^2 = 0,$$

since:

$$\omega(x) = 0, \qquad 2\,\omega(x \mid x') = \frac{d\,\omega(x)}{dt} = 0.$$

It might be the case that $\omega(x')$ is zero; however, as we will confirm later on, the ruled series would no longer constitute a skew surface then. Under the hypothesis that we have imposed, there is therefore only one solution, namely:

$$\mu = 0.$$

One says that two ruled surfaces agree along a common generator if their tangent planes are the same at each point of that generator, which demands that the Chasles correlation must be the same for the two surfaces. There is an infinitude of hyperboloids and paraboloids that satisfy this condition, which are the *quadrics of agreement*, which is a terminology whose use is widespread in descriptive geometry. Any quadric that is contained in the linear congruence of tangents will obviously be a quadric of agreement.

48. I denote that linear congruence by C_x. It is clear that the neighboring lines will give rise to another congruence $C_x + dx$, and one may prove that *these two congruences will have the same semi-quadric in common*.

Indeed, equations (1) represent C_x, and if one changes x into $x + x'\,dt$ and x' into $x' + x''\,dt$, where $x_i'' = dx_i'/dt$, then one will get the representation of $C_x + dx$; one then finds that:

$$(4) \qquad \begin{cases} \sum \left[\dfrac{\partial \omega(x)}{\partial x_i} + \dfrac{\partial \omega(x')}{\partial x'_i} dt \right] y_i = 0, \\ \sum \left[\dfrac{\partial \omega(x')}{\partial x'_i} + \dfrac{\partial \omega(x'')}{\partial x''_i} dt \right] y_i = 0. \end{cases}$$

These equations, when combined with equations (1), will give three equations, in all, namely:

$$(5) \qquad \sum \frac{\partial \omega(x)}{\partial x_i} y_i = 0, \qquad \sum \frac{\partial \omega(x')}{\partial x'_i} y_i = 0, \qquad \sum \frac{\partial \omega(x'')}{\partial x''_i} y_i = 0.$$

These three equations will be those of three complexes that have a semi-quadric $\mathcal{Q}$ in common.

From equations (5), an arbitrary line y of that semi-quadric must cut three consecutive lines of the ruled surface. These lines y will then be the asymptotic tangents of the second system that run through all of the points of the line x. The semi-quadric $\mathcal{Q}$ will then be composed of a system of generators for the *osculating hyperboloid* of the surface.

49. The *complementary semi-quadric* $\mathcal{Q}_0$ is found to be related to the general theory of contact for a ruled surface with a linear complex.

If we have a linear complex:

$$\sum \xi_i x_i = 0$$

then we will say that it has *contact of p^{th} order* with a given ruled surface if it contains (p + 1) consecutive generators of the surface.

The tangent complexes thus verify the two equations:

$$\sum \xi_i x_i = 0, \qquad \sum \xi_i x'_i = 0,$$

which indicate that these tangent linear complexes form a system with four terms that is complementary to the system of two terms that is represented by equation (2).

Now, consider the complexes that have second-order contact with the surface; they will be subject to the conditions:

$$(6) \qquad \sum \xi_i x_i = 0, \qquad \sum \xi_i x'_i = 0, \qquad \sum \xi_i x''_i = 0.$$

These three equations will reduce to two only if one verifies the six relations:

$$\alpha x''_i + \beta x'_i + \gamma x_i = 0 \qquad (i = 1, 2, \dots, 6);$$

now, in this case, since the x_i are solutions of the same second-order equation, one may set:

$$x_i = C_i T + C_i' T_0,$$

in which, the C_i, C_i' will denote constants and T, T_0 will be functions of t; the ruled series will then reduce to a plane pencil.

Since equations (6) are assumed to be distinct, the complexes that they define will define a system of *three* terms.

The complementary system is known to us; it is the system:

$$(7) \qquad \sum \left[\lambda \frac{\partial \omega(x)}{\partial x_i} + \mu \frac{\partial \omega(x')}{\partial x_i'} + \nu \frac{\partial \omega(x'')}{\partial x_i''} \right] y_i = 0,$$

which is composed of all the linear complexes that contain the semi-quadric $\mathcal{Q}$.

Indeed, the involution of the complex (7) with the complex ξ will be written:

$$(8) \qquad 2\Omega(\xi \mid u) = \sum \frac{\partial \Omega(u)}{\partial u_i} \xi_i = 0,$$

if one sets:

$$u_i = \lambda \frac{\partial \omega(x)}{\partial x_i} + \mu \frac{\partial \omega(x')}{\partial x_i'} + \nu \frac{\partial \omega(x'')}{\partial x_i''},$$

and, since one will deduce from that defining equation that:

$$\lambda x_i + \mu x_i' + \nu x_i'' = \frac{\partial \Omega(u)}{\partial u_i},$$

equation (8) will be written:

$$\lambda \sum \xi_i x_i + \mu \sum \xi_i x_i' + \nu \sum \xi_i x_i'' = 0.$$

All of the complexes (7) will then be in involution with the complexes x that verify equations (6).

The complexes ξ will thus indeed have the semi-quadric $\mathcal{Q}_0$ in common, which is complementary to the quadric $\mathcal{Q}$; i.e., the generators of the osculating hyperboloid of the same system as x.

50. Now consider the complexes that have a third-order contact with the surface; they will be defined by the four equations:

$$(9) \qquad \sum \xi_i x_i = 0, \quad \sum \xi_i x_i' = 0, \quad \sum \xi_i x_i'' = 0, \quad \sum \xi_i x_i''' = 0,$$

which will be distinct, at least when one has six equations of the form:

$$\alpha x_i''' + \beta x_i'' + \gamma x_i' + \delta x_i = 0.$$

Now, if this is true then the x_i will have the form:

(10) $$x_i = C_i\, T + C_i' T_0 + C_i'' T_{00}\,,$$

where the C will be constants, and T, T_0, T_{00} will be functions of t.

Form:

$$\omega(x) = \omega(C_i\, T + C_i' T_0 + C_i'' T_{00}) = 0,$$

or, upon developing this:

$$\omega(C)\, T^2 + \omega(C')T_0^2 + \omega(C'')T_{00}^2 + 2\omega(C \mid C')TT_{00} + 2\omega(C' \mid C'')T_0T_{00} = 0.$$

If the coefficients of that form in T, T_0, T_{00} are identically zero then the line x will be contained in a fixed hyper-pencil; the ruled series will be defined by the generators of a cone or the tangents to a curve. Excluding that case, it might be the case that the quadratic equation above is not true identically. However, one would then verify that equation by taking T, T_0, T_{00} to be second-degree polynomials in one parameter s, which one can substitute for the parameter t. Formulas (10) would take the form:

$$x_i = D_i\, s^2 + D_i' s + D_i''.$$

The ruled series would then be a semi-quadric.

Excluding this new class, equations (9) will then be distinct, and the complexes ξ that verify them will form a system with four terms. The complementary system with two terms will consist of two special complexes whose directrices Δ, Δ', by virtue of equations (9), will possess the property of intersecting four consecutive generators of the surface. These lines Δ and Δ' will then have third-order contact with the surface, and each of them will be at a point of the line x.

If one considers the osculating hyperboloid that relates to a line x and the osculating hyperboloid that relates to the neighboring line $x + dx$ then these two hyperboloids will intersect along two neighboring generators of x and along two other generators of the opposite system. These two generators will be the lines Δ and Δ'.

Finally, consider a linear complex that has fourth-order contact with the ruled surface. One must have:

$$\sum \xi_i x_i = 0, \quad \sum \xi_i x_i' = 0, \quad \sum \xi_i x_i'' = 0, \quad \sum \xi_i x_i''' = 0, \quad \sum \xi_i x_i^{\text{iv}} = 0,$$

and these five equations, if they are distinct, will define the ratios of ξ; i.e., the complex will be perfectly determined.

There will be no indeterminacy only if there exist six equations of the form:

$$\alpha x_i^{\text{iv}} + \beta x_i''' + \gamma x_i'' + \delta x_i' + \varepsilon x_i = 0,$$

which are equations that prove that there will exist at least two linear relations with constant coefficients between the x_i.

The surface or ruled series will thus belong to a linear congruence in this case. The ruled surfaces that are contained in a linear congruence play a very important role, and we shall return to them later on. For them, the *osculating complex* will be unavoidably indeterminate.

If one considers the osculating complexes that relate to two neighboring lines x and $x + dx$ on a ruled surface then the directrices of the common congruence will be the lines Δ and Δ' that were defined before.

Three consecutive osculating complexes will have the semi-quadric $\mathcal{Q}_0$ that was defined before in common.

Four consecutive osculating complexes will have two lines in common that are *infinitely close* to the line x.

The reader will easily prove these properties. The last one shows that if one takes a linear complex that depends upon one parameter arbitrarily then this complex will not always be the osculating complex of a ruled surface, because four consecutive complexes of the system will intersect along two lines that are generally distinct.

51. Up till now, we have excluded the hypothesis that $\omega(x') = 0$. Now, let:

$$\omega(x') = 0.$$

The complexes (2) will all be special, and the x_i' will be the coordinates of a line x' that is the directrix of one of these complexes. The lines x, x' will intersect at a point O and will have a plane π in common; the lines of the plane pencil (O, p) will be precisely the directrices of the complexes (2). The congruence C_x of the lines that cut two consecutive lines x and $x + x'\,dt$ will then decompose here into the set of lines in the plane π and the set of lines that issue from the point O.

One can say that the consecutive lines x and $x + x'\,dt$ will intersect at the point O and will have the plane π in common. One can likewise appreciate the infinitesimal order up to which this agreement is valid.

Indeed, the condition of agreement of two lines x and z can be written:

$$\omega(z - x) = \omega(z) + \omega(x) - 2\,\omega(z \mid x) = -\,2\,\omega(z \mid x) = 0,$$

if one considers that $\omega(x) = 0$, $\omega(z) = 0$. If one takes:

$$z_i = x_i + x_i'\,\Delta t + x_i''\frac{\Delta t^2}{2} + x_i'''\frac{\Delta t^3}{6} + \dots$$

then one will find painlessly:

$$\omega(z-x) = \omega(x')\,\Delta t^2 + \frac{1}{2}\frac{d\omega(x')}{dt}\Delta t^3 + \left[\frac{1}{6}\frac{d^2\omega(x')}{dt^2} - \frac{1}{12}\omega(x'')\right]\Delta t^4 + \ldots$$

Therefore, if $\omega(x')$ is zero for each line of the ruled series $\omega(z-x)$, which is generally of second order, then this will reduces to fourth order:

(11) $$\omega(z-x) = -\frac{1}{12}\omega(x'')\Delta t^4 + \ldots$$

One recovers it in another form, a property that was exhibited for the first time by Bouquet.

Indeed, we verify that if one makes use of metric elements then $\omega(z-x)$ will be proportional to the product of the shortest distance p between the lines x and z with the sine of their angle ε, or with that angle itself, namely:

$$p\varepsilon .$$

If one takes the element ε to be the infinitely small principal of the arc of the spherical indicatrix of the generators of the ruled series then $p\varepsilon$ will be an infinitesimal of order one higher than p. Thus, if $\omega(x')$ is zero and $p\varepsilon$ is of fourth order then p will be of third order, and this is Bouquet's theorem, precisely.

In general, the ruled series will be formed from the tangents of a skew curve. The point O of intersection of the consecutive lines will be the point of contact of the curve with x, and the plane π will be the osculating plane. From this, the lines of the plane pencil (O, π), which I will call the *osculating plane pencil*, will have a representation of the form:

(12) $$x_i + \lambda x_i' .$$

This representation will be very useful to us.

52. Meanwhile, it might be the case that the ruled series is defined by the generators of a cone or the tangents to a plane curve; however, the formulas would then take on a very special character. Indeed, one remarks that two arbitrary lines of the ruled series would intersect in this case, since they would all belong to the same hyper-pencil (spray or planar system). The expression $\omega(z-x)$ would then have to be rigorously zero, and, as a result, one would need to have:

$$\omega(x') = 0,$$

because the term Δt^4 must disappear. It is pointless to add that the terms in Δt^5, … would disappear. Indeed, I would like to prove that if one has:

$$\omega(x'') = 0$$

then the ruled series will be contained in a hyper-pencil.

Indeed, one deduces from the equations:

$$\begin{array}{lll} \omega(x \mid x) = 0, & \omega(x' \mid x) = 0, & \omega(x'' \mid x) = 0, \\ \omega(x \mid x') = 0, & \omega(x' \mid x') = 0, & \omega(x'' \mid x') = 0, \\ \omega(x \mid x'') = 0, & \omega(x' \mid x'') = 0, & \omega(x'' \mid x'') = 0, \\ \omega(x \mid x''') = 0, & \omega(x' \mid x''') = 0, & \omega(x'' \mid x''') = 0, \end{array}$$

which may be summarized by saying that $(x_1, x_2, \ldots, x_6)$, $(x_1', x_2', \ldots, x_6')$, $(x_1'', x_2'', \ldots, x_6'')$, $(x_1''', x_2''', \ldots, x_6''')$ are four systems of solutions of linear equations in $u_1, u_2, \ldots, u_6$, that:

$$(13) \qquad \begin{cases} 2\omega(x \mid u) = \sum \dfrac{\partial\omega(x)}{\partial x_i} u_i = 0, \\ 2\omega(x' \mid u) = \sum \dfrac{\partial\omega(x')}{\partial x_i'} u_i = 0, \\ 2\omega(x'' \mid u) = \sum \dfrac{\partial\omega(x'')}{\partial x_i''} u_i = 0. \end{cases}$$

These three equations in u_i will be distinct, because if there exists an identity of the form:

$$\lambda\, \omega(x \mid u) + \mu\, \omega(x' \mid u) + \nu\, \omega(x'' \mid u) = 0$$

then one will have:

$$\lambda \frac{\partial\omega(x)}{\partial x_i} + \mu \frac{\partial\omega(x')}{\partial x_i'} + \nu \frac{\partial\omega(x'')}{\partial x_i''} = 0,$$

or:

$$\frac{\partial\omega(\lambda x + \mu x' + \nu x'')}{\partial(\lambda x_i + \mu x_i' + \nu x_i'')} = 0 \qquad (i = 1, 2, \ldots, 6),$$

and, since ω has a non-zero discriminant, this will demand that one must have:

$$\lambda x_i + \mu x_i' + \nu x_i'' = 0 \qquad (i = 1, 2, \ldots, 6).$$

We have already seen that the ruled series will be a plane pencil in this case.

Moreover, since the three equations (13) are distinct and refer to six variables, any system of solutions of these equations will be deduced linearly from three other particular systems, which are nonetheless independent. There must therefore exist some relations of the form:

$$\alpha x_i''' + \beta x_i'' + \gamma x_i' + \delta x_i = 0,$$

which proves that the x_i will have the general expression:

$$x_i = a_i\, R + b_i\, S + c_i\, T,$$

where *R*, *S*, *T* will be three functions of *t*, and a_i , b_i , c_i will be constants. One indeed also recognizes that the ruled series will be contained in the hyper-pencil. It will therefore be a cone or the set of tangents to a planar curve.

53. As an application of the preceding remarks, we prove the theorem:

If the tangents to a curve belong to a linear complex then the osculating plane π at a point O of the curve will be the polar plane to that point in the complex.

It suffices to prove that the lines of the osculating plane pencil:

$$z_i = \lambda x_i + \mu x_i'$$

will all belong to the complex.

Now, upon taking:

$$A = \sum a_i x_i ,$$

where $\sum a_i x_i = 0$ is the equation of the linear complex, one will have, in fact, that:

$$\sum a_i x_i = \lambda A + \mu \frac{dA}{dt} = 0,$$

since $A = 0$ for all of the tangents to the curve.

54. In his research on the theory of contact, Lie introduced the notion that he called a *contact element* – i.e., the set that consists of a point and a plane through that point (viz., the united point and plane of no. 3).

We have encountered contact elements in the preceding chapters, whether in the form of a plane pencil of lines, or as a pair of corresponding elements that are defined by a correspondence between the points and the planes of a line.

We shall give some simple properties of plane pencils in space, while first supposing that the point and the plane depend upon the same parameter.

For example, if *x* is a variable line endowed with an envelope then we will know that:

$$x_i + \lambda x_i'$$

represents a variable plane pencil – viz., the osculating pencil.

More generally, let (O, π) be an arbitrary pencil and let *a*, *b* be two lines of that pencil that depend upon a parameter *t*. Any line *z* of the pencil will be represented by:

$$z_i = a_i \lambda + b_i \mu .$$

I consider an arbitrary variable line C that passes through the point O and also depends on the parameter t, so the spray of lines that issues from O will be represented by:

$$z_i = a_i\,\lambda + b_i\,\mu + c_i\,\nu\,.$$

To each value of $\lambda : \mu : \nu$, there will correspond a line z of the spray, and if $\lambda : \mu : \nu$ are functions of t then the line z will be displaced along with the point O. We seek the values of λ, μ, ν for which z will be precisely the tangent to the curve that is the locus of the point O.

It will suffice to write that $\lambda : \mu : \nu$ are functions of t such that the line z will have an envelope that touches at the point O or to express the idea that the osculating pencil has two of its lines in the spray. The line z will already be a line of that pencil. It will then suffice to write down that the line z' whose coordinates are $z'_1, \ldots, z'_6$ will belong to the spray, or that:

$$z'_i = a_i\,\varepsilon + b_i\,\varepsilon_1 + c_i\,\varepsilon_2\,;$$

i.e.:

$$a'_i\lambda + b'_i\mu + c'_i\nu + a_i\lambda' + b_i\mu' + c_i\nu' = a_i\,\varepsilon + b_i\,\varepsilon_1 + c_i\,\varepsilon_2,$$

which are equations of the form:

$$a'_i\lambda + b'_i\mu + c'_i\nu = a_i\,\rho + b_i\,\sigma + c_i\,\tau. \tag{14}$$

I multiply by $\partial\omega(a) / \partial a_i$ and I sum from $i = 1$ to $i = 6$; this will give:

$$\omega(a \mid a')\,\lambda + \omega(a \mid b')\,\mu + \omega(a \mid c')\,\nu = \omega(a)\,\rho + \omega(a \mid b)\,\sigma + \omega(a \mid c)\,\tau = 0,$$

because:

$$\omega(a) = 0, \qquad \omega(a \mid b) = 0, \qquad \omega(a \mid c) = 0.$$

One also has:

$$\omega(a \mid a') = 0,$$

so it thus remains that:

$$\omega(a \mid b')\,\mu + \omega(a \mid c')\,\nu = 0.$$

However, since $\omega(a \mid c) = 0$, one will have:

$$\frac{d\,\omega(a \mid c)}{dt} = \omega(a' \mid c) + \omega(a \mid c') = 0,$$

and, as a result, the equation that will be obtained can be written:

$$\omega(a \mid b')\,\mu - \omega(c \mid a')\,\nu = 0.$$

One will likewise find:

$$\omega(a \mid b')\,\lambda - \omega(b \mid c')\,\nu = 0.$$

$$\omega(c \mid a')\,\lambda - \omega(b \mid c')\,\mu = 0,$$

and, by definition:

(15)
$$\frac{\lambda}{\omega(b \mid c')} = \frac{\mu}{\omega(c \mid a')} = \frac{\nu}{\omega(a \mid b')}.$$

Likewise, if we take a line e in the plane π then any line of that plane will have a representation of the form:

$$z_i = \lambda\, a_i + \mu\, b_i + \nu\, e_i\, .$$

In particular, if one desires to know the $\lambda_1 : \mu_1 : \nu_1$ that give the line of contact of the plane π with its envelope (viz., its characteristic) then one will find that:

(15′)
$$\frac{\lambda_1}{\omega(b \mid e')} = \frac{\mu_1}{\omega(e \mid a')} = \frac{\nu_1}{\omega(a \mid b')}.$$

The lines c and e will only play an auxiliary role, here.

55. In general, the tangent D to the locus of the point O will not be in the plane π, and the characteristic Δ of the plane π will not pass through the point O.

In order for D to be in the plane π – i.e., to belong to the pencil (O, π) – it is necessary and sufficient that one have:

$$\nu = 0;$$

i.e.:

$$\omega(a \mid b') = 0.$$

However, this is also the condition for one to have $\nu_1 = 0$; i.e., for the line Δ to also belong to the pencil (O, π).

If a plane pencil that depends upon one parameter contains, at each instant, the tangent to the locus of its center O then it will also contain the characteristic of its plane π, and conversely.

This theorem is geometrically obvious: The plane p rolls along a curve C, which it touches at the point O, and the generator of the developable will be generated by the plane that passes through the point O. I will call the systems of plane pencils that are thus defined *bands* (bandeau). They are geometrically equivalent to the system that is defined by a curve and a developable through it.

Bands are characterized by the equation:

$$\omega(a \mid b') = 0.$$

Moreover, it is easy to obtain the lines D and Δ in this case by taking recourse to only the representation of the pencil, and without appealing to the auxiliary lines c and e.

Suppose that $\lambda : \mu$ have been chosen in such a way that:

$$\lambda\, a_i + \mu\, b_i$$

are the coordinates of the tangent to the curve that is the locus of the point O, and which, by hypothesis, belong to the pencil.

Since ν is zero, equations (14) will give:

$$a_i'\lambda + b_i'\mu = \rho\, a_i + \sigma\, b_i + \tau\, c_i\,,$$

so:

$$\omega(a'\,\lambda + b'\,\mu) = \omega(\rho\, a + \sigma\, b + \tau\, c) = 0\,;$$

one will thus have:

(16) $$\omega(a)\,\lambda^2 + 2\,\omega(a \mid b)\,\lambda\mu + \omega(b)\,\mu^2 = 0,$$

which is an equation that will give two values for $\lambda : \mu$: One of them will furnish the tangent to the curve that is the locus of the point O, and the other one, as one easily sees, will give the characteristic of the plane.

56. These lines will coincide if:

(17) $$[\omega(a \mid b)]^2 - \omega(a)\,\omega(b) = 0.$$

We may then regard the a_i' as the coordinates of a line a', which, as one sees, belongs to the pencil (O, π). Since $\omega(a') = 0$, equation (17) will give, in fact:

$$\omega(b' \mid a') = 0,$$

in such a way that the line a' will belong to the linear complex:

(18) $$\omega(b' \mid x) = 0.$$

This complex will be special only if one has $\omega(b') = 0$. If neither the point O nor the plane π are fixed – in which case, the lines of the pencil considered will intersect – then one may always suppose that the line b of the pencil is not the envelope and that $\omega(b')$ is not zero.

The equations:

$$\omega(b' \mid a) = 0, \quad \omega(b' \mid b) = 0$$

say that the lines a, b belong to the complex (18), and that π is the polar plane of the point O. The line a' of the complex that issues from O must then be in the plane π.

The pencil (O, π) *is then the osculating pencil of the curve that is the locus of the point O.*

If the a' are all zero or proportional to a then the reasoning will break down; however, the line a will then be fixedm and the point O and the plane π will be a point and a plane on that fixed line that, since they depend upon the same parameter, will constitute two homologous elements of a certain correspondence between the points and the planes of a

that will be given *a priori*. One must recall that such a set will possess the same properties as the osculating pencils of a skew curve.

Finally, it might be the case that the point O, or perhaps the plane π, is fixed. In these two cases, one will have:

$$\omega(a') = 0, \qquad \omega(b') = 0, \qquad \omega(a' \mid b') = 0.$$

57. I would now like to occupy myself with the plane pencils that depend upon several parameters.

I will first recall a general proposition that concerns these plane pencils.

Take a system of rectangular axes, and let x, y, z be the coordinates of a point O; the equation of a plane p through O will be:

$$Z - z = p(X - x) + q(Y - y),$$

in such a way that the system (O, π) will be defined by the five quantities x, y, z, p, q.

Suppose that these quantities depend upon *several* parameters and that, moreover, when these parameters vary, the displacement of the point O will be meaningful, up to second order, in the plane π; in other words, no matter what the law of variation of the parameters is, the tangents to the locus of points O will be in the plane π.

From that hypothesis, we must have:

$$dz - p\,dx - q\,dy = 0.$$

That equation proves that there exists at least one relation between x, y, z.

We thus successively imagine the hypotheses that there exist one, two, or three relations between x, y, z, resp.

If there is only one relation:

$$z = \varphi(x, y)$$

then one will infer that:

$$dz = \frac{\partial \varphi}{\partial x} dx + \frac{\partial \varphi}{\partial y} dy,$$

so:

$$\left(p - \frac{\partial \varphi}{\partial x}\right) dx + \left(q - \frac{\partial \varphi}{\partial y}\right) dy = 0.$$

If the coefficients of dx, dy are not zero then there will exist one relation between x, y, which will bring about a second relation between x, y ; one will then have:

$$p = \frac{\partial \varphi}{\partial x}, \qquad q = \frac{\partial \varphi}{\partial y},$$

here, which proves that the system of pencils (O, π) will be composed of the points of the surface and the tangent planes to each of these points.

Assume that there exist two relations:

$$z = \varphi(x), \qquad y = \psi(x),$$

so we will have:

$$dz = \varphi'(x)\, dx, \qquad dy = \psi'(x)\, dx,$$

which will imply that:

$$[\varphi'(x) - p - q\, \psi'(x)]\, dx = 0.$$

If dx were zero then there would be three relations between x, y, z, which is not the case. One will then have:

$$\varphi'(x) - p - q\, \psi'(x) = 0,$$

and there will be no other relation between x, y, z, p, q, because z, y, p, q would then be functions of the single variable x, and pencils do not depend on *several* parameters.

Here, we thus have the set of pencils that are obtained by associating each point of a curve with an *arbitrary* tangent plane to the curve at that point.

Finally, if three relations exist between x, y, z then one will have pencils whose point is fixed. The single plane must be variable and contain at least two parameters, since the plane must be an arbitrary one that passes through the fixed point.

As a particular case of such a surface, one has the developable that gives the pencils in which the plane depends only upon one parameter, and the plane that furnishes the pencils whose plane is fixed and whose point is arbitrary in that plane. These two cases will be duals of the two that one considered in the first place.

An intermediary case is that of the pencils that are obtained by associating each point of a line with a point on that line. In fact, in this case, the point will depend upon only one of the parameters, while the plane will depend upon the other, and these two parameters will be independent.

58. These facts find a very simple and elegant representation in line coordinates.

Indeed, take the plane pencil:

$$z_i = a_i + \lambda\, b_i\,,$$

where the lines a, b depend upon several parameters. Which of these pencils will be the ones that have an envelope? That is, how does on construct the tangent to the curve that is described by O, and, as a consequence, the characteristic of the plane of the pencil, in order it to belong to that pencil, no matter what the displacement of the pencil is?

Conforming to the results that were achieved before, it will be necessary and sufficient that one have:

(19) $$\omega(a \mid db) = 0$$

for all possible displacements, or, what amounts to the same thing:

(19′) $$\omega(b \mid da) = 0,$$

because:

$$0 = d\omega(a \mid b) = \omega(a \mid db) + \omega(b \mid da).$$

One thus sees that a pencil that depends upon several parameters, and which verifies the condition:

$$\omega(b \mid da) = 0,$$

must be composed of either:

1. A point of a surface and the tangent plane to that point.
2. A point of a curve and an arbitrary plane that is tangent to the curve at that point.
3. The tangent plane to a developable and an arbitrary point of contact of that plane with the developable.
4. A point and a plane of a line that are arbitrarily associated.
5. A plane through a point that is associated with that point.

In any case, we say that the pencil will have an envelope if:

$$\omega(b \mid da) = 0.$$

As one can see, the pencils with envelopes will depend upon only two parameters, in such a way that *if one finds oneself in the presence of a variable pencil that depends upon several parameters then the condition:*

$$\omega(b \mid da) = 0$$

will imply that the parameters that the pencil depends upon must be reducible to two.

We will soon make an application of that remark.

I now pass on to the study of the infinitesimal properties of the complexes of lines.

59. Suppose that one has a complex of lines:

$$f(x_1, x_2, \ldots, x_6) = f(x) = 0,$$

where x is a line of that complex, and $\omega(x)$ is the fundamental form, and consider the system of linear complexes in two terms:

$$\sum\left(\lambda\frac{\partial f}{\partial x_i}+\mu\frac{\partial \omega}{\partial x_i}\right)y_i = 0. \tag{20}$$

I shall give the name of *tangent linear complexes* to these complexes.

The following remark justifies that name: Let $x + dx + \frac{1}{2}d^2x + \ldots$ be a line of the neighboring complex to x, and replace y_i with $x_i + dx_i + \frac{1}{2}\,d^2x_i + \ldots$ in the left-hand side of (20), so one must have:

$$\sum\left(\lambda\frac{\partial f}{\partial x_i}+\mu\frac{\partial \omega}{\partial x_i}\right)y_i = \frac{1}{2}\sum\left(\lambda\frac{\partial f}{\partial x_i}+\mu\frac{\partial \omega}{\partial x_i}\right)d^2x_i + \ldots$$

upon taking into account that $df = 0$, $d\omega = 0$; i.e., one obtains a result of second order.

If the tangent linear complexes form a system of two terms then it will be obvious that such a complex cannot be represented by a single equation, but, in fact, by the two equations:

$$f(x) = 0, \qquad \omega(x) = 0.$$

We seek the special tangent complexes. We must write:

$$\Omega\left(\lambda\frac{\partial f}{\partial x} + \mu\frac{\partial \omega}{\partial x}\right) = 0;$$

i.e.:

$$\lambda^2\,\Omega\left(\frac{\partial f}{\partial x}\right) + 2\lambda\mu\,\Omega\left(\frac{\partial f}{\partial x}\middle|\frac{\partial \omega}{\partial x}\right) + \mu^2\,\Omega\left(\frac{\partial \omega}{\partial x}\right) = 0.$$

However, one has:

$$2\,\Omega\left(\frac{\partial f}{\partial x}\middle|\frac{\partial \omega}{\partial x}\right) = \sum \frac{\partial\Omega\left(\dfrac{\partial \omega}{\partial x}\right)}{\partial\dfrac{\partial \omega}{\partial x_i}}\frac{\partial f}{\partial x_i} = \sum x_i\frac{\partial f}{\partial x_i} = 0,$$

because $x_i = \partial\Omega\left(\dfrac{\partial \omega}{\partial x}\right) \Big/ \partial\dfrac{\partial \omega}{\partial x_i}$; one also has $\Omega\left(\dfrac{\partial \omega}{\partial x}\right) = \omega(x) = 0$; it thus remains that:

$$\lambda^2\,\Omega\left(\frac{\partial \omega}{\partial x}\right) = 0.$$

The equation that furnishes the special tangent complexes thus has equal roots, and, in turn, the tangent complexes will generally include only one special complex, which has x for its directrix. If one refers to no. 29 then one will see that all of the tangent linear complexes define the same normal correlation on x; for that reason, we give that correlation the name of *normal correlation of the complex* $f(x) = 0$ *on its line* x. One sees how that notion generalizes the notion of the normal correlation of a linear complex (no. 15).

We have seen (no. 53) that if the tangents x to a curve belong to a linear complex then the osculating plane pencil will belong to that complex, and, in turn, the point O of contact and the osculating plane π will be corresponding elements for the normal correlation of the complex on the line x. This important property will extend to the case of an arbitrary complex.

I would like to prove that *if the tangents* x *to a curve belong to a complex:*

$$f(x) = 0$$

then the osculating pencil of the curve will belong to the normal correlation of the complex $f = 0$ *on the line* x.

It obviously suffices to prove that this osculating pencil will belong to the normal correlation of the tangent complex:

$$\sum \frac{\partial f}{\partial x_i} y_i = 0,$$

since, by definition, that correlation will be the normal correlation of the complex $f(x) = 0$.

Indeed, the osculating pencil will be represented by:

$$\rho\, x_i + \sigma x_i',$$

upon supposing that the x_i are expressed as functions of the one parameter t and setting $x_i' = dx_i / dt$.

One must prove that:

$$\sum \frac{\partial f}{\partial x_i} (\rho x_i + \sigma x_i') = 0.$$

Now, this is obvious, since:

$$\sum \frac{\partial f}{\partial x_i} x_i = m\, f(x) = 0, \qquad \sum \frac{\partial f}{\partial x_i} x_i' = \frac{df(x)}{dt} = 0.$$

The theorem is thus proved.

60. Several special cases will exhibit the significance of this main theorem.

Consider an arbitrary plane π through x, so the lines of the complex $f = 0$ that are contained in this plane will envelop a curve, and the line x itself will touch that curve at a point O. It results from the preceding theorem that O and π will correspond under the normal correlation.

Therefore:

If one passes a plane π through a line x of a complex then the curve that is enveloped by the complex relative to the plane π will be touched by the line x at one point O; the point O and the plane π will correspond under the normal correlation of the complex.

Similarly:

If one takes a point O on a line x of a complex then the cone of the complex that has O for its summit will be tangent along x to a plane π that is homologous to O under the normal correlation.

61. One may generalize these results.

Consider a ruled surface that is generated by the lines of the complex.

In order to make this more precise, suppose that one has expressed the coordinates x_i of a line on that surface as functions of one parameter t. Let x and $x + x' \, dt$ be two neighboring lines on that surface.

The linear complex that is contained in the equation with two terms:

$$\sum \left[\rho \frac{\partial \omega(x)}{\partial x_i} + \sigma \frac{\partial \omega(x')}{\partial x_i'} \right] y_i = 0$$

will define a normal correlation on x (no. **47**) that is nothing but the Chasles correlation that relates to the ruled surface.

Now, compare these complexes with the tangent complexes to the proposed complex:

$$\sum \left[\lambda \frac{\partial f}{\partial x_i} + \mu \frac{\partial \omega(x)}{\partial x_i} \right] y_i = 0,$$

and form their simultaneous invariant:

$$\Omega \left[\rho \frac{\partial \omega(x)}{\partial x} + \sigma \frac{\partial \omega(x')}{\partial x'} \middle| \lambda \frac{\partial f}{\partial x} + \mu \frac{\partial \omega(x)}{\partial x} \right]$$

$$= \rho\lambda\,\Omega\left(\frac{\partial \omega}{\partial x} \middle| \frac{\partial f}{\partial x} \right) + \mu\rho\,\Omega\left(\frac{\partial \omega}{\partial x} \middle| \frac{\partial \omega}{\partial x} \right) + \sigma\lambda\,\Omega\left(\frac{\partial \omega}{\partial x'} \middle| \frac{\partial f}{\partial x} \right) + \sigma\mu\,\Omega\left(\frac{\partial \omega}{\partial x'} \middle| \frac{\partial \omega}{\partial x} \right).$$

That expression will be zero identically, because we have already seen that:

$$\Omega\left(\frac{\partial \omega}{\partial x} \middle| \frac{\partial f}{\partial x} \right) - m\, f(x) = 0,$$

$$\Omega\left(\frac{\partial \omega}{\partial x} \middle| \frac{\partial \omega}{\partial x} \right) = \Omega\left(\frac{\partial \omega}{\partial x} \right) = \omega(x) = 0.$$

$$\Omega\left(\frac{\partial \omega}{\partial x'} \middle| \frac{\partial f}{\partial x} \right) = \frac{1}{2} \sum \frac{\partial f}{\partial x_i} x_i' = \frac{1}{2} \frac{df}{dt} = 0,$$

$$\Omega\left(\frac{\partial \omega}{\partial x'} \middle| \frac{\partial \omega}{\partial x} \right) = \frac{1}{2} \sum \frac{\partial \omega}{\partial x_i} x_i' = \frac{1}{2} \frac{d\omega}{dt} = 0.$$

The two systems of two terms considered will then be composed of linear complexes in involution.

It follows from this (no. 33) that the two normal correlations that determine each of the two respective systems on x will be in involution. By employing the terminology that was introduced in no. 59, one can then say that:

If one considers the Chasles correlation of a ruled surface of a complex relative to one of its lines x then that correlation will be in involution with the normal correlation of the complex relative to x.

In other words, if one takes a point O on the line x and draws the plane τ that is tangent to the surface through O then that plane τ and the plane π that is homologous to O under the normal correlation will define a harmonic pencil with two fixed planes. Furthermore:

Let:

O	be a point on x,
τ	be the tangent plane to the ruled surface,
O'	be the point that corresponds to t under the normal correlation, and let
τ'	be the tangent plane to O'.

τ' will then be the plane that corresponds to O under the normal correlation to the complex.

If the surface considered is developable (or, more generally, composed of a ruled series with an envelope) then the Chasles correlation will be singular and its involution with the normal correlation will signify that its singular pair will belong to that correlation. That is precisely the theorem of no. 59.

62. Consider all of the lines x of a complex and all of the plane pencils (O, π) whose point and plane are corresponding elements under the normal correlation of the complex on a line x. I will call these plane pencils the *plane pencils of the complex.* One sees how this definition generalizes the one that we gave in no. 13 for the case of a linear complex.

Let (O, π) be a pencil of the complex and let x be the line of the complex whose normal correlation admits the point O and the plane π for its corresponding elements. The cone of the complex that has the point O for its summit will be tangent along x to the plane π, and the enveloping curve of the lines of the complex relative to the plane π will touch the line x at O. One may then state these theorems:

The planes π of the pencils of the complex whose point O is given will envelop the cone of the complex that has its summit at this point.

The locus of points O of the pencils of a complex whose plane is given will be the enveloping curve of the lines of the complex relative to that plane.

The cone of the complex will then be the enveloping locus of the planes whose enveloping curve passes through the summit of the cone. The enveloping curve relative to a plane will be the locus of summits of the cones of the tangent complex to that plane.

One observes that a plane pencil (O, π) of the complex can belong to the normal correlation of the two lines x, y of the complex only if the cone with its summit at O touches the plane π along x and y; moreover, in that case, the enveloping curve relative to the point O will have a double point at this point, where x and y are the tangents. I shall leave aside this exceptional case.

63. Recall the system with two terms of the tangent linear complex:

$$\sum\left(\lambda\frac{\partial f}{\partial x_i}+\mu\frac{\partial \omega}{\partial x_i}\right)y_i = 0.$$

As we have seen (no. 59), the special complexes to that system will be furnished by the equation:

$$\lambda^2\,\Omega\left(\frac{\partial f}{\partial x}\right) = 0.$$

They will thus all be special if:

$$\Omega\left(\frac{\partial f}{\partial x}\right) = 0.$$

We thus naturally encounter the remarkable expression:

$$\Omega\left(\frac{\partial f}{\partial x}\right)$$

that was introduced by Klein. I would like to show that it is a differential invariant of the complex.

Perform the linear transformation:

$$x_i = \sum_\rho A_{i\rho}x'_\rho \qquad (i, \rho = 1, 2, \dots, 6);$$

the coefficients a of the equation of linear complex are found to be coupled to those a' of its transformed equation by the formulas:

$$a'_\rho = \sum_i A_{i\rho}a_i\,,$$

and, by virtue of these latter formulas, one will have identically:

$$\Omega'(a') = \Delta^2\,\Omega(a),$$

where $\Omega(a)$ is the adjoint form to $\omega(x)$, $\Omega'(a')$ is the adjoint form to $\omega'(x')$, which is the transform of $\omega(x)$, and Δ is the discriminant of the substitution.

Having said this, suppose that the function $f(x)$ becomes $f'(x')$, in such a way that:

$$f'(x') = f(x),$$

so one will have:

$$\frac{\partial f'}{\partial x'_\rho} = \sum_i A_{i\rho} \frac{\partial f}{\partial x_i}.$$

Now, these equations imply the following one:

$$\Omega'\left(\frac{\partial f'}{\partial x'}\right) = \Delta^2\ \Omega\left(\frac{\partial f}{\partial x}\right),$$

which indeed proves the invariance of $\Omega\left(\frac{\partial f}{\partial x}\right)$.

One calls any line of a complex for which the invariant $\Omega\left(\frac{\partial f}{\partial x}\right)$ is zero a *singular line.*

It is interesting to examine the behavior of the normal correlation on a singular line.

Since the tangent complexes are all special and they define a system in two terms, one must conclude that their directrices will define a plane pencil (O, π) and, from the remark in no. 30, the normal correlation will be singular. Any homographic correlation in involution with it must then contain its singular pair (O, π). As a result, any non-developable ruled surface that is contained in the complex and passes through the singular line x must touch the plane π at O.

By contrast, any developable surface (of the complex) that passes through the line x must either admit O on its edge of regression or touch the plane π.

64. We have found an interesting application of the principles of the representation of surfaces by their tangents that were presented in no. 58.

Indeed, I would like to prove the following theorem, which is due to Pasch:

The plane pencils (O, π) *that pertain to all of the singular lines of a complex will have an envelope.*

Since one has:

$$\Omega\left(\frac{\partial f}{\partial x}\right) = 0,$$

if one sets:

$$v_i = \frac{\partial \Omega\left(\frac{\partial f}{\partial x}\right)}{\partial \frac{\partial f}{\partial x_i}}$$

then the y_i will be the coordinates of a line y, and that line will be the directrix of one of the tangent linear complexes.

The plane pencil of the directrices of these complexes – viz., the pencil (O, π) – will then be represented by the formulas:

$$z_i = \lambda\, x_i + \mu\, y_i\,.$$

The conditions:

$$\omega(x) = 0, \qquad \omega(y) = 0, \qquad \omega(x \mid y) = 0$$

will obviously be satisfied, so it will suffice to prove (no. 58) that:

$$\omega(y \mid dx) = 0,$$

or that:

$$\sum \frac{\partial \omega(y)}{\partial y_i}\, dx_i = 0.$$

However, since one has set:

$$y_i = \frac{\partial \Omega\left(\dfrac{\partial f}{\partial x}\right)}{\partial \dfrac{\partial f}{\partial x_i}},$$

one can then infer that:

$$\frac{\partial f}{\partial x_i} = \frac{\partial \omega(y)}{\partial y_i};$$

it will then suffice to prove that:

$$\sum \frac{\partial f}{\partial x_i}\, dx_i = df = 0,$$

which is obvious.

One gives the name of SURFACE OF SINGULARITIES to that remarkable surface ([7]).

The theorem that relates to the singular lines that are traced on the ruled surface of the complex shows that if one observes that a ruled surface of a complex generally contains singular lines then that would prove the following theorem:

Any ruled surface of the complex will generally touch the surface of singularities at a certain number of points.

65. Now, the theorem of no. 58 permits us to prove the following theorem, which was partially found by Cayley and completed by Klein:

If one has:

$$\Omega\left(\frac{\partial f}{\partial x}\right) = 0$$

([7]) One sees that it might happen that the surface of singularities reduces to a simple curve, or that the singular plane pencil generates one of the four other sets that were defined in no. 58, or a system of several of these sets.

identically, for a complex of lines, or by virtue of $f = 0$, $\omega = 0$, *then the lines of the complex will have an envelope* – i.e., they will touch a fixed surface, which will either be developable or not, or it might even intersect a fixed curve.

Indeed, recall the preceding notations:

$$v_i = \frac{\partial\Omega\left(\frac{\partial f}{\partial x}\right)}{\partial \frac{\partial f}{\partial x_i}};$$

we thus make the line y correspond to *any* line x of the complex that intersects it at a point O and has a plane π in common with it.

The pencil (O, p) will have the representation:

$$z_i = \lambda\, x_i + \mu\, y_i\,.$$

One might believe that it depends upon three parameters, like the line x, but in reality it depends upon only wo, although one does not know them *a priori;* however, be that as it may, since one has:

$$2\,\omega(y \mid dx) = \sum \frac{\partial f}{\partial x_i} dx_i = df = 0,$$

one will be assured that the pencil (O, p) will have an envelope (no. 58), *and furthermore, it will not depend upon more than two parameters.* If O describes a surface then π will touch the surface, and the complex will be that of the tangents to that surface. If, on the contrary, O describes a curve then all the lines of the complex will cut that curve and their set will be defined by that condition.

We will see later on how to differentiate these two cases.

66. The infinitesimal properties of congruences were known for quite a long time before those of complexes. They were presented in geometry in the earliest research into the theory of surfaces. In his *Traité de Géometrie,* G. Darboux gave them an important position and added to the interest that geometers already had in them from the research of Laplace on the linear second-order partial differential equations. We will have occasion to insist upon the role of these equations in the study of congruences. I would nonetheless like to recall the principal properties of congruences of lines.

The lines of a congruence are generally tangent to two surfaces; meanwhile, in certain cases, these surfaces might reduce to curves or coincide.

Let a congruence be common to two complexes A and B; let x be a line of that congruence. The normal correlations H_A, H_B of the complexes A, B on the line x will have two pairs (F, Φ'), (F', Φ) in common. The pairs (F, Φ), (F', Φ') that are inverse to these pairs (no. 21) will play a particularly important role; we shall call them *focal pairs:* F, F' will be the *foci* and Φ, Φ' will be the *focal planes* of the line x.

Focal pairs can be real or imaginary, or even merge together.

First, suppose that they are distinct. The lines x will depend upon two parameters (no. 9) – viz., the points F, F' – so the planes Φ, Φ' will depend upon two parameters, in general. The points F and F' will then generally describe two surfaces S and S' (respectively) that one calls *focal surfaces.* Meanwhile, it might be the case that the point F, for example, describes a curve, in which case we will say that the focal surface S reduces to a curve.

Consider a ruled surface that is contained in the congruence and passes through the line x; it will determine a Chasles correlation on x that must be (no. 61) in involution with each of the normal correlations H_A, H_B , and which, as a consequence (no. 25), must admit the pairs (F, Φ), (F', Φ') that are inverse to the pairs (F, Φ'), (F', Φ) that are common to H_A and H_B . In a word, *any Chasles correlation that is defined on x by a ruled surface of the congruence must admit focal pairs.* Or furthermore: *Any ruled surface of the congruence that passes through x will touch the plane* Φ *at F and the plane* Φ', *at F′.*

Consider the singular Chasles correlations that belong to the congruence. These correlations will be defined by the condition of being in involution with H_A and H_B . Thus, from no. 25, the singular pair will be (F, Φ') for the one and (F', Φ) for the other.

One may conclude that around each line x of the congruence there will be two neighboring lines $x + dx$, $x + d'x$ of the congruence that each form *an element* of the ruled series with envelope, along with x; to abbreviate, we say *an element of the developable.*

There are thus two ways of continuously displacing a line of a congruence when one starts with an arbitrary given position, in such a way that the line generates a developable. The congruence may then be decomposed into developables of a family in two ways. Two of these developables will pass through each line x of the congruence, and the osculating pencils of these developables will be (F, Φ') and (F', Φ), respectively. The two developables in question will then be real whenever these two pairs are.

First, imagine the case where S and S' are true surfaces. Consider a developable that is formed from lines of the congruence. The edge of that developable will be a locus of points F that is traced on S. We will thus have a family of curves C on S whose tangents will generate the congruence. Similarly, we will have a family of curves C' on S' whose tangents will likewise generate the congruence.

Any line x of the congruence will be tangent to a curve C at F and to a curve C' at F'; *it will then be tangent to the focal surfaces S and S′ at its foci.*

Any ruled surface that is composed of lines of the congruence is therefore found to circumscribe both of the surfaces S and S' at once. Now, its tangent plane at F will be the plane Φ, and its tangent plane at F' will be the plane Φ'.

The focal pairs (F, Φ), (F', Φ') *will then be tangents to the focal surfaces.*

In particular, consider the developable whose edge is C'; it will be circumscribed, with S, along a curve D. When C' describes the surface S', the curve D will generate a family of curves on S. I say that the *curves C and D will form a conjugate net on S.*

Indeed, the curve D will be the curve of contact of S with a developable whose rectilinear generator that passes through F will touch the curve C at that point. The tangent to D at the point F and the line x tangent to C at F will then be two conjugate tangents, in the sense of Dupin.

Similarly, the developables that have the curves C for their edges will be circumscribed with S' along the curves D' that will form a conjugate system with the curves C'.

One sees that a congruence establishes a point-wise correspondence between its focal surfaces. The point F on S will correspond to F' on S', and inversely. When two surfaces correspond point-wise, there will generally exist two families of conjugate curves on each of them whose image on the other one will be another family of conjugate curves. Here, these two families will be the curves C, D on S and the curves C', D' on S', because if F describes a curve C then F' will describe a curve C', and if F describes a curve D then F' will describes a curve C'.

There is nonetheless good reason to observe – and we shall return to this point – that if the asymptotes correspond on the two sheets S and S' then to any conjugate system that is traced on S there will correspond another conjugate system on S'.

67. The case where one of the surfaces S, S', or even both of them, become curves offers no difficulty. Suppose that F describes a curve V and F' describe a surface S', which is a surface that is, moreover, the locus of the curve C' that is the edge of the developables of a family that is defined by the lines of the congruence. These lines will then be subject to the double condition that they must cut V and touch S', except that here the developables of a family will reduce to cones whose summit F is taken to be on the curve V and which will be circumscribed by S'.

The curves D' will be the curves of contact of these cones.

As for the curves C', they will be the edges of the developables that pass through the curve V.

If the surface S' itself reduces to a curve V' then the congruence will be the set of lines that cut V and V'; the developables of the congruence will then be the cones that pass through V' whose summits are on V and the cones that pass through V' whose summits are on V'.

An interesting example is furnished by the lines that intersect both of two focal conics. All of the cones will then be cones of revolution.

We have already encountered an example of focal surfaces that reduce to lines in the linear congruence.

68. It would not be futile to recall the exposition of these results by another path.

Let the equations of the complexes A and B be:

$$f(x) = 0, \qquad g(x) = 0,$$

and let x be a line of the common congruence; I consider the equation:

(21) $$\sum \left(\lambda \frac{\partial f}{\partial x_i} + \mu \frac{\partial g}{\partial x_i} + \nu \frac{\partial \omega}{\partial x_i} \right) y_i = 0.$$

In that equation, y denotes a current line and λ, μ, ν are three arbitrary functions; as always, $\omega(x)$ is the fundamental form. This equation represents a system of linear complexes with three terms that have a common degenerate semi-quadric (no. 40).

Indeed, form the invariant:

$$\Omega\left(\lambda \frac{\partial f}{\partial x_i} + \mu \frac{\partial g}{\partial x_i} + \nu \frac{\partial \omega}{\partial x_i} \right)$$

$$= \Omega\left(\lambda \frac{\partial f}{\partial x} + \mu \frac{\partial g}{\partial x} \right) + \nu^2 \Omega\left(\frac{\partial \omega}{\partial x} \right) + \nu \sum \frac{\partial \Omega\left(\frac{\partial \omega}{\partial x} \right)}{\partial \frac{\partial \omega}{\partial x_i}} \left(\lambda \frac{\partial f}{\partial x_i} + \mu \frac{\partial g}{\partial x_i} \right).$$

One first has:

$$\Omega\left(\frac{\partial \omega}{\partial x} \right) = \omega(x) = 0,$$

$$x_i = \frac{\partial \Omega\left(\frac{\partial \omega}{\partial x} \right)}{\partial \frac{\partial \omega}{\partial x_i}},$$

and, as a result, the coefficient of ν will be written:

$$\sum y_i \left(\lambda \frac{\partial f}{\partial x_i} + \mu \frac{\partial g}{\partial x_i} \right) = \lambda\, m\, f(x) + \mu\, m'\, g(x),$$

where m, m' are the degrees of homogeneity of f and g. Since $f = g = 0$, one will thus have:

$$\Omega\left(\lambda \frac{\partial f}{\partial x} + \mu \frac{\partial g}{\partial x} + \nu \frac{\partial \omega}{\partial x} \right) = \Omega\left(\lambda \frac{\partial f}{\partial x} + \mu \frac{\partial g}{\partial x} \right),$$

or, upon developing this:

(22) $$= \Omega\left(\frac{\partial f}{\partial x} \right) \lambda^2 + 2\Omega\left(\frac{\partial f}{\partial x} \middle| \frac{\partial g}{\partial x} \right) \lambda\mu + \Omega\left(\frac{\partial g}{\partial x} \right) \mu^2 .$$

The special complexes of the system will be obtained by taking the values $\lambda_0 : \mu_0$ and $\lambda_0' : \mu_0'$ that annul that invariant for $\lambda : \mu$. These special complexes will then form the systems with two terms:

$$(23) \qquad \sum \left(\lambda_0 \frac{\partial f}{\partial x_i} + \mu_0 \frac{\partial g}{\partial x_i} + \nu \frac{\partial \omega}{\partial x_i} \right) y_i = 0,$$

$$(24) \qquad \sum \left(\lambda_0' \frac{\partial f}{\partial x_i} + \mu_0' \frac{\partial g}{\partial x_i} + \nu \frac{\partial \omega}{\partial x_i} \right) y_i = 0.$$

In these formulas, ν remains arbitrary. From no. 30, each of these systems (23), (24) will be composed of special complexes whose directions will form a plane pencil. We will thus have two plane pencils; I add that these two pencils will be (F', Φ), (F, Φ'). Indeed, let us seek the two plane pencils whose union comprises all of the lines common to the complexes (21). These common lines y will verify the equations:

$$\sum \frac{\partial f}{\partial x_i} y_i = 0, \qquad \sum \frac{\partial g}{\partial x_i} y_i = 0, \qquad \sum \frac{\partial \omega}{\partial x_i} y_i = 0.$$

From the last relation, they will cut x, and since x will belong to the two complexes:

$$\sum \frac{\partial f}{\partial x_i} y_i = 0, \qquad \sum \frac{\partial g}{\partial x_i} y_i = 0,$$

they may themselves belong to these two complexes only under the condition that they belong to one of the two pencils (F', Φ), (F, Φ'), which will have the two correlations H_A, H_B in common that were already defined above. Since the lines of the pencils (F', Φ), (F, Φ') will be the ones that have all of the complexes (21) in common, the directrices of the special complexes of this system will generate the focal pairs (F, Φ), (F', Φ') that are inverse to the first two.

Moreover, if we suppose that the complexes (23) generate the pencil (F, Φ) then any line of this pencil will thus be represented by:

$$z_i + \nu x_i ,$$

where we have set:

$$\frac{\partial \omega(x)}{\partial z_i} = \lambda_0 \frac{\partial f}{\partial x_i} + \mu_0 \frac{\partial g}{\partial x_i} .$$

It is obvious that the z_i thus defined will be the coordinates of a particular line of the pencil (F, Φ).

One verifies that the condition for (F, Φ) to have an envelope is found to be satisfied:

$$2\omega(z \mid dx) = \sum \frac{\partial \omega(z)}{\partial z_i} - \sum \left(\lambda_0 \frac{\partial f}{\partial x_i} - \mu_0 \frac{\partial g}{\partial x_i} \right) dx_i$$

$$= \lambda_0 \, df + \mu_0 \, dg = 0,$$

since $df = 0$, $dg = 0$.

It is then proved that the focal pairs possess an envelope.

The system of complexes (21) possesses a property that justifies the name of *tangent linear complexes* that one gives to these complexes. If one replaces the y_i in the left-hand side of (21) with the coordinates of a line of the congruence:

$$x_i + dx_i + \frac{1}{1\cdot 2} d^2 x_i + \frac{1}{1\cdot 2\cdot 3} d^3 x_i + \dots$$

that is infinitely close to x then one will find a result of order at most two. No other linear complex will present that peculiarity, which the reader may verify for himself.

69. Although we would not like to carry out a detailed study of the congruences of lines here, we would nonetheless like to give an account of the case in which the focal pairs coincide for all of the lines of the congruence.

This case is obviously characterized by the fact that the two roots of the equation (22) are equal, which gives:

$$\left[\Omega\left(\frac{\partial f}{\partial x}\middle|\frac{\partial g}{\partial x}\right)\right]^2 - \Omega\left(\frac{\partial f}{\partial x}\right)\Omega\left(\frac{\partial g}{\partial x}\right) = 0. \tag{25}$$

The left-hand side of that equation will be an invariant of the congruence; it will likewise be a combinant, because, if one sets:

$$f_1 = \mathcal{F}(f, g), \qquad g_1 = \mathcal{G}(f, g),$$

$$\Delta = \frac{\partial \mathcal{F}}{\partial f}\frac{\partial \mathcal{G}}{\partial g} - \frac{\partial \mathcal{F}}{\partial g}\frac{\partial \mathcal{G}}{\partial f}$$

then one will find:

$$\left[\Omega\left(\frac{\partial f_1}{\partial x}\middle|\frac{\partial g_1}{\partial x}\right)\right]^2 - \Omega\left(\frac{\partial f_1}{\partial x}\right)\Omega\left(\frac{\partial g_1}{\partial x}\right) = \Delta^2\left\{\left[\Omega\left(\frac{\partial f}{\partial x}\middle|\frac{\partial g}{\partial x}\right)\right]^2 - \Omega\left(\frac{\partial f}{\partial x}\right)\Omega\left(\frac{\partial g}{\partial x}\right)\right\}.$$

We seek to discover what the definition of the congruence would be in this case.

We have only one family of ruled series with an envelope, and the unique focal pair (F, Φ) will be the locus of directrices of the special tangent linear complex:

$$\sum\left(\lambda_0\frac{\partial f}{\partial x_i} + \mu_0\frac{\partial g}{\partial x_i} + \nu\frac{\partial \omega}{\partial x_i}\right)y_i = 0.$$

As before, one recognizes that the pair (F, Φ) will possess an envelope, which will generally be a surface S.

On the other hand, since the plane Φ' coincides here with the plane Φ, the pair (F, Φ) will, at the same time, constitutes the osculating pencil of the unique developable that one can form from the lines of the congruence, and which will pass through the line x. The edge C of that developable will be traced on S, and since the osculating plane Φ to C at the point F will, at the same, be tangent to S, it will then result that C is an asymptotic line of the surface S. Since this will be true for any developable of the congruence, it will appear to be *set of tangents to the asymptotic lines of a family on the surface S.*

Conversely, if one considers the asymptotes C of a family on a surface S then their tangents will constitute a congruence with coincident focal pairs.

Indeed, let there be a family of curves C on a surface S, and consider the congruence of tangents to these curves. Let x be one of these tangents that touch F along a curve C, let Φ be the plane tangent to the surface at F, and let Φ' be the osculating plane to C at F.

The planes Φ and Φ' are the focal planes of the line x and the second focal surface S' is the envelope of the plane Φ'. However, the lines C are asymptotes for S, so Φ' coincides with Φ and the focal pairs coincide.

70. It then remains for us to consider what happens if the point F no longer describes a surface, but a curve V when the focal pairs coincide. This case, which is rarely considered, nevertheless offers a certain interest.

The congruence will be composed of lines that cut the fixed curve V. An infinitude of lines will then pass through any point F of V. The lines issuing from F will form a hyper-pencil such that one may represent any one of these lines by the formulas:

$$z_i = \lambda\, a_i + \mu\, b_i + \nu\, c_i$$

or:

$$\omega(a) = \omega(b) = \omega(c) = \omega(a \mid b) = \omega(a \mid c) = \omega(b \mid c) = 0,$$

in which a_i, b_i, c_i are functions of the one parameter u. Moreover, there must exist a homogeneous relation between λ, μ, ν, which will be, in some sense, the equation of the cone that will be described by the lines of the congruence that issue from F.

One will have:

$$\begin{aligned} dz_i &= \lambda\, da_i + \mu\, db_i + \nu\, dc_i + a_i\, d\lambda + b_i\, d\mu + c_i\, d\nu \\ &= (\lambda a_i' + \mu b_i' + \nu c_i')\, du + (a_i\, d\lambda + b_i\, d\mu + c_i\, d\nu), \end{aligned}$$

in which a_i', b_i', c_i' are the derivatives of a_i, b_i, c_i with respect to u. Therefore:

$$\begin{aligned} \omega(dz) &= \omega(\lambda\, a' + \mu\, b' + \nu\, c')\, du^2 \\ &\quad + 2\omega(\lambda\, a' + \mu\, b' + \nu\, c' \mid a\, d\lambda + b\, d\mu + c\, d\nu)\, du + \omega(a\, d\lambda + b\, d\mu + c\, d\nu). \end{aligned}$$

The expression $\omega(a\, d\lambda + b\, d\mu + c\, d\nu)$ will be zero identically, and what will remain is:

$$\omega(dz) = \omega(\lambda\, a' + \mu\, b' + \nu\, c')\, du^2 + \omega(\lambda\, a' + \mu\, b' + \nu\, c' \mid a\, d\lambda + b\, d\mu + c\, d\nu)\, du.$$

The equation $\omega(dz) = 0$ will define the lines of the congruence that are close to z, which, along with z, will form an element of the developable. The solution $du = 0$ will give the cones whose summits are on the curve V. However, since the two families of developables will coincide here, the two solutions must give $du = 0$.

Suppose that λ, μ, ν are expressed as functions of u and a parameter v (*) that varies when the line describes the desired cone. It must then be the case that the term in $du\, dv$ will disappears, or that one has:

$$\omega\left(a'\lambda + b'\mu + c'\nu \,\middle|\, a\frac{\partial\lambda}{\partial v} + b\frac{\partial\mu}{\partial v} + c\frac{\partial\nu}{\partial v} \right) = 0$$

identically, which is written, upon developing it:

$$\omega(b \mid c)\left(\mu\frac{\partial\nu}{\partial v} - \nu\frac{\partial\mu}{\partial v} \right) + \omega(c \mid a')\left(\nu\frac{\partial\lambda}{\partial v} - \lambda\frac{\partial\nu}{\partial v} \right) + \omega(a \mid b')\left(\lambda\frac{\partial\mu}{\partial v} - \mu\frac{\partial\lambda}{\partial v} \right) = 0.$$

If one observes that $\omega(b \mid c')$, $\omega(b \mid c')$, $\omega(b \mid c')$ do not depend upon v then one will see that this equation is equivalent to the finite relation:

$$(26)\qquad \begin{vmatrix} \lambda & \mu & \nu \\ \omega(b \mid c') & \omega(a \mid c') & \omega(a \mid b') \\ \alpha & \beta & \gamma \end{vmatrix} = 0,$$

where α, β, γ are constants; i.e., as functions of u.

The linear form of that equation proves to us, first of all, that the lines of the congruence issuing from the point F of the curve V will generate a plane pencil. The plane Φ of that pencil will obviously be the focal plane. Moreover, in the spray that consists of the lines that issue from F, the tangent FT to the curve V will be represented by the following values of $\lambda : \mu : \nu$ (no. 54):

$$\frac{\lambda}{\omega(b \mid c')} = \frac{\mu}{\omega(c \mid a')} = \frac{\nu}{\omega(a \mid b')}.$$

Since these values of $\lambda : \mu : \nu$ will verify equation (26), one must conclude that the plane Φ will touch the curve V; i.e., conforming to the terminology of no. 55, the characteristics of the pencil (F, Φ) will belong to that pencil. From this, it will result immediately that, in general, *the congruence will be the locus of the lines that touch a given developable at the points of curve that is traced on that developable*, and that in the exceptional case, it will be the locus of plane pencils (F, Φ) such that the point and the plane constitute a pair of a correspondence that is determined between the points and planes with a fixed line (no. 56).

(*) [D. H. D. One must be careful to distinguish v ("vee") from ν ("nu") in these expressions. Generally, the "vee" is in the denominator.]

The singular linear congruence (no. 29) is the simplest case of this that one may cite.

71. It is often useful to represent a congruence by expressing the coordinates x_i of any of its lines x as functions of two parameters u, v. Likewise, it is often useful to represent the coordinates of a line of a complex by means of functions of three variables. We shall return later on to that representation of the complex; however, I would like to immediately present some remarks on this subject that concern congruences.

If one forms $\omega(dx)$ then since:

$$dx_i = \frac{\partial x_i}{\partial u}du + \frac{\partial x_i}{\partial v}dv ,$$

one will have:

$$\omega(dx) = E\,du^2 + 2F\,du\,dv + G\,dv^2, \tag{27}$$

where:

$$E = \omega\left(\frac{\partial x}{\partial u}\right), \qquad F = \omega\left(\frac{\partial x}{\partial u}\middle|\frac{\partial x}{\partial v}\right), \qquad G = \omega\left(\frac{\partial x}{\partial v}\right).$$

Any equation between u, v will furnish a ruled surface of the congruence; in particular, the integrals of the equation:

$$E\,du^2 + 2F\,du\,dv + G\,dv^2 = 0$$

will give the developables of the congruence. These developables will coincide if $EG - F^2 = 0$.

Consider a linear complex:

$$\sum a_i\,y_i = 0,$$

and replace the y_i with the coordinates of a line of the neighboring congruence to the line x of that same congruence. This will yield:

$$y_i = x_i + \frac{\partial x_i}{\partial u}du + \frac{\partial x_i}{\partial v}dv + \frac{1}{2}\left(\frac{\partial x_i}{\partial u}d^2u + \frac{\partial x_i}{\partial v}d^2v\right)$$
$$+ \frac{1}{2}\left(\frac{\partial^2 x_i}{\partial u^2}du^2 + 2\frac{\partial^2 x_i}{\partial u\,\partial v}du\,dv + \frac{\partial^2 x_i}{\partial v^2}dv^2\right) + \ldots$$

The equation of the complex will become:

$$\sum a_i y_i = \sum a_i x_i + \sum a_i \frac{\partial x_i}{\partial u}du + \sum a_i \frac{\partial x_i}{\partial v}dv + \frac{1}{2}\left(\sum a_i \frac{\partial x_i}{\partial u}d^2u + \sum a_i \frac{\partial x_i}{\partial v}d^2v\right)$$
$$+ \frac{1}{2}\sum a_i \frac{\partial^2 x_i}{\partial u^2}du^2 + \sum a_i \frac{\partial^2 x_i}{\partial u\,\partial v}du\,dv + \sum a_i \frac{\partial^2 x_i}{\partial v^2}dv^2 + \ldots$$

If one chooses the a in such a way that:

(28) $$\sum a_i y_i = 0, \qquad \sum a_i \frac{\partial x_i}{\partial u} = 0, \qquad \sum a_i \frac{\partial x_i}{\partial v} = 0$$

then the result of that substitution will be:

$$\sum a_i y_i = \frac{1}{2}\sum a_i \frac{\partial^2 x_i}{\partial u^2} du^2 + \sum a_i \frac{\partial^2 x_i}{\partial u\,\partial v} du\,dv + \sum a_i \frac{\partial^2 x_i}{\partial v^2} dv^2 + \ldots;$$

it will then reduce to second order.

For certain congruences, one may determine the complex $\sum a_i y_i = 0$ in such a manner that the second-order terms also disappear, in such a way that *up to third order* the neighboring lines to a line x in the congruence may be envisioned as being contained in a linear complex. In this case, we will say that the congruence possesses an *osculating linear complex* along each of its lines.

In order for the terms of second order to disappear, it is necessary that one must have, at the same time as equations (28), the following ones:

(29) $$\sum a_i \frac{\partial^2 x_i}{\partial u^2} = 0, \qquad \sum a_i \frac{\partial^2 x_i}{\partial u\,\partial v} = 0, \qquad \sum a_i \frac{\partial^2 x_i}{\partial v^2} = 0.$$

The compatibility of these equations may be expressed by writing that the following determinant is zero:

(30) $$\left\| \frac{\partial^2 x_i}{\partial u^2}, \frac{\partial^2 x_i}{\partial u\,\partial v}, \frac{\partial^2 x_i}{\partial v^2}, \frac{\partial x_i}{\partial u}, \frac{\partial x_i}{\partial v}, x_i \right\| = 0.$$

Now, the vanishing of this determinant obviously expresses the necessary and sufficient condition for the x_i to be solutions of the same equation of the Laplace form:

(31) $$A\frac{\partial^2 \theta}{\partial u^2} + 2B\frac{\partial^2 \theta}{\partial u\,\partial v} + C\frac{\partial^2 \theta}{\partial v^2} + D\frac{\partial \theta}{\partial u} + E\frac{\partial \theta}{\partial v} + G\theta = 0.$$

Therefore, *in order for a congruence to admit an osculating linear complex, it is necessary and sufficient that the coordinates of one of its lines verify an equation of the same form as* (31).

72. The congruences of coincident focal pairs are *always* of this kind.

Indeed, suppose that v = const. are the developables of the congruence, so the expression for $\omega(dx)$ must reduce to dv^2, and one will have:

$$\omega(dx) = \omega\left(\frac{\partial x}{\partial u}\right) du^2 + 2\omega\left(\frac{\partial x}{\partial u}\middle|\frac{\partial x}{\partial v}\right) du\,dv + \omega\left(\frac{\partial x}{\partial v}\right) dv^2;$$

one will thus have:

$$\omega\left(\frac{\partial x}{\partial u}\right) = 0, \quad \omega\left(\frac{\partial x}{\partial u}\middle|\frac{\partial x}{\partial v}\right) = 0.$$

I will consider the following expressions:

$$\varphi_i = A\frac{\partial^2 x_i}{\partial u^2} + 2B\frac{\partial^2 x_i}{\partial u\,\partial v} + C\frac{\partial^2 x_i}{\partial v^2} + D\frac{\partial x_i}{\partial u} + Gx_i,$$

and form:

$$\omega(x \mid \varphi) = A\,\omega\left(x\middle|\frac{\partial^2 x}{\partial u^2}\right) + B\omega\left(x\middle|\frac{\partial^2 x}{\partial u\,\partial v}\right) + C\omega\left(x\middle|\frac{\partial x}{\partial u}\right) + D\omega\left(x\middle|\frac{\partial x}{\partial u}\right) + G\omega(x),$$

$$\omega\left(\frac{\partial x}{\partial u}\middle|\varphi\right) = A\,\omega\left(\frac{\partial x}{\partial u}\middle|\frac{\partial^2 x}{\partial u^2}\right) + B\omega\left(\frac{\partial x}{\partial u}\middle|\frac{\partial^2 x}{\partial u\,\partial v}\right) + C\omega\left(\frac{\partial x}{\partial u}\right) + D\omega\left(\frac{\partial x}{\partial u}\middle|\frac{\partial x}{\partial v}\right) + G\omega\left(x\middle|\frac{\partial x}{\partial u}\right).$$

These two expressions will be zero. Indeed, one will have:

$$\omega(x) = 0, \qquad \frac{1}{2}\frac{\partial\omega(x)}{\partial u} = \omega\left(x\middle|\frac{\partial x}{\partial u}\right) = 0, \qquad \frac{1}{2}\frac{\partial\omega(x)}{\partial v} = \omega\left(x\middle|\frac{\partial x}{\partial v}\right) = 0,$$

$$\frac{\partial\omega\left(x\middle|\frac{\partial x}{\partial u}\right)}{\partial u} = \omega\left(x\middle|\frac{\partial^2 x}{\partial u^2}\right) + \omega\left(\frac{\partial x}{\partial u}\right) = 0,$$

and since $\omega\left(\frac{\partial x}{\partial u}\right) = 0$, one will have:

$$\omega\left(x\middle|\frac{\partial^2 x}{\partial u^2}\right) = 0.$$

Likewise:

$$\frac{\partial\omega\left(x\middle|\frac{\partial x}{\partial u}\right)}{\partial v} = \omega\left(x\middle|\frac{\partial^2 x}{\partial u\,\partial v}\right) + \omega\left(\frac{\partial x}{\partial u}\middle|\frac{\partial x}{\partial v}\right) = 0,$$

and since $\omega\left(\frac{\partial x}{\partial u}\middle|\frac{\partial x}{\partial v}\right) = 0$, one will have:

$$\omega\left(x\middle|\frac{\partial^2 x}{\partial u\,\partial v}\right) = 0.$$

One will then have:

(32)
$$\begin{cases} 2\omega(x\,|\,\varphi) &= \sum \dfrac{\partial\omega(x)}{\partial x_i}\varphi_i = 0, \\ 2\omega\left(\dfrac{\partial x}{\partial u}\,|\,\varphi\right) &= \sum \dfrac{\partial\omega(\xi)}{\partial x_i}\varphi_i = 0, \end{cases}$$

where one has set $\xi_i = \partial x_i / \partial u$, for the moment.

If one has:

$$\frac{\dfrac{\partial\omega(\xi)}{\partial\xi_1}}{\dfrac{\partial\omega(x)}{\partial x_1}} = \frac{\dfrac{\partial\omega(\xi)}{\partial\xi_2}}{\dfrac{\partial\omega(x)}{\partial x_2}} = \ldots = \frac{\dfrac{\partial\omega(\xi)}{\partial\xi_6}}{\dfrac{\partial\omega(x)}{\partial x_6}}$$

then, upon calling the common value of these ratios ρ, one will have:

$$\frac{\partial\omega(\xi-\rho x)}{\partial\omega(\xi_i-\rho x_i)} = 0 \qquad (i = 1, 2, \ldots, 6),$$

which will require that:

$$\xi_i - \rho\, x_i = 0,$$

or:

$$\frac{\partial x_i}{\partial u} = \rho\, x_i\,,$$

from which:

$$x_i = e^{\int \rho\, du} V_i\,.$$

The ratios of the x_i will depend upon only v, which is contrary to our hypothesis.

From the preceding, at least one of the determinants:

$$\frac{\partial\omega(\xi)}{\partial\xi_i}\frac{\partial\omega(x)}{\partial x_k} - \frac{\partial\omega(\xi)}{\partial\xi_k}\frac{\partial\omega(x)}{\partial x_i}$$

will be non-zero; for example, the one that corresponds to the indices $i = 1$, $k = 2$. One may then determine the equation:

(33)
$$A\frac{\partial^2\theta}{\partial u^2} + 2B\frac{\partial^2\theta}{\partial u\,\partial v} + C\frac{\partial^2\theta}{\partial v^2} + D\frac{\partial\theta}{\partial u} + E\frac{\partial\theta}{\partial v} + G\theta = 0$$

in such a fashion that it should admit the solutions x_3 , x_4 , x_5 , x_6 : since φ_3 , φ_4 , φ_5 , φ_6 are zero, equations (32) will give zero values for φ_1, φ_2, moreover, since the corresponding determinant will not be zero.

Since the six expressions φ_i will then be zero, the six coordinates x_i will verify equation (33). The congruence will thus admit an osculating linear complex.

73. However, this is not the only case.

For example, take a congruence G that is contained in a linear complex. The two focal surfaces will be reciprocal polars with respect to this complex. Indeed, let Δ be a line of the congruence and let (F, Φ), (F', Φ') be focal pairs. The pair (F, Φ') will be the osculating plane pencil of the curve C that is traced on the surface S (no. 66). This plane pencil will then belong to the linear complex, which will imply that the tangents to C must belong to that complex, as well (no. 53). The plane Φ' will then be the polar plane to F in the complex; likewise, the plane Φ will be the polar plane to F'. The two plane pencils (F, Φ), (F', Φ') will be polar to each other, and, as a result, the two focal surfaces S and S' that they envelop will be polar to each other with respect to the complex.

There is more: When the point F describes a curve on S, or, more precisely, when the plane pencil (F, Φ) describes a *band* circumscribed by S, the plane pencil (F', Φ') will describe the reciprocal band that is circumscribed by S'. In particular, if the band is an asymptotic band of S – i.e., a band in which the locus of the point F is an asymptote of S – then the corresponding band of S' will likewise be asymptotic. This amounts to saying that, by duality, the asymptotic band of a surface will transform into that of the transformed surface.

In other words, asymptotes will correspond on focal surfaces.

As was shown by G. Darboux, this remarkable property is general, and the preceding considerations permit us to given an immediate proof.

Let G be a congruence that admits an osculating linear complex on the line x. Let S be a focal surface, and consider the tangents to S that belong to the complex C_x. I shall refer to the reciprocal polar of S in the complex C_x by S_1, so the surface S_1 will be the second focal surface of the congruence G_1 of the tangent lines to S that belong to C_x. Finally, let S' be the second focal surface of the congruence G. Along the line x, the congruences G and G_1 will coincide, up to properties that depend upon third order. The two focal surfaces S' and S_1 must then be tangents to the point F' where x touches S', and moreover, the elements of second order of S' and S_1 must be the same along F'. The asymptotic tangents to S' and S must coincide.

However, if F is displaced along an asymptotic tangent of S then F' will be displaced along an asymptotic tangent, and therefore along an asymptotic tangent to S'. It is thus established that the asymptotic tangents on S and S' will correspond. The asymptotes to S and S' will then correspond.

We will have occasion to return to that question.

CHAPTER V.

KLEIN COORDINATES. – ANALLAGMATIC GEOMETRY.

Tetrahedral coordinates. – Characteristic form of $\omega(x)$; reciprocal. – Klein forms. – System of six linear complexes that are pair-wise in involution. – The remarkable configuration that they form. – Properties of the fifteen congruences C_{ij} . – Special notation for their directrices. – The semi-quadrics Q_{ijk} . – The ten fundamental quadrics. – Fundamental tetrahedra. – Remarkable relations between the quadrics and the tetrahedra. – Digression on a configuration that is offered by three linear complexes that are pair-wise in involution. – Grouping the summits and faces of the tetrahedra. – Properties of the permutations of six letters. – Desmic tetrahedra. – Distribution on a conic of the six poles of the same plane. – Configuration of sixteen points and sixteen planes. – Transformations that must take the fundamental form back to itself. – Some generalities on n-dimensional spaces. – Representation of a quadric on a plane. – Stereographic projection. – Correspondence between projective geometry on a quadric and anallagmatic geometry on a plane. – The geometry of ruled space is identical to the anallagmatic geometry of a four-dimensional space.

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74. In no. 3, we defined a special system of coordinates r_{ik} for the straight line, which is a notion that is found to be linked to that of a certain coordinate tetrahedron.

We then pointed out how one can substitute new coordinates for these coordinates by means of transformation formulas:

$$r_{ik} = A_{ik,1}\, x_1 + A_{ik,2}\, x_2 + \ldots + A_{ik,6}\, x_6 ,$$

in which the determinant of the transformation is not zero. The equations:

$$x_i = 0$$

each represent a linear complex, and these six complexes are obviously not part of the same system of five terms.

The variables r_{ik} verify the relation:

$$r_{12}\, r_{34} + r_{13}\, r_{42} + r_{14}\, r_{23} = 0,$$

and if one substitutes the variables x_i for them then the left-hand side of that equation becomes a quadratic form in $x_1, x_2, \ldots, x_6$:

$$\omega(x).$$

The form of the function $\omega(x)$ characterizes the coordinates. There are two particularly important types of them, and they have the closest links between them, moreover. The first one is the following:

$$x_1\, x_4 + x_2\, x_5 + x_3\, x_6 ,$$

and the second one, which was first considered by Klein, and which is the basis for his research into this geometry, consists of the sum of the squares:

$$x_1^2 + x_2^2 + \cdots + x_6^2 .$$

We shall study these two types in turn.

We first observe that the coordinates r_{ik} realize the former type, and we shall show that, conversely, *if the coordinates reduce the form* $\omega(x)$ *to the type:*

$$\omega(x) = x_1\, x_4 + x_2\, x_5 + x_3\, x_6$$

(i.e., the sum of three triangles) *then the* x_ρ *will be the coordinates* r_{ik} *with respect to a certain tetrahedron.*

Indeed, if we seek the adjoint form $\Omega(a)$ then we will find:

$$\Omega(a) = a_1\, a_4 + a_2\, a_5 + a_3\, a_6 \,;$$

i.e., $\omega(a)$. This is one of those cases in which the adjoint form reproduces the original form. For the complex $x_\rho = 0$, all of the coefficients a_i are zero, except for a_ρ, and in turn, $\Omega(a) = 0$; *the coordinate complexes are therefore all special*.

We now show that the directrices of these complexes are the edges of a tetrahedron.

The condition for the involution of the two complexes A, B is written:

$$a_4\, b_1 + b_4\, a_1 + a_5\, b_2 + b_5\, a_2 + a_6\, b_3 + b_6\, a_3 = 0$$

here; it is verified for every pair $x_\rho = 0$, $x_\sigma = 0$ of coordinate complexes, except for the three pairs of indices 1 and 4, 2 and 5, 3 and 6.

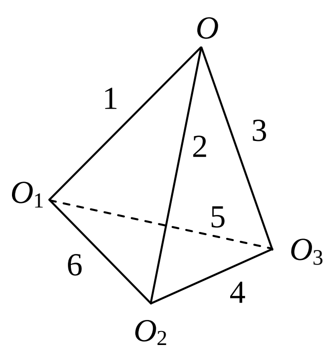

Figure 1.

For example, take the complexes with indices 1, 2, 3 (Fig. 1), so they are special and pair-wise in involution, and they are not part of the same system of two terms. It then results that their directrices define a trihedron or a triangle; for example, a trihedron with summit O.

The directrices of the complexes 2, 3, 4, likewise define a trihedron or a triangle. However, if they define a trihedron then the directrix of 4 must pass through the point of intersection O of the directrices 2 and 3. The directrix of 4 will then cut that of the complex 1 at O, which cannot happen, since 1 and 4 are not expected to be in involution. Therefore, the directrices of 2, 3, 4 define a triangle, and the directrix of 4 cuts that of 2 at a point O_2, and that of 3 at a point O_3 .

If one then takes the directrix of 5 then it will define a trihedron or triangle with those of 3 and 4. If it defines a triangle then it will be in the plane OO_2O_3 and will cut the directrix of 2, which cannot happen, since one expects that 2 and 5 will not be in involution. Therefore, the directrices of 5, 3, 4 will define a trihedron, and in turn, the directrix of 5 will pass through O_3 ; similarly, the directrix of 6 will pass through O_2 .

All that remains is to prove that the directrices of 5, 6, 1 intersect at the same point O_1; i.e., they define a trihedron. Now, in fact, these three directrices intersect pair-wise; they thus define a trihedron or a triangle. One cannot assume that they define a triangle because the directrix of 1 would then be in the plane of the directrices of 5 and 6 and would cut the directrix 4, but this cannot happen, since 1 and 4 are not in involution. It is therefore a trihedron that defines the directrix lines of the complexes 1, 6, 5.

One will obviously arrive at the same result if one starts with the hypothesis that the directrices of 1, 2, 3 define a triangle, and not a trihedron. One will have obtained a dualistic configuration from the viewpoint of the notations for what we have found.

Having said that, assign the index 1 to the point O_1, the index 2 to the point O_2, the index 3 to the point O_3, and the index 4 to the point O. Then, consider the coordinates r_{ik} that were defined in no. 4 and taken with respect to that tetrahedron.

The directrix of the complex:

1	is the line	OO_1	or	41,
2	"	OO_2	"	42,
3	"	OO_3	"	43,
4	"	O_2O_3	"	23,
5	"	O_3O_1	"	31
6	"	O_1O_2	"	12.

Now, the equation:

$$r_{ik} = 0$$

is the condition of intersection for a line with the line ik; therefore, with the system of r_{ik} the equation of the complex:

1	will be	$r_{41} = 0$,
2	"	$r_{42} = 0$,
3	"	$r_{43} = 0$,
4	"	$r_{23} = 0$,
5	"	$r_{31} = 0$,
6	"	$r_{12} = 0$,

and since the r_{ik} are linear functions of the x_ρ , one must have:

$$(1) \qquad \begin{cases} x_1 = \alpha_1 \, r_{41}, \\ x_2 = \alpha_2 \, r_{42}, \\ x_3 = \alpha_2 \, r_{43}, \\ x_4 = \alpha_1' \, r_{23}, \\ x_5 = \alpha_2' \, r_{31}, \\ x_6 = \alpha_3' \, r_{12}, \end{cases}$$

where the α, α' are constants. If one forms:

$$x_1\, x_4 + x_2\, x_5 + x_3\, x_6 = \alpha_1 \alpha_1' \, r_{41}\, r_{23} + \alpha_2 \alpha_2' \, r_{42}\, r_{31} + \alpha_3 \alpha_3' \, r_{43}\, r_{12}$$

then that form will differ only by a factor from:

$$r_{41}\, r_{23} + r_{42}\, r_{31} + r_{43}\, r_{12} \,,$$

and one will see that:

$$(2) \qquad \alpha_1 \alpha_1' = \alpha_2 \alpha_2' = \alpha_3 \alpha_3' .$$

However – and this is the essential point – formulas (1) indeed show that the x_ρ are the coordinates r_{ik}, when taken with respect to a certain tetrahedron, up to a factor.

The presence of the factors α is of no importance, since in regard to the relations (2), one can put them back into the x without changing the form:

$$x_1\, x_4 + x_2\, x_5 + x_3\, x_6 \,.$$

In summary, performing a transformation that takes the form above to the same form amounts to changing the tetrahedron of reference.

As an application of the formulas that define the r_{ik} in no. 4, the reader will easily verify that, conversely, any change of symbols or coordinates translates into a linear transformation of the coordinates r_{ik} .

75. The other coordinates that we shall speak of are due to Klein.

Suppose that one has a coordinate system of the preceding type – i.e., *tetrahedral* ones – and from now on denote these coordinates by the symbol r_{ik} , as in no. 4. We will have:

$$r_{41}\, r_{23} + r_{42}\, r_{31} + r_{43}\, r_{12} = 0,$$

or furthermore:

$$(r_{41} + r_{23})^2 + (r_{42} + r_{31})^2 + (r_{43} + r_{12})^2 - (r_{41} - r_{23})^2 - (r_{42} - r_{31})^2 - (r_{43} - r_{12})^2 = 0.$$

The fundamental form – *if one appeals to real numbers* – is therefore decomposable into six squares, three of which are positive and three of which are negative.

Perform the *real* transformation:

$$(3) \qquad \begin{cases} r_{41}+r_{23}=x_1, \\ r_{42}+r_{31}=x_2, \\ r_{43}+r_{12}=x_3, \\ r_{41}-r_{23}=x_4, \\ r_{42}-r_{31}=x_5, \\ r_{43}-r_{12}=x_6, \end{cases}$$

and the fundamental form becomes:

$$(4) \qquad x_1^2+x_2^2+x_3^2-x_4^2-x_5^2-x_6^2.$$

However, by reasons of *symmetry* that also present themselves in the theory of penta-spherical or hexa-spherical coordinates, one desires that (4) should be converted into a *sum* of squares ([8]). Obviously, this goal can be achieved only by an imaginary transformation.

For example, one may substitute the following equations for equations (3):

$$(5) \qquad \begin{cases} r_{41}+r_{23}=x_1, & r_{41}-r_{23}=x_4\sqrt{-1}, \\ r_{42}+r_{31}=x_2, & r_{42}-r_{31}=x_5\sqrt{-1}, \\ r_{43}+r_{12}=x_3, & r_{43}-r_{12}=x_6\sqrt{-1}, \end{cases}$$

and instead of (4) will then have:

$$(6) \qquad x_1^2+x_2^2+x_3^2+x_4^2+x_5^2+x_6^2,$$

a formula that is symmetric, but complicated by imaginaries.

Meanwhile, under the hypotheses that we have imposed, the six complex coordinates will be real, since x_1, x_2, x_3 are real and $\sqrt{-1}$ is a factor in x_4, x_5, x_6 .

However, this situation will not be *necessarily* true if we perform *any other* linear transformation that reduces the fundamental form to the sum of six squares.

76. If one is given a quadratic form that is the sum of squares (involving six squares):

$$x_1^2+x_2^2+\cdots+x_6^2$$

then one will say *orthogonal substitution* to mean any linear transformation that preserves the type of its form in such a way that by virtue of the transformation equations:

$$x_i = \alpha_{i,1}\, y_1 + \alpha_{i,2}\, y_2 + \ldots + \alpha_{i,6}\, y_6 \qquad (i = 1, 2, \ldots, 6),$$

([8]) In regard to that, consult the work of Darboux: "Sur une classe remarquable de courbes et de surfaces," "Sur les groupes de points, de cercles," and his *Leçons sur la théorie des surfaces.*

one must have:

$$x_1^2 + x_2^2 + \cdots + x_6^2 = y_1^2 + y_2^2 + \cdots + y_6^2 .$$

Consequently, upon performing an arbitrary orthogonal substitution on the variables x_i that are defined precisely by formulas (5), one will get the general type of coordinates that attribute the form of a sum of squares to the function ω.

The coordinates thus defined are those of Klein. However, it is easy to confirm that these coordinates are not essentially distinct from the coordinates that were defined in formulas (5), but which are more general in appearance.

Indeed, choose coordinates y_i that reduce ω to the form:

$$y_1^2 + y_2^2 + \cdots + y_6^2$$

and perform the linear transformation:

$$\text{(7)} \qquad \left\{ \begin{array}{ll} z_1 = y_1 + y_2\sqrt{-1}, & z_4 = y_1 - y_2\sqrt{-1}, \\ z_2 = y_4 + y_4\sqrt{-1}, & z_5 = y_3 - y_4\sqrt{-1}, \\ z_3 = y_5 + y_6\sqrt{-1}, & z_6 = y_5 - y_6\sqrt{-1}, \end{array} \right.$$

so the use of the variable z, thus defined, will reduce the form to the tetrahedral type:

$$z_1\, z_4 + z_2\, z_5 + z_3\, z_6 ,$$

in such a way that the z_i are the coordinates r_{ik} relative to a certain tetrahedron, while the y, from formulas (7), are the ones that one deduces precisely by applying formulas (5).

Meanwhile, there is a difference, because here the tetrahedron to which the coordinates z_i are referred *can very well be imaginary*. One agrees that this distinction implies nothing essential.

From that remark, the passage from a Klein coordinate system to another analogous system can be reduced to the passage from a tetrahedral coordinate system to another tetrahedral system that is preceded and followed by the transformation that is defined by formulas (5).

77. The system of Klein coordinates presents a remarkable configuration whose principal properties we shall discuss.

Six coordinate complexes $C_1, C_2, \ldots, C_6$ enter into it that are represented by the equations:

$$x_1 = 0, \qquad x_2 = 0, \qquad \ldots, \qquad x_6 = 0,$$

respectively.

None of these complexes are special, because the adjoint form to $\omega(x)$ is:

$$\Omega(a) = a_1^2 + a_2^2 + \cdots + a_6^2$$

here; it is not zero for any of the complexes C_i .

The condition for involution of the two complexes:

$$a_1\, x_1 + a_2\, x_2 + \ldots + a_6\, x_6 = 0,$$
$$b_1\, x_1 + b_2\, x_2 + \ldots + b_6\, x_6 = 0$$

is written:

$$a_1\, b_1 + a_2\, b_2 + \ldots + a_6\, b_6 = 0.$$

One thus recognizes that the complexes C_i, when taken two at a time, will be in involution or orthogonal. From the name *sextuply-orthogonal* that one sometimes gives to this coordinate system, Klein gave the name *fundamental system* to the set of six complexes C_i .

Conversely, suppose that the coordinate complexes $x_1, x_2, \ldots, x_6$ are pair-wise in involution.

When the adjoint form to the fundamental form is written:

$$\Omega(a) = \sum A_{ij}\, a_i\, a_j\, ,$$

the involution of the complexes:

$$x_i = 0, \qquad x_j = 0$$

will demand that $A_{ij} = 0$. All of the rectangles must be missing from $\Omega(a)$, and no coordinate complex can be special, because if $x_i = 0$ were special then one would have:

$$A_{ii} = 0,$$

and $\Omega(a)$ would be reducible to less than six squares. Upon once more introducing constant factors into the a, one can thus write:

$$\Omega(a) = a_1^2 + a_2^2 + \cdots + a_6^2\, ,$$

and one will then have:

$$\omega(x) = x_1^2 + x_2^2 + \cdots + x_6^2\, .$$

The coordinate system is then that of Klein.

This permits us to count the number of parameters that are contained in a fundamental system.

If we are given C_1 arbitrarily then we will introduce five parameters, because a linear complex depends upon five parameters. We take C_2 to be in involution with C_1, but otherwise arbitrary; we thus introduce four new parameters. C_3 must be in involution with $C_1$and C_2, but it still contains three new parameters. C_4 will contribute only two parameters, because it is subject to being in involution with C_1, C_2, C_3 . Finally, C_5 contains just one parameter, because it must be in involution with C_1, C_2, C_3, C_4 . As for C_6, it is defined uniquely by the condition of being in involution with C_1, C_2, C_3, C_4 , C_5 . We have thus constructed a fundamental system, and in truth, *the most general one.* We have thus introduced:

$$5 + 4 + 3 + 2 + 1 = 15$$

parameters in our construction. That is the number of parameters that the fundamental system contains.

One will observe that after having taken C_1, C_2, …, C_p to be real, we can then take C_{p+1} , …, C_6 to have arbitrary imaginary coefficients, in such a way that *the number of imaginary complexes in a fundamental system is arbitrary.* Nonetheless, it is impossible that one would have just one imaginary, because if C_1, C_2, …, C_5 were real then the complex C_6 would be unavoidably real. However, there can be two, three, four, five, or even six imaginaries.

In order to obtain such a system, one obviously requires an imaginary transformation.

78. Take p of the complexes C_i – say C_1, C_2, …, C_p – and define the system with p terms:

$$\lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_p x_p = 0.$$

It is clear that since $x_{p+1} = 0$, …, $x_6 = 0$ are $(6 - p)$ complexes in involution with C_1, C_2, …, C_p, the complementary system to the preceding system will be:

$$\lambda_{p+1} x_{p+1} + \ldots + \lambda_6 x_6 = 0.$$

This has numerous consequences, as we will see.

Let C_{ij} be the congruence that is common to the complex C_i and C_j . It is not singular, because its invariant is equal to unity. I shall denote its directrices by Δ_{ij} and Δ_{ji} , which are distinct. I observe that the system with two terms that is composed of the complexes that contain the congruence C_{ij} has the equation:

$$x_i + \lambda x_j = 0.$$

Its invariant is equal to $1 + \lambda^2$, so the special complexes of the system will have the equations:

$$\sqrt{-1}\ x_i + x_j = 0,$$
$$\sqrt{-1}\ x_j + x_i = 0.$$

The coordinates of the directrices of the congruence C_{ij} will all be zero, except for x_i and x_j , which will be proportional to $\pm\sqrt{-1}$ and to 1.

HERE IS HOW I FIX THE NOTATIONS:

I let Δ_{ij} denote the directix whose coordinates are $x_i = \sqrt{-1}$, $x_j = 1$, while the other coordinates are zero. Δ_{ji} will then have the coordinates $x_j = \sqrt{-1}$, $x_i = 1$, while the other coordinates are zero.

One will see how important this fixing of notations is from the standpoint of the correspondence that is established between the properties of the configuration of six fundamental complexes and those of the permutations of six letters.

There are fifteen combinations of six indices taken two at a time. There are thus fifteen congruences C_{ij} , and in turn, thirty lines $\Delta_{\rho\sigma}$.

One observes that: *Any line* Δ_{ij} *will belong to any complex* C_k *that has no common index with it.* There are four of these complexes C_k , C_l, C_m, C_n ; they have two lines in common, namely, Δ_{ij} and Δ_{ji} .

Take two congruences C_{ij} , C_{kl} that have no common index. *Their directrices form a skew quadrilateral.* Indeed, from the preceding, Δ_{ij} , for example, will belong to the complexes C_k and C_l . Therefore, Δ_{ij} will belong to the congruence, and consequently, so will Δ_{kl} and Δ_{lk} .

On the contrary, suppose that C_{ij} and C_{ik} have a common index i; in that case, the directrices cannot intersect. Indeed, let l, m, n be three indices other than i, j, k. The complexes C_l, C_m, C_n contain the directrices of C_{ij}, C_{ik} . Thus, these directrices belong to the semi-quadric Q_{lmn} that is common to these three complexes.

The complexes C_l, when taken three at a time, will give rise to twenty semi-quadrics. These semi-quadrics will be pairs of complementary semi-quadrics. Indeed, it is clear that the two semi-quadrics:

$$Q_{ijk} , \quad Q_{lmn} ,$$

with no common index, will be complementary. They will be carried by the same quadric that I shall represent by the symbol:

$$(Q_{ijk} , Q_{lmn}).$$

There are then ten of these quadrics. Klein gave them the name of *fundamental quadrics.*

Two semi-quadrics that have a common index:

$$Q_{ijl} , \quad Q_{lmn}$$

will have no line in common, because if a common line existed then it would be common to the five complexes C_i , C_k , C_l, C_m, C_n . The complements of these two semi-quadrics will be:

$$Q_{jmn} , \quad Q_{jki} ,$$

and they will also have a common index j.

On the contrary, consider two semi-quadrics that have two common indices:

$$Q_{ijk} , \quad Q_{ijl} ,$$

so these semi-quadrics will have the lines Δ_{mn}, Δ_{nm} in common, which are the directrices of C_{mn} . Their complements will be:

$$Q_{mnl} , \quad Q_{mnk} ,$$

and these complements will have Δ_{ij}, Δ_{ji} in common.

Therefore, the two fundamental quadrics:

$$(Q_{ijk} , Q_{mnl}), \quad (Q_{lji} , Q_{mnk})$$

will intersect along the skew quadrilateral that is defined by the lines Δ_{ij}, Δ_{ji} , Δ_{mn}, Δ_{nm} .

If a congruence C_{ij} has no common index with a semi-quadric Q_{lmn} then it will have two common indices with the complementary semi-quadric Q_{ijk} and its directrices will be carried by that semi-quadric. They will thus be traced on the quadric:

$$(Q_{ijk},\, Q_{lmn}).$$

Therefore, *in order for a congruence to have its directrices on a fundamental quadric, it is necessary and sufficient that it have two or zero indices in common with one or the other of the semi-quadrics that constitute the proposed fundamental quadric.*

However, it can happen that the congruence C_{ij} has an index in common with each of its semi-quadrics:

$$Q_{ikl}\,,\; Q_{lmn}\;.$$

One can prove that *in this case, the lines Δ_{ij}, Δ_{ji} will be conjugate with respect to the proposed fundamental quadric:*

$$(Q_{ikl},\, Q_{lmn}).$$

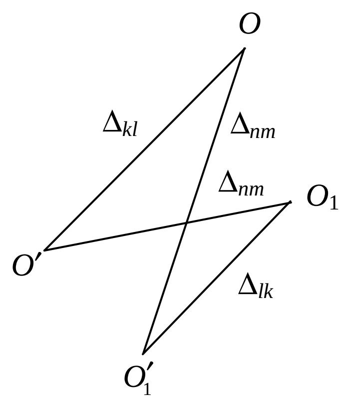

Figure 2.

Indeed, Δ_{mn}, Δ_{nm}, Δ_{kl}, Δ_{lk} define a skew quadrilateral on that quadric. Δ_{kl} cuts Δ_{mn} and Δ_{nm} at two points O, O', and Δ_{lk} cuts these same two lines at O_1, O'_1. The line Δ_{ij} cuts the three lines Δ_{kl}, Δ_{lk}, Δ_{mn}, Δ_{nm}. Therefore, since Δ_{ij} is not traced on the quadric, it is necessary that the points where they pierce that surface be two of the four points O, O', O_1, O'_1; they can be only O, O_1 or O', O'_1, and similarly, for Δ_{ji}. Therefore, the lines Δ_{ij}, Δ_{ji} will be precisely the ones that join O and O_1, O' and O'_1, resp. They will thus be the diagonals of the skew quadrilateral. Consequently, they will indeed be conjugate with respect to the proposed quadric.

However, our reasoning proves some other things.

The lines Δ_{ij}, Δ_{ji}, Δ_{kl}, Δ_{lk}, Δ_{mn}, Δ_{nm} will be the edges of a tetrahedron.

Therefore:

The directrices of three congruences C_{ij}, C_{kl}, C_{mn} with no common indices define a tetrahedron.

I shall denote that tetrahedron by $T(ij, kl, mn)$.

One can give this fact another proof.

I recall that the special complex whose directrix is $\Delta_{\rho\sigma}$ has the equation:

$$\sqrt{-1}\; x_\rho + x_\sigma = 0.$$

Consequently, set, in a general manner:

$$Z_{\rho\sigma} = \sqrt{-1}\; x_\rho + x_\sigma .$$

One will have:

$$Z_{ij}\, Z_{ji} + Z_{kl}\, Z_{lk} + Z_{mn}\, Z_{nm}$$

$$= (\sqrt{-1}\; x_i + x_j)(\sqrt{-1}\; x_j + x_i) + (\sqrt{-1}\; x_k + x_l)(\sqrt{-1}\; x_l + x_k) + (\sqrt{-1}\; x_m + x_n)(\sqrt{-1}\; x_n + x_m)$$
$$= -(x_i\, x_j + x_k\, x_l + x_m\, x_n) + \sqrt{-1}(x_i^2 + x_j^2 + x_k^2 + x_l^2 + x_m^2 + x_n^2) + (x_i\, x_j + x_k\, x_l + x_m\, x_n)$$
$$= \sqrt{-1}(x_i^2 + x_j^2 + x_k^2 + x_l^2 + x_m^2 + x_n^2).$$

The transformation formulas:

$$\begin{aligned} Z_{ij} &= \sqrt{-1}\; x_i + x_j , \\ Z_{ji} &= \sqrt{-1}\; x_j + x_i , \\ Z_{kl} &= \sqrt{-1}\; x_k + x_l , \\ Z_{lk} &= \sqrt{-1}\; x_l + x_k , \\ Z_{mn} &= \sqrt{-1}\; x_m + x_n , \\ Z_{nm} &= \sqrt{-1}\; x_n + x_m \end{aligned}$$

thus associate the quadratic form $\omega(x)$ with the *tetrahedral* form:

$$Z_{ij}\, Z_{ji} + Z_{kl}\, Z_{lk} + Z_{mn}\, Z_{nm} ,$$

and in turn, conforming to no. 74, the axes of the six special complexes:

$$Z_{ij} = 0, \quad Z_{ji} = 0, \quad Z_{kl} = 0, \quad Z_{lk} = 0, \quad Z_{mn} = 0, \quad Z_{nm} = 0,$$

i.e., the directrices of the congruences C_{ij}, C_{kl}, C_{mn} – form a tetrahedron.

79. Arrange the directrices of these congruences into the matrix below:

$$\begin{array}{ccc} \Delta_{ij}, & \Delta_{kl}, & \Delta_{mn}, \\ \Delta_{ji}, & \Delta_{lk}, & \Delta_{nm}. \end{array}$$

It is clear that all of the pairs of elements from different rows in this matrix will intersect, except for the ones that are in the same column, which then constitute pairs of opposite edges of the tetrahedron $T(ij, kl, mn)$, precisely.

By grouping the elements of the preceding matrix by threes, but without ever taking two of them from the same column, one can proceed in several manners. Once take all three of them from the first row, or two from the first and one from the second, or one from the first and two from the second, or even finally all three of them from the second row. We thus obtain eight different groups of three lines that intersect pair-wise and consequently define either a trihedron or a triangle.

In this way, we will realize the four trihedra and the four triangles whose faces are on our tetrahedron $T(ij, kl, mn)$.

Suppose, to fix ideas, that the lines that are placed in the first row:

$$(\Delta_{ij}, \Delta_{kl}, \Delta_{mn})$$

define a trihedron with summit O. The other three lines, namely, the ones in the second row:

$$(\Delta_{ji}, \Delta_{lk}, \Delta_{nm}),$$

will obviously define a triangle that constitutes the opposite face to the concurrence point of the first three edges.

If we now replace one of the lines in the symbol:

$$(\Delta_{ij}, \Delta_{kl}, \Delta_{mn}),$$

for example, Δ_{mn} – with the line Δ_{nm} in the second row, which is placed beneath it, then we will get three lines:

$$(\Delta_{ij}, \Delta_{kl}, \Delta_{nm})$$

that define one of the faces that meet at the point O.

One thus sees that one will obtain the four faces of the tetrahedron by taking an odd number (viz., 1 or 3) of lines from the second row and an even number (viz., 2 or 0) from the first one.

On the contrary, one will get the four trihedra of the tetrahedron by taking an odd number (1 or 3) of lines from the first row and an even number (2 or 0) from the second.

Therefore, if the lines:

$$(\Delta_{ij}, \Delta_{kl}, \Delta_{mn})$$

form a trihedron then the same thing will be true for the triples of lines:

$$(\Delta_{ij}, \Delta_{kl}, \Delta_{nm}),$$
$$(\Delta_{ji}, \Delta_{kl}, \Delta_{nm}),$$
$$(\Delta_{ji}, \Delta_{lk}, \Delta_{mn}),$$

while the triples of lines:

$$(\Delta_{ji}, \Delta_{lk}, \Delta_{nm}),$$
$$(\Delta_{ji}, \Delta_{kl}, \Delta_{mn}),$$

$$(\Delta_{ij}, \Delta_{lk}, \Delta_{mn}),$$
$$(\Delta_{ij}, \Delta_{kl}, \Delta_{nm})$$

will form triangles.

One can summarize these facts in a very laconic statement:

Let there be a triple of lines:

$$(\Delta_{ij}, \Delta_{kl}, \Delta_{mn}).$$

These lines will form a trihedron or a triangle. If one permutes the two indices in one of these lines then one will again have three lines that intersect pair-wise and again form a trihedron or a triangle, *and only the type of the configuration will be changed – i.e., if the first triple forms a triangle then the new one will form a trihedron, and conversely.*

We shall call the tetrahedra $T(ij, kl, mn)$ *fundamental*. There are fifteen of these tetrahedra. Indeed, each of them is characterized by a distribution of three pairs:

$$(ij), \quad (kl), \quad (mn)$$

of indices 1, 2, …, 6. The order of these pairs matters little, as well as the order of the indices within a pair.

Observe that the index 1 figures in one of these pairs. Let $i = 1$, so j can be 2, 3, 4, 5, or 6, which already gives us five classes of groupings. If the index that is associated with the index 1 has been chosen then it remains for one to distribute the other four indices into two pairs. The number of possible dispositions is equal to one-half the number of combination of four objects taken two at a time; i.e., $\frac{1}{2} \cdot \frac{4 \cdot 3}{2} = 3$. Each class is then comprised of three dispositions. There are five classes, so there will be $3 \times 5 = 15$ tetrahedra.

Consider a directrix Δ_{ij} of the congruence C_{ij}. This directrix is cut by the directrices of the congruences:

$$C_{kl} \quad \text{and} \quad C_{mn}, \qquad C_{km} \quad \text{and} \quad C_{ln}, \qquad C_{kn} \quad \text{and} \quad C_{lm}.$$

We have grouped these congruences in pairs, because the directrices of C_{kl} and C_{mn}, for example, cut Δ_{ij} at the same two points. On each line Δ_{ij} we will then have three pairs of summits of fundamental tetrahedra. *These three pairs, when taken two at a time, will be harmonically related.*

For example, the two points where Δ_{kl}, Δ_{lk} cut define a harmonic proportion with the ones where Δ_{ij} is cut by Δ_{km} and Δ_{mk}. Indeed, the first two points are two summits of the tetrahedron $T(ij, kl, mn)$, and the other two are the ones where the edge Δ_{ij}, which carries them, pierce the quadric:

$$(Q_{ikl}, Q_{jmn}).$$

Therefore, since the tetrahedron $T(ij, kl, mn)$ is conjugate with respect to that conic, the harmonic property is indeed true.

80. We have seen that the tetrahedron $T(ij, kl, mn)$ has two of its pairs of opposite edges Δ_{kl}, Δ_{lk}, Δ_{mn}, Δ_{nm} on the quadric:

$$(Q_{ikl}, Q_{jmn}),$$

while the opposite edges Δ_{ij}, Δ_{ji} are conjugate with respect to that quadric.

One formed that quadric by means of the grouping of indices into three pairs:

$$ij, \quad kl, \quad mn,$$

by taking Q_{ikl} to be one index in one of these pairs (e.g., the index i), two from another (k and l), and none from the last one (m and n). Q_{jmn} is likewise formed by taking one index from one of the two pairs, two from a second one, and none from the third.

There are obviously six fundamental quadrics that thus each contain two pairs of opposite edges of $T(ij, kl, mn)$; all the pairs of opposite edges, except for Δ_{ij}, Δ_{ji}, are contained in the two quadrics:

$$(Q_{ikl}, Q_{jmn}), \qquad (Q_{lmn}, Q_{jkl}).$$

(ditto), except for Δ_{kl}, Δ_{lk}, the quadrics:

$$(Q_{kij}, Q_{lmn}), \qquad (Q_{kmn}, Q_{ilj}).$$

(ditto), except for Δ_{mn}, Δ_{nm}, the quadrics:

$$(Q_{mij}, Q_{nki}), \qquad (Q_{mkl}, Q_{nij}).$$

Four other fundamental quadrics remain that contain no edge of the proposed tetrahedron. They are the quadrics that one obtains by taking each of the component semi-quadrics to be one (and only one) of the three indices in each of the pairs:

$$ij, \quad kl, \quad mn\,.$$

One thus finds the fundamental quadrics:

$$\begin{array}{l}(Q_{ikm}\,,\,Q_{jlm}),\\(Q_{ilm}\,,\,Q_{jkm}),\\(Q_{ikn}\,,\,Q_{jlm}),\\(Q_{ilm}\,,\,Q_{jkm}).\end{array}$$

These four quadrics admit the tetrahedron $T(ij, kl, mn)$ as a common conjugate tetrahedron.

Indeed, we have seen that if a fundamental quadric is given – for example:

$$(Q_{ikm}\,,\,Q_{jlm}),$$

then any congruence that has a common index with its two component semi-quadrics Q_{ikm}, Q_{jlm} – for example, C_{ij} – will have its two directrices conjugate with respect to the quadric.

Therefore, in regard to the mode of formation of our four quadrics, precisely, one sees that the directrices of C_{ij}, C_{kl}, C_{mn} form many pairs of conjugate lines that are common to these four quadrics. Since these lines form the tetrahedron $T(ij, kl, mn)$, one indeed sees that this tetrahedron will be conjugate with respect to all of these four quadrics at once.

One can attach this property to another one that concerns three linear complexes in involution.

Let C_i, C_j, C_k be three linear complexes that are pair-wise in involution. Let O be a point of space, let π_i, π_j, π_k be its polar planes in the three complexes, and let α_{ij}, α_{jk}, α_{ik} be the intersections of these planes.

On the line α_{ij} one finds the point O_k, which is the pole of π_i in the complex C_j, and on α_{ik} one finds the point O_j, which is the pole of π_i in the complex C_k.

Since O and O_k are poles of the same plane π_i in C_i and C_j, respectively, it results that (in regard to involution) O_k and O will be poles of the same plane on these two complexes C_i and C_j, respectively (Fig. 3).

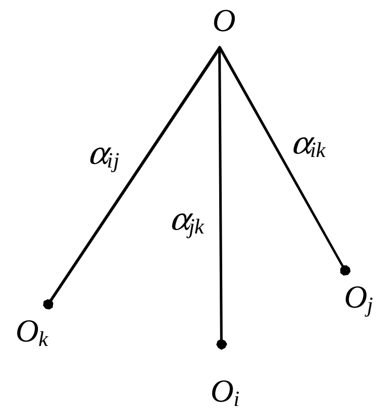

Figure 3.

Since π_j is the polar plane to O in C_j, one sees that π_j will be the polar plane to O_k in C_i. Therefore, O_k is the pole of π_i in C_j and of π_j in C_i. Similarly, the point O_j is the pole of π_k in C_i and of π_i in C_k.

One likewise confirms that there exists a point O_i on α_{jk} that is at the same time the pole of π_k in C_j and of π_j in C_k. Finally, the plane π of the points O_i, O_j, O_k is the pole of these points in each of the complexes C_i, C_j, C_k, respectively.

Indeed, we shall prove by example that O_i is the pole of the plane π in the complex C_i. It will suffice to prove that $O_i\,O_k$ and $O_i\,O_j$ are two lines of the complex. Now, in effect, $O_i\,O_k$ issues from the point O_k in the plane π_j that is polar to O_k in C_i, and $O_i\,O_j$ issues from O_j in the plane π_k that is polar to O_j in C_i.

We have thus formed a tetrahedron such that each plane of the faces admits precisely the three summits that it contains as poles in the three complexes.

The law of distribution of the poles and polar planes gives rise to the following table:

	O	O_i	O_j	O_k
π	*	C_i	C_j	C_k
π_i	C_i	*	C_k	C_j
π_j	C_j	C_k	*	C_i
π_k	C_k	C_j	C_i	*

Take a plane in the left-hand column and a summit in the row above – for example, π_i and O_j . At the intersection of the row π_i and the column O_j , one will finds C_k . That will be the complex, with respect to which, π_i and O_j are conjugate.

Had I taken π_i and O_i , I would have arrived at a vacuous case. Indeed, O_i would have to be the summit that is opposite to π_i , and in turn, that point and that plane cannot be conjugate to any complex.

Let Δ_{ij}, Δ_{ji} be the directrices of the congruence C_{ij} that is common to the complexes C_i and C_j . These lines cut OO_k and O_iO_j , because the latter lines belong to both of the complexes C_i and C_j at once. However, there is more: Since O_i and O_j are the poles of the same plane π in the complexes C_i and C_j , respectively, one will see that the line segment O_iO_j is divided harmonically by the lines Δ_{ij}, Δ_{ji} , by virtue of the properties of complexes in involution that were proved already.

Now, imagine the semi-quadric:

$$Q_{ijk}$$

that is common to the complexes C_i , C_j , C_k .The complementary semi-quadric obviously contains Δ_{ij}, Δ_{ji} . Therefore, the points where the line O_iO_j cuts Δ_{ij}, Δ_{ji} are also the ones where it cuts the quadric that carries the semi-quadric Q_{ijk} . The same reasoning is then applied to the other edges of the tetrahedron. One then sees that:

The tetrahedron $OO_iO_jO_k$ is conjugate with respect to the quadric that carries the semi-quadric Q_{ijk} .

For example, take the tetrahedron:

$$T(ij, kl, mn)$$

and one of the four fundamental quadrics that were considered already; e.g.:

$$(Q_{lkm}, Q_{jin}).$$

The tetrahedron $T(ij, kl, mn)$ is such that the planes of its faces have the three summits that are situated in each of these planes for poles with respect to the complexes C_i, C_j, C_k .

Indeed, assume that Δ_{ij}, Δ_{kl}, Δ_{mn} form a triangle. Call the plane of that triangle π, so any line that issues from the point $(\Delta_{ij}, \Delta_{kl})$ (viz., the intersection of Δ_{ij} and Δ_{kl}) and is traced in the plane π will cut Δ_{mn} and Δ_{nm} . It will thus belongs to C_{mn} and, in turn, to C_{nm}. Therefore, the point (Δ_{ij}, Δ_{kl}) will be the pole of π in C_m . Similarly, the point (Δ_{ij}, Δ_{mn}) will be the pole of π in C_k , and finally, (Δ_{mn}, Δ_{kl}) will be the pole of π in C_i . In the case of the tetrahedron $OO_iO_jO_k$ that was just discussed, the tetrahedron T(ij, kl, mn) will therefore indeed be conjugate with respect to the quadric that carries Q_{ikm} and Q_{jln} .

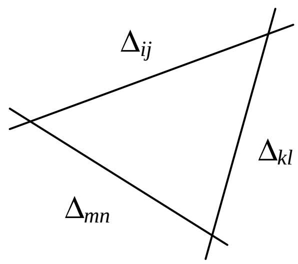

Figure 4.

We complete the discussion of the set of these four quadrics that admit the tetrahedron T(ij, kl, mn) as their conjugate by proving an interesting property in regard to them:

Each of these quadrics is its own proper reciprocal polar with respect to any of the other three.

One first observes that these four quadrics intersect pair-wise along four lines. Take any two of them; e.g.:

$$(Q_{ikm}, Q_{jln}),$$
$$(Q_{ikm}, Q_{jln}).$$

They will intersect along the directrices of C_{im} and C_{jn} , which form a tetrahedron with Δ_{kl} and Δ_{lk} .

Therefore, Δ_{kl} and Δ_{lk} will intersect these two quadrics at the same points. Then, take one of the other four edges of the tetrahedron T(ij, kl, mn) – for example, Δ_{ij} – so the two line segments that these two quadrics determine on Δ_{ij} will be the ones that determine the directrices of the congruences C_{km} and C_{kn} . From the remark that concluded no. 79, these two pairs of points will be harmonically related.

Therefore, there are two quadrics Q, Q' that have a common conjugate tetrahedron T(ij, kl, mn) that intersect along four lines that form a skew quadrilateral whose diagonals are Δ_{kl} , Δ_{lk} and which ultimately determines some pairs of harmonically-related line segments on the other four edges of the tetrahedron that one envisions.

When referred to the common conjugate tetrahedron, the equation of Q will be:

$$X^2 + Y^2 + Z^2 + T^2 = 0,$$

and if $X = 0$, $Y = 0$ are the equations of Δ_{kl} then those of Δ_{lk} will be:

$$Z = 0, \quad T = 0.$$

The harmonic properties that were established then prove that Q' will have an equation of the form:

$$X^2 + Y^2 - Z^2 - T^2 = 0,$$

and one thus indeed recognizes that the two quadrics will be the proper reciprocal polars of each other.

81. In order to represent the configuration of lines Δ_{ij} more completely, one can introduce a symbol that exhibits an interesting correspondence between the properties of that configuration and those of the permutations of six letters.

Three lines Δ_{ij} , Δ_{kl} , Δ_{mn} with no common index will always form a hyper-sheaf (viz., a triangle or trihedron).

I shall represent that hyper-sheaf by the notation:

$$(ij, kl, mn).$$

There are as many symbols of that form as there are permutations of six letters, namely, 720. However, I observe that one can permute the pairs of indices *ij, kl, mn* without the symbol ceasing to apply to the set of three lines Δ_{ij} , Δ_{kl} , Δ_{mn} . The six permutations:

$$\begin{array}{lll} (ij, kl, mn), & (kl, ij, mn), & (mn, kl, ij), \\ (ij, mn, kl), & (mn, ij, kl), & (kl, mn, ij), \end{array}$$

are applied to the same three lines. We will then have, in reality, 720 / 6 = 120 hyper-sheaves. Sixty of them are sprays [*gerbes*] (viz., the summits of the fundamental tetrahedra). Sixty of them are planes (viz., the faces of the tetrahedra).

We shall now establish a rule for distinguishing them.

From what was said in no. 79, it first results that if one permutes i and j, or k and l, or even m and n in the symbol:

$$(ij, kl, mn)$$

then the nature of the hyper-sheaf will change.

One can similarly add that upon performing several of these permutations, one obtains eight hyper-sheaves, four of which will be the summits of the fundamental tetrahedron:

$$T(ij, kl, mn),$$

while the other four will be the faces of that same tetrahedron.

I now add that *for any* two indices that one permutes in the symbol (ij, kl, mn) the hyper-sheaf that it represents will *always* change in nature.

Since the permutation of two arbitrary indices results from an odd number of permutations of successive indices (*see*, the theory of determinants), and since that fact has already been established for the two indices of the same pair *ij, kl, mn*, it will suffice to establish that it is true for two consecutive indices of two different pairs – for example, m and l.

Therefore, one considers the hyper-sheaf:

$$(ij,\ km,\ ln),$$

and establishes that it is of a different type then that of the original hypersheaf:

$$(ij,\ kl,\ mn).$$

One then seeks to determine whether these hyper-sheaves have any common lines.

The hyper-sheaf (ij, kl, mn) is formed from lines that intersect Δ_{ij}, Δ_{kl}, Δ_{mn}, so the lines of that hyper-sheaf will therefore verify the equations:

$$\begin{aligned} Z_{ij} &= x_i\sqrt{-1} + x_j = 0, \\ Z_{kl} &= x_k\sqrt{-1} + x_l = 0, \\ Z_{mn} &= x_m\sqrt{-1} + x_n = 0; \end{aligned}$$

i.e.:

(8) $$\frac{x_i}{\sqrt{-1}} = x_j\,, \qquad \frac{x_k}{\sqrt{-1}} = x_l\,, \qquad \frac{x_m}{\sqrt{-1}} = x_n\,.$$

Analogously, the hyper-sheaf (ij, km, ln) will be defined by the equations:

(8) $$\frac{x_i}{\sqrt{-1}} = x_j\,, \qquad \frac{x_k}{\sqrt{-1}} = x_m\,, \qquad \frac{x_l}{\sqrt{-1}} = x_n\,.$$

The set of equations (8) and (9) will then reduce to:

(10) $$\frac{x_i}{\sqrt{-1}} = \frac{x_j}{1}, \qquad \frac{x_k}{-1} = \frac{x_l}{\sqrt{-1}} = \frac{x_m}{\sqrt{-1}} = \frac{x_n}{1}.$$

The ratio $\frac{x_j}{x_i}$ is uniquely-defined, and similarly the ratios $\frac{x_l}{x_n}, \frac{x_m}{x_n}, \frac{x_n}{x_k}$ are, as well, but the ratio $\frac{x_k}{x_i}$ remains arbitrary. Moreover, our two hyper-sheaves have a planar sheaf of lines in common. This obviously demands that they be of opposite types, and furthermore, it is necessary that these hyper-sheaves be *united*; i.e., that the one that is a spray has its summit in the plane of the one that consists of a planar system of lines. One can therefore affirm that if one permutes two *arbitrary* consecutive indices in the symbol:

$$(ij,\ kl,\ mn)$$

then the hyper-sheaf that it represents will change type.

Moreover, as one does in the theory of determinants, one counts the number of inversions that are presented by the permutation:

$$(ij, kl, mn).$$

Since the permutation of two consecutive indices will change the parity of the number of inversions, one can say this:

Two symbols represent two hyper-sheaves of the same or different type, according to whether the two numbers of inversions that they present are of the same or opposite parity, respectively.

It is indeed evident that nothing will indicate *a priori* what type of parity applies to the sprays or planar systems, but *it will suffice that the choice be fixed in a symbol* – for example, in:

$$(1, 2, 3, 4, 5, 6),$$

in order for one to know what it is for all of the other ones. Therefore, if the preceding symbol agrees with a spray, since it contains zero inversions, then all of the symbols that contain an even number of inversions will agree with sprays. On the contrary, the planar systems will agree with the ones that present an odd number.

We believe that the introduction of that representation into the study of the configuration of the fundamental system will shed some light on it, since it defines a link between that configuration and the system of permutations that one can define with six distinct indices.

82. The sixty summits and the sixty faces of the fifteen fundamental tetrahedra present a remarkable grouping.

Consider the tetrahedra:

$$T(ij, kl, mn), \quad T(ij, km, ln), \quad T(ij, kn, ln)$$

that one obtains by grouping the indices k, l, m, n into two pairs in all possible ways (which gives three such dispositions). These three tetrahedra will obviously have the pair of opposite edges Δ_{ij}, Δ_{ji} in common.

For example, there are thus three pairs of summits on Δ_{ij} that each belong to one of the three tetrahedra. These pairs will be pair-wise harmonic with respect to each other. Indeed, the trihedron $T(ij, kl, mn)$ is conjugate with respect to the quadric:

$$Q = Q(Q_{ikl}, Q_{jin}),$$

and that will contain two pairs of opposite edges of the tetrahedron $T(ij, km, ln)$.

The line Δ_{ij} cuts that quadric Q at two points, which are the two summits of $T(ij, km, ln)$ that are carried by Δ_{ij}, precisely. These points thus divide the edge Δ_{ij} of the first tetrahedron $T(ij, kl, mn)$ harmonically.

One will likewise prove that the three pairs of faces of the three tetrahedra:

$$T(ij, kl, mn), \quad T(ij, km, ln), \quad T(ij, kn, lm)$$

that pass through Δ_{ij} will divide it harmonically.

Suppose that the lines Δ_{ij}, Δ_{kl}, Δ_{mn} define a trihedron; we call their point of concurrence O. The trihedron of these lines belongs to the tetrahedron $T(ij, kl, mn)$. Through the edge Δ_{ij}, in addition to the faces of the tetrahedron, there will pass a pair of faces of the tetrahedron $T(ij, km, ln)$ and a pair of faces of the tetrahedron $T(ij, kn, lm)$.

The symbols of these faces are easy to define.

First, let:

$$(ij, kl, mn)$$

be the symbol of the trihedron of lines Δ_{ij}, Δ_{kl}, Δ_{mn}, so the symbols:

$$(ij, km, ln),$$
$$(ij, mk, nl)$$

will be those of the two faces of the tetrahedron $T(ij, kl, mn)$ that contain Δ_{ij}. Analogously:

$$(ij, kn, ml),$$
$$(ij, nk, lm)$$

will be the symbols of the two faces of the tetrahedron $T(ij, kn, ml)$ that contain Δ_{ij}.

The parity rule for permutations that was given in the preceding number permits one to define these four symbols with no hesitation.

Similarly, relative to Δ_{kl}, we will have:

$$\left.\begin{array}{c}(kl, im, jn)\\(kl, mi, nj)\end{array}\right\} \text{ faces of } T(kl, im, jn) \text{ that go through } \Delta_{kl},$$

$$\left.\begin{array}{c}(kl, in, mj)\\(kl, ni, jm)\end{array}\right\} \text{ faces of } T(kl, in, mj) \text{ that go through } \Delta_{kl},$$

and finally, relative to Δ_{mn}:

$$\left.\begin{array}{c}(mn, ki, lj)\\(mn, ik, jl)\end{array}\right\} \text{ faces of } T(mn, ki, lj) \text{ that go through } \Delta_{mn},$$

$$\left.\begin{array}{c}(mn, kj, il)\\(mn, jk, li)\end{array}\right\} \text{ faces of } T(mn, kj, il) \text{ that go through } \Delta_{mn}.$$

We thus see that through any summit O of a fundamental tetrahedron $T(ij, kl, mn)$ there will pass, not only three faces of that tetrahedron, but also twelve faces of the other six fundamental tetrahedra, each of which has a pair of common opposite edges with the proposed tetrahedron.

I add that *these twelve faces will intersect along sixteen lines* that issue from O; i.e., that any plane among the four that are drawn through Δ_{ij}, and any plane among the four that are drawn through Δ_{kl} will intersect in one of the four planes that are drawn through Δ_{mn}.

For example, take the face:

$$(ij, km, ln)$$

that is drawn through Δ_{ij}, and associate it with the four that are drawn through Δ_{kl}.

We can define the four groups:

$$\text{I.} \qquad \left\{ \begin{array}{l} (ij, km, ln), \\ (kl, im, jn), \\ (mn, kj, il); \end{array} \right.$$

$$\text{II.} \qquad \left\{ \begin{array}{l} (ij, km, ln), \\ (kl, nj, mi), \\ (mn, li, jk); \end{array} \right.$$

$$\text{III.} \qquad \left\{ \begin{array}{l} (ij, km, ln), \\ (kl, mj, in), \\ (mn, lj, ki); \end{array} \right.$$

$$\text{IV.} \qquad \left\{ \begin{array}{l} (ij, km, ln), \\ (kl, ni, jm), \\ (mn, ik, jl). \end{array} \right.$$

Among these groups, the first plane is always that of the face (ij, km, ln) that is drawn through Δ_{ij}. The second plane is one of the four that are drawn through Δ_{kl}. As for the third plane in each group, it is one of the four that are drawn through Δ_{mn}. In each group, the third plane will depend upon the first two.

It is easy to confirm that the three planes of the same pair have a line in common. Here is the representation of these lines for each of the four triples of faces above:

$$\text{I.} \qquad \frac{x_i}{-1} = \frac{x_j}{\sqrt{-1}} = \frac{x_k}{-1} = \frac{x_l}{\sqrt{-1}} = \frac{x_m}{\sqrt{-1}} = \frac{x_n}{1},$$

$$\text{II.} \qquad \frac{x_i}{+1} = \frac{x_j}{-\sqrt{-1}} = \frac{x_k}{-1} = \frac{x_l}{\sqrt{-1}} = \frac{x_m}{\sqrt{-1}} = \frac{x_n}{1},$$

III. $$\frac{x_i}{\sqrt{-1}} = \frac{x_j}{1} = \frac{x_k}{-1} = \frac{x_l}{\sqrt{-1}} = \frac{x_m}{\sqrt{-1}} = \frac{x_n}{1},$$

IV. $$\frac{x_i}{-\sqrt{-1}} = \frac{x_j}{-1} = \frac{x_k}{-1} = \frac{x_l}{\sqrt{-1}} = \frac{x_m}{\sqrt{-1}} = \frac{x_n}{1}.$$

We get four lines on the face (*ij*, *km*, *ln*) that is drawn through Δ_{ij} in this way, and in turn, there are indeed sixteen of these lines around *O*.

One can give a regular process for defining the coordinates of these lines.

Observe that if one starts with the spray:

$$(ij, kl, mn)$$

then these sixteen lines will, *a fortiori*, verify the equations:

$$\frac{x_i}{\sqrt{-1}} = \frac{x_j}{1}, \qquad \frac{x_k}{\sqrt{-1}} = \frac{x_l}{1}, \qquad \frac{x_m}{\sqrt{-1}} = \frac{x_n}{1},$$

which one can presently write by simply reading the symbol (*ij*, *kl*, *mn*).

Each of the lines of the spray will thus be defined by a system of values of the two ratios:

$$\frac{x_j}{x_n}, \qquad \frac{x_l}{x_n}.$$

Now, these ratios can only be +1, –1, $\sqrt{-1}$, $-\sqrt{-1}$, because in all of the hyper-sheaves that we shall consider the equations that we will have to write will always be of the form:

$$x_\alpha = \varepsilon \cdot x_\beta,$$

where ε is equal to one of the four quantities above, and since the multiplication or division will only permute these quantities, one indeed sees that:

$$\frac{x_j}{x_n}, \qquad \frac{x_l}{x_n}.$$

can only be +1, –1, $\sqrt{-1}$, $-\sqrt{-1}$.

That will then give us sixteen possible combinations, and since we have sixteen lines, these sixteen combinations will all be realizable.

In that way, one will obtain the sixteen lines of the spray by taking $x_1, x_2, \ldots, x_6$ to be proportional to +1, –1, $\sqrt{-1}$, $-\sqrt{-1}$ in all possible ways, *such that the equations of the spray:*

$$\frac{x_i}{\sqrt{-1}} = \frac{x_j}{1}, \qquad \frac{x_k}{\sqrt{-1}} = \frac{x_l}{1}, \qquad \frac{x_m}{\sqrt{-1}} = \frac{x_n}{1}$$

will nonetheless be preserved.

One observes that the ratios of the coordinates x_k, x_l, x_m, x_n will be the same for the four lines that are situated in *the same* plane through Δ_{ij} .

I will call the lines that we just defined Ξ.

We will get an analogous result if the symbol:

$$(ij, kl, mn)$$

is consistent with a planar system. There will then be twelve summits of the fundamental tetrahedra that are situated in that plane, in addition to the three summits of the face, and these summits will be distributed with four of them on the edges of that face. Moreover, there will be groups of three on sixteen lines Ξ′, whose analytical representation will be the same as that of the sixteen lines Ξ.

However, there is more: *These new lines* Ξ′ *that we just obtained do not define a set that is different from the one that is defined by the lines* Ξ.

For example, consider the line that is common to the three planar systems:

$$(ij, km, ln),$$
$$(kl, im, jn),$$
$$(mn, kj, il),$$

which has the coordinates:

$$\frac{x_i}{-1} = \frac{x_j}{\sqrt{-1}} = \frac{x_k}{-1} = \frac{x_l}{\sqrt{-1}} = \frac{x_m}{\sqrt{-1}} = \frac{x_n}{1}.$$

It already passes through the summit of the spray:

$$(ij, kl, mn).$$

One confirms that it also passes through the sprays:

$$(il, km, jn).$$
$$(im, kj, ln).$$

Therefore, any line Ξ that is common to three planes of the faces will serve as a junction of the three summits ([9]).

([9]) One glimpses the possibility of establishing a complete correspondence between the groups of permutations of six letters and the properties of the configuration of the fundamental system. Here, I will content myself by giving some general indications, while reserving the more fundamental development of these new remarks for a special effort.

The number of these lines is, moreover, very easy to evaluate. There are 60 summits that each carry 16 lines Ξ. However, since each line is counted three times in this enumeration (since each of them contains three summits) there will be:

$$\frac{60 \cdot 16}{3} = 320$$

lines Ξ.

83. One can join the 60 points pair-wise in a number of ways, namely:

$$\frac{60 \cdot 59}{2} = 1770$$

ways.

However, each of the lines Ξ that are represented has just three lines that are joined pair-wise at the summits, namely, 960 lines.

There will thus remain 1770 – 960 = 810 lines that are joined pair-wise.

Now, take the edges Δ_{ij}. There are six summits of the tetrahedron on them. Each of them thus represents a number of lines that are joined pair-wise, namely:

$$\frac{5 \cdot 6}{2} = 15,$$

and since there are 30 of these edges, that would make $30 \times 15 = 450$ joined lines. What then remains are:

$$810 - 450 = 360$$

lines, which are neither edges nor lines Ξ, and which join the summits pair-wise.

It is easy to see how one can obtain 360 new lines, which I denote by Ξ_0.

Take the summit (ij, kl, mn), which is the intersection of the lines $\Delta_{ij}, \Delta_{kl}, \Delta_{mn}$.

There are six tetrahedra:

$$\begin{array}{ll} T(ij, kl, mn), & T(ij, kn, ln), \\ T(kl, im, jn), & T(kl, in, jm), \\ T(mn, ik, jl), & T(mn, il, jk), \end{array}$$

which each have a pair of edges in common with the tetrahedron $T(ij, kl, mn)$.

The first two each have two summits on Δ_{ji}, which makes four of them, and similarly the second two give four on Δ_{lk}, and the last give four on Δ_{mn}. In all: $3 \times 4 = 12$ points.

Having said that, join the summit:

$$(ij, kl, mn)$$

of the tetrahedron $T(ij, kl, mn)$ to these twelve summits.

We will thus have twelve lines Ξ_0, and we will have all of them in this way, because the number of lines thus obtained will be equal to exactly:

$$\frac{60 \times 12}{2} = 360.$$

One thus obtains the lines Ξ_0 by joining the summit of a tetrahedron $T(ij, kl, mn)$ that is taken from an edge Δ_{ij} to a summit that is taken from the opposite edge Δ_{ji} of another fundamental tetrahedron that is subject to having Δ_{ij} and Δ_{ji} for opposite edges.

One should not fail to observe that any line Ξ_0 is also the intersection of two planes of the faces of the tetrahedron that was considered above.

We seek to represent the lines Ξ_0 . In order to do this, we shall take a hyper-sheaf:

$$(ij, kl, mn);$$

say, a spray, to be precise. Take a summit of one of the tetrahedra $T(ij, kl, mn)$, $T(ij, km, ln)$ on Δ_{ji} and join it to the summit of the proposed spray.

The sprays of the tetrahedron $T(ij, kn, lm)$ that contains Δ_{ij} are the following two:

$$(ji, ml, kn),$$
$$(ji, lm, nk).$$

Similarly, the sprays of the tetrahedron $T(ij, km, ln)$ that contains Δ_{ij} are:

$$(ji, km, ln),$$
$$(ji, mk, nl).$$

Some very simple calculations give:

The line that is common to the sprays (ij, kl, mn), (ji, ml, kn):

$$\frac{x_i}{0} = \frac{x_j}{0} = \frac{x_k}{\sqrt{-1}} = \frac{x_l}{1} = \frac{x_m}{\sqrt{-1}} = \frac{x_n}{1}.$$

The line that is common to the sprays (ij, kl, mn), (ji, lm, nk):

$$\frac{x_i}{0} = \frac{x_j}{0} = \frac{x_k}{\sqrt{-1}} = \frac{x_l}{1} = \frac{x_m}{-\sqrt{-1}} = \frac{x_n}{-1}.$$

The line that is common to the sprays (ij, kl, mn), (ji, km, ln):

$$\frac{x_i}{0} = \frac{x_j}{0} = \frac{x_k}{\sqrt{-1}} = \frac{x_l}{1} = \frac{x_m}{1} = \frac{x_n}{-\sqrt{-1}}.$$

The line that is common to the sprays (ij, kl, mn), (ji, km, ln):

$$\frac{x_i}{0} = \frac{x_j}{0} = \frac{x_k}{\sqrt{-1}} = \frac{x_l}{1} = \frac{x_m}{-1} = \frac{x_n}{\sqrt{-1}}.$$

One easily sees that all of the lines Ξ_0 are obtained by annulling two of the coordinates and taking the other four to be proportional to one of the four quantities +1, –1, $+\sqrt{-1}$, $-\sqrt{-1}$ in all possible ways, in such a way that if $x_i = 0$, $x_j = 0$ then one will nonetheless have two relations of the form:

$$\frac{x_k}{\sqrt{-1}} = x_l\,, \qquad \frac{x_m}{\sqrt{-1}} = x_n\,.$$

84. We cannot leave this subject without exhibiting a very curious property of fundamental tetrahedra.

Take a summit (*ij*, *kl*, *mn*) of the tetrahedron *T*(*ij*, *kl*, *mn*). There are eight tetrahedra that have no edge in common with it. Take one of these tetrahedra – for example:

$$T(ik, jm, ln),$$

and join the point (*ij*, *kl*, *mn*) to the summits of that tetrahedron. We will then have four of the sixteen lines Ξ. On each of these four lines there is thus once more a summit, which makes four summits. I say that these four summits belong to the same fundamental tetrahedron.

Indeed, the four summits of the tetrahedron *T*(*ik*, *jm*, *ln*) have the symbols:

$$\begin{array}{ll} (ik, jm, ln), & (ik, mj, nl), \\ (ki, mj, ln), & (ki, jm, nl). \end{array}$$

Now, one easily confirms that the three points:

$$(ij, kl, mn), \qquad (ik, jm, ln), \qquad (km, ni, jl)$$

are in a straight line. Similarly, the points:

$$\begin{array}{llll} (ij, kl, mn), & (ik, mj, nl), & \text{and} & (mk, in, jl), \\ (ij, kl, mn), & (ki, mj, ln), & \text{and} & (km, lj, in), \\ (ij, kl, mn), & (ki, jm, nl), & \text{and} & (mk, ni, lj), \end{array}$$

are, as well.

One indeed sees that the four new points are the summits of the tetrahedron:

$$T(mk, in, lj).$$

Therefore: *Relative to each of the summits of a fundamental tetrahedron T*(*ij*, *kl*, *mn*), *the eight fundamental tetrahedra that have no common edge with the preceding one are pair-wise homologous.*

From this, one can conclude that three fundamental tetrahedra that have no common edge define a *desmic* system of three tetrahedra; i.e., the faces of the one pass through the sixteen lines of intersection of the faces of the other two, and the summits of one are on the sixteen joined lines of the summits of the other two.

On the subject of these desmic systems, one can consult a paper by Stephanos that was included in the *Bulletin des Sciences mathématiques*, t. XIV, of that collection.

85. The fundamental system gives rise to a remarkable correspondence between the points and planes of space.

First, consider a plane π. *The poles of that plane in the six fundamental complexes are on the same conic.*

Indeed, let O_i be the pole of the plane π in the complex C_i, and take three of these points O_1, O_2, O_3. As one knows, the complexes C_1, C_2, C_3 permit one to associate these three points with a fourth point O such that the plane OO_2O_3 is the polar plane to O in C_3, OO_3O_1 is the polar plane to O in C_2, and OO_1O_2 is the polar plane to O in C_0 (*see*, no. 80). The tetrahedron $OO_1O_2O_3$ is conjugate with respect to the quadric Q that carries the semi-quadric Q_{123}. Therefore, the triangle $O_1O_2O_3$ is conjugate with respect to the conic K that is the trace of Q on the plane π. Now, the quadric Q also carries the semi-quadric Q_{456}. Therefore, the triangle $O_4O_5O_6$ is also conjugate with respect to the conic K. Since the two triangles $O_1O_2O_3$ and $O_4O_5O_6$ are conjugate with respect to the same conic, their summits are on the same conic.

One can even add that their edges touch the same conic.

Analogously: If one distributes the polar planes π_1, π_2, π_3, π_4, π_5, π_6 to the same point O in the six fundamental complexes into two trihedra then the two trihedra will be conjugate with respect to the same second-degree cone. Their edges will be on the same second-degree cone, and *their faces will touch a third second-degree cone.*

86. We preserve the preceding notations.

Since the point O_i and the point O_j are the poles of the same plane π in C_i and C_j, respectively, the point O_i and the point O_j will also be the poles of the same plane π_{ij} with respect to C_i and C_j. That plane π_{ij} will pass through the lines $O_i\,O_j$; there will be then fifteen planes π_{ij}.

Take the three planes:

$$\pi_{ij}, \quad \pi_{jk}, \quad \pi_{kl}.$$

The point O_{ijk} where they intersect is the one that we considered above, and which is the pole:

$$\begin{array}{lll} \text{of } \pi_{ij} & \text{in } C_k, \\ \text{“} \ \pi_{jk} & \text{“} \ C_i, \\ \text{“} \ \pi_{kl} & \text{“} \ C_j. \end{array}$$

Now, take the other three indices l, m, n. We will likewise have three planes π_{lm}, π_{mn}, π_{ml} that intersect at a point O_{lmn}.

However, it is clear that O_{ijk} and O_{lmn} coincide. Indeed, we know that the tetrahedra $O_{ijk}O_iO_jO_k$, $O_{lmn}O_lO_mO_n$ are conjugate with respect to the quadric Q, which contains the complementary semi-quadrics Q_{ijk}, Q_{lmn}. Therefore, O_{ijk} and O_{lmn} are the poles of the same plane π with respect to Q.

There are $\frac{6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3} = 20$ combinations of indices taken three at a time, and in turn, there are twenty tetrahedra:

$$O_{ijk} O_i O_j O_k ,$$

but there are only ten points O_{ijk}, since O_{ijk} *is identical to* O_{lmn}.

From that, one sees that the point ($O_{ijk} = O_{lmn}$) is the pole:

$$\begin{array}{llll} \text{of the plane} & \pi_{jk} & \text{in} & C_i , \\ \text{“} & \pi_{ki} & \text{“} & C_j , \\ \text{“} & \pi_{ij} & \text{“} & C_k , \\ \text{“} & \pi_{mk} & \text{“} & C_l , \\ \text{“} & \pi_{nl} & \text{“} & C_m , \\ \text{“} & \pi_{lm} & \text{“} & C_n . \end{array}$$

In summation, we have a configuration of sixteen points:

$$O_1, \quad O_2, \quad O_3, \quad O_4, \quad O_5, \quad O_6 \qquad (O_{123} = O_{456}), \quad (O_{134} = O_{256}), \quad (O_{124} = O_{356}), \ldots,$$

and sixteen planes:

$$\pi, \pi_{12}, \pi_{13}, \pi_{14}, \ldots$$

such that the poles of the sixteen planes in the six fundamental complexes are part of the sixteen points, and the polar planes to the sixteen points are part of the sixteen planes.

Each of the sixteen planes thus contains six of the sixteen points, and six of the sixteen planes pass through each of the sixteen points.

If one takes the poles of one of the sixteen planes with respect to the ten fundamental quadrics then one will obtain the ten points of a system that is not situated in that plane, and if one takes the polar planes of one of the sixteen points with respect to the fundamental quadrics then one will obtain ten planes of the system that do not pass through that point.

87. One can associate this remarkable configuration with an important correspondence that the fundamental system gives rise to.

We just saw that any plane π is found to be part of a configuration of sixteen planes and sixteen points that it defines completely. Consequently, we can say that the knowledge of a spray will define a configuration of sixteen sprays and sixteen planar systems that are part of the proposed spray.

More generally: Any hyper-sheaf is part of a configuration of thirty-two hyper-sheaves, sixteen of which are sprays and sixteen of which are planar systems.

Let $x_1, x_2, x_3, x_4, x_5, x_6$ be a line. If we let $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6$ denote the symbol +1 or the symbol –1 then the expressions:

$$\varepsilon_1 x_1, \ \varepsilon_2 x_2, \ \varepsilon_3 x_3, \ \varepsilon_4 x_4, \ \varepsilon_5 x_5, \ \varepsilon_6 x_6$$

will be the coordinates of $2^5 = 32$ lines, among which, will be the lines $x_1, x_2, \ldots, x_6$, and which will define a special configuration with them. First, two lines of the configuration do not intersect, *in general*, because:

$$\varepsilon_1 x_1^2 + \varepsilon_2 x_2^2 + \cdots + \varepsilon_6 x_6^2$$

can be a consequence of $x_1^2 + x_2^2 + \cdots + x_6^2 = 0$ only if $\varepsilon_1 = \varepsilon_2 = \ldots = \varepsilon_6$, in which case, the two lines will not be distinct.

It comes to mind that if the line $x_1, x_2, \ldots, x_6$ generates a hyper-sheaf:

$$x_i = a_i\,\lambda + b_i\,\mu + c_i\,\nu$$

then the same thing will be true for the other thirty-one lines of the configuration. For these lines, one will have:

$$x_i' = \varepsilon_i(a_i\lambda' + b_i\mu' + c_i\nu').$$

Are the hyper-sheaves of the same nature?

If we suppose that the number of positive ε is even – for example, 2μ – then there will be $6 - 2\mu$ negative ones. By a change of all the signs, one can always suppose that $2\mu = 4$, because if $2\mu = 2$ then one will have $6 - 2\mu = 4$.

Therefore, let:

$$\begin{aligned}
x_1' &= a_1\lambda' + b_1\mu' + c_1\nu', \\
x_2' &= a_2\lambda' + b_2\mu' + c_2\nu', \\
x_3' &= a_3\lambda' + b_3\mu' + c_3\nu', \\
x_4' &= a_4\lambda' + b_4\mu' + c_4\nu', \\
-\,x_5' &= a_5\lambda' + b_5\mu' + c_5\nu', \\
-\,x_6' &= a_6\lambda' + b_6\mu' + c_6\nu'.
\end{aligned}$$

If one writes that:

$$x_1 = x_1', \qquad x_2 = x_2', \qquad x_3 = x_3', \qquad x_4 = x_4', \qquad x_5 = x_5', \qquad x_6 = x_6'$$

then one will find that:

$$\begin{aligned}
a_1(\lambda - \lambda') + b_1(\mu - \mu') + c_1(\nu - \nu') &= 0, \\
a_2(\lambda - \lambda') + b_2(\mu - \mu') + c_2(\nu - \nu') &= 0, \\
a_3(\lambda - \lambda') + b_3(\mu - \mu') + c_3(\nu - \nu') &= 0, \\
a_4(\lambda - \lambda') + b_4(\mu - \mu') + c_4(\nu - \nu') &= 0, \\
a_5(\lambda - \lambda') + b_5(\mu - \mu') + c_5(\nu - \nu') &= 0, \\
a_6(\lambda - \lambda') + b_6(\mu - \mu') + c_6(\nu - \nu') &= 0.
\end{aligned}$$

The first four equations demand that:

$$\lambda - \lambda' = \mu - \mu' = \nu - \nu' = 0,$$

and the other two give:

$$a_5\lambda + b_5\,\mu + c_5\nu = 0,$$
$$a_6\lambda + b_6\,\mu + c_6\nu = 0$$

in order to determine $\lambda : \mu : \nu$. In this case, the two hyper-sheaves will have a line in common. They will have the same type.

On the contrary, if there are an odd number of ε that are equal to + 1 then one can always suppose that are five or three of them. If there are three then instead of the six equations above, one will have the system:

$$\begin{aligned}
a_1(\lambda - \lambda') + b_1(\mu - \mu') + c_1(\nu - \nu') &= 0,\\
a_2(\lambda - \lambda') + b_2(\mu - \mu') + c_2(\nu - \nu') &= 0,\\
a_3(\lambda - \lambda') + b_3(\mu - \mu') + c_3(\nu - \nu') &= 0,\\
a_4(\lambda + \lambda') + b_4(\mu + \mu') + c_4(\nu + \nu') &= 0,\\
a_5(\lambda + \lambda') + b_5(\mu + \mu') + c_5(\nu + \nu') &= 0,\\
a_6(\lambda + \lambda') + b_6(\mu + \mu') + c_6(\nu + \nu') &= 0,
\end{aligned}$$

and the existence of a common line will be impossible, because the first three equations give:

$$\lambda - \lambda' = \mu - \mu' = \nu - \nu' = 0,$$

and the others give:

$$\lambda + \lambda' = \mu + \mu' = \nu + \nu' = 0,$$

so:

$$\lambda = \lambda' = \mu = \mu' = \nu = \nu' = 0.$$

The hyper-sheaves will then have different types.

Finally, if there is only one negative ε then one will have five equations of the form:

$$a_i(\lambda - \lambda') + b_i(\mu - \mu') + c_i(\nu - \nu') = 0,$$

which gives:

$$\lambda = \lambda', \quad \mu = \mu', \quad \nu = \nu',$$

and in turn, a unique equation of the form:

$$a_i\,\lambda + b_i\,\mu + c_i\,\nu = 0.$$

In this case, the hyper-sheaves will thus have a plane sheaf of lines in common. They will again be of different types, but they will be *united*, moreover.

The unique equation:

$$a_j\,\lambda + b_j\,\mu + c_j\,\nu = 0$$

expresses the idea that $x_j = 0$; i.e., that the sheaf that is common to our two hyper-sheaves will be a sheaf of complexes C_j .

It is, moreover, easy to recover the preceding results that were obtained.

Suppose, to fix ideas, that the line $x_1, x_2, \ldots, x_6$ generates a plane system π, so the other thirty-one lines:

$$\varepsilon_1 x_1,\ \varepsilon_2 x_2,\ \ldots,\ \varepsilon_6 x_6$$

must generate a hyper-sheaf.

The fifteen of them for which there is an even number of positive ε's will again generate planar systems. The other sixteen will generate sprays, and of these sprays there will be six of them that have just one negative ε and for which the six summits O_1, O_2, …, O_6 will be in the plane π. The lines of the planar sheaf (π, O_i) will belong to the complex C_i and O_i will therefore be the pole of the plane π in the complex C_i. One sees how we recover the configuration of sixteen points and sixteen planes that we already described.

We will have occasion to return to these questions in the context of the theory of second-degree complexes and Kummer surfaces.

88. From the standpoint of the transformation of coordinates, we had to occupy ourselves with those transformations that preserve the type of the fundamental form, or, as one says, makes it go back to itself. Instead of taking the viewpoint of transformations of coordinates, one can pose another problem that I would like to treat.

Let x_1, x_2, …, x_6 be the linear coordinates of a line – i.e., there are deduced linearly from arbitrary tetrahedral coordinates, as we have seen – and let:

$$\omega(x)$$

be the corresponding linear form.

There exist linear transformations:

$$x_i' = a_{i1}\,x_1 + a_{i2}\,x_2 + \ldots + a_{i6}\,x_6 \tag{11}$$

that preserve the expression of the fundamental form in such a way that by virtue of equations (11), one will have:

$$\omega(x) = \omega(x'). \tag{12}$$

These transformations can be considered as making a line in the coordinate system x_1, x_2, …, x_6 correspond to another line x' *in the same coordinate system*, since the coordinates x_i' annul the same form as the x_i.

What is the nature of this transformation?

If x describes a planar sheaf then one will have:

$$x_i = a_i\,\lambda + b_i\,\mu,$$

and in turn, in regard to the linear form of the x_i', when expressed as functions of the x_i, one will also have:

$$x_i' = a_i'\lambda + b_i'\mu.$$

The line x' will thus also describe a planar sheaf.

One has the same proof for the hyper-sheaf. If x describes a hyper-sheaf then x' will describe another one.

However, here a paramount distinction arises here.

The hyper-sheaves that are generated by x and x' can have the same name (i.e., spray and spray or plane and plane), or they can even have opposite names (i.e., spray and plane or plane and spray).

In the first case, all of the lines x that issue from a point P will correspond to all of the x' that issue from a point P'. All of the lines x in a plane π will correspond to all of the lines x' in a plane π'. Moreover, if P is in the plane π then P' will be in the plane π', because the planar sheaf (P, π) will correspond to the planar sheaf (P', π'). From all of this behavior, one recognizes a homographic transformation of space.

In the second case, all of the lines that issue from a point P will correspond to the lines in a plane π', and all of the lines in a plane π will correspond to the lines that issue from a point P'. Moreover, if the plane π and the point P are united then the plane π' and the point P' will also be united, which is once more due to the conservation of sheaves.

The transformation thus consists of a *dualistic* correspondence between the figures that are loci of lines and the figures that are loci of lines x'.

The solution to our problem is the following:

If the equations of the linear transformation:

$$x'_i = a_{i1}\,x_1 + a_{i2}\,x_2 + \ldots + a_{i6}\,x_6 \qquad (i = 1, 2, \ldots, 6)$$

give:

$$\omega(x) = \omega(x')$$

then they will establish either a homographic correspondence between the lines x and x′ or a dualistic correspondence.

89. For Klein, that remark was the point of departure for a curious encounter between the geometry of lines in space and the geometry of the metric properties in a four-dimensional space.

Today, the notion of spaces of more than three dimensions has won the right to be mentioned in geometry. We would not like to say that this implies that a systematic and complete study of n-dimensional spaces would be of genuine geometric interest; the interest that would be attached to such a study would be entirely philosophical and speculative. Nonetheless, *certain properties* of n-dimensional spaces find a useful interpretation in the figures of ordinary geometry. Thanks to these properties, the facts of Euclidian geometry can often take on a more rational and illuminating form. From that viewpoint, the language of n-dimensional geometry can be of great service, and it will be devoid of any auxiliary affectation that one would reject without further examination. It is within these limits that the study of n-dimensional geometry deserves to be confined. One will find an example in the study of the straight line.

Let $x_1, x_2, \ldots, x_n, x_{n+1}$ be $n + 1$ homogeneous variables; i.e., ones that involve only their ratios. We regard these parameters as the homogeneous coordinates in an n-dimensional space E_n .

A homogeneous equations of degree m in $x_1, x_2, \ldots, x_{n+1}$ represents a space of degree m that is contained in the space E_n^1 and is endowed with only $n - 1$ dimensions. We represent such a space by:

$$E_{n-1}^m .$$

In particular, a linear relation represents an $(n - 1)$-dimensional linear space:

$$E_{n-1}^1$$

that is contained in E_n^1.

If one is given k linear equations – i.e., k spaces E_{n-1}^1 – then they will have an $(n - k)$-dimensional space in common that we will again qualify by the word "linear."

More generally, if one is given k equations in the x_i then one will define an $(n - k)$-dimensional space E_{n-1}^μ.

The degree μ of that space is defined to be the number of points that it has in common with an arbitrarily-chosen k-dimensional linear space E_k^1.

If $\mu = 2$ then we will say that the space is *quadratic*.

For example, a second-degree equation in $x_1, x_2, \ldots, x_{n+1}$ defines an $(n - 1)$-dimensional quadratic space E_{n-1}^2. To abbreviate, we also say that it is an $(n - 1)$-dimensional quadric. The intersection of an $(n - 1)$-dimensional quadric and k- 1 $(n - 1)$-dimensional linear spaces is obviously an $(n - k)$-dimensional quadric space.

The quadratic spaces give rise to the same theories that the quadrics, cones, and conics do.

For example, let an $(n - 1)$-dimensional quadratic space be given in n-dimensional space by:

$$\omega(x) = 0,$$

and let:

$$x_1, x_2, \ldots, x_{n+1}; \qquad y_1, y_2, \ldots, y_{n+1}$$

be two points of that space. One says that the points are conjugate if:

$$\omega(x \mid y) = 0.$$

The locus of points x that are conjugate to a fixed point y is an $(n - 1)$-dimensional linear space. That linear space generalizes the notion of polar plane or polar line for the quadrics and conics, respectively.

Let $\Omega(a)$ be the adjoint form to $\omega(x)$. The equation:

$$\Omega(a) = 0$$

expresses the idea that the linear space:

$$\sum a_i x_i = 0$$

is tangent ([10]) to the quadratic space $\omega(x) = 0$. Similarly, the equation:

$$\Omega(a \mid b) = 0$$

expresses the idea that the two linear spaces:

$$\sum a_i x_i = 0, \qquad \sum b_i x_i = 0$$

are conjugate; i.e., each of them contains the pole of the other one.

An $(n - 1)$-dimensional quadric space is the locus of an infinitude of lower-dimensional linear spaces.

90. The geometry of quadric spaces holds a special interest for us.

Indeed, we have seen that one can define any line in space by means of six homogeneous coordinates $x_1, x_2, \ldots, x_6$ that are linked by a second-degree equation:

$$\omega(x) = 0.$$

Moreover, if one considers the x_i to be the coordinates of a point in a five-dimensional space E_5^1 then the equation $\omega = 0$ will represent a four-dimensional quadratic space E_4^2 in that space. One can therefore say that *the geometry of the lines in ordinary space is identical to that of a point on a four-dimensional quadric E_4^2 that is contained in a five-dimensional space.*

The lines of a linear complex:

$$\sum a_i x_i = 0$$

are represented by the points of intersection of the linear space E_4^1 that is represented by that equation with the fundamental quadric E_4^2. The equation:

$$\Omega(a) = 0,$$

which expresses the idea that the complex is special, expresses the idea that the space E_4^1 is tangent to the quadratic space E_4^2.

If one considers two linear complexes:

$$\sum a_i x_i = 0, \qquad \sum a_i' x_i = 0$$

then the condition of involution:

$$\Omega(a \mid a') = 0$$

([10]) I.e., its pole is on the quadric.

expresses the idea that the corresponding linear spaces E_4^1, $E_4'^1$ will be *conjugate* with respect to the fundamental quadric E_4^2.

The quadric E_4^2 contains linear spaces of dimensions one and two.

Indeed, we know that if x^0 and x^{00} are two lines that intersect then the expressions:

(13) $$x_i = x_i^0 \lambda + x_i^{00} \mu$$

will be the coordinates of a line of the planar sheaf that is defined by these two lines. It will then result immediately that when $\lambda : \mu$ varies one will always have:

$$\omega(x) = \omega(x^0 \lambda + x^{00} \mu) = 0.$$

Now, when they are interpreted in five-dimensional space, equations (13) will represent a one-dimensional linear space E_1^1 that is contained in E_4^2.

Conversely, let E_1^1 be a one-dimensional linear space in E_4^2, so the coordinates x_i of a point in that linear space will be represented by formulas such as (13), where one must have:

$$\omega(x) = \omega(x^0 \lambda + x^{00} \mu) = 0$$

for any λ, μ. In the geometry of lines, we will thus have a planar sheaf. One can, moreover, state this proposition:

There are an infinitude of one-dimensional linear spaces on E_4^2. In line geometry, these spaces correspond to the planar sheaves in Euclidian space, in such a way that there is a quintuple infinitude of these linear spaces on E_4^2.

One confirms in the same way that there is an infinitude of two-dimensional linear space in E_4^2 that correspond to the hyper-sheaves of linear geometry.

However, there are two kinds of hyper-sheaves: viz., sprays and planar systems. One can thus predict that will be two distinct families of two-dimensional linear spaces in E_4^2.

That fact, which is completely analogous to the fact that there is a double system of rectilinear generators of the quadrics in ordinary space, can be exhibited directly. As one will see, it presents *an essential difference* from the example that I have compared it to.

For ordinary quadrics, two rectilinear generators always intersect if they are from different systems, and never if they are from the same system.

The opposite is true here, because two linear spaces E_2^1 from the same family will always have a point in common; this amounts to saying that two sprays or even two planes will always have a common line.

That amounts to saying that a spray and a plane do not generally have a common line, and that if this is the case then they will have a planar sheaf of lines in common.

A complex of lines that is defined by an equation:

$$f(x_1, x_2, \ldots, x_6) = 0$$

will be represented by the trace of the space E_4 that is represented by the equation $f = 0$ on the quadric E_4^2.

We thus obtain a three-dimensional space E_2 on E_4^2.

Similarly, a two-dimensional space E_2 that is traced on E_4^2 will represent a congruence, and a one-dimensional space will represent a ruled surface.

91. This agreement between ruled geometry and that of the point on a four-dimensional quadric in a five-dimensional space will still have no great utility if one does not heed a remark that concerns the geometry of quadratic spaces.

I will first take the example of an ordinary quadric on ordinary space.

Let Q be such a quadric, let O be a point on it, and let π be an arbitrary plane.

Imagine that one makes any point M of the plane correspond to a point P of the quadric by taking the intersection of the latter with the line OM. Conversely, a point P of the quadric will correspond to one and only one point M. The correspondence is *single-valued* in both directions. One expresses that by saying that the quadric is *representable on the plane* ([11]).

One can give a concrete analytical representation to that representation and associate it with an old method that Chasles imagined for the study of curves that were traced on quadrics ([12]).

Let OG_0, OH_0 be two rectilinear generators of the quadric that issue from the point O. Two generators pass through the point P of the quadric. One of them – viz., G – comes from the same system as OG_0, while the second one – viz., H – comes from the same system as OH_0. G cuts OH_0 at a point P', and H cuts OG_0 at a point P''. In order to define the position of P' on OH_0, one can take the anharmonic ratio that it defines with the point O and two other fixed points on OH_0, in such a way that if A', B' denote these fixed points then one will have:

$$u = \frac{A'O}{P'O} : \frac{A'B'}{P'B'}.$$

Similarly, if A'', B'' denote two fixed points on OG_0 then one will define the point P'' by the parameter:

$$v = \frac{A''O}{P''O} : \frac{A''B''}{P''B''}.$$

Once u and v are known, the points P', P'' will result, as well as the point P, and consequently, the point M in the plane π.

([11]) For the question of which surfaces are representable on the plane, one can consult several notes that Darboux dedicated to that question in the *Bulletin des Sciences mathématiques*. The original papers of Clebsch appeared in the *Mathematischen Annalen*. Today, that theory is very well-developed and deserves a special study.

([12]) *Comptes rendus des seances de l'Academie des Sciences*, t. LIII.

Call the traces of OG_0 and OH_0 on that plane G_0, H_0, resp. The line G_0M is the trace of the plane OG_0M on the plane π. This plane OG_0M is obviously tangent to the quadric at P'', and due to Chasles's theorem, the line is traced on a plane that is tangent to a ruled surface. The parameter v is equal to the anharmonic ratio:

$$v = (G_0\alpha'', G_0M, G_0H_0, G_0\beta''),$$

where $G_0\alpha''$ is the trace of the tangent plane at A'' on the plane π, and $G_0\beta''$ is the trace of the tangent plane at B''.

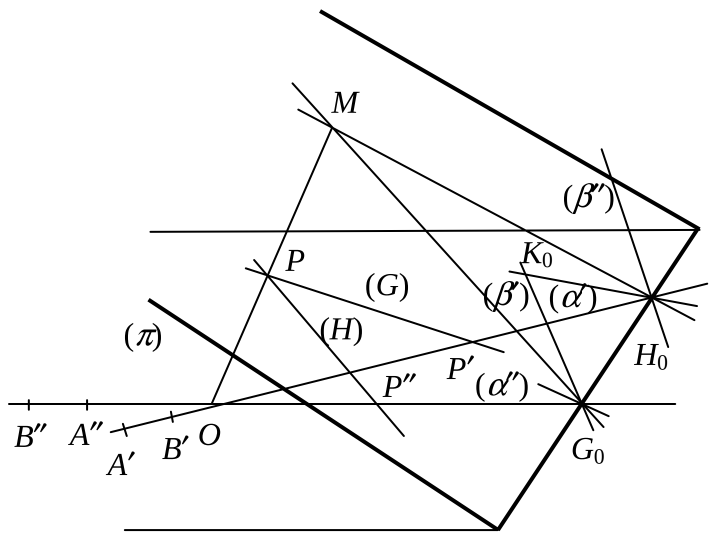

Figure 5.

Similarly, since $H_0\alpha'$, $H_0\beta'$ are traces of the tangent planes at A', B', resp., one will have:

$$u = (H_0\alpha', H_0M, H_0G_0, H_0\beta').$$

Take the triangle of reference on the plane π to be the triangle that is defined by the lines $H_0\beta'$, $G_0\beta''$, and G_0H_0, and one will immediately see that if:

$$X = 0, \quad Y = 0, \quad Z = 0$$

represent the equations of these three lines then, upon introducing constant factors into X, Y, Z, one will have:

$$u = \frac{X}{Z}, \qquad v = \frac{Y}{Z}.$$

If one lets K_0 denote the intersection point of the lines $G_0\beta''$ and $H_0\beta'$ then one will see that the quantities u, v are *the Chasles coordinates of the point P on the quadric*, and that they are also the triangular coordinates of the point M with respect to the triangle of reference $G_0H_0K_0$.

The points H_0, G_0 play an essential role in this representation. Any point of OG_0 will project onto G_0 and any point of OH_0 will project onto H_0 . These points H_0 , G_0 are thus points of indeterminacy, in the sense that each of them is the projection of an infinitude of points of the quadric.

There is also a point of indeterminacy on the quadric. Indeed, it is clear that if the point P of the quadric tends to the point O then the point M will be placed on the line G_0H_0 , and that the position of the point M will be the trace on G_0H_0 of the limiting position of OP when OP becomes tangent to the surface at O.

We thus see that there are two remarkable points G_0, H_0 on the plane and one remarkable line, which is the line that joins them. There is one remarkable point O on the surface and two remarkable lines, namely, the generators that issue from that point.

In representations of this kind, the points G_0, H_0 are given the name of *base points* for the representations and the line G_0H_0 is given the name of *fundamental line.*

In the general case of the representation of surfaces on the plane, the nature of the base points and fundamental lines – or, *generically*, FUNDAMENTAL ELEMENTS – characterizes the representation.

One proves that, in general, the curves in the plane that represent plane sections of the surface pass through the base points or fundamental lines.

Here, this is obvious, because any plane section cuts OG_0 at one point and OH_0 at another, and the perspective is therefore a conic that passes through the two points G_0, H_0.

One knows that the metric properties of plane figures are defined as relations between those figures and two remarkable points in the plane, namely, the circular points at infinity. From the projective viewpoint, one can thus regard all of the properties of the relation between a figure and two points in the plane as being metric.

The conics that pass through these two fixed points will be called *circles*. From this standpoint, one can say that the plane sections of the quadric will be represented by circles in the plane.

Moreover, one recognizes that in order to realize that representation effectively, it will suffice to take the point O to be an umbilic of the quadric and take the plane π to be a plane that is parallel to the tangent plane at the point O.

One then finds that one has generalized a very old transformation, namely, the *stereographic transformation.*

However, such a restriction is useless to us, since we are always free to take two arbitrary points of the plane to be the basis for the metric properties.

92. One can exhibit this representation of quadrics on the plane in a more analytical form that lends itself better to the generalization that we have in mind.

Indeed, take two points O and O' on the quadric that are not situated on the same rectilinear generator. Draw two conjugate planes through the line OO', and let Δ be the intersection of the planes tangent to O and O'. That line will cut the conjugate planes at two points O'', O'''; we take the tetrahedron $OO'O''O'''$ to be the tetrahedron of reference. The quadric will have an equation of the form:

$$x^2 + y^2 - zt = 0 \tag{14}$$

by introducing numerical constants into x, y, z, t, which are pointless to specify explicitly. I then set:

$$
(15) \qquad \begin{cases} \rho x = XZ, \\ \rho y = YZ, \\ \rho z = Z^2, \end{cases}
$$

and I observe that equation (14) then gives:

$$
(16) \qquad \rho t = X^2 + Y^2.
$$

We have thus expressed x, y, z as functions of three homogeneous parameters X, Y, Z.

We can regard X, Y, Z as the triangular coordinates of a point in a plane, and we will have thus realized a representation of the quadric in the plane *analytically*.

I shall not stop to prove that the representation is realized geometrically by the stereographic projection that I defined above.

Observe that any plane section:

$$ax + by + cz + dt = 0$$

is represented on the plane by the conic:

$$
(17) \qquad (aX + bY + cZ) - dt\,(X^2 + Y^2) = 0
$$

that passes through the two fixed points:

$$Z = 0, \qquad X \pm iY = 0.$$

If one regards these two points as the circular points at infinity in the plane then equation (17) will be the general equation of the circles in the plane.

93. Having said that, we seek to answer the following question:

What exactly are the properties of plane figures that correspond to the projective properties of the quadric?

In order to resolve this question with any precision, we shall look for the plane transformation that corresponds to a homographic transformation that preserves the proposed quadric.

Let x, y, z, t be the coordinates of a point P of the quadric and let x', y', z', t' be those of the corresponding point P'. One has:

$$(18)\qquad \begin{cases} x' = ax + by + cz + dt, \\ y' = a'x + b'y + c'z + d't, \\ x'' = a''x + b''y + c''z + d''t, \\ x''' = a'''x + b'''y + c'''z + d''t, \end{cases}$$

and one must have:

$$(19)\qquad x'^2 + y'^2 - z'\,t' = k(x^2 + y^2 - zt).$$

Let (X, Y, Z) be the coordinates of the point M that corresponds to the point P, and let (X', Y', Z') be those of the point M' that corresponds to the point P'.

Upon replacing x, y, z, t, y', z', t' in (18) with their values in terms of X, Y, Z, X', Y', Z', one gets:

$$(20)\qquad \begin{cases} \sigma X'Z' = aXZ + bYZ + cZ^2 + d(X^2 + Y^2), \\ \sigma Y'Z' = a'XZ + b'YZ + c'Z^2 + d'(X^2 + Y^2), \\ \sigma Z'^2 = a''XZ + b''YZ + c''Z^2 + d''(X^2 + Y^2), \\ \sigma(X'^2 + Z'^2) = a'''XZ + b'''YZ + c'''Z^2 + d'''(X^2 + Y^2). \end{cases}$$

There are obviously too many of these equations for one to define X', Y', Z' as functions of X, Y, Z. However, from the identity (19), they are compatible; i.e., when $a, b, c, d, a', b', c', d', \ldots, c''', d'''$ obey that identity.

In order to simplify the interpretation of the formulas, I will make $Z' = Z = 1$, and I will write the formulas in the form:

$$(T)\qquad \begin{cases} X' = \dfrac{aX + bY + c + d(X^2 + Y^2)}{a''X + b''Y + c'' + d''(X^2 + Y^2)}, \\[2ex] Y' = \dfrac{a'X + b'Y + c' + d'(X^2 + Y^2)}{a''X + b''Y + c' + d''(X^2 + Y^2)}, \\[2ex] X'^2 + Y'^2 = \dfrac{a'''X + b'''Y + c''' + d'''(X^2 + Y^2)}{a''X + b''Y + c'' + d''(X^2 + Y^2)}. \end{cases}$$

X, Y will then be the rectangular coordinates of a point, and X', Y' will be those of its transform.

Suppose that one performs a first transformation on that form T, and then another T' that has other coefficients, so the linear nature of these formulas will show us that the resulting transformation $T'T$ will again be a transformation of the same form.

In a word, these transformations define what Lie called a *group*.

A homothetic transformation around an arbitrary point, an arbitrary displacement, a symmetry transformation with respect to an arbitrary line, and more generally, an inversion with respect to an arbitrary circle define elements of the group, as one will recognize immediately from the formulas that express these various transformations. I would like to prove that, conversely: *Any transformation that is defined by formulas* (T)

will result from the successive application of a certain number of these transformations ([13]).

Indeed, let T_1 denote the translation that changes the point X, Y into the point X'', Y'', and which is represented by the formulas:

$$(T_1) \qquad X'' = X + h, \qquad Y'' = Y + k,$$

where h and k are two constants; then envision the transformation:

$$(T_2) \qquad \begin{cases} X' = \dfrac{a_1 X'' + b_1 Y'' + c_1 + d_1 (X''^2 + Y''^2)}{a_1'' X'' + b_1'' Y'' + c_1'' + d_1'' (X''^2 + Y''^2)}, \\[2ex] Y' = \dfrac{a_1' X'' + b_1' Y'' + c_1' + d_1' (X''^2 + Y''^2)}{a_1'' X'' + b_1'' Y'' + c_1'' + d_1'' (X''^2 + Y''^2)}, \\[2ex] X'^2 + Y'^2 = \dfrac{a_1''' X'' + b_1''' Y'' + c_1''' + d_1''' (X''^2 + Y''^2)}{a_1'' X'' + b_1'' Y'' + c_1'' + d_1'' (X''^2 + Y''^2)}. \end{cases}$$

The composition of the two operations (T_1) and (T_2) is equivalent to the general transformation (T), where c, c' are not zero; one can then set $T = T_2 \, T_1$.

Now, if one considers the identity:

$$(a_1 X'' + \ldots)^2 + (a_1' X'' + \ldots)^2 = (a_1'' X'' + \ldots)^2 + (a_1''' X'' + \ldots)^2$$

then one will see that the left-hand side is annulled with X'', Y'', so the same must be true for the right-hand side. One thus has:

$$c_1'' c_1''' = 0.$$

First, suppose that $c_1'' = 0$. Then, upon performing the inversion:

$$(T_0) \qquad X'' = \frac{X'''}{X'''^2 + Y'''^2}, \qquad Y'' = \frac{Y'''}{X'''^2 + Y'''^2},$$

the operation T_2 will appear to be the product $T_3 \, T_0$ of the operations T_3 and T_0 , where T_3 is defined by the formulas:

([13]) KLEIN, *Mathematischen Annalen*, t. V.

$$(T_3)\qquad \begin{cases} X' = \dfrac{a_1 X''' + b_1 Y''' + d_1}{a_1'' X''' + b_1'' Y''' + d_1''}, \\ Y' = \dfrac{a_1' X''' + b_1' Y''' d_1'}{a_1'' X''' + b_1'' Y''' + d_1''}, \\ X'^2 + Y'^2 = \dfrac{a_1''' X'' + b_1''' Y'' + c_1'''(X'''^2 + Y'''^2) + d_1'''}{a_1'' X'' + b_1'' Y'' + d_1''}, \end{cases}$$

and one will have:

$$T = T_2\, T_1 = T_3\, T_0\, T_1\, .$$

On the contrary, suppose that it is the c_1''' that are zero. Then, upon once more performing the operation T_0, T_3 will appear to be the product $T_2' T_0$ of the two operations T_0 and T_2', where T_2' is then defined by:

$$(T_2')\qquad \begin{cases} X' = \dfrac{a_1 X''' + b_1 Y''' + d_1}{a_1'' X''' + b_1'' Y''' + c_1''(X'''^2 + Y'''^2) + d_1''}, \\ Y' = \dfrac{a_1' X''' + b_1' Y''' d_1'}{a_1'' X''' + b_1'' Y''' + c_1''(X'''^2 + Y'''^2) + d_1''}, \\ X'^2 + Y'^2 = \dfrac{a_1''' X'' + b_1''' Y'' + d_1'''}{a_1'' X''' + b_1'' Y''' + c_1''(X'''^2 + Y'''^2) + d''}. \end{cases}$$

Now, in order to perform the transformation (T_2'), one can perform the transformation:

$$(T_3')\qquad \begin{cases} X_1 = \dfrac{a_1 X''' + b_1 Y''' + d_1}{a_1'' X''' + b_1'' Y''' + d_1''}, \\ Y_1 = \dfrac{a_1' X''' + b_1' Y''' d_1'}{a_1'' X''' + b_1'' Y''' + d_1''}, \\ X_1^2 + Y_1^2 = \dfrac{a_1'' X''' + b_1'' Y''' + c_1''(X'''^2 + Y'''^2) + d_1''}{a_1''' X'' + b_1''' Y'' + d_1'''}, \end{cases}$$

and follow it with the inversion T_0:

$$X' = \frac{X_1}{X_1^2 + Y_1^2}, \qquad Y' = \frac{Y_1}{X_1^2 + Y_1^2}.$$

The transformations T_3 and T_3' have the same character. The have the general form:

$$X' = \frac{\alpha X + \beta Y + \gamma}{\alpha'' X + \beta'' Y + \gamma''},$$
$$Y' = \frac{\alpha' X + \beta' Y + \gamma'}{\alpha'' X + \beta'' Y + \gamma''},$$
$$X'^2 + Y'^2 = \frac{\alpha''' X + \beta''' Y + \gamma''' + \delta'''(X^2 + Y^2)}{\alpha'' X + \beta'' Y + \gamma''}.$$

We write down that one has identically:

$$(\alpha X + \beta Y + \gamma)^2 + (\alpha' X + \beta' Y + \gamma')^2$$
$$= (\alpha'' X + \beta'' Y + \gamma'')^2 [(\alpha'' X + \beta'' Y + \gamma'' + \delta(X^2 + Y^2)].$$

One first sees that α'', β'' must be zero, which then permits one to set $\gamma'' = 1$. What then remains is:

$$X' = \alpha X + \beta Y + \gamma,$$
$$Y' = \alpha' X + \beta' Y + \gamma',$$
$$X'^2 + Y'^2 = \alpha'' X + \beta'' Y + \gamma'' + \delta''(X^2 + Y^2),$$

with the identity:

$$(\alpha X + \beta Y + \gamma)^2 + (\alpha' X + \beta' Y + \gamma')^2 = \alpha'' X + \beta'' Y + \gamma'' + \delta''(X^2 + Y^2).$$

One must then have, in particular:

$$(\alpha X + \beta Y)^2 + (\alpha' X + \beta' Y)^2 = \delta''(X^2 + Y^2).$$

Now, this identity proves that one can set either:

$$\text{(21)} \qquad \begin{cases} \alpha = \sqrt{\delta''}\cos\theta, & \beta = \sqrt{\delta''}\sin\theta, \\ \alpha' = -\sqrt{\delta''}\sin\theta, & \beta' = \sqrt{\delta''}\cos\theta, \end{cases}$$

or

$$\text{(22)} \qquad \begin{cases} \alpha = \sqrt{\delta''}\cos\theta, & \beta = \sqrt{\delta''}\sin\theta, \\ \alpha' = \sqrt{\delta''}\sin\theta, & \beta' = -\sqrt{\delta''}\cos\theta. \end{cases}$$

In the first case, the transformation T_3 will represent an arbitrary displacement D in the plane that is preceded by a homothety H; one will then have:

$$T_3 = D \cdot H.$$

In the second case, the homothety is accompanied by a symmetry transformation S with respect to a line, and one then has:

$$T_3 = D \cdot H \cdot S.$$

Therefore, in summation, one will have:

$$T = \begin{cases} T_3 T_0 T_1 \\ \text{or even} \\ T_0 T_3 T_0 T_1, \end{cases}$$

where

$$T_3 = \begin{cases} D \cdot H, \\ \text{or even} \\ D \cdot H \cdot S. \end{cases}$$

Therefore, T indeed reduces to a superposition of operations of the following nature:

Motions, homotheties, inversions, and symmetries with respect to lines.

All of these transformations have a common property: They transform any circle in the plane into another one, or in other words, the group of transformations preserves the family of circles in the plane. One can then give these transformations the name of *anallagmatic transformations*.

Consequently, one sees that, when interpreted on the representative plane, the homographic transformations of a quadric to itself will have the group of anallagmatic transformations of the plane for their images.

The projective properties of the quadric then correspond to the anallagmatic properties in the plane.

91. All of what we just said about the representation of ordinary quadrics on a plane extend to the case of $(n - 1)$-dimensional quadrics in n-dimensional space.

For example, take the quadric:

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 - x_5 x_6 = 0 \tag{23}$$

in five-dimensional space. We set:

$$\rho x_1 = X_1 X_5 ,$$
$$\rho x_2 = X_2 X_5 ,$$
$$\rho x_3 = X_3 X_5 ,$$
$$\rho x_4 = X_4 X_5 ,$$
$$\rho x_5 = X_5^2 ,$$

and equation (23) will give:

$$\rho x_5 = X_1^2 + X_2^2 + X_3^2 + X_4^2 .$$

We have represented our quadric on a four-dimensional linear space in which X_1, X_2, X_3, X_4, X_5 are the homogeneous coordinates of a point.

Here, we have a *fundamental figure*, or figure of indeterminacy. It is represented by the equations:

$$X_4 = 0, \qquad X_1^2 + X_2^2 + X_3^2 + X_4^2 = 0.$$

It constitutes a two-dimensional quadratic space that I shall represent by I_2 .

Call any quadric in four-dimensional space that contain I_2 a *sphere*, so the equation of a sphere will be:

$$(a\,X_1 + b\,X_2 + c\,X_3 + d\,X_4 + c\,X_5)\,X_5 + f(X_1^2 + X_2^2 + X_3^2 + X_4^2) = 0.$$

It is convenient to reduce the variable X_5 to unity, and when it is equated to zero, it will represent the infinity in our four-dimensional space, so the equation of our sphere would have the form:

$$a_1\,X_1 + a_2\,X_2 + a_3\,X_3 + a_4\,X_4 + a_5 + a_6\,(X_1^2 + X_2^2 + X_3^2 + X_4^2) = 0. \tag{24}$$

The distance between two points will be:

$$\sqrt{(X_1 - X_1')^2 + \ldots + (X_4 - X_4')^2}\,.$$

A displacement, a symmetry, a homothety, and an inversion are defined as they are in the case of ordinary space, and by the same argument as was presented above, we recognize that any linear transformation that preserves the form:

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 - x_5 x_6\,,$$

i.e., any homographic or dualistic transformation of ruled space, translates in the representative four-dimensional space into a succession of operations such as:

1. Homothety.
2. Symmetry.
3. Inversion.
4. Displacements.

These are all transformations that leave the notion of sphere invariant.

From this viewpoint, we can say that:

From the dualistic and projective viewpoint, ruled geometry is identical to the anallagmatic geometry of a four-dimensional space.

95. One sees that in the representations that we occupy ourselves with a linear complex (i.e., a section of the four-dimensional quadric by a four-dimensional linear space) is found to be represented by a sphere in four-dimensional space.

If:

$$a_1\,x_1 + a_2\,x_2 + \ldots + a_6\,x_6 = 0$$

is the equation of the linear complex then that of the sphere will be equation (24), precisely.

The equation of the sphere takes on the form:

$$\left(X_1+\frac{a_1}{2a_4}\right)^2+\left(X_2+\frac{a_2}{2a_4}\right)^2+\left(X_3+\frac{a_3}{2a_4}\right)^2+\left(X_4+\frac{a_4}{2a_4}\right)^2=\frac{a_1^2+a_2^2+a_3^2+a_4^2-4a_5a_6}{a_6^2}.$$

The expression in the right-hand side represent the square of the radius of the sphere, and $-\frac{a_1}{2a_4}, -\frac{a_2}{2a_4}, -\frac{a_3}{2a_4}, -\frac{a_4}{2a_4}$ are the coordinates of its center. The radius is zero if:

(25) $$a_1^2+a_2^2+a_3^2+a_4^2-4a_5a_6 = 0.$$

Now, since the fundamental form is:

$$x_1^2+x_2^2+x_3^2+x_4^2-x_5x_6$$

here, the invariant of the complex will be precisely the left-hand side of (25). *The spheres of radius zero will thus correspond to the special complexes.*

Similarly, the equation:

$$a_1\,b_1 + a_2\,b_2 + a_3\,b_3 + a_4\,b_4 - 2a_5\,b_6 - 2a_6\,b_5 = 0$$

expresses the *involution of the two complexes:*

$$a_1\,x_1 + a_2\,x_2 + \ldots = 0, \qquad b_1\,x_1 + b_2\,x_2 + \ldots = 0;$$

it also expresses the *orthogonality* of the two corresponding spheres.

A linear congruence is represented by the intersection of two spheres. One can make two spheres of radius zero pass through that intersection that each represent one of the special complexes that have the directrices of the congruence for their directrices.

The intersection of two spheres in four-dimensional space is, in addition to I_2, which is set apart, a two-dimensional quadratic space that one can call a two-dimensional sphere.

If one denotes the three-dimensional spheres by S_3 then I will denote the two-dimensional ones by S_2 .

The intersection of three three-dimensional spheres is a circle S_1 or one-dimensional quadratic space of a special kind, because it always has two points in common with the two-dimensional space I_2 .

An infinitude (viz., a double infinitude) of spheres pass through a circle S_1 that are images of the system with three terms of linear complexes that are drawn through the semi-quadric whose image is S_1 . An infinitude of these complexes are special. Their directrices, which generate the complementary semi-quadric, have the points of a second circle S_1' for their images, a circle that is the locus of the centers of the spheres of radius zero that are drawn through S_1 . The correspondence between S_1 and S_1' is obviously reciprocal.

96. The planar sheaves of lines and the hyper-sheaves in ruled space also have a very simple representation.

If the line x generates a planar sheaf of lines then, as we know, one can write:

$$x_1 = a_1 + \rho\, b_1, \qquad x_2 = a_2 + \rho\, b_2\,, \qquad x_3 = a_3 + \rho\, b_3\,,$$
$$x_4 = a_4 + \rho\, b_4, \qquad x_5 = 1 + \rho, \qquad x_6 = a_6 + \rho\, b_6\,.$$

The coordinates X_1, X_2 X_3, X_4 of the corresponding point in four-dimensional space will be:

(26) $$X_1 = \frac{x_1}{x_5} = \frac{a_1 + \rho b_1}{1+\rho}, \quad X_2 = \frac{a_2 + \rho b_2}{1+\rho}, \quad X_3 = \frac{a_3 + \rho b_3}{1+\rho}, \quad X_4 = \frac{a_4 + \rho b_4}{1+\rho}.$$

Moreover, one will have:

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 - x_5 x_6 = 0,$$

i.e.:

$$(a_1 + \rho\, b_1)^2 + (a_2 + \rho\, b_2)^2 + (a_3 + \rho\, b_3)^2 + (a_4 + \rho\, b_4)^2 = (1 + \rho)\,(a_6 + \rho\, b_6).$$

This must be true for any ρ, so one gets:

$$a_1^2 + a_2^2 + a_3^2 + a_4^2 = a_6\,,$$
$$b_1^2 + b_2^2 + b_3^2 + b_4^2 = b_6\,,$$
$$2a_1\, b_1 + 2a_2\, b_2 + 2a_3\, b_3 + 2a_4\, b_4 = a_6 + b_6\,,$$

so, upon eliminating a_6 and b_6, one gets:

(27) $$(a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2 + (a_4 - b_4)^2 = 0.$$

The binomials $a_i - b_i$ are the director coefficients α_i of the line that is represented by equations (26), which can be written, upon setting:

$$\alpha_i = a_i - b_i\,,$$

as

(28) $$\frac{X_1 - a_1}{\alpha_1} = \frac{X_2 - a_2}{\alpha_2} = \frac{X_3 - a_3}{\alpha_3} = \frac{X_4 - a_4}{\alpha_4}.$$

Equation (27), which is written:

(29) $$\alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \alpha_4^2 = 0,$$

obviously expresses the idea that the point of the line (28) that is at infinity belongs to the quadratic space I_2 . It also expresses the idea that the distance between two arbitrary points of the line is zero. The lines considered are lines of length zero, and can be defined by the property that they have a point in common with I_2 .

Moreover, it is natural to introduce the Chasles coordinate of the point of intersection by setting:

$$(30)\qquad \frac{\alpha_1}{\lambda_0+\mu_0} = \frac{\alpha_2}{(\lambda_0-\mu_0)\sqrt{-1}} = \frac{\alpha_3}{\lambda_0\mu_0-1} = \frac{\alpha_4}{(\lambda_0\mu_0+1)\sqrt{-1}},$$

and furthermore, the general representation of our lines (and in turn, of the planar sheaves of ruled space) will be:

$$(31)\qquad \frac{X_1-a_1}{\lambda_0+\mu_0} = \frac{X_2-a_2}{(\lambda_0-\mu_0)\sqrt{-1}} = \frac{X_3-a_3}{\lambda_0\mu_0-1} = \frac{X_4-a_4}{(\lambda_0\mu_0+1)\sqrt{-1}}.$$

It is clear that if λ_0 and μ_0 remain fixed then the point of intersection with I_2 will also remain fixed.

If one leaves λ_0 fixed then when μ_0 varies the point in question will describe a rectilinear generator of a system I_2. On the contrary, when λ_0 varies, while μ_0 remains fixed, it will describe a rectilinear generator of the second system.

The representation goes much deeper than one might first believe.

Indeed, let us seek to represent a hyper-sheaf.

If the line x generates a hyper-sheaf then one can set:

$$\begin{aligned}
x_1 &= a_1 + \rho\, b_1 + \rho'\, b_1',\\
x_2 &= a_2 + \rho\, b_2 + \rho'\, b_2',\\
x_3 &= a_3 + \rho\, b_3 + \rho'\, b_3',\\
x_4 &= a_4 + \rho\, b_4 + \rho'\, b_4',\\
x_5 &= 1 + \rho + \rho',\\
x_6 &= a_6 + \rho\, b_6 + \rho'\, b_6',
\end{aligned}$$

with the relation:

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 - x_5 x_6 = 0,$$

and since this must be true for any ρ, ρ', we will get:

$$\begin{aligned}
a_1^2 + a_2^2 + a_3^2 + a_4^2 &= a_6\,,\\
b_1^2 + b_2^2 + b_3^2 + b_4^2 &= b_6\,,\\
b_1'^2 + b_2'^2 + b_3'^2 + b_4'^2 &= b_6',\\
2(a_1\, b_1 + a_2\, b_2 + a_3\, b_3 + a_4\, b_4) &= a_6 + b_6\,,\\
2(a_1 b_1' + a_2\, b_2' + a_3\, b_3' + a_4\, b_4') &= a_6 + b_6',\\
2(b_1 b_1' + b_2\, b_2' + b_3\, b_3' + b_4\, b_4') &= b_6 + b_6';
\end{aligned}$$

thus, by eliminating a_6, b_6, b_6', one will get:

$$(a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2 + (a_4 - b_4)^2 = 0,$$

$$(a_1 - b_1')^2 + (a_2 - b_2')^2 + (a_3 - b_3')^2 + (a_4 - b_4')^2 = 0,$$
$$(b_1 - b_1')^2 + (b_2 - b_2')^2 + (b_3 - b_3')^2 + (b_4 - b_4')^2 = 0.$$

One verifies the first two equations by setting:

$$\frac{b_1 - a_1}{\lambda_0 + \mu_0} = \frac{b_2 - a_2}{(\lambda_0 - \mu_0)\sqrt{-1}} = \frac{b_3 - a_3}{\lambda_0\mu_0 - 1} = \frac{b_4 - a_4}{(\lambda_0\mu_0 + 1)\sqrt{-1}} = \theta,$$
$$\frac{b_1' - a_1}{\lambda_0' + \mu_0'} = \frac{b_2' - a_2}{(\lambda_0' - \mu_0')\sqrt{-1}} = \frac{b_3' - a_3}{\lambda_0'\mu_0' - 1} = \frac{b_4' - a_4}{(\lambda_0'\mu_0' + 1)\sqrt{-1}} = \theta',$$

in which λ_0, μ_0, λ_0', μ_0', θ, θ' are arbitrary.

One infers from this that:

$$\begin{aligned}
b_1' - b_1 &= [\theta'(\lambda_0' + \mu_0') - \theta(\lambda_0 + \mu_0)],\\
b_2' - b_2 &= [\theta'(\lambda_0' - \mu_0') - \theta(\lambda_0 - \mu_0)]\sqrt{-1},\\
b_3' - b_3 &= [\theta'(\lambda_0'\mu_0' - 1) - \theta(\lambda_0\mu_0 - 1)],\\
b_4' - b_4 &= [\theta'(\lambda_0'\mu_0' + 1) - \theta(\lambda_0\mu_0 + 1)]\sqrt{-1},
\end{aligned}$$

so:

$$\begin{aligned}
0 &= (b_1' - b_1)^2 + (b_2' - b_2)^2 + (b_3' - b_3)^2 + (b_4' - b_4)^2\\
&= -2\theta\theta'[(\lambda_0' + \mu_0')(\lambda_0 + \mu_0) - (\lambda_0' - \mu_0')(\lambda_0 - \mu_0)\\
&\qquad + (\lambda_0'\mu_0' - 1)(\lambda_0\mu_0 - 1) - (\lambda_0'\mu_0' + 1)(\lambda_0\mu_0 + 1)]\\
&= -4\theta\theta'(\lambda_0' - \lambda_0)(\mu_0' - \mu_0).
\end{aligned}$$

One sees that one must have either:

$$\lambda_0' = \lambda_0$$

or

$$\mu_0' = \mu_0.$$

For example, take $\lambda_0' = \lambda_0$.

The corresponding hyper-sheaf is represented in four-dimensional space by the equations:

$$X_i = \frac{a_i + \rho\, b_i + \rho' b_i'}{1 + \rho + \rho'},$$

or again by the equations:

$$\frac{X_1 - a_1}{\rho(b_1 - a_1) + \rho'(b_1' - a_1)} = \frac{X_2 - a_2}{\rho(b_2 - a_2) + \rho'(b_2' - a_2)}$$
$$= \frac{X_3 - a_3}{\rho(b_3 - a_3) + \rho'(b_3' - a_3)} = \frac{X_4 - a_4}{\rho(b_4 - a_4) + \rho'(b_4' - a_4)};$$

i.e., upon inserting θ into ρ and θ' into ρ':

$$\frac{X_1 - a_1}{\rho(\lambda_0 + \mu_0) + \rho'(\lambda'_0 + \mu'_0)} = \frac{X_2 - a_2}{\rho(\lambda_0 - \mu_0)\sqrt{-1} + \rho'(\lambda'_0 - \mu'_0)\sqrt{-1}}$$
$$= \frac{X_3 - a_3}{\rho(\lambda_0\mu_0 - 1) + \rho'(\lambda'_0\mu'_0 - 1)} = \frac{X_4 - a_4}{[\rho(\lambda_0\mu_0 + 1) + \rho'(\lambda'_0\mu'_0 + 1)]\sqrt{-1}}.$$

These equations, where $\rho : \rho'$ is arbitrary and even variable, define a two-dimensional linear space that represents the hyper-sheaf in question. Now – and this is a very remarkable fact – upon setting:

$$\frac{\mu_0\,\rho + \mu'_0\,\rho'}{1 + \rho'} = \mu,$$

these equations can take on the form:

$$(32)\qquad \frac{X_1 - a_1}{\lambda_0 + \mu} = \frac{X_2 - a_2}{(\lambda_0 - \mu)\sqrt{-1}} = \frac{X_3 - a_3}{\lambda_0\mu - 1} = \frac{X_4 - a_4}{(\lambda_0\mu + 1)\sqrt{-1}}.$$

These equations are deduced from equations (30) by replacing the constant parameter μ_0 with a variable parameter μ.

If one has adopted the solution $\mu'_0 = \mu_0$ then one will arrive at the formula:

$$(33)\qquad \frac{X_1 - a_1}{\lambda_0 + \mu_0} = \frac{X_2 - a_2}{(\lambda_0 - \mu_0)\sqrt{-1}} = \frac{X_3 - a_3}{\lambda_0\mu_0 - 1} = \frac{X_4 - a_4}{(\lambda_0\mu_0 + 1)\sqrt{-1}},$$

which is deduced from equations (30) by varying λ.

The linear spaces (32) and (33) are two-dimensional, since μ is variable in (32) and λ is variable in (33). In some way, they are *isotropic* two-dimensional linear spaces. They possess the property of intersecting each plane at infinity along a rectilinear generator of I_2. However, one of them intersects I_2 along a generator of one system [equation (32)], while the other one, along a generator of the opposite system [equation (33)].

We thus have two types of isotropic linear spaces E'_2 ([14]).

The one type corresponds to hyper-sheaves that are sprays, while the other one, to the hyper-sheaves that are planar systems.

It is assuredly quite curious that the separation of the two systems of generators of I_2 amounts to the distinction between the geometry of points in three-dimensional space, which is the domain of ruled figures, and that of the planes.

For example, if we take the equations:

$$(34)\qquad \frac{X_1 - a_1}{\lambda_0 + \mu_0} = \frac{X_2 - a_2}{(\lambda_0 - \mu_0)\sqrt{-1}} = \frac{X_3 - a_3}{\lambda_0\mu_0 - 1} = \frac{X_4 - a_4}{(\lambda_0\mu_0 + 1)\sqrt{-1}}$$

([14]) This fact is not new. The isotropic lines in the plane already define two distinct families.

then we will have the representation of a planar sheaf (O, π) that includes the line Λ that has the point a_1, a_2, a_3, a_4 for its image in four-dimensional space. When λ_0, μ_0 take on all possible values, *we will get all of the planar sheaves that contain* Λ.

If λ_0 remains fixed then, as we know, the line X will generate a hyper-sheaf, one of whose elements O or π remains fixed – for example, O – and then equations (34) will represent all of the lines that issue from O.

On the contrary, if it is μ_0 that remains fixed then it will be the plane π that is found to be fixed and represented as the support of a planar system of lines.

Therefore, in summation, when a sheaf is represented by formulas such as (34) then a_1, a_2, a_3, a_4 will represent a line of that sheaf, λ_0, the point O, and μ_0, the plane π of the sheaf on that line ([15]).

If one relates λ_0, μ_0 homographically then the locus of the line X will be a *singular* linear congruence that admits Λ for its directrix ([16]).

There exist other coordinates systems, but their study naturally leads into a series of infinitesimal properties.

I will add that the coordinates that I have defined projectively at the beginning can take on an important metric form. We shall return to the metric properties of ruled systems at some other time.

([15]) One can compare this with the representation that I gave in 1882 in my paper "Sur les propriétés infinitésimales de l'espace réglé," pp. 23.

([16]) The reader can compare the preceding with the chapter on penta-spherical or hexaspherical coordinates in Tome I of the *Leçons* of G. Darboux. The sphere in Euclidian space gives rise to a theory that is entirely similar to that of the line.