THE PRINCIPLES OF MECHANICS

MATHEMATICAL INVESTIGATIONS

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FOREWORD

The investigations of **Helmholtz** regarding the "Prinzipien der Statik monocyklischer Systeme" and "die physikalische Bedeutung des Princips der kleinsten Wirkung" have led me to generalize the definition of force and its measurement that is given in the mechanics of ponderable masses, and on the basis of that extension, to exhibit the analytical form of the more general principles of mechanics that it implies, which subsume the known principles as special cases. However, I would like to regard all of the extended mechanical principles as only mathematical truths that, it seems to me, might allow the essence and meanings of the theorems of the mechanics of ponderable masses to emerge a bit clearer than when one determines them directly from experiments on the basis of **Newton**'s laws. However, I shall keep myself fundamentally distant from the discussion of the question of whether the more general treatment of the theorems of mechanics is in any way suitable for representing physical processes of a complicated nature, just as **Helmholtz** succeeded in describing physical processes by not assuming that a separation of actual and potential energy was given in the expression for the first-order kinetic potential. However, in extending the concept of kinetic potential, it was essential for me to investigate the extension of the principles of "hidden motion" and "incomplete problems" that **Helmholtz** introduced into the mechanics of ponderable masses and **Hertz** made the foundation of his mechanics, and to discuss the question in general of when a mechanical problem with a certain number of parameters and under the influence of forces of any order can be reduced to a problem with a larger or smaller number of parameters under the action of forces of lower or higher order, by which, among other things, the motion of two mass-points that move according to **Weber**'s law can be described by the motion of three points, two of which attract each other according to **Newton's law**, while the third one is coupled to the other two in a well-defined way and acts only by way of its inertia.

Finally, it seems to me to be essential to point out that the **Laplace-Poisson** partial differential equation also has its analogue in the mechanics of higher-order forces, and that, just as in the theory of the ordinary **Newtonian** potential, the extended **Newtonian** potential finds a variety of applications in the treatment of problems of motion under the influence of higher-order forces.

For the extension of the principle of least action while preserving the law of energy, and for presentation of the general system of **Hamilton**'s total differential equations, as well as the associated partial differential equation, I refer to the works of:

Ostrogradsky, "Mémoires sur les équations différentielles relative au problème des isopérimètres," Mém. de l'acad. de St. Pétersbourg, sc. math. et phys. **4** (1850).

and

Jacobi, "De aequationum differentialium isoperimetricarum transformationibus earumque reductione ad aequationem differentialem partialem primi ordinis non linearem," *Gesammelte Werke* V.

For the treatment of the principle of least action in the mechanics of ponderable masses, the works of:

A. Mayer, "Die beiden allgemeinen Sätze der Variationsrechnung, welche den beiden Formen des Princips der kleinsten Action in der Dynamik entsprechen," Verh. kön. Ges. Wiss. Leipzig (1886).

and

Helmholtz, "Ueber die physikalische Bedeutung des Princips der kleinsten Action." *Wissenschaftliche Abhandlungen*, Bd. III.

should come under consideration, to which I add:

Réthy, "Ueber das Princip der kleinsten Action," Math. Ann., Bd. 48,

in which the validity of the action principle based on the aforementioned works of **Helmholtz** is proved without the assistance of the law of *vis viva*. Finally, in relation to that, I should refer to the works of:

Hölder, "Die Principien von Hamilton und Maupertuis," Nach. kön. Ges. Wiss. zu Göttingen, math.-phys. Cl. (1900).

For works that were included in my first publications on the principle of mechanics, in regard to the proof of the existence of the kinetic potential, I would like to stress:

A. Mayer, "Die Existenzbedingungen eines kinetischen Potentials," Ber. kön. Ges. Wiss. zu Leipzig (1896).

A. Hirsch, "Ueber eine charakteristische Eigenschaft der Differentialgleichungen der Variationsrechnung," Math. Ann., Bd. 49.

K. Boehm, "Die Existenzbedingungen eines kinetischen Potentials höherer Ordnung," J. f. Mathematik," Bd. 121.

Heidelberg in November 1900.

The author.

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§ 1. – The extended d'Alembert principle.

When one generalizes the known principles of mechanics, one must anticipate the validity of **Newton**'s laws, namely:

1, A body will remain in a state of rest or uniform rectilinear motion as long as it is not compelled to change its state by the application of external forces (Law of Inertia),

2. The change in the motion of a body is proportional to the applied force and points in the same direction, which implies a measure of the force as the product of the mass and the acceleration.

However, one maintains the viewpoint that the specialization of those general principle that comes about when one establishes **Newton**'s laws should lead to the known laws of the mechanics of ponderable masses.

If a point moves along a straight line whose arc-length from a fixed starting point might be measured by s, and if S is a certain function of the motion whose properties will be given later, and which will be called the *force* (for reasons that will be given later), then the *work* done by the displacement of the point through the distance ds by the motion or the forces in that direction might be defined by the product:

$$S\,ds\,,$$

although nothing further is initially assumed regarding the measure of the forces that is defined by the motion. If we now define a fixed coordinate system and, while assuming the decomposability of the forces, denote the components of the forces along the three axes by X, Y, Z, and the infinitely-small increments that correspond to ds by dx, dy, dz then the work done can also be represented by:

$$X\,dx + Y\,dy + Z\,dz\,.$$

If an arbitrary system of n points is now supposed to move then when we denote the forces that act upon the i^{th} point along the coordinates axes by X_i, Y_i, Z_i, the *total work* done on the system will be defined by the sum:

$$\sum_{i=1}^{n}(X_i\,dx_i + Y_i\,dy_i + Z_i\,dz_i)\,,$$

when we denote the simultaneous variations of the coordinates of the point by dx_i, dy_i, dz_i. If we subject the system to arbitrary constraints, such that its motion will be a different one, then the forces that must act upon the individual points of the system along the coordinate directions in order to allow the motion that now takes place must be different from the previous ones, and the total work that is done on the system will be represented by:

$$\sum_{i=1}^{n}(X_i' \, dx_i + Y_i' dy_i + Z_i' \, dz_i) \, ,$$

in which the dx_i, dy_i, dz_i can be different from before, in general.

We shall now state the following demand as the extended **d'Alembert** principle for the general mechanics:

The total work that is done on the newly-constrained system is equal to the total work done on the original system under the same displacements, and indeed for all of the ones that points of the newly-restricted system experience at all.

When all possible or *virtual* displacements of the coordinates are denoted by δx_i, δy_i, δz_i , that principle can be represented by the equation:

(1) $$\sum_{i=1}^{n}(X_i' \, dx_i + Y_i' dy_i + Z_i' \, dz_i) = \sum_{i=1}^{n}(X_i \, dx_i + Y_i \, dy_i + Z_i \, dz_i) \, .$$

Now, if the new constraint state is characterized by the fact that the x_i, y_i, z_i depend upon μ mutually-independent quantities p_1, p_2, …, p_μ then for $s = 1, 2, \ldots, \mu$, a virtual motion (among others) will be represented by:

$$\delta p_1 = \delta p_2 = \ldots = \delta p_{s-1} = \delta p_{s+1} = \ldots = \delta p_\mu = 0$$

while δp_s remains arbitrary, and the relations will then follow from (1):

(2) $$\sum_{i=1}^{n}\left(X_i' \frac{\partial x_i}{\partial p_s} + Y_i' \frac{\partial y_i}{\partial p_s} + Z_i' \frac{\partial z_i}{\partial p_s} \right) = \sum_{i=1}^{n}\left(X_i \frac{\partial x_i}{\partial p_s} + Y_i \frac{\partial y_i}{\partial p_s} + Z_i \frac{\partial z_i}{\partial p_s} \right) \qquad (s = 1, 2, \ldots, \mu).$$

If one now calls the mutually-independent quantities p_1, p_2, …, p_μ the *free* coordinates then one can speak of a force P_s that must act in the direction of the coordinate p_s in order for the motion of the system to proceed in the assumed way, and since all of the δp with the exception of δp_s are zero, the total work done on the new system that is restricted by constraints, in that sense, will be defined by the expression:

$$P_s \, \delta P_s \, ,$$

such that when one uses (2), the extended **d'Alembert** principle can also be put into the form:

(3) $$P_s = \sum_{i=1}^{n}\left(X_i \frac{\partial x_i}{\partial p_s} + Y_i \frac{\partial y_i}{\partial p_s} + Z_i \frac{\partial z_i}{\partial p_s} \right) \qquad (s = 1, 2, \ldots, \mu),$$

and when one lets $\frac{\partial x_i}{\partial p_s}$, $\frac{\partial y_i}{\partial p_s}$, $\frac{\partial z_i}{\partial p_s}$ denote the products of those quantities with the projections of the forces that act along the directions of the coordinate axes X_i, Y_i, Z_i onto the direction of the coordinate p_s , one can also interpret this by saying:

During the duration of the motion, the force that acts upon the coordinate p_s is equal to the sum of the projections of all forces that act upon the points of the original system onto the direction of p_s.

§ 2. – Analytical expression for the measure of force.

In order to arrive at an expression for the measure of force, it will be necessary to satisfy the relation:

$$(1) \qquad P_s = \sum_{i=1}^{n} \left(X_i \frac{\partial x_i}{\partial p_s} + Y_i \frac{\partial y_i}{\partial p_s} + Z_i \frac{\partial z_i}{\partial p_s} \right),$$

which is valid from the conventions that were made above under the assumption of the decomposability of the forces, in the most general way, and indeed in such a way that the measures of the forces that point along the coordinate axes depend upon only the corresponding coordinates and their derivatives with respect to time. However, we must preface a few remarks in regard to the handling of that problem that will also be applied many times later on.

Lemma 1:

Let $p_1, p_2, \ldots, p_\mu$ be quantities that depend upon time t, and let:

$$R = f(t, p_1, p_1', \ldots, p_1'^{(\nu)}, p_2, p_2', \ldots, p_2'^{(\nu)}, \ldots, p_\mu, p_\mu', \ldots, p_\mu'^{(\nu)}) ,$$

in which n represents the highest-order of derivatives with respect to t that are taken. It will then follow from the fact that:

$$\delta R^{(\rho)} = \frac{d^\rho \delta R}{dt^\rho},$$

in which $R^{(\rho)}$ means the ρ^{th} derivatives with respect to *t*, that one has:

$$\delta R^{(\rho)} = \sum_{\lambda=1}^{\mu} \left\{ \frac{\partial R^{(\rho)}}{\partial p_\lambda} \delta p_\lambda + \frac{\partial R^{(\rho)}}{\partial p_\lambda'} \delta p_\lambda' + \cdots + \frac{\partial R^{(\rho)}}{\partial p_\lambda^{(\nu+\rho)}} \delta p_\lambda^{(\nu+\rho)} \right\}$$

and

$$\frac{d^\rho \delta R}{dt^\rho} = \frac{d^\rho}{dt^\rho} \sum_{\lambda=1}^{\mu} \sum_{\alpha=0}^{\nu} \frac{\partial R}{\partial p_\lambda^{(\alpha)}} \delta p_\lambda^{(\alpha)}$$

$$= \sum_{\lambda=1}^{\mu} \sum_{\alpha=0}^{\nu} \left\{ \frac{d^\rho}{dt^\rho} \frac{\partial R}{\partial p_\lambda^{(\alpha)}} \delta p_\lambda^{(\alpha)} + \rho_1 \frac{d^{\rho-1}}{dt^{\rho-1}} \frac{\partial R}{\partial p_\lambda^{(\alpha)}} \delta p_\lambda^{(\alpha+1)} + \cdots + \frac{\partial R}{\partial p_\lambda^{(\alpha)}} \delta p_\lambda^{(\alpha+\rho)} \right\},$$

when one sets the coefficients of the corresponding variations equal to each other, one will have:

$$(2) \quad \frac{\partial R^{(\rho)}}{\partial p_\lambda^{(\rho-\kappa)}} = \frac{d^\rho}{dt^\rho} \frac{\partial R}{\partial p_\lambda^{(\alpha)}} + \rho_1 \frac{d^{\rho-1}}{dt^{\rho-1}} \frac{\partial R}{\partial p_\lambda^{(\alpha)}} + \rho_2 \frac{d^{\rho-2}}{dt^{\rho-2}} \frac{\partial R}{\partial p_\lambda^{(\rho-\kappa-2)}} + \cdots + \rho_{\rho-\kappa} \frac{d^\alpha}{dt^\alpha} \frac{\partial R}{\partial p_\lambda} \quad (\kappa = 1, 2, \ldots, \rho)$$

and

(3) $$\frac{\partial R^{(\rho)}}{\partial p_\lambda^{(\rho+\kappa)}} = \frac{d^\rho}{dt^\rho}\frac{\partial R}{\partial p_\lambda^{(\rho+\kappa)}} + \rho_1 \frac{d^{\rho-1}}{dt^{\rho-1}}\frac{\partial R}{\partial p_\lambda^{(\rho+\kappa-1)}} + \rho_2 \frac{d^{\rho-2}}{dt^{\rho-2}}\frac{\partial R}{\partial p_\lambda^{(\rho+\kappa-2)}} + \cdots + \frac{\partial R}{\partial p_\lambda^{(\kappa)}} \qquad (\kappa = 1, 2, \ldots, \nu).$$

If R depends upon only t, $p_1, \ldots, p_\mu$ *, and not their derivatives, then it will follow from* (2) that for $\kappa = \rho$ and $\kappa = \rho - \sigma$, when $\sigma \le \rho$, that:

$$\frac{\partial R^{(\rho)}}{\partial p_\lambda} = \frac{d^\rho}{dt^\rho}\frac{\partial R}{\partial p_\lambda} \qquad \text{and} \qquad \frac{\partial R^{(\rho)}}{\partial p_\lambda^{(\sigma)}} = \rho_\sigma \frac{d^{\rho-\sigma}}{dt^{\rho-\sigma}}\frac{\partial R}{\partial p_\lambda},$$

and when the latter equation is differentiated σ times with respect to t, those two equations will imply the *frequently-applied relation:*

(4) $$\frac{d^\sigma}{dt^\sigma}\frac{\partial R^{(\rho)}}{\partial p_\lambda^{(\sigma)}} = \rho_\sigma \frac{\partial R^{(\rho)}}{\partial p_\lambda}.$$

If R depends upon not only $t, p_1, \ldots, p_\mu$, but also the first derivatives of those quantities, then (2) will imply the following relations for $\kappa = \rho$, $\kappa = \rho - 1$, and $\kappa = \rho - \sigma$:

$$\frac{\partial R^{(\rho)}}{\partial p_\lambda} = \frac{d^\sigma}{dt^\sigma}\frac{\partial R}{\partial p_\lambda}, \qquad \frac{\partial R^{(\rho)}}{\partial p'_\lambda} = \frac{d^\sigma}{dt^\sigma}\frac{\partial R}{\partial p'_\lambda} + \rho \frac{d^{\sigma-1}}{dt^{\sigma-1}}\frac{\partial R}{\partial p_\lambda},$$

$$\frac{\partial R^{(\rho)}}{\partial p_\lambda^{(\sigma)}} = \rho_{\sigma-1} \frac{d^{\rho-\sigma+1}}{dt^{\rho-\sigma+1}}\frac{\partial R^{(\rho)}}{\partial p_\lambda} + \rho_\sigma \frac{d^{\rho-\sigma}}{dt^{\rho-\sigma}}\frac{\partial R}{\partial p_\lambda},$$

and the relation will once more follow from this that:

(5) $$\frac{d^\sigma}{dt^\sigma}\frac{\partial R^{(\rho)}}{\partial p_\lambda^{(\sigma)}} = (\rho_\sigma - \rho_1 \rho_{\sigma-1})\frac{\partial R^{(\rho)}}{\partial p_\lambda} + \rho_{\sigma-1}\frac{d}{dt}\frac{\partial R^{(\rho)}}{\partial p'_\lambda},$$

while the first two of them and the equation:

$$\frac{\partial R^{(\rho)}}{\partial p_\lambda^{(\rho+1)}} = \frac{\partial R}{\partial p'_\lambda},$$

which follows from (3) for $\kappa = 1$, will imply the relation:

(6) $$\frac{d^{\rho+1}}{dt^{\rho+1}}\frac{\partial R^{(\rho)}}{\partial p_\lambda^{(\rho+1)}} = -\rho \frac{\partial R^{(\rho)}}{\partial p_\lambda} + \frac{d}{dt}\frac{\partial R^{(\rho)}}{\partial p'_\lambda},$$

and similar formulas that express:

$$\frac{d^\sigma}{dt^\sigma}\frac{\partial R^{(\rho)}}{\partial p_\lambda^{(\sigma)}} \qquad \text{in terms of} \qquad \frac{\partial R^{(\rho)}}{\partial p_\lambda}, \ \frac{d}{dt}\frac{\partial R^{(\rho)}}{\partial p'_\lambda}, \ \ldots, \ \frac{d^\alpha}{dt^\alpha}\frac{\partial R^{(\rho)}}{\partial p_\lambda^{(\alpha)}}$$

linearly and homogeneously when R also includes the 2nd, 3rd, … αth derivatives of the p.

In order to establish the relations between the partial and derivatives with respect to t of the function R, which depends upon t, p_1, …, p_μ, and their derivatives up to order ν, one remarks that:

$$\frac{dR}{dt} = \frac{\partial R}{\partial t} + \sum_{\lambda=1}^{\mu}\frac{\partial R}{\partial p_\lambda}p'_\lambda + \sum_{\lambda=1}^{\mu}\frac{\partial R}{\partial p'_\lambda}p''_\lambda + \cdots + \sum_{\lambda=1}^{\mu}\frac{\partial R}{\partial p_\lambda^{(\nu)}}p_\lambda^{(\nu+1)}$$

will imply that:

$$\frac{\partial}{\partial t}\frac{dR}{dt} = \frac{\partial^2 R}{\partial t^2} + \sum_{\lambda=1}^{\mu}\frac{\partial^2 R}{\partial p_\lambda\,\partial t}p'_\lambda + \sum_{\lambda=1}^{\mu}\frac{\partial^2 R}{\partial p'_\lambda\,\partial t}p''_\lambda + \cdots + \sum_{\lambda=1}^{\mu}\frac{\partial^2 R}{\partial p_\lambda^{(\nu)}\,\partial t}p_\lambda^{(\nu+1)},$$

and since:

$$\frac{\partial}{\partial t}\frac{dR}{dt} = \frac{\partial^2 R}{\partial t^2} + \sum_{\lambda=1}^{\mu}\frac{\partial^2 R}{\partial t\,\partial p_\lambda}p'_\lambda + \sum_{\lambda=1}^{\mu}\frac{\partial^2 R}{\partial t\,\partial p'_\lambda}p''_\lambda + \cdots + \sum_{\lambda=1}^{\mu}\frac{\partial^2 R}{\partial t\,\partial p_\lambda^{(\nu)}}p_\lambda^{(\nu+1)},$$

the equation:

(7)
$$\frac{\partial}{\partial t}\frac{dR}{dt} = \frac{d}{dt}\frac{\partial R}{\partial t}$$

will follow, and from that, one will have:

(8)
$$\frac{\partial^\lambda}{\partial t^\lambda}\frac{dR}{dt} = \frac{d}{dt}\frac{\partial^\lambda R}{\partial t^\lambda}$$

in general, or when R is replaced with R', R'', … :

(9)
$$\frac{\partial^\lambda}{\partial t^\lambda}\frac{d^\mu R}{dt^\mu} = \frac{d^\mu}{dt^\mu}\frac{\partial^\lambda R}{\partial t^\lambda}.$$

Lemma 2:

If R_1, R_2, … *are functions of* t, p_1, p_2, …, p_μ, *and* V *is a function that depends upon* t, R_1, R'_1, …, $R_1^{(\nu_1)}$, R_2, R'_2, …, $R_2^{(\nu_2)}$, … then from the principles of the calculus of variations, under the assumption that the variations of p_1, p_2, …, p_μ, as well as their derivatives up to orders $\nu_1 - 1$, $\nu_2 - 1$, …, vanish for $t = t_0$ and $t = t_1$, one will have:

$$\delta \int_{t_0}^{t_1} V(t, R_1, R_1', \ldots, R_1^{(\nu_1)}, R_2, R_2', \ldots, R_2^{(\nu_1)}, \ldots)\, dt$$

$$= \int_{t_0}^{t_1} \sum_{\alpha=1,2,\ldots} \left\{ \frac{\partial V}{\partial R_\alpha} - \frac{d}{dt}\frac{\partial V}{\partial R_\alpha'} + \cdots + (-1)^{\nu_\alpha} \frac{d^{\nu_\alpha}}{dt^{\nu_\alpha}} \frac{\partial V}{\partial R_\alpha^{(\nu_\alpha)}} \right\} \delta R_\alpha \, dt$$

$$= \int_{t_0}^{t_1} \sum_{\alpha=1,2,\ldots} \left\{ \frac{\partial V}{\partial R_\alpha} - \frac{d}{dt}\frac{\partial V}{\partial R_\alpha'} + \cdots + (-1)^{\nu_\alpha} \frac{d^{\nu_\alpha}}{dt^{\nu_\alpha}} \frac{\partial V}{\partial R_\alpha^{(\nu_\alpha)}} \right\}$$
$$\times \left(\frac{\partial R_\alpha}{\partial p_1} \delta p_1 + \frac{\partial R_\alpha}{\partial p_2} \delta p_2 + \cdots + \frac{\partial R_\alpha}{\partial p_\mu} \delta p_\mu \right) dt .$$

However, when the greatest of the numbers ν_1, ν_2, … is denoted by ν, since one further has:

$$\delta \int_{t_0}^{t_1} V(t, R_1, R_1', \ldots, R_1^{(\nu_1)}, R_2, R_2', \ldots, R_2^{(\nu_1)}, \ldots)\, dt$$
$$= \int_{t_0}^{t_1} \sum_{\lambda=1}^{\mu} \left\{ \frac{\partial V}{\partial p_\lambda} - \frac{d}{dt}\frac{\partial V}{\partial p_\lambda'} + \cdots + (-1)^{\nu} \frac{d^{\nu}}{dt^{\nu}} \frac{\partial V}{\partial p_\lambda^{(\nu)}} \right\} \delta p_\lambda \, dt ,$$

a comparison of the two expressions will give the relation:

$$\text{(10)} \qquad \frac{\partial V}{\partial p_\lambda} - \frac{d}{dt}\frac{\partial V}{\partial p_\lambda'} + \cdots + (-1)^{\nu} \frac{d^{\nu}}{dt^{\nu}} \frac{\partial V}{\partial p_\lambda^{(\nu)}} = \sum_{\alpha=1,2,\ldots} \left\{ \frac{\partial V}{\partial R_\alpha} - \frac{d}{dt}\frac{\partial V}{\partial R_\alpha'} + \cdots + (-1)^{\nu_\alpha} \frac{d^{\nu_\alpha}}{dt^{\nu_\alpha}} \frac{\partial V}{\partial R_\alpha^{(\nu_\alpha)}} \right\} \frac{\partial R_\alpha}{\partial p_\lambda} ,$$

which will define the foundation for all further considerations when $\nu_1 = \nu_2 = \ldots = \nu$.

That second lemma will next allow one to find solutions to equation (1) that are already very general. Namely, let:

$$T_x\,(t, x, x', x'', \ldots, x^{(\nu)}) ,$$

in which ν is an arbitrary positive whole number, be an arbitrary function of its arguments and set:

$$T = \sum_{i=1}^{n} T_{x_i}\,(t, x_i, x_i', \ldots, x_i^{(\nu)}) + \sum_{i=1}^{n} T_{y_i}\,(t, y_i, y_i', \ldots, y_i^{(\nu)}) + \sum_{i=1}^{n} T_{z_i}\,(t, z_i, z_i', \ldots, z_i^{(\nu)}) .$$

When the rectangular coordinates that are introduced by arbitrary constraints are expressed in terms of μ free coordinates $p_1, p_2, \ldots, p_\mu$, and one further denotes the resulting value of T by (T) then (10) will imply that the following relation must exist:

$$-\frac{\partial(T)}{\partial p_s}+\frac{d}{dt}\frac{\partial(T)}{\partial p_s'}-\cdots+(-1)^{\nu-1}\frac{d^\nu}{dt^\nu}\frac{\partial(T)}{\partial p_s^{(\nu)}}$$

$$= \sum_{i=1}^{n}\left\{\left(-\frac{\partial T}{\partial x_i}+\frac{d}{dt}\frac{\partial T}{\partial x_i'}-\cdots+(-1)^{\nu-1}\frac{d^\nu}{dt^\nu}\frac{\partial T}{\partial x_s^{(\nu)}}\right)\frac{\partial x_i}{\partial p_s}\right.$$
$$+\left(-\frac{\partial T}{\partial y_i}+\frac{d}{dt}\frac{\partial T}{\partial y_i'}-\cdots+(-1)^{\nu-1}\frac{d^\nu}{dt^\nu}\frac{\partial T}{\partial y_s^{(\nu)}}\right)\frac{\partial y_i}{\partial p_s}$$
$$\left.+\left(-\frac{\partial T}{\partial z_i}+\frac{d}{dt}\frac{\partial T}{\partial z_i'}-\cdots+(-1)^{\nu-1}\frac{d^\nu}{dt^\nu}\frac{\partial T}{\partial z_s^{(\nu)}}\right)\frac{\partial z_i}{\partial p_s}\right\}$$

$$= \sum_{i=1}^{n}\left\{\left(-\frac{\partial T_{x_i}}{\partial x_i}+\frac{d}{dt}\frac{\partial T_{x_i}}{\partial x_i'}-\cdots+(-1)^{\nu-1}\frac{d^\nu}{dt^\nu}\frac{\partial T_{x_i}}{\partial x_s^{(\nu)}}\right)\frac{\partial x_i}{\partial p_s}\right.$$
$$+\left(-\frac{\partial T_{y_i}}{\partial y_i}+\frac{d}{dt}\frac{\partial T_{y_i}}{\partial y_i'}-\cdots+(-1)^{\nu-1}\frac{d^\nu}{dt^\nu}\frac{\partial T_{y_i}}{\partial y_s^{(\nu)}}\right)\frac{\partial y_i}{\partial p_s}$$
$$\left.+\left(-\frac{\partial T_{z_i}}{\partial z_i}+\frac{d}{dt}\frac{\partial T_{z_i}}{\partial z_i'}-\cdots+(-1)^{\nu-1}\frac{d^\nu}{dt^\nu}\frac{\partial T_{z_i}}{\partial z_s^{(\nu)}}\right)\frac{\partial z_i}{\partial p_s}\right\},$$

and *therefore, one of the solutions of equation* (1) *will be a measure of the force that a point that moves along the X-axis in the sense above exerts and is expressed by:*

(11) $$X = -\frac{\partial T}{\partial x}+\frac{d}{dt}\frac{\partial T}{\partial x'}-\cdots+(-1)^{\nu-1}\frac{d^\nu}{dt^\nu}\frac{\partial T}{\partial x^{(\nu)}},$$

in which n is an arbitrary positive whole number, and T is an arbitrary function of t, x, x', x'', …, $x^{(\nu)}$.

In the mechanics of ponderable masses, for which $\nu = 1$, it will then follow that a measure of the force is:

$$X = -\frac{\partial T}{\partial x}+\frac{d}{dt}\frac{\partial T}{\partial x'}.$$

However, in order to find all solutions of equation (1), as we will need to do later in our treatment of the kinetic potential, the question might be posed in the somewhat more general form:

Suppose that one is given an arbitrarily function:

$$T(r_1, r_1', \ldots, r_1^{(\nu)}, r_2, r_2', \ldots, r_2^{(\nu)}, \ldots, r_\kappa, r_\kappa', \ldots, r_\kappa^{(\nu)}),$$

in which $r_1, r_2, \ldots, r_\kappa$ are arbitrary quantities that depend upon a variable t, and $r_1^{(\alpha)}, r_2^{(\alpha)}, \ldots, r_\kappa^{(\alpha)}$ mean the α^{th} derivatives of those quantities with respect to t. What is most general form of the function f that one can compose from the partial differential quotients:

$$\frac{\partial T}{\partial r_s}, \quad \frac{\partial T}{\partial r_s'}, \quad \ldots, \quad \frac{\partial T}{\partial r_s^{(\nu)}},$$

and their total derivatives with respect to t of any order that will possess the property that when $r_1, r_2, \ldots, r_\kappa$ are made to depend in any way upon μ mutually-independent quantities $p_1, p_2, \ldots, p_\mu$ in which the variable t does not occur explicitly, and the function of $p_1, \ldots, p_\mu$ that results from substituting those values in T is denoted by (T), the same function f that is defined by:

$$\frac{\partial (T)}{\partial p_s}, \quad \frac{\partial (T)}{\partial p_s'}, \quad \ldots, \quad \frac{\partial (T)}{\partial p_s^{(\nu)}},$$

and the total differential quotients of those quantities with respect to t will be equal to the sum of the products of the f-function for the variables r_s multiplied by $\frac{\partial r_s}{\partial p_s}$, or that the following equation will be satisfied:

$$f\left(\frac{\partial (T)}{\partial p_s}, \frac{d}{dt}\frac{\partial (T)}{\partial p_s}, \frac{d^2}{dt^2}\frac{\partial (T)}{\partial p_s}, \ldots, \frac{\partial (T)}{\partial p_s'}, \frac{d}{dt}\frac{\partial (T)}{\partial p_s'}, \ldots, \frac{\partial (T)}{\partial p_s^{(\nu)}}, \frac{d}{dt}\frac{\partial (T)}{\partial p_s^{(\nu)}}, \ldots\right) \tag{12}$$

$$= \sum_{s=1}^{\kappa} f\left(\frac{\partial T}{\partial r_s}, \frac{d}{dt}\frac{\partial T}{\partial r_s}, \frac{d^2}{dt^2}\frac{\partial T}{\partial r_s}, \ldots, \frac{\partial T}{\partial r_s'}, \frac{d}{dt}\frac{\partial T}{\partial r_s'}, \ldots, \frac{\partial T}{\partial r_s^{(\nu)}}, \frac{d}{dt}\frac{\partial T}{\partial r_s^{(\nu)}}, \ldots\right)\frac{\partial r_s}{\partial p_s}$$

for every choice of the function T and any relationship between the r and the p?

Here, as we will often do in what follows in the proofs of lemmas when the principle of the proof remains precisely the same, for the sake of brevity, we would like to assume that ν has a certain value (but one that is always greater than 1, in order to not remain in the mechanics of ponderable masses), and in order to answer the question that was just posed, it will be sufficient to assume that the derivatives of the quantities r go only as far as second order.

Since the relation (2) must be true for any dependency that might exist between the r and p, it must also be true when r_1 is chosen to be an arbitrary function of p_1. One further sets $\mu = \kappa$ and $r_2 = p_2, \ldots, r_\mu = p_\mu$ such that when one chooses $s = 1$ and denotes r_1, p_1 by r and p, resp., the equation:

$$f\left(\frac{\partial (T)}{\partial p}, \frac{d}{dt}\frac{\partial (T)}{\partial p}, \ldots, \frac{\partial (T)}{\partial p'}, \frac{d}{dt}\frac{\partial (T)}{\partial p'}, \ldots, \frac{\partial (T)}{\partial p''}, \frac{d}{dt}\frac{\partial (T)}{\partial p''}, \ldots\right) \tag{13}$$

$$= f\left(\frac{\partial T}{\partial r}, \frac{d}{dt}\frac{\partial T}{\partial r}, \dots, \frac{\partial T}{\partial r'}, \frac{d}{dt}\frac{\partial T}{\partial r'}, \dots, \frac{\partial T}{\partial r''}, \frac{d}{dt}\frac{\partial T}{\partial r''}, \dots\right)\frac{\partial r}{\partial p}$$

must be satisfied for every choice of T and every relation between r and p. On the one hand, the latter demand will specify the dependency of $\mathfrak{r}_1$, $\mathfrak{r}_2$, … and r', r'', … upon each other, when one sets:

$$\frac{\partial r}{\partial p} = \mathfrak{r}_1 , \qquad \frac{\partial^2 r}{\partial p^2} = \mathfrak{r}_2 , \qquad \frac{\partial^3 r}{\partial p^3} = \mathfrak{r}_3 , \qquad \dots,$$

so:

$$r' = \mathfrak{r}_1\, p' , \qquad r'' = \mathfrak{r}_1\, p'' + \mathfrak{r}_2\, p'^2 , \qquad r''' = \mathfrak{r}_1\, p''' + 3\mathfrak{r}_2\, p''\, p' + r^3 p'^3 , \qquad \dots$$

On the other hand, it is clear that when one sets:

$$\frac{\partial T}{\partial r} = u , \qquad \frac{\partial T}{\partial r'} = v , \qquad \frac{\partial T}{\partial r''} = w ,$$

since equation (13) must exist for an arbitrary choice of T, not only will the quantities u, v, w be independent of each other, but so are the quantities:

$$\frac{d}{dt}\frac{\partial T}{\partial r} = u' = u_1 , \qquad \frac{d^2}{dt^2}\frac{\partial T}{\partial r} = u'' = u_2 , \qquad \dots$$

$$\frac{d}{dt}\frac{\partial T}{\partial r'} = v' = v_1 , \qquad \frac{d^2}{dt^2}\frac{\partial T}{\partial r'} = v'' = v_2 , \qquad \dots$$

but neither will a connection exist between the u, v, w, since the higher partial differential quotients of T with respect to r, r', r'' are independent of each other. If one now sets:

$$\frac{\partial r}{\partial p} = \mathfrak{r}_1 = \rho_1 , \quad \frac{\partial r'}{\partial p} = \frac{\partial \mathfrak{r}_1}{\partial p}\, p' = \mathfrak{r}_2\, p' = \rho_2 , \qquad \frac{\partial r''}{\partial p} = \mathfrak{r}_2\, p'' + \frac{\partial \mathfrak{r}_2}{\partial p}\, p'^2 = \mathfrak{r}_2\, p'' + \mathfrak{r}_3\, p'^2 = \rho_3 , \qquad \dots,$$

in which the quantities ρ_1, ρ_2, ρ_3, … are, in turn, mutually independent, then with the use of the relation:

$$\frac{\partial r^{(\sigma)}}{\partial p^{(\lambda)}} = \sigma_\lambda \frac{\partial r^{(\sigma-\lambda)}}{\partial p} \qquad (\lambda \leq \sigma),$$

which is implied by equation (2), and with the help of the notations that were introduced, it will follow that:

$$\frac{\partial (T)}{\partial p} = u\,\rho_1 + v\,\rho_2 + w\,\rho_3\,, \qquad \frac{\partial (T)}{\partial p'} = v\,\rho_1 + 2\,w\,\rho_2\,, \qquad \frac{\partial (T)}{\partial p''} = w\,\rho_1\,,$$

$$\frac{d}{dt}\frac{\partial (T)}{\partial p} = u_1\,\rho_1 + (u + v_1)\,\rho_2 + (v + w_1)\,\rho_3 + w\,\rho_4\,,\ \ldots,$$

$$\frac{d^2}{dt^2}\frac{\partial (T)}{\partial p} = u_2\,\rho_1 + (2u_1 + v_2)\,\rho_2 + (u + 2v_1 + w_2)\,\rho_3 + (v + 2w_1)\,\rho_4 + (v + 2w_1)\,\rho_3 + w\,\rho_5\,, \ldots,$$

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Therefore, (13) would demand the following identity for arbitrary values of all the quantities that enter into it:

$$(14) \qquad \begin{aligned} f\,(&u\,\rho_1 + v\,\rho_2 + w\,\rho_3\,,\ u_1\,\rho_1 + (u + v_1)\,\rho_2 + (v + w_1)\,\rho_3 + \rho_4\,,\\ &u_2\,\rho_1 + (2u_1 + v_2)\,\rho_2 + (u + 2\,v_1 + w_2)\,\rho_3 + (v + 2\,w_1)\,\rho_4 + w\,\rho_5\,,\ \ldots,\\ &v\,\rho_1 + 2w\,\rho_2, \qquad v_1\,\rho_1 + (v + 2\,w_2)\,\rho_2 + 2w\,\rho_3\,,\\ &v_2\,\rho_1 + (2v_1 + 2\,w_1)\,\rho_2 + (v + 2 \cdot 2\,w_1)\,\rho_3 + 2\,w\,\rho_4\,,\ \ldots,\\ &w\,\rho_1, \qquad w_1\,\rho_1 + w\,\rho_2\,,\ \ w_2\,\rho_1 + 2\,w_1\,\rho_2 + w\,\rho_3\,,\ \ldots)\\ &= \rho_1\,f\,(u, u_1, u_2, u_3, \ldots, v, v_1, v_2, v_3, \ldots, w, w_1, w_2, w_3, \ldots)\,. \end{aligned}$$

If one sets:

$$v = v_1 = v_2 = \ldots = w = w_1 = w_2 = \ldots = 0\,, \qquad \rho_1 = 1$$

in that then it will follow from the resulting equation:

$$f\,(u, u_1 + u\,\rho_2\,, u_2 + 2\,u_1\,\rho_2 + u\,\rho_3\,, u_3 + 3\,u_2\,\rho_2 + 3\,u_1\,\rho_3 + u\,\rho_4\,, \ldots, 0, 0, \ldots)$$

$$= f\,(u, u_1, u_2, u_3, \ldots, 0, 0, \ldots)$$

that of the quantities u, u_1, u_2, u_3, … in the function f, only the quantity u can occur, since the arguments on the left-hand side from the second one onwards can assume arbitrary values that are independent of u, u_1, u_2, …, due to the arbitrariness in the quantities ρ_2, ρ_3, ... Moreover, if one sets:

$$w = w_1 = w_2 = \ldots = 0\,, \qquad \rho_1 = 1\,, \qquad \rho_2 = 0\,,$$

in (14) then it will likewise follow from the equation:

$$f\,(u, v, v_1, v_1 + v\,\rho_3\,, v_3 + 3\,v_1\,\rho_3 + v\,\rho_4\,, \ldots, 0, 0, \ldots)$$

$$= f\,(u, v, v_1, v_2, v_3, \ldots, 0, 0, \ldots)$$

that the function *f* can no longer depend upon v_2, v_3, …, nor can it depend upon w_2, w_3, …, such that the necessary form of *f* that will fulfill equation (14) identically will go to:

$$(15) \qquad f\,(u\,\rho_1 + v\,\rho_2 + w\,\rho_3\,,\; v\,\rho_1 + 2\,w\,\rho_2\,,\; v_1\,\rho_1 + (v + 2\,w_1)\,\rho_2 + 2\,w\,\rho_3\,,$$
$$w\,\rho_1\,,\; w_1\,\rho_1 + w\,\rho_2\,,\; w_2\,\rho_1 + 2\,w_1\,\rho_2 + w\,\rho_3)$$
$$= \rho_1\,f\,(u, v, v_1, w, w_1, w_2)\;.$$

If we now return to the original demand that was posed by equation (12) and assume that *the function T depends upon at least two r-quantities* then if we recall (15), the equation to be fulfilled identically will assume the form:

$$(16) \qquad f\left\{\sum_\varepsilon\left(u^{(\varepsilon)}\,\rho_1^{(\varepsilon)} + v^{(\varepsilon)}\,\rho_2^{(\varepsilon)} + w^{(\varepsilon)}\,\rho_3^{(\varepsilon)}\right), \sum_\varepsilon\left(v^{(\varepsilon)}\,\rho_1^{(\varepsilon)} + 2w^{(\varepsilon)}\,\rho_2^{(\varepsilon)}\right),\right.$$
$$\sum_\varepsilon\left(v_1^{(\varepsilon)}\,\rho_1^{(\varepsilon)} + (v^{(\varepsilon)} + 2\,w_1^{(\varepsilon)})\,\rho_2^{(\varepsilon)} + 2w^{(\varepsilon)}\,\rho_3^{(\varepsilon)}\right), \sum_\varepsilon w^{(\varepsilon)}\,\rho_1^{(\varepsilon)},$$
$$\left.\sum_\varepsilon\left(w_1^{(\varepsilon)}\,\rho_1^{(\varepsilon)} + w^{(\varepsilon)}\,\rho_2^{(\varepsilon)}\right), \sum_\varepsilon\left(w_2^{(\varepsilon)}\,\rho_1^{(\varepsilon)} + 2w^{(\varepsilon)}\rho_2^{(\varepsilon)} + w^{(\varepsilon)}\,\rho_3^{(\varepsilon)}\right)\right\}$$
$$= \sum_\varepsilon \rho_1^{(\varepsilon)}\,f\,(u^{(\varepsilon)}, v^{(\varepsilon)}, v_1^{(\varepsilon)}, w^{(\varepsilon)}, w_1^{(\varepsilon)}, w_2^{(\varepsilon)})\;,$$

in which we have provided all of the corresponding quantities with the superscript (ε). However, since when it will follow from the equation that results by partial differentiation with respect to the arguments when one sets $\rho_1^{(\varepsilon)} = 1$, $\rho_2^{(\varepsilon)} = 0$, $\rho_3^{(\varepsilon)} = 0$ that the differential quotients are independent of the arguments *f* will then be a linear function of the arguments with constant coefficients that has the form:

$$f\,(u^{(\varepsilon)}, v^{(\varepsilon)}, w^{(\varepsilon)}, w_1^{(\varepsilon)}, w_2^{(\varepsilon)})$$
$$= \alpha_0 + \alpha\,u^{(\varepsilon)} + \beta\,v^{(\varepsilon)} + \beta_1\,v_1^{(\varepsilon)} + \gamma\,w^{(\varepsilon)} + \gamma_1\,w_1^{(\varepsilon)} + \gamma_2\,w_2^{(\varepsilon)}.$$

Upon substituting that expression in (16) and identifying the coefficients of $u^{(\varepsilon)}$, $v^{(\varepsilon)}$, …, and again identifying the coefficients of $\rho_1^{(\varepsilon)}$, $\rho_2^{(\varepsilon)}$, … in that, the defining equations will follow:

$$\alpha_0 = 0\,, \qquad \alpha + \beta_1 = 0\,, \qquad \beta_1 + \gamma_2 = 0\,, \qquad 2\,\beta + \beta_2 = 0\,,$$

such that *f* will assume the form:

$$f\,(u^{(\varepsilon)}, v^{(\varepsilon)}, w^{(\varepsilon)}, w_1^{(\varepsilon)}, w_2^{(\varepsilon)})$$
$$= \alpha\left(u^{(\varepsilon)} - v_1^{(\varepsilon)} + w_3^{(\varepsilon)}\right) - \beta\left(v^{(\varepsilon)} - 2\,w_1^{(\varepsilon)}\right) + \gamma\,w^{(\varepsilon)}\;,$$

in which α, β, γ remain arbitrary. Since two of the quantities α, β, γ can always be assumed to be equal to zero, that will then yield the three necessary and sufficient forms for f as solutions of equation (12) in the case of $\nu = 2$:

$$-\frac{\partial T}{\partial r_\varepsilon} + \frac{d}{dt}\frac{\partial T}{\partial r'_\varepsilon} - \frac{d^2}{dt^2}\frac{\partial T}{\partial r''_\varepsilon}, \qquad \frac{\partial T}{\partial r} - \frac{d}{dt}\frac{\partial T}{\partial r'}, \qquad -\frac{\partial T}{\partial r''},$$

and the theorem will follow for arbitrary ν in precisely the same way:

All of the functions f that satisfy equation (12) *for an arbitrary function T of* $r_\varepsilon, r'_\varepsilon, \dots, r_\varepsilon^{(\nu)}$, *in which* $\varepsilon = 1, 2, \dots, \kappa$, *and for an arbitrary dependency between* $r_1, r_2, \dots, r_\kappa$ *and* $p_1, p_2, \dots, p_\mu$, *will be included in the form:*

$$f = (-1)^{\nu-\lambda+1}\left[\frac{\partial T}{\partial r_\varepsilon^{(\nu-\lambda)}} - (\nu-\lambda+1)\frac{d}{dt}\frac{\partial T}{\partial r_\varepsilon^{(\nu-\lambda+1)}} + \frac{(\nu-\lambda+2)(\nu-\lambda+1)}{1\cdot 2}\frac{d^2}{dt^2}\frac{\partial T}{\partial r_\varepsilon^{(\nu-\lambda+2)}} - \cdots \right.$$
$$\left. + (-1)^\lambda \frac{\nu(\nu-1)\cdots(\nu-\lambda+1)}{1\cdot 2\cdots\lambda}\frac{d^\lambda}{dt^\lambda}\frac{\partial T}{\partial r_\varepsilon^{(\nu)}}\right],$$

in which λ *assumes the values* $1, 2, \dots, \nu$.

In order to remain in agreement with the mechanics of ponderable masses, if we define:

$$B_\lambda = (-1)^{\nu-\lambda+1}\left[\frac{\partial T}{\partial x^{(\nu-\lambda)}} - (\nu-\lambda+1)_1\frac{d}{dt}\frac{\partial T}{\partial x^{(\nu-\lambda+1)}} + (\nu-\lambda+2)_2\frac{d^2}{dt^2}\frac{\partial T}{\partial x^{(\nu-\lambda+2)}} - \cdots + (-1)^\lambda \nu_\lambda \frac{d^\lambda}{dt^\lambda}\frac{\partial T}{\partial x^{(\lambda)}}\right]$$

$$(\lambda = 0, 1, 2, \dots, \nu - 1)$$

to be the moment of motion of order λ *for a point that moves along the X-axis in the case of* $\nu > 1$ *then that will yield the form that was found above for general expression for the measure of the force:*

$$X = -\frac{\partial T}{\partial x} + \frac{d}{dt}\frac{\partial T}{\partial x'} + \frac{d^2}{dt^2}\frac{\partial T}{\partial x''} - \cdots + (-1)^{\nu-1}\frac{d^\nu}{dt^\nu}\frac{\partial T}{\partial x^{(\nu)}},$$

in which T is initially an arbitrary function of $x, x', x'', \dots, x^{(\nu)}$, *as the only possible one.*

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§ 3. – Analytical expression for the *vis viva*.

The expression that was obtained above for the measure of the force for the mechanics of ponderable masses was:

$$X = -\frac{\partial T}{\partial x} + \frac{d}{dt}\frac{\partial T}{\partial x'} \;,$$

in which T means an arbitrary function of x and x'. When one imposes the condition that it should be independent of x and x', in harmony with **Newton**'s laws, that equation will imply that:

$$-\frac{\partial^2 T}{\partial x^2} + \frac{d}{dt}\frac{\partial^2 T}{\partial x\,\partial x'} = 0\,, \qquad \frac{d}{dt}\frac{\partial^2 T}{\partial x'^2} = 0\,,$$

and T will then assume the form:

$$T = a\,x'^2 + f(x)\,x' + \alpha\,x + \beta\,,$$

in which a, α, β are constants, and $f(x)$ can be a function of x that is still arbitrary, while the measure of the force will be given in the form:

$$X = -\,\alpha + 2\,a\,x''.$$

If the two laws of **Newton** are to be satisfied then obviously one must have $\alpha = 0$ and $a = m / 2$. Therefore T will go to:

$$T = \tfrac{1}{2}\,m\,x'^2$$

when $\beta = 0$, which is an expression that one defines to be the *vis viva*. We would now like to also specialize the expression for the measure of the force that was found above in the general case by similar conditions in order to be led to the generalization of the expression for the *vis viva*.

To that end, in order to not interrupt the exposition of the general principles of mechanics later on, we shall already treat some lemmas at this point that will play an essential role in the general formulation of mechanics.

Lemma 3:

If V is a function of t, p_1, p_2, …, p_μ, and their derivatives with respect to t up to order ν then it is known that:

$$(1)\qquad \delta\int_{t_0}^{t_1} V\,dt = \sum_{\lambda=1}^{\mu}\left[\left(\frac{\partial V}{\partial p'_\lambda} - \frac{d}{dt}\frac{\partial V}{\partial p''_\lambda} + \cdots + (-1)^{\nu-1}\frac{d^{\nu-1}}{dt^{\nu-1}}\frac{\partial V}{\partial p^{(\nu)}_\lambda}\right)\delta p_\lambda\right]_{t_0}^{t_1}$$

$$
\begin{aligned}
&+ \sum_{\lambda=1}^{\mu}\left[\left(\frac{\partial V}{\partial p''_\lambda} - \cdots + (-1)^{\nu-2}\frac{d^{\nu-2}}{dt^{\nu-2}}\frac{\partial V}{\partial p_\lambda^{(\nu)}}\right)\delta p'_\lambda\right]_{t_0}^{t_1} \\
&+ \sum_{\lambda=1}^{\mu}\left[\frac{\partial V}{\partial p_\lambda^{(\nu)}}\,\delta p_\lambda^{(\nu-1)}\right]_{t_0}^{t_1} \\
&+ \int_{t_0}^{t_1}\sum_{\lambda=1}^{\mu}\left(\frac{\partial V}{\partial p_\lambda} - \frac{d}{dt}\frac{\partial V}{\partial p'_\lambda} + \cdots + (-1)^{\nu}\frac{d^{\nu}}{dt^{\nu}}\frac{\partial V}{\partial p_\lambda^{(\nu)}}\right)\delta p_\lambda\,dt\,.
\end{aligned}
$$

If one assumes that V is the differential quotient of a function $f(t, p_1, p'_1, \ldots, p_1^{(\nu-1)}, \ldots, p_\mu, p'_\mu, \ldots, p_\mu^{(\nu-1)})$ with respect to t then when one chooses a certain value t_1 for the upper limit of t and agrees that δp_λ, $\delta p'_\lambda$, …, $\delta p_\lambda^{(\nu-1)}$ should vanish for t_0 and t_1, then because one has:

$$
\delta\int_{t_0}^{t_1} V\,dt = \delta\int_{t_0}^{t_1}\frac{df}{dt}\,dt = \delta\,[\,f\,]_{t_0}^{t_1} = 0\,,
$$

from (1), the following identity relation should exist:

$$
\frac{\partial V}{\partial p_\lambda} - \frac{d}{dt}\frac{\partial V}{\partial p'_\lambda} + \cdots + (-1)^{\nu}\frac{d^{\nu}}{dt^{\nu}}\frac{\partial V}{\partial p_\lambda^{(\nu)}} = 0\,. \tag{2}
$$

That is then the necessary condition for V to be representable as the differential quotient with respect to t of a function, and in that case, under the assumption that δp_λ, $\delta p'_\lambda$, …, $\delta p_\lambda^{(\nu-1)}$ vanish for t_0, equation (1) will yield the variational expression:

$$
\begin{aligned}
\delta\int_{t_0}^{t_1} V\,dt &= \sum_{\lambda=1}^{\mu}\left(\frac{\partial V}{\partial p'_\lambda} - \frac{d}{dt}\frac{\partial V}{\partial p''_\lambda} + \cdots + (-1)^{\nu-1}\frac{d^{\nu-1}}{dt^{\nu-1}}\frac{\partial V}{\partial p_\lambda^{(\nu)}}\right)\delta p_\lambda \\
&+ \sum_{\lambda=1}^{\mu}\left(\frac{\partial V}{\partial p''_\lambda} - \frac{d}{dt}\frac{\partial V}{\partial p'''_\lambda} + \cdots + (-1)^{\nu-2}\frac{d^{\nu-2}}{dt^{\nu-2}}\frac{\partial V}{\partial p_\lambda^{(\nu)}}\right)\delta p'_\lambda + \ldots \\
&+ \sum_{\lambda=1}^{\mu}\frac{\partial V}{\partial p_\lambda^{(\nu)}}\,\delta p_\lambda^{(\nu-1)}\,.
\end{aligned} \tag{3}
$$

The fact that equation (3) implies an immediately-obvious identity follows from (2) and (3) of **§ 2**, from which:

$$
\frac{\partial f'}{\partial p'_\lambda} - \frac{d}{dt}\frac{\partial f'}{\partial p''_\lambda} + \frac{d^2}{dt^2}\frac{\partial f'}{\partial p'''_\lambda} - \cdots = \left(\frac{d}{dt}\frac{\partial f'}{\partial p'_\lambda} + \frac{\partial f}{\partial p_\lambda}\right),
$$

$$-\frac{d}{dt}\left(\frac{d}{dt}\frac{\partial f}{\partial p''_\lambda}+\frac{\partial f}{\partial p'_\lambda}\right)+\frac{d^2}{dt^2}\left(\frac{d}{dt}\frac{\partial f}{\partial p'''_\lambda}-\frac{\partial f}{\partial p''_\lambda}\right)-\cdots = \frac{\partial f}{\partial p_\lambda}$$

$$\frac{\partial f}{\partial p''_\lambda}-\frac{d}{dt}\frac{\partial f}{\partial p'''_\lambda}+\cdots = \left(\frac{d}{dt}\frac{\partial f}{\partial p''_\lambda}+\frac{\partial f}{\partial p'_\lambda}\right)+\frac{d}{dt}\left(\frac{d}{dt}\frac{\partial f}{\partial p'''_\lambda}-\frac{\partial f}{\partial p''_\lambda}\right)+\cdots = \frac{\partial f}{\partial p'_\lambda},$$

etc., such that equation (3) will go to:

$$\delta f = \sum_{\lambda=1}^{\mu}\left(\frac{\partial f}{\partial p_\lambda}\,\delta p_\lambda+\frac{\partial f}{\partial p'_\lambda}\,\delta p'_\lambda+\cdots+\frac{\partial f}{\partial p_\lambda^{(\nu-1)}}\,\delta p_\lambda^{(\nu-1)}\right),$$

due to $V = f'$.

Now, the converse of that theorem, which is known already, namely, that the existence of the identity (2) is also the sufficient condition for V to be representable as the differential quotient with respect to t of a function of t, $p_1, \ldots, p_\mu$, and its derivatives, will be established with the help of the relations (2) and (3) of § **2**, to the extent that would seem appropriate for the further applications of that theorem.

If, in fact, equation (2) is satisfied identically, such that equations (3) will be true under the assumption that the variations δp_λ, $\delta p'_\lambda$, …, $\delta p_\lambda^{(\nu-1)}$ vanish for $t = t_0$, or when one sets:

(4) $$\frac{\partial V}{\partial p_\lambda^{(\rho)}}-\frac{d}{dt}\frac{\partial V}{\partial p_\lambda^{(\rho+1)}}+\cdots+(-1)^{\nu-\rho}\frac{d^{\nu-\rho}}{dt^{\nu-\rho}}\frac{\partial V}{\partial p_\lambda^{(\nu)}} = V_{\rho\lambda},$$

such that:

(5) $$\delta\int_{t_0}^{t} V\,dt = \sum_{\lambda=1}^{\mu} V_{1\lambda}\,\delta p_\lambda+\sum_{\lambda=1}^{\mu} V_{2\lambda}\,\delta p'_\lambda+\cdots+\sum_{\lambda=1}^{\mu} V_{\nu\lambda}\,\delta p_\lambda^{(\nu-1)},$$

then by means of (2) and (4):

(6) $$\frac{\partial V}{\partial p_\lambda}-\frac{dV_{1\lambda}}{dt} = 0, \quad \frac{\partial V}{\partial p'_\lambda}-\frac{dV_{2\lambda}}{dt} = V_{1\lambda}, \quad \frac{\partial V}{\partial p''_\lambda}-\frac{dV_{3\lambda}}{dt} = V_{2\lambda}, \quad \ldots, \quad \frac{\partial V}{\partial p_\lambda^{(\nu-1)}}-\frac{dV_{\nu\lambda}}{dt} = V_{\nu-1\,\lambda}$$

will be fulfilled identically, and partially-differentiating the first of those equations for λ_1 and λ_2 with respect to p_{λ_2} and p_{λ_1} will next give the identity:

(7) $$\frac{d}{dt}\left(\frac{\partial V_{1\lambda_1}}{\partial p_{\lambda_2}}-\frac{\partial V_{1\lambda_2}}{\partial p_{\lambda_1}}\right) = 0.$$

It likewise follows from the first and second of equations (6) with the use of the cited auxiliary formulas that:

$$\frac{\partial^2 V}{\partial p_{\lambda_1}\,\partial p'_{\lambda_2}} - \frac{d}{dt}\frac{\partial V_{1\lambda_1}}{\partial p'_{\lambda_2}} - \frac{\partial V_{1\lambda_1}}{\partial p_{\lambda_2}} = 0$$

and

$$\frac{\partial^2 V}{\partial p'_{\lambda_2}\,\partial p_{\lambda_1}} - \frac{d}{dt}\frac{\partial V_{2\lambda_2}}{\partial p'_{\lambda_1}} - \frac{\partial V_{1\lambda_2}}{\partial p_{\lambda_1}} = 0\,,$$

and therefore, due to (7):

(8)
$$\frac{d^2}{dt^2}\left(\frac{\partial V_{1\lambda_1}}{\partial p'_{\lambda_2}} - \frac{\partial V_{2\lambda_2}}{\partial p_{\lambda_1}}\right) = 0\,,$$

whereas from the second of equations (6):

$$\frac{\partial^2 V}{\partial p'_{\lambda_1}\,\partial p'_{\lambda_2}} - \frac{d}{dt}\frac{\partial V_{2\lambda_1}}{\partial p'_{\lambda_2}} - \frac{\partial V_{2\lambda_1}}{\partial p_{\lambda_2}} - \frac{\partial V_{1\lambda_1}}{\partial p'_{\lambda_2}} = 0\,,$$

$$\frac{\partial^2 V}{\partial p'_{\lambda_2}\,\partial p'_{\lambda_1}} - \frac{d}{dt}\frac{\partial V_{2\lambda_2}}{\partial p'_{\lambda_1}} - \frac{\partial V_{2\lambda_2}}{\partial p_{\lambda_1}} - \frac{\partial V_{1\lambda_2}}{\partial p'_{\lambda_1}} = 0\,.$$

Therefore, according to (8), that will give:

(9)
$$\frac{d^3}{dt^3}\left(\frac{\partial V_{2\lambda_1}}{\partial p'_{\lambda_2}} - \frac{\partial V_{2\lambda_2}}{\partial p'_{\lambda_1}}\right) = 0\,.$$

Thus, as is immediately clear, one will generally have:

(10)
$$\frac{d^{\alpha+\beta-1}}{dt^{\alpha+\beta-1}}\left(\frac{\partial V_{\alpha\lambda_1}}{\partial p^{(\beta-1)}_{\lambda_2}} - \frac{\partial V_{\beta\lambda_2}}{\partial p^{(\alpha-1)}_{\lambda_1}}\right) = 0\,,$$

in which one can also have $\alpha = \beta$ and $\lambda_1 = \lambda_2$. Now since α and β will have a value of at most ν, so one will have:

(11)
$$\frac{d^{2\nu-1}}{dt^{2\nu-1}}\left(\frac{\partial V_{\alpha\lambda_1}}{\partial p^{(\beta-1)}_{\lambda_2}} - \frac{\partial V_{\beta\lambda_2}}{\partial p^{(\alpha-1)}_{\lambda_1}}\right) = 0\,,$$

or

$$\frac{\partial V_{\alpha\lambda_1}}{\partial p^{(\beta-1)}_{\lambda_2}} - \frac{\partial V_{\beta\lambda_2}}{\partial p^{(\alpha-1)}_{\lambda_1}} = c_0 + c_1\,t + \ldots + c_{2\nu-2}\,t^{2\nu-2},$$

identically in any case, in which $c_0, c_1, \ldots, c_{2\nu-2}$ are constants, then when one sets:

(12) $$\frac{\partial V}{\partial p_\kappa} = W^{(\kappa)},$$

as will become clear upon partial differentiation of the equation (2) with respect to p_κ while recalling the equation (2) of § **2**, the identity will follow that:

$$\frac{\partial W^{(\kappa)}}{\partial p_\lambda} - \frac{d}{dt}\frac{\partial W^{(\kappa)}}{\partial p'_\lambda} + \cdots + (-1)^\nu \frac{d^\nu}{dt^\nu}\frac{\partial W^{(\kappa)}}{\partial p_\lambda^{(\nu)}} = 0 .$$

Therefore from (3) and (4), when one sets:

$$\frac{\partial V_{\delta\varepsilon}}{\partial p_\kappa} = W_{\delta\varepsilon}^{(\kappa)},$$

that will give:

(13) $$\delta \int_{t_0}^{t} W^{(\kappa)}\, dt = \sum_{\lambda=1}^{\mu} W_{1\lambda}^{(\kappa)} \delta p_\lambda + \sum_{\lambda=1}^{\mu} W_{2\lambda}^{(\kappa)} \delta p'_\lambda + \cdots + \sum_{\lambda=1}^{\mu} W_{\nu\lambda}^{(\kappa)} \delta p_\lambda^{(\nu-1)} .$$

However, since equation (11) implies the identity relation:

$$\frac{\partial W_{\alpha\lambda_1}}{\partial p_{\lambda_2}^{(\rho-1)}} = \frac{\partial W_{\rho\lambda_2}}{\partial p_{\lambda_1}^{(\alpha-1)}},$$

the variation (13) will be the complete variation of a function f_κ of t, p_1, p_2, …, p_μ, and its derivatives up to order $\nu - 1$. Therefore, $W^{(\kappa)}$ will be the total differential quotient of one such function F_κ with respect to t. If one now sets:

$$\frac{\partial V}{\partial p_1} = \frac{dF_1}{dt},$$

so

$$V = \frac{d}{dt}\int F_1\, dp_1 + V^{(1)}(t, p_2, p_3, \ldots, p_\mu, p'_1, \ldots, p'_\mu, \ldots, p_1^{(\nu)}, \ldots, p_\mu^{(\nu)}),$$

then it will follow by partial differentiation with respect to p_2 that:

$$\frac{\partial V}{\partial p_2} = \frac{d}{dt}\int \frac{\partial F_1}{\partial p_2}\, dp_1 + \frac{\partial V^{(1)}}{\partial p_2} = \frac{dF_2}{dt} .$$

Therefore:

$$V^{(1)} = \frac{d}{dt}\left[\int\left(F_2 - \int \frac{\partial F_1}{\partial p_2}\, dp_1\right) dp_2\right] + V^{(2)}(t, p_3, \ldots, p_\mu, p_1', \ldots, p_\mu', \ldots, p_1^{(\nu)}, \ldots, p_\mu^{(\nu)}),$$

so

$$V = \frac{d}{dt}\left[\int F_1\, dp_1 + \int\left(F_2 - \int \frac{\partial F_1}{\partial p_2}\, dp_1\right) dp_2\right] + V^{(2)},$$

and if one proceeds in that manner then it will follow that:

$$V = \frac{d\Phi}{dt} + V^{(\mu)}(p_1', \ldots, p_\mu', \ldots, p_1^{(\nu)}, \ldots, p_\mu^{(\nu)}) \;,$$

in which Φ represents a function of t, p_1, p_2, …, p_μ, and their derivatives up to order $\nu - 1$.

If one now sets:

$$V - \frac{d\Phi}{dt} = \bar{V}$$

then because when one sets:

$$\frac{\partial \Phi}{\partial p_\lambda^{(\rho)}} = \Phi_{\rho+1,\, \lambda}\,,$$

in addition to equation (5), the relation:

$$\delta\int_{t_0}^{t} \frac{d\Phi}{dt}\, dt = \sum_{\lambda=1}^{\mu} \Phi_{1\lambda}\, \delta p_\lambda + \sum_{\lambda=1}^{\mu} \Phi_{2\lambda}\, \delta p_\lambda' + \cdots + \sum_{\lambda=1}^{\mu} \Phi_{\nu\lambda}\, \delta p_\lambda^{(\nu-1)}$$

will also exist, $\bar{V}$ will, in turn, satisfy the variational equation:

$$\delta\int_{t_0}^{t} \bar{V}\, dt = \sum_{\lambda=1}^{\mu} \bar{V}_{2\lambda}\, \delta p_\lambda' + \sum_{\lambda=1}^{\mu} \bar{V}_{3\lambda}\, \delta p_\lambda'' + \cdots + \sum_{\lambda=1}^{\mu} \bar{V}_{\nu\lambda}\, \delta p_\lambda^{(\nu-1)}.$$

That is analogous to equation (5), by means of the relations (2), (3) in **§ 2**, since $\bar{V}$ no longer includes the quantities p_1, p_2, …, p_μ, and as with (5), that will imply that:

$$\bar{V} - \frac{d\Psi}{dt} = \bar{V}(t, p_1'', \ldots, p_\mu'', \ldots, p_1^{(\nu)}, \ldots, p_\mu^{(\nu)}),$$

so

$$V = \frac{d\Phi}{dt} + \frac{d\Psi}{dt} + \bar{V}\,.$$

If one continues to reason in that way then V will be represented as the complete differential quotient with respect to t of a function of t, p_1, …, p_μ, and their derivatives up to order $\nu - 1$, and we will get the theorem:

The identical fulfillment of the equation:

$$\frac{\partial V}{\partial p_\lambda} - \frac{d}{dt}\frac{\partial V}{\partial p'_\lambda} + \frac{d^2}{dt^2}\frac{\partial V}{\partial p''_\lambda} - \cdots + (-1)^\nu \frac{d^\nu}{dt^\nu}\frac{\partial V}{\partial p_\lambda^{(\nu)}} = 0$$

is the necessary and sufficient condition for V to be representable as the differential quotient with respect to t of a function of t, p_1, …, p_μ, *and their derivatives up to order* $\nu - 1$.

We shall now apply the theorem that was just proved to the proof of a lemma that plays an essential role in mechanics, as we will see, and its derivation shall be given in a form that will be the basis for our later investigations.

Lemma 4. – One addresses the problem of ascertaining the necessary and sufficient conditions for the μ functions N_1, N_2, …, N_μ of t, p_1, …, p_μ, and their derivatives up to order ν to be representable in the form:

$$N_\kappa = \frac{\partial M}{\partial p_\kappa} - \frac{d}{dt}\frac{\partial M}{\partial p'_\kappa} + \frac{d^2}{dt^2}\frac{\partial M}{\partial p''_\kappa} - \cdots + (-1)^\nu \frac{d^\nu}{dt^\nu}\frac{\partial M}{\partial p_\kappa^{(\nu)}}. \tag{14}$$

In order to next find the necessary conditions for that representation, one partially-differentiates (14) with respect to $p_\lambda^{(2\nu)}$, and with the use of the relations (2) and (3) of **§ 2** (which will always find application in what follows), that will give:

$$\frac{\partial N_\kappa}{\partial p_\lambda^{(2\nu)}} = (-1)^\nu \frac{\partial^2 M}{\partial p_\kappa^{(\nu)}\,\partial p_\lambda^{(\nu)}} = \frac{\partial N_\lambda}{\partial p_\kappa^{(2\nu)}}. \tag{15}$$

In precisely the same way, one will get the two equations:

$$\frac{\partial N_\kappa}{\partial p_\lambda^{(2\nu-1)}} = (-1)^{\nu-1} \frac{\partial^2 M}{\partial p_\kappa^{(\nu-1)}\,\partial p_\lambda^{(\nu)}} + (-1)^\nu \left(\nu \frac{d}{dt}\frac{\partial^2 M}{\partial p_\kappa^{(\nu)}\,\partial p_\lambda^{(\nu)}} + \frac{\partial^2 M}{\partial p_\kappa^{(\nu)}\,\partial p_\lambda^{(\nu-1)}} \right),$$

$$\frac{\partial N_\lambda}{\partial p_\kappa^{(2\nu-1)}} = (-1)^{\nu-1} \frac{\partial^2 M}{\partial p_\lambda^{(\nu-1)}\,\partial p_\kappa^{(\nu)}} + (-1)^\nu \left(\nu \frac{d}{dt}\frac{\partial^2 M}{\partial p_\kappa^{(\nu)}\,\partial p_\lambda^{(\nu)}} + \frac{\partial^2 M}{\partial p_\lambda^{(\nu)}\,\partial p_\kappa^{(\nu-1)}} \right),$$

whose combination will imply:

$$\frac{\partial N_\kappa}{\partial p_\lambda^{(2\nu-1)}} - 2\nu \frac{d}{dt}\frac{\partial N_\kappa}{\partial p_\lambda^{(2\nu)}} = -\frac{\partial N_\lambda}{\partial p_\kappa^{(2\nu-1)}},$$

when one recalls (15). The partial differentiation of (14) with respect to $p_\lambda^{(2\nu-2)}$ ($p_\kappa^{(2\nu-2)}$, resp.) likewise yields:

$$\frac{\partial N_\kappa}{\partial p_\lambda^{(2\nu-2)}} - (2\nu-1)_1 \frac{d}{dt}\frac{\partial N_\kappa}{\partial p_\lambda^{(2\nu-1)}} + (2\nu)_2 \frac{d^2}{dt^2}\frac{\partial N_\kappa}{\partial p_\lambda^{(2\nu)}} = \frac{\partial N_\lambda}{\partial p_\kappa^{(2\nu-2)}}.$$

Therefore, in general:

$$(16) \qquad \frac{\partial N_\kappa}{\partial p_\lambda^{(\rho)}} - (\rho-1)_1 \frac{d}{dt}\frac{\partial N_\kappa}{\partial p_\lambda^{(\rho+1)}} + (\rho+2)_2 \frac{d^2}{dt^2}\frac{\partial N_\kappa}{\partial p_\lambda^{(\rho+2)}} - \cdots + (-1)^{2\nu-\rho}(2\nu)_{2\nu-\rho} \frac{d^{2\nu-\rho}}{dt^{2\nu-\rho}}\frac{\partial N_\kappa}{\partial p_\lambda^{(2\nu)}}$$
$$= (-1)^\rho \frac{\partial N_\lambda}{\partial p_\kappa^{(\rho)}},$$

in which ρ assumes the values 0, 1, 2, …, 2ν, and κ and λ assume the values 1, 2, …, μ, whereas when M includes only one variable, along with its derivatives, in addition to t, since $k = l$, the necessary condition for N will go to:

$$(17) \qquad (1-(-1)^\rho)\frac{\partial N_\kappa}{\partial p_\lambda^{(\rho)}} - (\rho+1)_1 \frac{d}{dt}\frac{\partial N_\kappa}{\partial p_\lambda^{(\rho+1)}} + (\rho+2)_2 \frac{d^2}{dt^2}\frac{\partial N_\kappa}{\partial p_\lambda^{(\rho+2)}} - \cdots$$
$$+ (-1)^{2\nu-\rho}(2\nu)_{2\nu-\rho} \frac{d^{2\nu-\rho}}{dt^{2\nu-\rho}}\frac{\partial N_\kappa}{\partial p_\lambda^{(2\nu)}} = 0,$$

from (16). One then sees that for $\rho = 2\nu$, equation (17) is an identity, and that for $\rho = 2\nu - 1$, one further has:

$$(18) \qquad \frac{\partial N}{\partial p^{(2\nu-1)}} - \nu \frac{d}{dt}\frac{\partial N}{\partial p^{(2\nu)}} = 0,$$

which will also lead to $\rho = 2\nu - 2$. The assumptions that $\rho = 2\nu - 3$ and $\rho = 2\nu - 4$ will yield the two relations:

$$2\frac{\partial N}{\partial p^{(2\nu-3)}} - (2\nu-2)_1 \frac{d}{dt}\frac{\partial N}{\partial p^{(2\nu-2)}} + (2\nu-1)_2 \frac{d^2}{dt^2}\frac{\partial N}{\partial p^{(2\nu-1)}} - (2\nu)_3 \frac{d^3}{dt^3}\frac{\partial N}{\partial p^{(2\nu)}} = 0,$$

$$-(2\nu-3)_1 \frac{d}{dt}\frac{\partial N}{\partial p^{(2\nu-3)}} + (2\nu-2)_2 \frac{d^2}{dt^2}\frac{\partial N}{\partial p^{(2\nu-2)}} - (2\nu-1)_3 \frac{d^3}{dt^3}\frac{\partial N}{\partial p^{(2\nu-1)}} + (2\nu)_4 \frac{d^4}{dt^4}\frac{\partial N}{\partial p^{(2\nu)}} = 0.$$

The fact that those two relations will, in turn, go to just one by means of (18) is obvious from the fact that when the second of them is divided by $2\nu - 3$ and the first one is differentiated with respect to t, multiplied by 2, and added to the latter result, that will produce a homogeneous linear equation in $\frac{d^3}{dt^3}\frac{\partial N}{\partial p^{(2\nu-1)}}$ and $\frac{d^4}{dt^4}\frac{\partial N}{\partial p^{(2\nu)}}$ that must necessarily be equation (18), when it is differentiated with respect to t three times. It will then follow, as one can see immediately, that since the first two terms in (17) for two successive values of ρ will emerge from each other upon differentiation with respect to t, except for a common numerical factor, the condition equations that N must necessarily fulfill will be obtained when one sets $\rho = 1, 3, 5, \ldots, 2\nu - 1$.

It might be remarked that since equation (17) yields the relations:

$$2\frac{\partial N}{\partial p'} - 2\frac{d}{dt}\frac{\partial N}{\partial p''} + 3\frac{d^2}{dt^2}\frac{\partial N}{\partial p'''} - 4\frac{d^3}{dt^3}\frac{\partial N}{\partial p''''} + \cdots + 2\nu\frac{d^{2\nu-1}}{dt^{2\nu-1}}\frac{\partial N}{\partial p^{(2\nu)}} = 0 ,$$

$$\frac{d}{dt}\left\{\frac{\partial N}{\partial p'} - \frac{d}{dt}\frac{\partial N}{\partial p''} + \frac{d^2}{dt^2}\frac{\partial N}{\partial p'''} - \cdots + \frac{d^{2\nu-1}}{dt^{2\nu-1}}\frac{\partial N}{\partial p^{(2\nu)}}\right\} = 0$$

for $\rho = 1$ and $\rho = 0$, the bracket in the second equation will be a constant that will have the value zero, since when it is coupled with the first equation, it must yield a sequence of the remaining equations that are homogeneous linear in the differential quotients for $\rho = 3, 5, \ldots, 2\nu - 1$, and that will generally give the relation:

(19) $$\frac{\partial N}{\partial p'} - \frac{d}{dt}\frac{\partial N}{\partial p''} + \frac{d^2}{dt^2}\frac{\partial N}{\partial p'''} - \cdots + \frac{d^{2\nu-1}}{dt^{2\nu-1}}\frac{\partial N}{\partial p^{(2\nu)}} = 0 .$$

For the sake of what follows, it would not be trivial to point out that this relation can also be derived directly from equation (14) when it is partially-differentiated with respect to p', p'', …, $p^{(2\nu)}$ and one applies formulas (2) and (3) of **§ 2** and forms the indicated algebraic sum.

However, we would now like to show that the conditions on N_κ (N, resp.) that are expressed by equations (16) and (17) are also sufficient for the existence of a function M of $t, p_1, p_2, \ldots, p_\mu$, $p_1', \ldots, p_\mu', \ldots, p_1^{(\nu)}, \ldots, p_\mu^{(\nu)}$ by which N_κ (N, resp.) can be represented in the form (14), and that proof shall be carried out by a method that will find frequent application in the following using Lemma 1.

For $\mu = 1$, $\nu = 1$, it is immediately obvious from the only identity condition equation that is true in this case:

(20) $$\frac{\partial N}{\partial p'} - \frac{d}{dt}\frac{\partial N}{\partial p''} = 0$$

that one must have:

$$\frac{\partial^2 N}{\partial p''^2} = 0\,, \qquad \text{so} \qquad N = p''\varphi(t, p, p') + \psi(t, p, p')\,,$$

and that when one sets:

$$Q = \int N\,dp = p''\int \varphi(t, p, p')\,dp + \int \psi(t, p, p')\,dp = p''\,\Phi(t, p, p') + \Psi(t, p, p')\,,$$

as a result of (20), the following relation will exist:

(21) $$\frac{\partial Q}{\partial p'} - \frac{d}{dt}\frac{\partial Q}{\partial p''} = \Omega(t, p') = \frac{\partial \omega(t, p')}{\partial p'}\,,$$

which is independent of p. However, under the assumption that the variations δp and $\delta p'$ vanish at the limits of the integral t_0 and t_1 , by using the relation (21), one will get:

(22) $$\int_{t_0}^{t_1} N\,\delta p\,dt = \int_{t_0}^{t_1} \frac{\partial Q}{\partial p}\,\delta p\,dt = \delta\int_{t_0}^{t_1} Q\,dt - \int_{t_0}^{t_1} \Omega(t, p')\,\delta p'\,dt = \delta\int_{t_0}^{t_1} (Q - \omega(t, p'))\,dt\,.$$

If one now determines a function f that is linear in p'' and is the differential quotient with respect to t of a function:

$$F(t, p, p') = \int \Phi(t, p, p')\,dp' + \Phi_1(t, p)\,,$$

so

$$f = \frac{dF}{dt} = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial p}\,p' + \Phi(t, p, p')\,p''\,,$$

then when one subtracts the identity that is demanded by the first lemma:

$$\delta\int_{t_0}^{t_1} f\,dt = 0$$

from equation (22), that will imply the relation:

$$\int_{t_0}^{t_1} N\,\delta p\,dt = \delta\int_{t_0}^{t_1} (Q - \omega - f)\,dt = \delta\int_{t_0}^{t_1} M\,dt = \int_{t_0}^{t_1}\left(\frac{\partial M}{\partial p} - \frac{d}{dt}\frac{\partial M}{\partial p'}\right)\delta p\,dt\,,$$

in which M depends upon only t, p, p'. That will therefore prove the existence of a function M that depends upon only t, p, p' by which N can be expressed in the form (*):

(*) It would not be superfluous to explain the details of the proof by way of example.

$$N = \frac{\partial M}{\partial p} - \frac{d}{dt}\frac{\partial M}{\partial p'} .$$

If $\mu = 1$, $\nu = 2$ then the two identically-fulfilled condition equations will exist:

(23) $$\frac{\partial N}{\partial p'} - \frac{d}{dt}\frac{\partial N}{\partial p''} + \frac{d^2}{dt^2}\frac{\partial N}{\partial p'''} - \frac{d^3}{dt^3}\frac{\partial N}{\partial p^{IV}} = 0 ,$$

(24) $$\frac{\partial N}{\partial p'''} - 2\frac{d}{dt}\frac{\partial N}{\partial p^{IV}} = 0 .$$

That will next imply that:

$$\frac{\partial^2 N}{\partial p^{IV\,2}} = 0 , \qquad \text{so} \qquad N = p^{IV}\,\varphi(t, p, p', p'', p''') + \psi(t, p, p', p'', p''') ,$$

and upon substituting that value in (24), one will get:

$$\frac{\partial \varphi}{\partial p'''} = 0 , \qquad \frac{\partial \psi}{\partial p'''} = 2\left(\frac{\partial \varphi}{\partial t} + \frac{\partial \varphi}{\partial p}\,p' + \frac{\partial \varphi}{\partial p'}\,p'' + \frac{\partial \varphi}{\partial p''}\,p'''\right) ,$$

so it will then follow that:

$$N = p^{IV} \cdot \varphi(t, p, p', p'') + \frac{\partial \varphi}{\partial p''}\,p'''^{\,2} + 2\left(\frac{\partial \varphi}{\partial t} + \frac{\partial \varphi}{\partial p}\,p' + \frac{\partial \varphi}{\partial p'}\,p''\right)p''' + \chi(t, p, p', p'') .$$

If one once more sets:

(25) $\quad Q = \int N\,dp$

Let the function that satisfies equation (20) be:

$$N = -p^2 - 6\,p'\,p'' .$$

That will imply that:

$$\varphi = -6\,p' , \qquad \psi = -p^2 , \qquad Q = -6\,p\,p'\,p'' - \tfrac{1}{3}\,p^3 ,$$

$$\frac{\partial Q}{\partial p'} - \frac{d}{dt}\frac{\partial Q}{\partial p''} = 6\,p'^2 , \qquad \Omega = 6\,p'^2 , \qquad \omega = 2\,p'^2 , \qquad \Phi = -6\,p\,p' ,$$

$$F = -3\,p\,p'^2 , \qquad f = -3\,p'^2 - 6\,p\,p'\,p'' ,$$

and therefore:

$$M = p'^3 - \tfrac{1}{3}\,p^3 .$$

$$= p^{IV}\int \varphi\,dp + p'''^{\,2}\int \frac{\partial\varphi}{\partial p''}\,dp + 2p'''\int\left(\frac{\partial\varphi}{\partial t} + \frac{\partial\varphi}{\partial p}\,p' + \frac{\partial\varphi}{\partial p'}\,p''\right)dp + \int \chi(t, p, p', p'')\,dp$$

then by means of (23), one will see that:

$$\text{(26)}\qquad \frac{\partial Q}{\partial p'} - \frac{d}{dt}\frac{\partial Q}{\partial p''} + \frac{d^2}{dt^2}\frac{\partial Q}{\partial p'''} - \frac{d^3}{dt^3}\frac{\partial Q}{\partial p^{IV}} = \Omega(t, p', p'', p''', p^{IV}, p^{V})$$

is independent of *p*. However, if one partially-differentiates that equation with respect to p^V and p^{IV} then it will once more follow with the help of formulas (2) and (3) of **§ 2**, while considering the form (25) for *Q*, that:

$$\frac{\partial\Omega}{\partial p^{V}} = 0\,, \qquad \frac{\partial\Omega}{\partial p^{IV}} = 0\,,$$

which is immediately obvious, such that (26) will be independent of *p*, p^{IV}, and p^V, and assume the form:

$$\text{(27)}\qquad \frac{\partial Q}{\partial p'} - \frac{d}{dt}\frac{\partial Q}{\partial p''} + \frac{d^2}{dt^2}\frac{\partial Q}{\partial p'''} - \frac{d^3}{dt^3}\frac{\partial Q}{\partial p^{IV}} = \Omega(t, p', p'', p''')\ .$$

That amounts to developing the characteristic property of the function Ω. However, it is easy to see, in turn, that by means of (2) and (3) in **§ 2**, one has:

$$\begin{aligned}
\frac{\partial\Omega}{\partial p''} &= \frac{\partial^2 Q}{\partial p'\,\partial p''} \\
&\quad - \left[\frac{d}{dt}\frac{\partial^2 Q}{\partial p''^{\,2}} + \frac{\partial^2 Q}{\partial p''\,\partial p'}\right] + \left[\frac{d^2}{dt^2}\frac{\partial^2 Q}{\partial p'''\,\partial p''} + 2\frac{d}{dt}\frac{\partial^2 Q}{\partial p'''\,\partial p'} + \frac{\partial^2 Q}{\partial p'''\,\partial p}\right] \\
&\quad - \left[\frac{d^3}{dt^3}\frac{\partial^2 Q}{\partial p^{IV}\,\partial p''} + 3\frac{d^2}{dt^2}\frac{\partial^2 Q}{\partial p^{IV}\,\partial p'} + 3\frac{d}{dt}\frac{\partial^2 Q}{\partial p^{IV}\,\partial p}\right],
\end{aligned}$$

$$\begin{aligned}
\frac{\partial\Omega}{\partial p'''} &= \frac{\partial^2 Q}{\partial p'\,\partial p'''} \\
&\quad - \left[\frac{d}{dt}\frac{\partial^2 Q}{\partial p''\,\partial p'''} + \frac{\partial^2 Q}{\partial p''^{\,2}}\right] + \left[\frac{d^2}{dt^2}\frac{\partial^2 Q}{\partial p'''^{\,2}} + 2\frac{d}{dt}\frac{\partial^2 Q}{\partial p'''\,\partial p''} + \frac{\partial^2 Q}{\partial p'''\,\partial p'}\right] \\
&\quad - \left[\frac{d^3}{dt^3}\frac{\partial^2 Q}{\partial p^{IV}\,\partial p'''} + 3\frac{d^2}{dt^2}\frac{\partial^2 Q}{\partial p^{IV}\,\partial p''} + 3\frac{d}{dt}\frac{\partial^2 Q}{\partial p^{IV}\,\partial p'} + \frac{\partial^2 Q}{\partial p^{IV}\,\partial p}\right],
\end{aligned}$$

and therefore:

$$\frac{\partial\Omega}{\partial p''}-\frac{d}{dt}\frac{\partial\Omega}{\partial p'''} = 2\frac{d^3}{dt^3}\frac{\partial^2 Q}{\partial p^{IV}\,\partial p''}-2\frac{d}{dt}\frac{\partial^2 Q}{\partial p^{IV}\,\partial p}+\frac{\partial^2 Q}{\partial p'''\,\partial p}-\frac{d^3}{dt^3}\frac{\partial^2 Q}{\partial p'''^{\,2}}+\frac{d^4}{dt^4}\frac{\partial^2 Q}{\partial p^{IV}\,\partial p'''},$$

which will imply that:

(28) $$\frac{\partial\Omega}{\partial p''}-\frac{d}{dt}\frac{\partial\Omega}{\partial p'''} = 0$$

with the use of the form for Q that was found in (25). However, since equations (25) and (27) yield the relation:

(29) $$\int_{t_0}^{t_1} N\,\delta p\,dt = \delta\int_{t_0}^{t_1} Q\,dt-\int_{t_0}^{t_1}\Omega(t, p', p'', p''')\,\delta p'\,dt$$

precisely as above, for a function (which is, however, a function of $t,\ p',\ p'',\ p'''$) that satisfies the relation (28) that is analogous to equation (20), from what was proved before, there will exist a function K of $t,\ p',\ p''$ that satisfies the equation:

(30) $$\int_{t_0}^{t_1}\Omega(t, p', p'', p''')\,\delta p'\,dt = \delta\int_{t_0}^{t_1} K\,dt\,.$$

The combination of (29) and (30) will then imply that:

(31) $$\int_{t_0}^{t_1} N\,\delta p\,dt = \delta\int_{t_0}^{t_1}(Q-K)\,dt = \delta\int_{t_0}^{t_1} R\,dt\ ,$$

in which, from (25), R will have the form:

(32) $$R = p^{IV}\int\varphi\,dp+p'''^{\,2}\int\frac{\partial\varphi}{\partial p''}\,dp+2p'''\int\left(\frac{\partial\varphi}{\partial t}+\frac{\partial\varphi}{\partial p}\,p'+\frac{\partial\varphi}{\partial p'}\,p''\right)dp+\omega(t, p, p', p'')\,.$$

If one now sets:

$$S = p'''\int\varphi\,dp+\Omega_1(t, p, p')$$

then:

(33) $$\Psi = \frac{\partial S}{\partial t}$$

$$= p^{IV}\int\varphi\,dp+p'''\int\left(\frac{\partial\varphi}{\partial t}+\frac{\partial\varphi}{\partial p}\,p'+\frac{\partial\varphi}{\partial p'}\,p''+\frac{\partial\varphi}{\partial p''}\,p'''\right)dp+\frac{\partial\Omega_1}{\partial t}+\frac{\partial\Omega_1}{\partial p}\,p'+\frac{\partial\Omega_1}{\partial p'}\,p''+\frac{\partial\Omega_1}{\partial p''}\,p'''\,,$$

and in that way, one will determine Ω_1 in such a way that:

$$\frac{\partial\varphi}{\partial p''} = \int\left(\frac{\partial\varphi}{\partial t} + \frac{\partial\varphi}{\partial p}\,p' + \frac{\partial\varphi}{\partial p'}\,p''\right)dp\,.$$

Since Ψ is the differential quotient of S with respect to t, one will have:

$$\delta\int_{t_0}^{t_1}\Psi\,dt = 0\,, \tag{34}$$

and when one sets:

$$R - \Psi = M\,,$$

in which M depends upon only t, p, p', p'', from (32) and (33), the difference of (31) and (34) will yield the relation:

$$\int_{t_0}^{t_1} N\,\delta p\,dt = \delta\int_{t_0}^{t_1} M\,dt\,.$$

Will therefore imply the existence of a function M of t, p, p', p'' by which N can be represented in the form (*):

$$N = \frac{\partial M}{\partial p} - \frac{d}{dt}\frac{\partial M}{\partial p'} + \frac{d^2}{dt^2}\frac{\partial M}{\partial p''}\,.$$

For $\mu = 1$, $\nu = 3$, the question will, in turn, reduce to the determination of a function $\Omega(t, p', p'', p''', p^{IV}, p^{V})$ that will exhibit both of them for $\nu = 2$ and prove to be a necessary and sufficient condition to be satisfied, etc.

(*) For example, if $N = -p + 4p''p''' + 2p'p^{IV}$, which will make the two equations (23) and (24) sufficient, then that will give:

$$\varphi = 2p'\,, \qquad \psi = -p + 4p''p'''\,, \qquad \chi = -p\,,$$

$$Q = -\frac{1}{2}p^2 + 4p\,p''p''' + 2p\,p'p^{IV}\,, \qquad \Omega = -4p'p''' - 2p''^2\,,$$

$$K = 2p'p''^2\,, \quad R = 2p\,p'p^{IV} + 4p\,p''p''' - \frac{1}{2}p^2 - 2p'p''^2\,, \quad \Omega_1 = p\,p''^2 - 2p'^2p''\,,$$

$$S = 2p\,p'p''' + p\,p''^2 - 2p'^2p''\,, \quad \Psi = 2p\,p'p^{IV} + 4p\,p''p''' - 3p'p''^2\,.$$

Therefore:

$$M = R - \Psi = -\frac{1}{2}p^2 + p'p''^2\,,$$

from which, one will in fact have the identity:

$$-p + 4p''p''' + 2p'p^{IV} = \frac{\partial M}{\partial p} - \frac{d}{dt}\frac{\partial M}{\partial p'} + \frac{d^2}{dt^2}\frac{\partial M}{\partial p''} = -p + \frac{d}{dt}(p''^2) + \frac{d^2}{dt^2}(2p'p'')\,.$$

We then find that:

The necessary and sufficient conditions for a function N of t, p, p', …, $p^{(2\nu)}$ *to be representable in terms of a function M of t, p,* p', …, $p^{(\nu)}$ *in the form:*

$$N = \frac{\partial M}{\partial p} - \frac{d}{dt}\frac{\partial M}{\partial p'} + \frac{d^2}{dt^2}\frac{\partial M}{\partial p''} - \cdots + (-1)^\nu \frac{d^\nu}{dt^\nu}\frac{\partial M}{\partial p^{(\nu)}} \tag{35}$$

are given by the equations:

$$(1-(-1)^\rho)\frac{\partial N}{\partial p^{(\rho)}} - (\rho+1)_1 \frac{d}{dt}\frac{\partial N}{\partial p^{(\rho+1)}} + (\rho+2)_2 \frac{d^2}{dt^2}\frac{\partial N}{\partial p^{(\rho+1)}} - \cdots + (-1)^{2\nu-\rho}(2\nu)_{2\nu-\rho}\frac{d^{2\nu-\rho}}{dt^{2\nu-\rho}}\frac{\partial N}{\partial p^{(2\nu)}} = 0$$

$$(\rho = 1, 3, 5, \ldots, 2\nu - 1),$$

which must be fulfilled identically.

However, one also soon sees that there are infinitely-many such functions M, since when M_1 is the differential quotient with respect to t of an *arbitrary* function of t, p, p', …, $p^{(\nu-1)}$, from Lemma 3, the identity will exist:

$$0 = \frac{\partial M_1}{\partial p} - \frac{d}{dt}\frac{\partial M_1}{\partial p'} + \frac{d^2}{dt^2}\frac{\partial M_1}{\partial p''} - \cdots + (-1)^\nu \frac{d^\nu}{dt^\nu}\frac{\partial M_1}{\partial p^{(\nu)}}, \tag{36}$$

and therefore, when one sets:

$$M - M_1 = L,$$

it will follow from (35) and (36) that:

$$N = \frac{\partial L}{\partial p} - \frac{d}{dt}\frac{\partial L}{\partial p'} + \frac{d^2}{dt^2}\frac{\partial L}{\partial p''} - \cdots + (-1)^\nu \frac{d^\nu}{dt^\nu}\frac{\partial L}{\partial p^{(\nu)}} .$$

However, *all* functions M that satisfy (35) will be determined in that way since if M_2 means any such function then subtracting (35) and M_2 would yield the identity:

$$\frac{\partial(M-M_2)}{\partial p} - \frac{d}{dt}\frac{\partial(M-M_2)}{\partial p'} + \cdots + (-1)^\nu \frac{d^\nu}{dt^\nu}\frac{\partial(M-M_2)}{\partial p^{(\nu)}} = 0 .$$

From Lemma 3, that would demand that $M - M_2$ must be the differential quotient of a function of t, p, p', …, $p^{(\nu-1)}$ with respect to t. It will then follow that:

When N satisfies the conditions that were given above, infinitely-many solutions M will satisfy equation (25), *but they will all differ by only differential quotients of functions of t, p,* p', …, $p^{(\nu-1)}$ *with respect to t.*

In order to prove the existence of a function *M* by which the quantity *N* can be represented in the form that was given in (14) for the case in which $\mu > 1$ and the functions $N_1, N_2, \ldots, N_\mu$ satisfy the equations (16) and to explain what is applicable, we would like to again restrict the principle that is completely analogous to the one that was used before to the case $\mu = 2$, $\nu = 1$, for which the condition equations (16) will assume the form:

(37) $$\frac{d}{dt}\frac{\partial N_1}{\partial p_1'} - \frac{d^2}{dt^2}\frac{\partial N_1}{\partial p_1''} = 0 ,$$

(38) $$\frac{d}{dt}\frac{\partial N_2}{\partial p_2'} - \frac{d^2}{dt^2}\frac{\partial N_2}{\partial p_2''} = 0 ,$$

(39) $$\frac{\partial N_1}{\partial p_1'} - \frac{d}{dt}\frac{\partial N_1}{\partial p_1''} = 0 ,$$

(40) $$\frac{\partial N_2}{\partial p_2'} - \frac{d}{dt}\frac{\partial N_2}{\partial p_2''} = 0 ,$$

(41) $$\frac{\partial N_1}{\partial p_2} - \frac{d}{dt}\frac{\partial N_1}{\partial p_2'} + \frac{d^2}{dt^2}\frac{\partial N_1}{\partial p_2''} = \frac{\partial N_2}{\partial p_1} ,$$

(42) $$\frac{\partial N_2}{\partial p_1} - \frac{d}{dt}\frac{\partial N_2}{\partial p_1'} + \frac{d^2}{dt^2}\frac{\partial N_2}{\partial p_1''} = \frac{\partial N_1}{\partial p_2} ,$$

(43) $$\frac{\partial N_1}{\partial p_2'} - 2\frac{d}{dt}\frac{\partial N_1}{\partial p_2''} = -\frac{\partial N_2}{\partial p_1'} ,$$

(44) $$\frac{\partial N_2}{\partial p_1'} - 2\frac{d}{dt}\frac{\partial N_2}{\partial p_1''} = -\frac{\partial N_1}{\partial p_2'} ,$$

(45) $$\frac{\partial N_1}{\partial p_2''} = \frac{\partial N_2}{\partial p_1''} .$$

Only equations (39), (40), (41), (43), and (45) can be considered to be independent of each other, as is immediately obvious. Now, one can once more see immediately from equations (39), (40), and (43), (44) that the functions N_1 and N_2 must be linear functions of p_1'' and p_2''. Therefore, by means of equation (45), they will have the form:

(46) $$N_1 = p_1''\varphi_{11}(t, p_1, p_2, p_1', p_2') + p_2''\varphi_{12}(t, p_1, p_2, p_1', p_2') + \chi_1(t, p_1, p_2, p_1', p_2') ,$$

(47) $$N_2 = p_1''\varphi_{12}(t, p_1, p_2, p_1', p_2') + p_2''\varphi_{22}(t, p_1, p_2, p_1', p_2') + \chi_2(t, p_1, p_2, p_1', p_2') ,$$

in which equations (39) and (40) imply that the functions φ_{11}, φ_{12}, φ_{22}, χ_1 and χ_2 are subject to the conditions:

(48) $$\frac{\partial \varphi_{12}}{\partial p_1'} = \frac{\partial \varphi_{11}}{\partial p_2'} ,$$

(49) $$\frac{\partial \varphi_{12}}{\partial p_2'} = \frac{\partial \varphi_{22}}{\partial p_1'} ,$$

(50) $$\frac{\partial \chi_1}{\partial p_1'} = \frac{\partial \varphi_{11}}{\partial t} + \frac{\partial \varphi_{11}}{\partial p_1} p_1' + \frac{\partial \varphi_{11}}{\partial p_2} p_2' ,$$

(51) $$\frac{\partial \chi_2}{\partial p_2'} = \frac{\partial \varphi_{22}}{\partial t} + \frac{\partial \varphi_{22}}{\partial p_1} p_1' + \frac{\partial \varphi_{22}}{\partial p_2} p_2' ,$$

while from (41) and (42):

(52) $$\frac{\partial \varphi_{11}}{\partial p_2} - \frac{\partial^2 \chi_1}{\partial p_2' \, \partial p_1'} + \frac{\partial^2 \varphi_{12}}{\partial t \, \partial p_1'} + p_1' \frac{\partial^2 \varphi_{12}}{\partial p_1 \, \partial p_1'} + p_2' \frac{\partial^2 \varphi_{12}}{\partial p_2 \, \partial p_1'} = 0 ,$$

(53) $$2\frac{\partial \varphi_{12}}{\partial p_2} - \frac{\partial^2 \chi_1}{\partial p_2'^2} + \frac{\partial^2 \varphi_{12}}{\partial t \, \partial p_2'} + p_1' \frac{\partial^2 \varphi_{12}}{\partial p_1 \, \partial p_2'} + p_2' \frac{\partial^2 \varphi_{12}}{\partial p_2 \, \partial p_2'} = \frac{\partial \varphi_{22}}{\partial p_1} ,$$

(54) $$\frac{\partial \chi_1}{\partial p_2} - \frac{\partial^2 \chi_1}{\partial p_2' \, \partial t} - \frac{\partial^2 \chi_1}{\partial p_2' \, \partial p_1'} p_1' - \frac{\partial^2 \varphi_{12}}{\partial p_2' \, \partial p_2} p_2' + \frac{\partial^2 \varphi_{12}}{\partial t^2} + \frac{\partial^2 \varphi_{12}}{\partial t \, \partial p_1} p_1' + \frac{\partial^2 \varphi_{12}}{\partial t \, \partial p_2} p_2'$$
$$+ p_1' \left(\frac{\partial^2 \varphi_{12}}{\partial p_2 \, \partial t} + \frac{\partial^2 \varphi_{12}}{\partial p_1^2} p_1' + \frac{\partial^2 \varphi_{12}}{\partial p_1 \, \partial p_2} p_2' \right) + p_2' \left(\frac{\partial^2 \varphi_{12}}{\partial p_2 \, \partial t} + \frac{\partial^2 \varphi_{12}}{\partial p_2 \, \partial p_1} p_1' + \frac{\partial^2 \varphi_{12}}{\partial p_2^2} p_2' \right) = \frac{\partial \chi_2}{\partial p_1} ,$$

along with the ones that emerge by switching the indices 1 and 2 in them. It will finally follow from (43) and (44) that:

(55) $$\frac{\partial \chi_1}{\partial p_2'} - 2\left(\frac{\partial \varphi_{12}}{\partial t} + \frac{\partial \varphi_{12}}{\partial p_1} p_1' + \frac{\partial \varphi_{12}}{\partial p_2} p_2' \right) = -\frac{\partial \chi_2}{\partial p_1} .$$

If one now sets:

(56) $$N_1 = \frac{\partial Q}{\partial p_1} , \qquad N_2 = \frac{\partial Q}{\partial p_2} + f ,$$

in analogy with the previous methods, and in which, from (46):

(57) $$Q = p_1'' \int \varphi_{11}\, dp_1 + p_2'' \int \varphi_{12}\, dp_1 + \int \chi_1\, dp_1 ,$$

then:

$$-f = -N_2 + \frac{\partial Q}{\partial p_2} = -\,p_1''\left[\varphi_{12} - \int \frac{\partial \varphi_{11}}{\partial p_2}\, dp_1\right] - p_2''\left[\varphi_{22} - \int \frac{\partial \varphi_{12}}{\partial p_2}\, dp_1\right] - \chi_2 + \int \frac{\partial \chi_1}{\partial p_2}\, dp_1 ,$$

and due to the fact that:

$$\frac{\partial Q}{\partial p_2'} - \frac{d}{dt}\frac{\partial Q}{\partial p_2''} = -\,p_1'\,\varphi_{12} - p_2' \int \frac{\partial \varphi_{12}}{\partial p_2}\, dp_1 + \int \frac{\partial \chi_1}{\partial p_2'}\, dp_1 - \int \frac{\partial \varphi_{12}}{\partial t}\, dp_1 ,$$

as is easy to see, from (52) and (53), one will have:

$$-f - \frac{d}{dt}\left[\frac{\partial Q}{\partial p_2'} - \frac{d}{dt}\frac{\partial Q}{\partial p_2''}\right]$$

$$= -\chi_2 + \int \frac{\partial \chi_1}{\partial p_2}\, dp_1 + p_1'\left[\frac{\partial \varphi_{12}}{\partial t} + \frac{\partial \varphi_{12}}{\partial p_1}\, p_1' + \frac{\partial \varphi_{12}}{\partial p_2}\, p_2'\right] + p_2'\left[\int \frac{\partial^2 \varphi_{12}}{\partial p_2\, \partial t}\, dp_1 + p_1' \frac{\partial \varphi_{12}}{\partial p_1} + p_2' \int \frac{\partial^2 \varphi_{12}}{\partial p_2^2}\, dp_1\right]$$

$$-\int \frac{\partial^2 \chi_1}{\partial p_2'\, \partial t}\, dp_1 - p_1' \frac{\partial \chi_1}{\partial p_2'} - p_2' \int \frac{\partial^2 \chi_1}{\partial p_2'\, \partial p_2}\, dp_1 + \int \frac{\partial^2 \varphi_{12}}{\partial t^2}\, dp_1 + \frac{\partial \varphi_{12}}{\partial t}\, p_1' + p_2' \int \frac{\partial^2 \varphi_{12}}{\partial t\, \partial p_2}\, dp_1 ,$$

and one sees immediately that the right-hand side (so the left-hand side, as well) of that equation is independent of p_1 and p_2'. Namely, if one takes the partial differential quotient of the right-hand side with respect to p_1 then that will be zero identically as a result of equation (54), while the differential quotient with respect to p_2' will vanish as a result of equations (51) and (53), and one will then get:

(58) $$f + \frac{d}{dt}\left[\frac{\partial Q}{\partial p_2'} - \frac{d}{dt}\frac{\partial Q}{\partial p_2''}\right] = \omega(t, p_2, p_1') .$$

It is just as simple to see that:

$$\frac{\partial Q}{\partial p_1'} - \frac{d}{dt}\frac{\partial Q}{\partial p_1''} = p_1'' \int \frac{\partial \varphi_{11}}{\partial p_1'}\, dp_1 + p_2'' \int \frac{\partial \varphi_{12}}{\partial p_1'}\, dp_1 + \int \frac{\partial \chi_1}{\partial p_1'}\, dp_1$$

$$-\int \left(\frac{\partial \varphi_{11}}{\partial t} + \frac{\partial \varphi_{11}}{\partial p_1}\, p_1' + \frac{\partial \varphi_{11}}{\partial p_2}\, p_2' + \frac{\partial \varphi_{11}}{\partial p_1'}\, p_1'' + \frac{\partial \varphi_{11}}{\partial p_2'}\, p_2''\right) dp_1 ,$$

or from (48):

$$\frac{\partial Q}{\partial p_1'} - \frac{d}{dt}\frac{\partial Q}{\partial p_1''} = \int \left(\frac{\partial \chi_1}{\partial p_1'} - p_2' \frac{\partial \varphi_{11}}{\partial p_2} - \frac{\partial \varphi_{11}}{\partial t} \right) dp_1 - p_1' \varphi_{11} ,$$

from which it will follow that when this expression is differentiated with respect to p_1 by means of (50) and differentiated with respect to p_2' by means of (48) and (52), it will vanish, and therefore:

$$\frac{\partial Q}{\partial p_1'} - \frac{d}{dt}\frac{\partial Q}{\partial p_1''} = \omega_1 (t, p_2, p_1') \tag{59}$$

will depend upon only p_2 and p_1', if one overlooks t.

Now, if one makes the convention that the variations δp_1, δp_2, $\delta p_1'$, $\delta p_2'$ must vanish for $t = t_0$ and $t = t_1$ and forms the integral:

$$\begin{aligned}
\int_{t_0}^{t_1} (N_1 \, \delta p_1 + N_2 \, \delta p_2) \, dt &= \int_{t_0}^{t_1} \left(\frac{\partial Q}{\partial p_1} \delta p_1 + \frac{\partial Q}{\partial p_2} \delta p_2 + f \, \delta p_2 \right) dt \\
&= \delta \int_{t_0}^{t_1} Q \, dt + \int_{t_0}^{t_1} \left(f \, \delta p_2 - \frac{\partial Q}{\partial p_1'} \delta p_1' - \frac{\partial Q}{\partial p_2'} \delta p_2' - \frac{\partial Q}{\partial p_1''} \delta p_1'' - \frac{\partial Q}{\partial p_2''} \delta p_2'' \right) dt \\
&= \delta \int_{t_0}^{t_1} Q \, dt + \int_{t_0}^{t_1} \left(f \, \delta p_2 - \left[\frac{\partial Q}{\partial p_2'} - \frac{d}{dt} \frac{\partial Q}{\partial p_2''} \right] \delta p_2' \right) dt - \int_{t_0}^{t_1} \left(\frac{\partial Q}{\partial p_1'} - \frac{d}{dt} \frac{\partial Q}{\partial p_1''} \right) \delta p_1' \, dt
\end{aligned}$$

then from (58) and (59), that will go to:

$$\begin{aligned}
&\int_{t_0}^{t_1} (N_1 \, \delta p_1 + N_2 \, \delta p_2) \, dt \\
&\quad = \delta \int_{t_0}^{t_1} Q \, dt + \int_{t_0}^{t_1} \frac{d}{dt} \left[\left(\frac{\partial Q}{\partial p_2'} - \frac{d}{dt} \frac{\partial Q}{\partial p_2''} \right) \delta p_2' \right] dt + \int_{t_0}^{t_1} \omega (t, p_2, p_1') \, \delta p_2 \, dt - \int_{t_0}^{t_1} \omega_1 (t, p_2, p_1') \, \delta p_1' \, dt ,
\end{aligned}$$

or when one uses the limit conditions that were established:

$$\int_{t_0}^{t_1} (N_1 \, \delta p_1 + N_2 \, \delta p_2) \, dt = \delta \int_{t_0}^{t_1} Q \, dt + \int_{t_0}^{t_1} (\omega \, \delta p_2 - \omega_1 \, \delta p_1') \, dt . \tag{60}$$

However, from (58), one has:

$$\frac{\partial \omega}{\partial p_1'} = \frac{\partial f}{\partial p_1'} + \frac{d}{dt} \left(\frac{\partial^2 Q}{\partial p_2' \, \partial p_1'} - \frac{d}{dt} \frac{\partial^2 Q}{\partial p_2'' \, \partial p_1'} - 2 \frac{\partial^2 Q}{\partial p_2'' \, \partial p_1} \right) + \frac{\partial^2 Q}{\partial p_2' \, \partial p_1} ,$$

and from (59):

$$\frac{\partial \omega}{\partial p_2} = \frac{\partial^2 Q}{\partial p_1' \, \partial p_2} - \frac{d}{dt} \frac{\partial^2 Q}{\partial p_1'' \partial p_2} ,$$

and furthermore, from (56) and (43):

$$\frac{\partial f}{\partial p_1'} = \frac{\partial N_2}{\partial p_1'} - \frac{\partial^2 Q}{\partial p_2 \, \partial p_1'} = 2 \frac{d}{dt} \frac{\partial N_2}{\partial p_1''} - \frac{\partial N_1}{\partial p_2'} - \frac{\partial^2 Q}{\partial p_2 \, \partial p_1'} = 2 \frac{d}{dt} \frac{\partial^2 Q}{\partial p_1 \, \partial p_2''} - \frac{\partial^2 Q}{\partial p_2' \, \partial p_1} - \frac{\partial^2 Q}{\partial p_2 \, \partial p_1'} ,$$

from which, it will follow that:

$$\frac{\partial \omega}{\partial p_1'} + \frac{\partial \omega}{\partial p_2} = - \frac{d}{dt} \left(\frac{d}{dt} \frac{\partial^2 Q}{\partial p_2'' \, \partial p_1'} - \frac{\partial^2 Q}{\partial p_1' \, \partial p_2'} + \frac{\partial^2 Q}{\partial p_1'' \partial p_2} \right) .$$

However, from (57):

$$\frac{d}{dt} \frac{\partial^2 Q}{\partial p_2'' \, \partial p_1'} - \frac{\partial^2 Q}{\partial p_1' \, \partial p_2'} + \frac{\partial^2 Q}{\partial p_1'' \partial p_2}$$
$$= \frac{d}{dt} \int \frac{\partial \varphi_{12}}{\partial p_1'} dp_1 - p_1'' \int \frac{\partial^2 \varphi_{11}}{\partial p_1' \, \partial p_2'} dp_1 - p_2'' \int \frac{\partial^2 \varphi_{11}}{\partial p_1' \, \partial p_2'} dp_1 - \int \frac{\partial^2 \chi_1}{\partial p_1' \, \partial p_2'} dp_1 + \int \frac{\partial \varphi_{12}}{\partial p_2} dp_1$$

will vanish identically due to (48) and (52), so it will follow that:

$$\frac{\partial \omega}{\partial p_1'} = \frac{\partial (-\omega_1)}{\partial p_2} .$$

Therefore:

$$\int_{t_0}^{t_1} (\omega \, \delta p_2 - \omega_1 \, \delta p_1') \, dt = \delta \int_{t_0}^{t_1} \Omega(t, p_2, p_1') \, dt .$$

Thus, when one sets:

$$Q + \Omega = R ,$$

equation (60) will go to:

(61) $$\int_{t_0}^{t_1} (N_1 \, \delta p_1 + N_2 \, \delta p_2) \, dt = \delta \int_{t_0}^{t_1} R \, dt ,$$

in which one has:

(62) $$R = p_1'' \int_{t_0}^{t_1} \varphi_{11} \, dp_1 + p_2'' \int_{t_0}^{t_1} \varphi_{12} \, dp_1 + \int_{t_0}^{t_1} \chi_1 \, dp_1 + \Omega(t, p_2, p_1') .$$

If one now determines a function $F(t, p_1, p_2, p_1', p_2')$ for which:

$$\frac{\partial F}{\partial p_1'} = \int \varphi_{11}\, dp_1 \qquad \text{and} \qquad \frac{\partial F}{\partial p_2'} = \int \varphi_{12}\, dp_1\,,$$

which is possible because of the relation (48), and sets:

(63) $$f_1 = \frac{dF}{dt} = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial p_1} p' + \frac{\partial F}{\partial p_2} p_2' + p_1'' \int_{t_0}^{t_1} \varphi_{11}\, dp_1 + p_2'' \int_{t_0}^{t_1} \varphi_{12}\, dp_1\,,$$

then by means of Lemma 3:

(64) $$\delta \int_{t_0}^{t_1} f_1\, dt = 0\,.$$

The difference of (61) and (64) will then yield:

(65) $$\int_{t_0}^{t_1} (N_1\, \delta p_1 + N_2\, \delta p_2)\, dt = \delta \int_{t_0}^{t_1} (R - f_1)\, dt = \delta \int_{t_0}^{t_1} M\, dt\,,$$

in which M depends upon only t, p_1, p_2, p_1', p_2' because of (62) and (63), and from (65), it will have the property that:

$$N_1 = \frac{\partial M}{\partial p_1} - \frac{d}{dt}\frac{\partial M}{\partial p_1'}\,, \qquad N_2 = \frac{\partial M}{\partial p_2} - \frac{d}{dt}\frac{\partial M}{\partial p_2'}\,.$$

The infinitude of different values for M again differ from each other by only functions that are complete differential quotients with respect to t of arbitrary functions of t, p_1, and p_2 .

The basic principle of the proof will also remain valid for the general theorem:

The necessary and sufficient conditions for μ functions $N_1, N_2, \ldots, N_\mu$ of $t, p_1, p_2, \ldots, p_\mu$, and their derivatives up to order 2ν to be represented by ***one*** *function M of $t, p_1, p_2, \ldots, p_\mu$, and their derivatives up to order ν in the form:*

$$N_\kappa = \frac{\partial M}{\partial p_\kappa} - \frac{d}{dt}\frac{\partial M}{\partial p_\kappa'} + \frac{d^2}{dt^2}\frac{\partial M}{\partial p_\kappa''} - \cdots + (-1)^\nu \frac{d^\nu}{dt^\nu}\frac{\partial M}{\partial p_\kappa^{(\nu)}} \qquad (\kappa = 1, 2, \ldots, \mu)$$

is given by the equations:

$$\frac{\partial N_\kappa}{\partial p_\lambda^{(\rho)}} - (\rho+1)_1 \frac{d}{dt}\frac{\partial N_\kappa}{\partial p_\lambda^{(\rho+1)}} + (\rho+2)_2 \frac{d^2}{dt^2}\frac{\partial N_\kappa}{\partial p_\lambda^{(\rho+2)}} - \cdots + (-1)^{2\nu-\rho}(2\nu)_{2\nu-\rho} \frac{d^{2\nu-\rho}}{dt^{2\nu-\rho}}\frac{\partial N_\kappa}{\partial p_\lambda^{(2\nu)}} = (-1)^\rho \frac{dN_\kappa}{dp_\lambda^{(\rho)}}\,,$$

which must be fulfilled identically, and in which ρ *assumes the values* $0, 1, 2, \ldots, 2\nu$, *while* κ *and* λ *assume the values* $1, 2, \ldots, \mu$ (*).

That method likewise gives the way by which one can exhibit the analytical expression for M.

That lemma, which will be exploited in full generality later, shall next be employed to specialize the expression for the measure of the force.

If we imitate what we did before in the mechanics of ponderable masses and impose the demand that for an arbitrary value of ν, the measure of the force that is represented by the expression:

$$X = -\frac{\partial T}{\partial x} + \frac{d}{dt}\frac{\partial T}{\partial x'} - \cdots + (-1)^{\nu-1}\frac{d^\nu}{dt^\nu}\frac{\partial T}{\partial x^{(\nu)}},$$

in which T means an arbitrary function $t, x, x', \ldots, x^{(\nu)}$, must be independent of $x, x', \ldots, x^{(\nu)}$ then it should next be remarked that for every function T, from Lemma 4, X will be subject to the condition that:

(*) For example, let:

$$N_1 = 6p_2^3 p_1' p_1'' - p_2'' + 2p_1 p_2'^2 + 9p_2^2 p_2' p_1'^2,$$
$$N_2 = -p_1'' - p_2'' - 2p_1'^2 p_2'' - 3p_2^2 p_1'^2 - 4p_1 p_1' p_2',$$

which satisfy equations (37) to (45) identically. One will then have:

$$\varphi_{11} = 6p_2^3 p_1', \qquad \varphi_{11} = -1, \qquad \varphi_{11} = -2p_1^2,$$
$$\chi_1 = 2p_1 p_2'^2 + 9p_2^2 p_2' p_1'^2, \qquad \chi_1 = -3p_2^2 p_1'^2 - 4p_1 p_1' p_2',$$

and therefore:

$$Q = p_1'' \cdot 6p_2^3 p_1 p_1' - p_2'' \cdot p_1 + p_1^2 p_2'^2 + 9p_1 p_2^2 p_2' p_1'^2,$$
$$f = -p_1''(1 + 18p_1 p_2^2 p_1') - 2p_1^2 p_2'' - 3p_2 p_2'^3 - 4p_1 p_1' p_2' - 18p_1 p_2 p_2' p_1'^2,$$

from which it will follow that:

$$\omega = 6p_2^2 p_2'^3, \qquad \omega_1 = -6p_2^3 p_1'^2, \quad \Omega = 2p_2^3 p_1'^3.$$

Furthermore, since:

$$F = -3p_1 p_2^3 p_1'^2 - p_1' p_2',$$

so

$$f_1 = 3p_1'^3 p_2^3 + 9p_1 p_2^2 p_2' p_1'^2 + 6p_1 p_2^3 p_1' p_1'' - p_1 p_2'' - p_1' p_2',$$
$$R = p_1'' \cdot 6p_2^3 p_1' p_1 - p_2'' \cdot p_1 + p_1^2 p_2'^2 + 9p_1 p_2^2 p_2' p_1'^2 + 2p_2^3 p_1'^3,$$

one will get M in the form:

$$M = p_1^2 p_2'^2 - p_1'^3 p_2^3 + p_1' p_2',$$

so one will in fact have:

$$N_1 = \frac{\partial M}{\partial p_1} - \frac{d}{dt}\frac{\partial M}{\partial p_1'}, \qquad N_2 = \frac{\partial M}{\partial p_2} - \frac{d}{dt}\frac{\partial M}{\partial p_2'}.$$

$$(66)\qquad (1-(-1)^{\rho})\frac{\partial X}{\partial x^{(\rho)}}-(\rho+1)_1\frac{d}{dt}\frac{\partial X}{\partial x^{(\rho+1)}}+(\rho+2)_2\frac{d^2}{dt^2}\frac{\partial X}{\partial x^{(\rho+2)}}-\cdots+(-1)^{2\nu-\rho}(2\nu)_{2\nu-2}\frac{d^{2\nu-\rho}}{dt^{2\nu-\rho}}\frac{\partial X}{\partial x^{(2\nu)}}=0$$

$$(\rho = 1, 3, \ldots, 2\nu - 1),$$

which must be fulfilled identically.

Now let ν be an odd number, and let X be independent of $x, x', \ldots, x^{(\nu)}$, so equations (66), for $\rho = 1, 3, 5, \ldots, \nu, \nu + 2, \nu + 4, \ldots, 2\nu - 3, 2\nu - 1$, will go to:

$$(67)\qquad \begin{cases} (1+\nu)_{\nu}\dfrac{d^{\nu}}{dt^{\nu}}\dfrac{\partial X}{\partial x^{(\nu+1)}}-(2+\nu)_{\nu}\dfrac{d^{\nu+1}}{dt^{\nu+1}}\dfrac{\partial X}{\partial x^{(\nu+2)}}+\cdots+(2\nu)_{2\nu-1}\dfrac{d^{2\nu-1}}{dt^{2\nu-1}}\dfrac{\partial X}{\partial x^{(2\nu)}}=0, \\ (1+\nu)_{\nu-2}\dfrac{d^{\nu-2}}{dt^{\nu-2}}\dfrac{\partial X}{\partial x^{(\nu+1)}}-(2+\nu)_{\nu-1}\dfrac{d^{\nu-1}}{dt^{\nu-1}}\dfrac{\partial X}{\partial x^{(\nu+2)}}+\cdots+(2\nu)_{2\nu-3}\dfrac{d^{2\nu-1}}{dt^{2\nu-1}}\dfrac{\partial X}{\partial x^{(2\nu)}}=0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ (1+\nu)_1\dfrac{d}{dt}\dfrac{\partial X}{\partial x^{(\nu+1)}}-(2+\nu)_2\dfrac{d^2}{dt^2}\dfrac{\partial X}{\partial x^{(\nu+2)}}+\cdots+(2\nu)_{\nu}\dfrac{d^{\nu}}{dt^{\nu}}\dfrac{\partial X}{\partial x^{(2\nu)}}=0, \\ 2\dfrac{\partial X}{\partial x^{(\nu+2)}}-(\nu+3)_1\dfrac{d}{dt}\dfrac{\partial X}{\partial x^{(\nu+3)}}+\cdots-(2\nu)_{\nu-2}\dfrac{d^{\nu-2}}{dt^{\nu-2}}\dfrac{\partial X}{\partial x^{(2\nu)}}=0, \\ 2\dfrac{\partial X}{\partial x^{(\nu+4)}}-(\nu+5)_1\dfrac{d}{dt}\dfrac{\partial X}{\partial x^{(\nu+5)}}+\cdots-(2\nu)_{\nu-4}\dfrac{d^{\nu-4}}{dt^{\nu-4}}\dfrac{\partial X}{\partial x^{(2\nu)}}=0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 2\dfrac{\partial X}{\partial x^{(2\nu-3)}}-(2\nu-2)_1\dfrac{d}{dt}\dfrac{\partial X}{\partial x^{(2\nu-2)}}+(2\nu-1)_2\dfrac{d^2}{dt^2}\dfrac{\partial X}{\partial x^{(2\nu-1)}}-(2\nu)_3\dfrac{d^3}{dt^3}\dfrac{\partial X}{\partial x^{(2\nu)}}=0, \\ 2\dfrac{\partial X}{\partial x^{(2\nu-3)}}-(2\nu)_1\dfrac{d}{dt}\dfrac{\partial X}{\partial x^{(2\nu)}}=0. \end{cases}$$

When one differentiates the successive equation $0, 2, 3, \ldots, \nu - 1, \nu + 1, \nu + 3, \ldots, 2\nu - 2$ times with respect to t, one will get ν homogeneous linear equations in the ν quantities:

$$\frac{d^{2\nu-1}}{dt^{2\nu-1}}\frac{\partial X}{\partial x^{(2\nu)}}, \qquad \frac{d^{2\nu-2}}{dt^{2\nu-2}}\frac{\partial X}{\partial x^{(2\nu-1)}}, \qquad \ldots, \qquad \frac{d^{\nu}}{dt^{\nu}}\frac{\partial X}{\partial x^{(\nu+1)}},$$

with a non-zero determinant. Therefore, one will have:

$$\frac{d^{2\nu-1}}{dt^{2\nu-1}}\frac{\partial X}{\partial x^{(2\nu)}}=0, \qquad \frac{d^{2\nu-2}}{dt^{2\nu-2}}\frac{\partial X}{\partial x^{(2\nu-1)}}=0, \quad \ldots, \qquad \frac{d^{\nu}}{dt^{\nu}}\frac{\partial X}{\partial x^{(\nu+1)}}=0,$$

identically. If X does not contain the variable t explicitly, either, then it is immediately clear that it will follow that the quantities:

$$\frac{\partial X}{\partial x^{(2\nu)}}, \qquad \frac{\partial X}{\partial x^{(2\nu-1)}}, \qquad \ldots, \qquad \frac{\partial X}{\partial x^{(\nu+1)}}$$

are constants. Therefore, X will necessarily have the form:

$$X = A_0\,x^{(2\nu)} + A_1\,x^{(2\nu-1)} + \cdots + A_{\nu-1}\,x^{(\nu+1)} + A_\nu\,,$$

since X should not depend upon t, x, x', …, $x^{(\nu)}$ explicitly. However, since the last $(\nu - 1) / 2$ equations in (67) imply that:

$$A_1 = A_3 = \ldots = A_{\nu-4} = A_{\nu-2} = 0\,,$$

and since precisely the same argument is valid for even ν, we will find that:

The necessary and sufficient condition for the expression:

$$X = -\frac{\partial T}{\partial x} + \frac{d}{dt}\frac{\partial T}{\partial x'} - \cdots + (-1)^{\nu-1}\frac{d^\nu}{dt^\nu}\frac{\partial T}{\partial x^{(\nu)}}$$

for the measure of the force that is exerted on a point that moves along the X-axis to be independent of t, x, x', x'', …, $x^{(\nu)}$ *has the form:*

$$X = A_0\,x^{(2\nu)} + A_2\,x^{(2\nu-2)} + A_4\,x^{(2\nu-4)} + \cdots + A_{\nu-1}\,x^{\nu+1} \qquad \textit{for odd } \nu,$$

and

$$X = A_0\,x^{(2\nu)} + A_2\,x^{(2\nu-2)} + A_4\,x^{(2\nu-4)} + \cdots + A_{\nu-2}\,x^{\nu+2} \qquad \textit{for odd } \nu,$$

for which T can correspondingly be chosen to be:

$$T = -\tfrac{1}{2}\left\{(-1)^\nu A_0\,x^{(\nu)^2} + (-1)^{\nu-1}A_2\,x^{(\nu-1)^2} + (-1)^{\nu-2}A_4\,x^{(\nu-2)^2} + \cdots + (-1)^{\frac{\nu+1}{2}}A_{\nu-1}\,x^{\left(\frac{\nu+1}{2}\right)^2}\right\}$$

and

$$T = -\tfrac{1}{2}\left\{(-1)^\nu A_0\,x^{(\nu)^2} + (-1)^{\nu-1}A_2\,x^{(\nu-1)^2} + (-1)^{\nu-2}A_4\,x^{(\nu-2)^2} + \cdots + (-1)^{\frac{\nu}{2}+1}A_{\nu-1}\,x^{\left(\frac{\nu}{2}+1\right)^2}\right\},$$

respectively.

When the function T is based upon either of the last two forms, we will call it the *vis viva* of the moving point.

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§ 4. – The first form of the extended Lagrange equations.

Once the measure of the force has been found in the previously-established form, **d'Alembert**'s principle, as represented by equation (1) of **§ 1**, will imply the relation:

$$\begin{aligned}(1)\qquad \sum_{i=1}^{n}\Bigg\{&\left(-\frac{\partial T}{\partial x_i}+\frac{d}{dt}\frac{\partial T}{\partial x_i'}-\cdots+(-1)^{\nu-1}\frac{d^\nu}{dt^\nu}\frac{\partial T}{\partial x_i^{(\nu)}}\right)\delta x_i\\
&+\left(-\frac{\partial T}{\partial y_i}+\frac{d}{dt}\frac{\partial T}{\partial y_i'}-\cdots+(-1)^{\nu-1}\frac{d^\nu}{dt^\nu}\frac{\partial T}{\partial y_i^{(\nu)}}\right)\delta y_i\\
&+\left(-\frac{\partial T}{\partial z_i}+\frac{d}{dt}\frac{\partial T}{\partial z_i'}-\cdots+(-1)^{\nu-1}\frac{d^\nu}{dt^\nu}\frac{\partial T}{\partial z_i^{(\nu)}}\right)\delta z_i\Bigg\}\\
&=\sum_{i=1}^{n}(X_i\,\delta x_i+Y_i\,\delta y_i+Z_i\,\delta z_i),\end{aligned}$$

in which X_i, Y_i, Z_i mean the applied forces on the system, and one assumes that the restriction on the degrees of freedom of the systems is given by m equations that are linear and homogeneous in the virtual displacements and take the forms:

$$(2)\qquad \begin{cases}\displaystyle\sum_{i=1}^{n}(f_{1i}\,\delta x_i+\varphi_{1i}\,\delta x_i+\psi_{1i}\,\delta x_i)=0,\\ \displaystyle\sum_{i=1}^{n}(f_{2i}\,\delta x_i+\varphi_{2i}\,\delta x_i+\psi_{2i}\,\delta x_i)=0,\\ \displaystyle\sum_{i=1}^{n}(f_{3i}\,\delta x_i+\varphi_{3i}\,\delta x_i+\psi_{3i}\,\delta x_i)=0,\end{cases}$$

in which the functions f_{ki}, φ_{ki}, ψ_{ki} should depend upon time and the coordinates, but not upon their derivatives, and whose *integrability or non-integrability characterizes the holonomic or non-holonomic systems*, resp. From a well-known argument, multiplying equations (2) by the quantities λ_1, λ_2, …, λ_m and adding them to (1) will yield the ***Lagrange** equations of motion in their first form:*

$$(3)\quad \begin{cases} -\dfrac{\partial T}{\partial x_i}+\dfrac{d}{dt}\dfrac{\partial T}{\partial x_i'}-\cdots+(-1)^{\nu-1}\dfrac{d^\nu}{dt^\nu}\dfrac{\partial T}{\partial x_i^{(\nu)}}=X_i+\lambda_1 f_{1i}+\lambda_2 f_{2i}+\cdots+\lambda_m f_{mi}, \\ -\dfrac{\partial T}{\partial y_i}+\dfrac{d}{dt}\dfrac{\partial T}{\partial y_i''}-\cdots+(-1)^{\nu-1}\dfrac{d^\nu}{dt^\nu}\dfrac{\partial T}{\partial y_i^{(\nu)}}=Y_i+\lambda_1 \varphi_{1i}+\lambda_2 \varphi_{2i}+\cdots+\lambda_m \varphi_{mi},\ (i=1,2,\dots,n). \\ -\dfrac{\partial T}{\partial z_i}+\dfrac{d}{dt}\dfrac{\partial T}{\partial z_i'}-\cdots+(-1)^{\nu-1}\dfrac{d^\nu}{dt^\nu}\dfrac{\partial T}{\partial z_i^{(\nu)}}=Z_i+\lambda_1 \psi_{1i}+\lambda_2 \psi_{2i}+\cdots+\lambda_m \psi_{mi} \end{cases}$$

We will now say that *a force system possesses a force function of order* ν *when a function U of t, the coordinates* x_i, y_i, z_i, *and the derivatives with respect to time up to order* ν *exists such that* X_i, Y_i, Z_i *are defined by the expressions:*

$$X_i=\frac{\partial U}{\partial x_i}-\frac{d}{dt}\frac{\partial U}{\partial x_i'}+\cdots+(-1)^\nu\frac{d^\nu}{dt^\nu}\frac{\partial U}{\partial x_i^{(\nu)}},$$
$$Y_i=\frac{\partial U}{\partial y_i}-\frac{d}{dt}\frac{\partial U}{\partial y_i'}+\cdots+(-1)^\nu\frac{d^\nu}{dt^\nu}\frac{\partial U}{\partial y_i^{(\nu)}},$$
$$Z_i=\frac{\partial U}{\partial z_i}-\frac{d}{dt}\frac{\partial U}{\partial z_i'}+\cdots+(-1)^\nu\frac{d^\nu}{dt^\nu}\frac{\partial U}{\partial z_i^{(\nu)}}.$$

Those forces will then be referred to as *internal forces*. From Lemma 4, the necessary and sufficient conditions for X_i, Y_i, Z_i, which are functions of time, the coordinates, and their derivatives up to order 2ν, and when x_i, y_i, z_i are denoted by $p_1, p_2, \dots, p_{3n}$, while X_i, Y_i, Z_i are denoted by $N_1, N_2, \dots, N_{3n}$, are represented by the equations:

$$\frac{\partial N_\kappa}{\partial p_\lambda^{(\rho)}}-(\rho+1)_1\frac{d}{dt}\frac{\partial N_\kappa}{\partial p_\lambda^{(\rho+1)}}+(\rho+2)_2\frac{d^2}{dt^2}\frac{\partial N_\kappa}{\partial p_\lambda^{(\rho+2)}}-\cdots+(-1)^{2\nu-\rho}(2\nu)_{2\nu-\rho}\frac{d^{2\nu-\rho}}{dt^{2\nu-\rho}}\frac{\partial N_\kappa}{\partial p_\lambda^{(2\nu)}}=(-1)^\rho\frac{\partial N_\lambda}{\partial p_\kappa^{(\rho)}},$$

which must be satisfied identically, in which κ, λ assume the values $1, 2, \dots, 3n$, and ρ assumes the values $1, 2, \dots, 2\nu$. Thus, if the force R that acts between two points is a function of the distance between them and the derivatives with respect to time up to order 2ν then, under the assumption that the force can be decomposed along the three components with use of formula (10) in **§ 2**, we will get the following theorem:

If a system of forces $R_{\lambda\mu}$ *acts upon the n points of a system that depends upon the distance* $r_{\lambda\mu}$ *from point* λ *to point* μ *and their derivatives with respect to time up to order* 2ν, *and if the equations:*

$$(1-(-1)^\rho)\frac{\partial R_{\lambda\mu}}{\partial r_{\lambda\mu}^{(\rho)}}-(\rho+1)_1\frac{d}{dt}\frac{\partial R_{\lambda\mu}}{\partial r_{\lambda\mu}^{(\rho+1)}}+(\rho+2)_2\frac{d^2}{dt^2}\frac{\partial R_{\lambda\mu}}{\partial r_{\lambda\mu}^{(\rho+2)}}-\cdots+(-1)^{2\nu-\rho}\frac{d^{2\nu-\rho}}{dt^{2\nu-}}\frac{\partial R_{\lambda\mu}}{\partial r_{\lambda\mu}^{(2\nu)}}=0$$

$$(\rho = 1, 3, \ldots, 2\nu - 1)$$

are satisfied identically, moreover, then the force system will possess a force function of order ν, *and indeed, when* $W_{\lambda\mu}$ *means a function of* $r_{\lambda\mu}$ *and the derivatives of those quantities with respect to t up to order* ν *that satisfies the equations:*

$$R_{\lambda\mu} = \frac{\partial W_{\lambda\mu}}{\partial r_{\lambda\mu}} - \frac{d}{dt}\frac{\partial W_{\lambda\mu}}{\partial r'_{\lambda\mu}} + \cdots + (-1)^{\nu}\frac{d^{\nu}}{dt^{\nu}}\frac{\partial W_{\lambda\mu}}{\partial r^{(2\nu)}_{\lambda\mu}},$$

it will be represented by:

$$W_{12} + W_{13} + \ldots + W_{1n} + W_{23} + \ldots + W_{2n} + \ldots + W_{n-1,n} .$$

Thus, if the force that acts between two electric mass-points is given by **Weber**'s law as:

$$R = -\frac{m\,m_1}{r^2} + \frac{m\,m_1}{r^2}\frac{r'^2}{k^2} - \frac{2\,m\,m_1}{k^2\,r^2}\,r''$$

then since the condition:

$$\frac{\partial R}{\partial r'} - \frac{d}{dt}\frac{\partial R}{\partial r''} = 0$$

is fulfilled identically, it will have a force function, and indeed, it will take the form:

$$W = \frac{m\,m_1}{r^2}\left(1 + \frac{r'^2}{k^2}\right),$$

such that:

$$R = \frac{\partial W}{\partial r} - \frac{d}{dt}\frac{\partial W}{\partial r'} .$$

Therefore, the forces on the systems whose motion is defined by equations (1) or (3) will consist of internal and external forces. If we denote the components of the external forces along the axes by Q_i, R_i, S_i then if we set:

$$-T - U = H, \tag{4}$$

those equations of motion will assume the form:

$$\sum_{i=1}^{n}\left\{\left(\frac{\partial H}{\partial x_i} - \frac{d}{dt}\frac{\partial H}{\partial x'_i} + \cdots + (-1)^{\nu}\frac{d^{\nu}}{dt^{\nu}}\frac{\partial H}{\partial x^{(\nu)}_i} - Q_i\right)\delta x_i\right. \tag{5}$$

$$+\left(\frac{\partial H}{\partial y_i}-\frac{d}{dt}\frac{\partial H}{\partial y_i'}+\cdots+(-1)^\nu\frac{d^\nu}{dt^\nu}\frac{\partial H}{\partial y_i^{(\nu)}}-R_i\right)\delta y_i$$
$$\left.+\left(\frac{\partial H}{\partial z_i}-\frac{d}{dt}\frac{\partial H}{\partial z_i'}+\cdots+(-1)^\nu\frac{d^\nu}{dt^\nu}\frac{\partial H}{\partial z_i^{(\nu)}}-S_i\right)\delta z_i\right\}=0\,,$$

or

$$\text{(6)}\qquad\left\{\begin{aligned}
&\frac{\partial H}{\partial x_i}-\frac{d}{dt}\frac{\partial H}{\partial x_i}+\cdots+(-1)^\nu\frac{d^\nu}{dt^\nu}\frac{\partial H}{\partial x_i^{(\nu)}}=Q_i+\lambda_1 f_{1i}+\cdots+\lambda_m f_{mi}\,,\\
&\frac{\partial H}{\partial y_i}-\frac{d}{dt}\frac{\partial H}{\partial y_i'}+\cdots+(-1)^\nu\frac{d^\nu}{dt^\nu}\frac{\partial H}{\partial y_i^{(\nu)}}=R_i+\lambda_1 \varphi_{1i}+\cdots+\lambda_m \varphi_{mi}\,,\\
&\frac{\partial H}{\partial z_i}-\frac{d}{dt}\frac{\partial H}{\partial z_i'}+\cdots+(-1)^\nu\frac{d^\nu}{dt^\nu}\frac{\partial H}{\partial z_i^{(\nu)}}=S_i+\lambda_1 \psi_{1i}+\cdots+\lambda_m \psi_{mi}\,,
\end{aligned}\right.$$

in which the function H of t, x_i, y_i, z_i, x_i', y_i', z_i', …, $x_i^{(\nu)}$, $y_i^{(\nu)}$, $z_i^{(\nu)}$ that is defined by equation (4) shall be called the *kinetic potential of order* ν.

Equations (6) represent a system of $3n$ total differential equations of order 2ν, in which we assume that the external forces are either purely functions of time or functions of time, the coordinates, and their derivatives, but not the ones of order greater than 2ν. Now, if the system is holonomic, so the equations will be complete variations of the equations:

$$\text{(7)}\qquad\begin{aligned}
&F_1(t, x_1,\dots,x_n, y_1,\dots,y_n, z_1,\dots,z_n)=0, \quad \dots\\
&F_m(t, x_1,\dots,x_n, y_1,\dots,y_n, z_1,\dots,z_n)=0,
\end{aligned}$$

relative to the coordinates, then the $3n + m$ quantities:

$$x_1,\ \dots,\ x_n,\ y_1,\ \dots,\ y_n,\ z_1,\ \dots,\ z_n,\ \lambda_1,\ \dots,\ \lambda_m$$

can be obtained from the $3n + m$ equations (6) and (7) by integrating the differential equations as functions of time t and $6\nu m$ arbitrary constants. However, if the system is not holonomic, but arranged such that time t does not occur in the condition equations (2) explicitly, then one can also replace the virtual displacements with actual ones, and one will then get m differential equations of the form:

$$\text{(8)}\qquad\left\{\begin{aligned}
&\sum_{i=1}^n(f_{1i}\,x_i'+\varphi_{1i}\,y_i'+\psi_{1i}\,z_i')=0\,,\\
&\sum_{i=1}^n(f_{2i}\,x_i'+\varphi_{2i}\,y_i'+\psi_{2i}\,z_i')=0\,, \quad \cdots\\
&\sum_{i=1}^n(f_{mi}\,x_i'+\varphi_{mi}\,y_i'+\psi_{mi}\,z_i')=0\,,
\end{aligned}\right.$$

in addition to the $3n$ differential equations (6) that are given by **Lagrange**'s equations, such that the coordinates will, in turn, be given in terms of time by integrating the $3n + m$ differential equations (6) and (8). However, if t occurs in the condition equations of the non-holonomic system then the treatment of the problem will have to be adapted to the constraints in each individual problem.

If a point it attracted to the origin as the center according to **Weber**'s law then, from (6), if W is the **Weber** force function, and the *vis viva* is set equal to:

$$T = \tfrac{1}{2} m(x'^2 + y'^2 + z'^2),$$

the differential equations of motion will be represented by the equations:

$$m \frac{d^2 x}{dt^2} = \frac{\partial W}{\partial x} - \frac{d}{dt} \frac{\partial W}{\partial x'},$$

$$m \frac{d^2 y}{dt^2} = \frac{\partial W}{\partial y} - \frac{d}{dt} \frac{\partial W}{\partial y'},$$

$$m \frac{d^2 z}{dt^2} = \frac{\partial W}{\partial z} - \frac{d}{dt} \frac{\partial W}{\partial z'}.$$

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§ 5. – The second form of the extended Lagrange equations.

If the system is holonomic, so the $3n$ *coordinates* x_i, y_i, z_i *are given functions of* μ *mutually-independent quantities* $p_1, p_2, \ldots, p_\mu$*, which will be called the* ***free*** *coordinates, then* ***d'Alembert****'s principle, in the form that given by* (3) *of* **§ 1**, *will also imply the* μ ***Lagrange*** *equations of motion in the second form:*

$$-\frac{\partial T}{\partial p_s}+\frac{d}{dt}\frac{\partial T}{\partial p_s'}+\cdots+(-1)^{\nu-1}\frac{d^\nu}{dt^\nu}\frac{\partial T}{\partial p_s^{(\nu)}} = \sum_{i=1}^{n}\left(X_i\frac{\partial x_i}{\partial p_s}+Y_i\frac{\partial y_i}{\partial p_s}+Z_i\frac{\partial z_i}{\partial p_s}\right) \qquad (s = 1, 2, \ldots, \mu) \tag{1}$$

for this case, as well, or in turn, by separating the internal and external forces:

$$\frac{\partial H}{\partial p_s}-\frac{d}{dt}\frac{\partial H}{\partial p_s'}+\cdots+(-1)^{\nu}\frac{d^\nu}{dt^\nu}\frac{\partial H}{\partial p_s^{(\nu)}} = P_s \qquad (s = 1, 2, \ldots, \mu) \tag{2}$$

when one sets:

$$\sum_{i=1}^{n}\left(Q_i\frac{\partial x_i}{\partial p_s}+R_i\frac{\partial y_i}{\partial p_s}+S_i\frac{\partial z_i}{\partial p_s}\right) = P_s\,. \tag{3}$$

Those equations are implied immediately by (5) in **§ 4** by means of Lemma 2, and their integration will yield the free coordinates $p_1, p_2, \ldots, p_\mu$ as functions of t and $2\nu\mu$ arbitrary coordinates.

In what follows, we shall assume that the equations of motion take the form of (6) in **§ 4** or (2) above without assuming that H is separated into the two summands – T and – U, on the basis of things that we shall go into in detail about later on.

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§ 6. – The extended Hamilton principle.

If one forms the integral:

$$\int_{t_0}^{t_1}\left(H-\sum_{\lambda=1}^{\mu}P_\lambda\, p_\lambda\right)dt$$

and assumes that the external forces P_λ are given as functions of time, but independent of the coordinates, during the arbitrary, but well-defined, time interval from t_0 to t_1, and one further assumes that H is finite for all values of the coordinates that come under consideration and their derivatives during that time period, along with all of its partial differential quotients with respect to just those quantities up to order $\nu + 1$, then under the assumption that all δp_s , $\delta p_s'$, …, $\delta p_s^{(\nu-1)}$ vanish for $t = t_0$ and $t = t_1$, one will have:

$$(1) \qquad \delta\int_{t_0}^{t_1}\left(H-\sum_{\lambda=1}^{\mu}P_\lambda\, p_\lambda\right)dt = \int_{t_0}^{t_1}\sum_{s=1}^{\mu}\left(\frac{\partial H}{\partial p_s}-\frac{d}{dt}\frac{\partial H}{\partial p_s'}+\cdots+(-1)^{\nu}\frac{d^{\nu}}{dt^{\nu}}\frac{\partial H}{\partial p_s^{(\nu)}}-P_s\right)\delta p_s\, dt\,.$$

Since the variations δp_s are mutually independent, it will follow that the relation:

$$(2) \qquad \delta\int_{t_0}^{t_1}\left(H-\sum_{\lambda=1}^{\mu}P_\lambda\, p_\lambda\right)dt = 0$$

implies equations (2) of **§ 5**, and conversely that:

The ***Hamiltonian*** *principle that is represented by equation* (2) *is therefore equivalent to the second form of the Lagrange equations*.

By means of the assumption that was made about the variations for t_0 and t_1, when the quantities p in them are expressed as functions of time by integrating the equations of motion, and the integration constants in them are expressed in terms of the initial and final coordinates and their derivatives of order one up to $\nu - 1$, one compares them to infinitely-close functions of time that assume the same values for t_0 and t_1, along with their first $\nu - 1$ derivatives. In that way, the transition time for the system to go from its initial configuration to its final one will be given, and it will be the same for all systems that one compares it to.

If the external forces are all zero then **Hamilton**'s principle will go to:

$$(3) \qquad \delta\int_{t_0}^{t_1}H\, dt = 0\,,$$

and that will say that:

The mean values of the kinetic potential that are calculated over equal time intervals under normal motion between given initial and final configurations (which is defined by the same values of the coordinates and their first $\nu - 1$ *derivatives) will be a limiting value.*

One then sees immediately, in turn, the identity of **Hamilton**'s principle (3) with **d'Alembert**'s principle from the relation:

$$\delta \int_{t_0}^{t_1} H\,dt = \int_{t_0}^{t_1} \sum_{i=1}^{n} \left\{ \left(\frac{\partial H}{\partial x_i} - \frac{d}{dt}\frac{\partial H}{\partial x_i'} + \cdots + (-1)^\nu \frac{d^\nu}{dt^\nu}\frac{\partial H}{\partial x_i^{(\nu)}} \right) \delta x_i \right.$$
$$+ \left(\frac{\partial H}{\partial y_i} - \frac{d}{dt}\frac{\partial H}{\partial y_i'} + \cdots + (-1)^\nu \frac{d^\nu}{dt^\nu}\frac{\partial H}{\partial y_i^{(\nu)}} \right) \delta y_i$$
$$\left. + \left(\frac{\partial H}{\partial z_i} - \frac{d}{dt}\frac{\partial H}{\partial z_i'} + \cdots + (-1)^\nu \frac{d^\nu}{dt^\nu}\frac{\partial H}{\partial z_i^{(\nu)}} \right) \delta z_i \right\} dt\,.$$

Therefore, it is also equivalent to the first form of the **Lagrange** equations, whereas when the external forces do not vanish, but are, in turn, purely functions of time, that identity will require that **Hamilton**'s principle must take the form:

$$\delta \int_{t_0}^{t_1} \left[H - \sum_{\lambda=1}^{\mu} (x_i\,Q_i + y_i\,R_i + z_i\,S_i) \right] dt = 0\,.$$

Therefore, it will remain valid for holonomic and non-holonomic systems, but, as will usually be assumed in the following investigation, it will initially be assumed that the derivatives of the coordinates with respect to time will enter into the constraint equations of the problem. However, if the latter is the case then the constraint equations will read:

$$\begin{aligned} F_1\,(t, x_i, y_i, z_i, x_i', y_i', z_i', \ldots, x_i^{(\nu)}, y_i^{(\nu)}, z_i^{(\nu)}) &= 0, \quad \cdots \\ F_m\,(t, x_i, y_i, z_i, x_i', y_i', z_i', \ldots, x_i^{(\nu)}, y_i^{(\nu)}, z_i^{(\nu)}) &= 0 \end{aligned} \tag{4}$$

(in which case, we would also like to call the system a *holonomic* one). The variations of the coordinates and their derivatives for every t will then be subject to the m constraint equations:

$$\sum_{i=1}^{n} \left\{ \frac{\partial F_r}{\partial x_i}\delta x_i + \frac{\partial F_r}{\partial y_i}\delta y_i + \frac{\partial F_r}{\partial z_i}\delta z_i \right.$$
$$+ \frac{\partial F_r}{\partial x_i'}\delta x_i' + \frac{\partial F_r}{\partial y_i'}\delta y_i' + \frac{\partial F_r}{\partial z_i'}\delta z_i' + \cdots \tag{5}$$

$$+\ \frac{\partial F_r}{\partial x_i^{(\nu)}}\delta x_i^{(\nu)} + \frac{\partial F_r}{\partial y_i^{(\nu)}}\delta y_i^{(\nu)} + \frac{\partial F_r}{\partial z_i^{(\nu)}}\delta y_i^{(\nu)} \Bigg\} = 0$$

$$(r = 1, 2, \ldots, m),$$

from which, since those relations should be fulfilled for the entire course of the motion during the time interval $t_1 - t_0$ that one employs in **Hamilton**'s principle and under the assumption that was made for the validity of **Hamilton**'s principle, namely, that the variations of the coordinates and their derivatives up to order $\nu - 1$ must vanish at the limits t_0 and t_1, one will get:

$$\int_{t_0}^{t_1}\sum_{i=1}^{n}\Bigg\{\left(\frac{\partial F_r}{\partial x_i} - \frac{d}{dt}\frac{\partial F_r}{\partial x_i'} + \cdots + (-1)^\nu \frac{d^\nu}{dt^\nu}\frac{\partial F_r}{\partial x_i^{(\nu)}}\right)\delta x_i + \cdots$$

$$+ \left(\frac{\partial F_r}{\partial z_i} - \frac{d}{dt}\frac{\partial F_r}{\partial z_i'} + \cdots + (-1)^\nu \frac{d^\nu}{dt^\nu}\frac{\partial F_r}{\partial z_i^{(\nu)}}\right)\delta z_i \Bigg\}\, dt = 0$$

$$(r = 1, 2, \ldots, m).$$

Therefore, the following relations will exist for the variations of the coordinates themselves, which are homogeneous and linear in them:

$$\sum_{i=1}^{n}\Bigg\{\left(\frac{\partial F_r}{\partial x_i} - \frac{d}{dt}\frac{\partial F_r}{\partial x_i'} + \cdots + (-1)^\nu \frac{d^\nu}{dt^\nu}\frac{\partial F_r}{\partial x_i^{(\nu)}}\right)\delta x_i \tag{6}$$

$$+ \left(\frac{\partial F_r}{\partial y_i} - \frac{d}{dt}\frac{\partial F_r}{\partial y_i'} + \cdots + (-1)^\nu \frac{d^\nu}{dt^\nu}\frac{\partial F_r}{\partial y_i^{(\nu)}}\right)\delta y_i$$

$$+ \left(\frac{\partial F_r}{\partial z_i} - \frac{d}{dt}\frac{\partial F_r}{\partial z_i'} + \cdots + (-1)^\nu \frac{d^\nu}{dt^\nu}\frac{\partial F_r}{\partial z_i^{(\nu)}}\right)\delta z_i \Bigg\} = 0$$

$$(r = 1, 2, \ldots, m).$$

When one again endows those m constraint equations with multipliers $\lambda_1, \ldots, \lambda_m$ and adds them to equation (4) in **§ 4**, which expresses **d'Alembert**'s principle, that will give the $3n$ differential equations:

$$(7)\qquad \begin{cases} \dfrac{\partial H}{\partial x_i}-\dfrac{d}{dt}\dfrac{\partial H}{\partial x_i'}+\cdots+(-1)^\nu\dfrac{d^\nu}{dt^\nu}\dfrac{\partial H}{\partial x_i^{(\nu)}}=Q_i+\sum_{r=1}^m\lambda_r\left(\dfrac{\partial F_r}{\partial x_i}-\dfrac{d}{dt}\dfrac{\partial F_r}{\partial x_i'}+\cdots+(-1)^\nu\dfrac{d^\nu}{dt^\nu}\dfrac{\partial F_r}{\partial x_i^{(\nu)}}\right),\\[2ex] \dfrac{\partial H}{\partial y_i}-\dfrac{d}{dt}\dfrac{\partial H}{\partial y_i'}+\cdots+(-1)^\nu\dfrac{d^\nu}{dt^\nu}\dfrac{\partial H}{\partial y_i^{(\nu)}}=R_i+\sum_{r=1}^m\lambda_r\left(\dfrac{\partial F_r}{\partial y_i}-\dfrac{d}{dt}\dfrac{\partial F_r}{\partial y_i'}+\cdots+(-1)^\nu\dfrac{d^\nu}{dt^\nu}\dfrac{\partial F_r}{\partial y_i^{(\nu)}}\right),\\[2ex] \dfrac{\partial H}{\partial z_i}-\dfrac{d}{dt}\dfrac{\partial H}{\partial z_i'}+\cdots+(-1)^\nu\dfrac{d^\nu}{dt^\nu}\dfrac{\partial H}{\partial z_i^{(\nu)}}=S_i+\sum_{r=1}^m\lambda_r\left(\dfrac{\partial F_r}{\partial z_i}-\dfrac{d}{dt}\dfrac{\partial F_r}{\partial z_i'}+\cdots+(-1)^\nu\dfrac{d^\nu}{dt^\nu}\dfrac{\partial F_r}{\partial z_i^{(\nu)}}\right). \end{cases}$$

In conjunction with the differential equations (4), they will produce a simultaneous system of $3n + m$ differential equations in the $3n + m$ functions x_i, y_i, z_i, λ_r (*).

However, if the system of constraint equations that also contains the derivatives of the coordinates is *non-holonomic*, so it might have the form:

(*) Therefore, let a constraint equation between, e.g., two coordinates x, y, and their first derivatives be given in the form:

$$(\alpha)\qquad 2y'(x^2-x-1)-x\,y(2x-1)=0,$$

such that the relation between the variations will read:

$$[2y'(2x-1)-2x\,y]\,\delta x-x'(2x-1)\,\delta y-y(2x-1)\,\delta x'-2(x^2-x-1)\,\delta y'=0.$$

Upon integrating that between the limits t_0 and t_1, and under the assumption that the variations of the coordinates x and y vanish at the limits t_0 and t_1, that will then go to:

$$\int_{t_0}^{t_1}(3y'(2x-1)\,\delta x+x'(-6x+2)\,\delta y)\,dt=0,$$

which will therefore produce the relation between the variations of the coordinates:

$$y'(2x-1)\,\delta x-x'(2x-1)\,\delta y=0,$$

or by means of (α):

$$(\beta)\qquad y(2x-1)\,\delta x-2\,(x^2-x-1)\,\delta y=0.$$

However, if one remarks that the general integral of the differential equation (α) is represented by:

$$(\gamma)\qquad x^2-x-1=c\,y^2,$$

from which the relation between the variations will yield:

$$(\delta)\qquad (2x-1)\,\delta x=2\,c\,y\,\delta y,$$

then eliminating c from (γ) and (δ) will, in turn, lead to the relation (β).

$$
(8) \qquad
\begin{aligned}
&\sum_{i=1}^{n}\sum_{\lambda=0}^{\nu}\Big(f_{1i}^{(\lambda)}\,\delta x_i^{(\lambda)} + \varphi_{1i}^{(\lambda)}\,\delta y_i^{(\lambda)} + \psi_{1i}^{(\lambda)}\,\delta z_i^{(\lambda)}\Big) = 0\,, \quad \cdots \\
&\sum_{i=1}^{n}\sum_{\lambda=0}^{\nu}\Big(f_{mi}^{(\lambda)}\,\delta x_i^{(\lambda)} + \varphi_{mi}^{(\lambda)}\,\delta y_i^{(\lambda)} + \psi_{mi}^{(\lambda)}\,\delta z_i^{(\lambda)}\Big) = 0\,,
\end{aligned}
$$

then once more by integrating between the limits t_0 and t_1, reducing to the variations δx_i, δy_i, δz_i with the help of m **Lagrange** multipliers $\lambda_1, \dots, \lambda_m$, and adding them to **d'Alembert**'s principle, that will next produce the system of differential equations:

$$
(9) \quad \left\{
\begin{aligned}
&\frac{\partial H}{\partial x_i} - \frac{d}{dt}\frac{\partial H}{\partial x_i'} + \cdots + (-1)^\nu \frac{d^\nu}{dt^\nu}\frac{\partial H}{\partial x_i^{(\nu)}} = Q_i + \sum_{r=1}^{m}\lambda_r \left(f_{ri}^{(0)} - \frac{df_{ri}^{(1)}}{dt} + \frac{d^2 f_{ri}^{(1)}}{dt^2} + \cdots + (-1)^\nu \frac{d^\nu f_{ri}^{(\nu)}}{dt^\nu}\right), \\
&\frac{\partial H}{\partial y_i} - \frac{d}{dt}\frac{\partial H}{\partial y_i'} + \cdots + (-1)^\nu \frac{d^\nu}{dt^\nu}\frac{\partial H}{\partial y_i^{(\nu)}} = R_i + \sum_{r=1}^{m}\lambda_r \left(\varphi_{ri}^{(0)} - \frac{d\varphi_{ri}^{(1)}}{dt} + \frac{d^2 \varphi_{ri}^{(1)}}{dt^2} + \cdots + (-1)^\nu \frac{d^\nu \varphi_{ri}^{(\nu)}}{dt^\nu}\right), \\
&\frac{\partial H}{\partial z_i} - \frac{d}{dt}\frac{\partial H}{\partial z_i'} + \cdots + (-1)^\nu \frac{d^\nu}{dt^\nu}\frac{\partial H}{\partial z_i^{(\nu)}} = S_i + \sum_{r=1}^{m}\lambda_r \left(\psi_{ri}^{(0)} - \frac{d\psi_{ri}^{(1)}}{dt} + \frac{d^2 \psi_{ri}^{(1)}}{dt^2} + \cdots + (-1)^\nu \frac{d^\nu \psi_{ri}^{(\nu)}}{dt^\nu}\right).
\end{aligned}
\right.
$$

Now, if the constraint equations (8) once more do not include t explicitly then they will produce m differential equations between the coordinates, and together with the $3n$ differential equations (9), they will define a system of simultaneous differential equations for determining x_i, y_i, z_i, λ_1, …, λ_m as functions of time. However, for the case in which equations (8) are not free of t, the method for treating the problem must again be adapted to the demands of the problem.

However, one can also put the extended **Hamilton** principle into a more general form. Namely, if one replaces the quantities $p_s^{(k)}$ with p_{sk} in the function H and thus regards H as a function of:

$$
t, p_1, p_2, \dots, p_\mu, p_{11}, p_{21}, \dots, p_{\mu 1}, \dots, p_{1\nu}, p_{2\nu}, \dots, p_{\mu\nu},
$$

then the variation will be:

$$
\delta\int_{t_0}^{t_1}\left\{ H - \sum_{\lambda=1}^{\mu}\left[P_\lambda\, p_\lambda + (p_{\lambda 1} - p_\lambda')\frac{\partial H}{\partial p_{\lambda 1}} + (p_{\lambda 2} - p_\lambda'')\frac{\partial H}{\partial p_{\lambda 2}} + \cdots + (p_{\lambda o} - p_\lambda^{(\nu)})\frac{\partial H}{\partial p_{\lambda o}}\right]\right\} dt
$$

$$
\begin{aligned}
= \delta\int_{t_0}^{t_1}\sum_{s=1}^{\mu}\Bigg\{&\left[\frac{\partial H}{\partial p_s} - P_s - \sum_{\lambda=1}^{\mu}(p_{\lambda 1} - p_\lambda')\frac{\partial^2 H}{\partial p_{\lambda 1}\,\partial p_s} - \sum_{\lambda=1}^{\mu}(p_{\lambda 1} - p_\lambda'')\frac{\partial^2 H}{\partial p_{\lambda 2}\,\partial p_s} - \cdots - \sum_{\lambda=1}^{\mu}(p_{\lambda\nu} - p_\lambda^{(\nu)})\frac{\partial^2 H}{\partial p_{\lambda\nu}\,\partial p_s}\right]\delta p_s \\
&+ \frac{\partial H}{\partial p_{s1}}\delta p_s' + \frac{\partial H}{\partial p_{s2}}\delta p_s'' + \cdots + \frac{\partial H}{\partial p_{s\nu}}\delta p_s^{(\nu)} \\
&- \left[\sum_{\lambda=1}^{\mu}\left\{(p_{\lambda 1} - p_\lambda')\frac{\partial^2 H}{\partial p_{\lambda 1}\,\partial p_{s1}} + (p_{\lambda 1} - p_\lambda'')\frac{\partial^2 H}{\partial p_{\lambda 2}\,\partial p_{s1}} + \cdots + (p_{\lambda\nu} - p_\lambda^{(\nu)})\frac{\partial^2 H}{\partial p_{\lambda\nu}\,\partial p_{s1}}\right\}\right]\delta p_{s1} \\
&- \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots
\end{aligned}
$$

$$- \left[\sum_{\lambda=1}^{\mu} \left\{ (p_{\lambda 1} - p'_\lambda) \frac{\partial^2 H}{\partial p_{\lambda 1}\, \partial p_{s\nu}} + (p_{\lambda 1} - p''_\lambda) \frac{\partial^2 H}{\partial p_{\lambda 2}\, \partial p_{s\nu}} + \cdots + (p_{\lambda\nu} - p_\lambda^{(\nu)}) \frac{\partial^2 H}{\partial p_{\lambda\nu}\, \partial p_{s\nu}} \right\} \right] \delta p_{s\nu} \Bigg\} dt \;.$$

When one makes the convention that only the variations:

$$\delta p_s\,,\; \delta p'_s,\; \ldots,\; \delta p_s^{(\nu-1)}$$

should vanish for $t = t_0$ and $t = t_1$, it will then follow that the equation:

$$\text{(10)} \qquad \delta \int_{t_0}^{t_1} \left\{ H - \sum_{\lambda=1}^{\mu} \left[P_\lambda\, p_\lambda + (p_{\lambda 1} - p'_\lambda) \frac{\partial H}{\partial p_{\lambda 1}} + (p_{\lambda 2} - p''_\lambda) \frac{\partial H}{\partial p_{\lambda 2}} + \cdots + (p_{\lambda o} - p_\lambda^{(\nu)}) \frac{\partial H}{\partial p_{\lambda o}} \right] \right\} dt = 0$$

is equivalent to the relations:

$$\text{(11)} \qquad \begin{aligned} &\frac{\partial H}{\partial p_s} - \frac{d}{dt} \frac{\partial H}{\partial p_{s1}} + \cdots + (-1)^\nu \frac{d^\nu}{dt^\nu} \frac{\partial H}{\partial p_{s\nu}} - P_s \\ &- \sum_{\lambda=1}^{\mu} (p_{\lambda 1} - p'_\lambda) \frac{\partial^2 H}{\partial p_{\lambda 1}\, \partial p_s} - \cdots - \sum_{\lambda=1}^{\mu} (p_{\lambda\nu} - p_\lambda^{(\nu)}) \frac{\partial^2 H}{\partial p_{\lambda\nu}\, \partial p_s} = 0\,, \end{aligned}$$

$$\text{(12)} \qquad \left\{ \begin{array}{l} \displaystyle\sum_{\lambda=1}^{\mu} \left\{ (p'_{\lambda 1} - p'_\lambda) \frac{\partial^2 H}{\partial p_{\lambda 1}\, \partial p_{s1}} + (p'_{\lambda 2} - p''_\lambda) \frac{\partial^2 H}{\partial p_{\lambda 2}\, \partial p_{s1}} + \cdots + (p'_{\lambda\nu} - p_\lambda^{(\nu)}) \frac{\partial^2 H}{\partial p_{\lambda\nu}\, \partial p_{s1}} \right\} = 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \displaystyle\sum_{\lambda=1}^{\mu} \left\{ (p'_{\lambda 1} - p'_\lambda) \frac{\partial^2 H}{\partial p_{\lambda 1}\, \partial p_{s\nu}} + (p'_{\lambda 2} - p''_\lambda) \frac{\partial^2 H}{\partial p_{\lambda 2}\, \partial p_{s\nu}} + \cdots + (p'_{\lambda\nu} - p_\lambda^{(\nu)}) \frac{\partial^2 H}{\partial p_{\lambda\nu}\, \partial p_{s\nu}} \right\} = 0 \end{array} \right.$$

for $s = 1, 2, \ldots, \mu$. Now, if the determinant of the second differential quotients:

$$\frac{\partial^2 H}{\partial p_{\alpha\beta}\, \partial p_{\gamma\delta}}\,,$$

in which α and γ assume the values $1, 2, \ldots, \mu$, while β and δ assume the values $1, 2, \ldots, \nu$, is not identically zero then equations (12) will imply that:

$$p_{\lambda r} = p_\lambda^{(r)}$$

for all values of λ and r from the sequence of numbers $1, 2, \ldots, \mu$ ($1, 2, \ldots, \nu$, resp.), and equations (11) will once more go to the **Lagrange** equations then, so:

Under the given conditions, the generalized ***Hamilton*** *principle* (10) *is equivalent to the second form of the* ***Lagrange*** *equations*.

§ 7. – The extended principle of the conservation of *vis viva*

If we once more start with the equation:

$$\frac{\partial H}{\partial p_s} - \frac{d}{dt}\frac{\partial H}{\partial p_s'} + \cdots + (-1)^\nu \frac{d^\nu}{dt^\nu}\frac{\partial H}{\partial p_s^{(\nu)}} = P_s$$

then when we multiply it by p_s' and sum over s from 1 to μ, that will give:

$$(1) \qquad \sum_{s=1}^{\mu} p_s' \frac{\partial H}{\partial p_s} - \sum_{s=1}^{\mu} p_s' \frac{d}{dt}\frac{\partial H}{\partial p_s'} + \cdots + (-1)^\nu \sum_{s=1}^{\mu} p_s' \frac{d^\nu}{dt^\nu}\frac{\partial H}{\partial p_s^{(\nu)}} = \sum_{s=1}^{\mu} P_s \, p_s' \ .$$

If we assume that t does not enter into H explicitly, such that:

$$(2) \qquad \frac{dH}{dt} = \sum_{s=1}^{\mu} p_s' \frac{\partial H}{\partial p_s} + \sum_{s=1}^{\mu} p_s'' \frac{\partial H}{\partial p_s'} + \cdots + (-1)^\nu \sum_{s=1}^{\mu} p_s^{(\nu+1)} \frac{\partial H}{\partial p_s^{(\nu)}} ,$$

then it will follow upon substituting the expression for $\sum_{s=1}^{\mu} p_s' \frac{\partial H}{\partial p_s}$ from (1) in (2) that:

$$\frac{dH}{dt} - \sum_{s=1}^{\mu}\left\{ p_s' \frac{d}{dt}\frac{\partial H}{\partial p_s'} + p_s'' \frac{\partial H}{\partial p_s'} \right\} + \sum_{s=1}^{\mu}\left\{ p_s' \frac{d^2}{dt^2}\frac{\partial H}{\partial p_s''} - p_s''' \frac{\partial H}{\partial p_s'} \right\} + \cdots + (-1)^\nu \sum_{s=1}^{\mu}\left\{ p_s' \frac{d^\nu}{dt^\nu}\frac{\partial H}{\partial p_s^{(\nu)}} - p_s^{(\nu+1)} \frac{\partial H}{\partial p_s^{(\nu)}} \right\}$$
$$= \sum_{s=1}^{\mu} P_s \, p_s'$$

or

$$\frac{dH}{dt} - \frac{d}{dt}\sum_{s=1}^{\mu} p_s' \frac{\partial H}{\partial p_s'} + \frac{d}{dt}\sum_{s=1}^{\mu}\left\{ p_s' \frac{d}{dt}\frac{\partial H}{\partial p_s''} - p_s''' \frac{\partial H}{\partial p_s'} \right\} + \cdots + (-1)^\nu \frac{d}{dt}\sum_{s=1}^{\mu}\left\{ p_s' \frac{d^{\nu-1}}{dt^{\nu-1}}\frac{\partial H}{\partial p_s^{(\nu)}} - p_s'' \frac{d^{\nu-2}}{dt^{\nu-2}}\frac{\partial H}{\partial p_s^{(\nu)}} + \cdots \right.$$
$$\left. - (-1)^\nu p_s^{(\nu)} \frac{\partial H}{\partial p_s^{(\nu)}} \right\} = \sum_{s=1}^{\mu} P_s \, p_s' \ .$$

Finally, upon integrating over t:

$$H - \sum_{s=1}^{\mu} p_s' \frac{\partial H}{\partial p_s'} + \sum_{s=1}^{\mu}\left\{ p_s' \frac{d}{dt}\frac{\partial H}{\partial p_s''} - p_s''' \frac{\partial H}{\partial p_s'} \right\} + \cdots + (-1)^\nu \sum_{s=1}^{\mu}\left\{ p_s' \frac{d^{\nu-1}}{dt^{\nu-1}}\frac{\partial H}{\partial p_s^{(\nu)}} - p_s'' \frac{d^{\nu-2}}{dt^{\nu-2}}\frac{\partial H}{\partial p_s^{(\nu)}} + \cdots \right.$$
$$\left. - (-1)^\nu p_s^{(\nu)} \frac{\partial H}{\partial p_s^{(\nu)}} \right\} = \sum_{s=1}^{\mu} \int P_s \, p_s' \, dt + h \ ,$$

in which h means a constant, or:

(3)
$$\begin{aligned} H - \sum_{s=1}^{\mu} p_s' \left(\frac{\partial H}{\partial p_s'} - \frac{d}{dt} \frac{\partial H}{\partial p_s''} + \cdots + (-1)^{\nu-1} \frac{d^{\nu-1}}{dt^{\nu-1}} \frac{\partial H}{\partial p_s^{(\nu)}} \right) \\ - \sum_{s=1}^{\mu} p_s'' \left(\frac{\partial H}{\partial p_s''} - \frac{d}{dt} \frac{\partial H}{\partial p_s'''} + \cdots + (-1)^{\nu-2} \frac{d^{\nu-2}}{dt^{\nu-2}} \frac{\partial H}{\partial p_s^{(\nu)}} \right) \\ - \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ - \sum_{s=1}^{\mu} p_s^{(\nu)} \frac{\partial H}{\partial p_s^{(\nu)}} = \sum_{s=1}^{\mu} \int P_s \, p_s' \, dt + h , \end{aligned}$$

which expresses *the principle of conservation of vis viva*.

If one sets:

(4)
$$\begin{aligned} H - \sum_{s=1}^{\mu} p_s' \left(\frac{\partial H}{\partial p_s'} - \frac{d}{dt} \frac{\partial H}{\partial p_s''} + \cdots + (-1)^{\nu-1} \frac{d^{\nu-1}}{dt^{\nu-1}} \frac{\partial H}{\partial p_s^{(\nu)}} \right) \\ - \sum_{s=1}^{\mu} p_s'' \left(\frac{\partial H}{\partial p_s''} - \frac{d}{dt} \frac{\partial H}{\partial p_s'''} + \cdots + (-1)^{\nu-2} \frac{d^{\nu-2}}{dt^{\nu-2}} \frac{\partial H}{\partial p_s^{(\nu)}} \right) \\ - \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ - \sum_{s=1}^{\mu} p_s^{(\nu)} \frac{\partial H}{\partial p_s^{(\nu)}} = E , \end{aligned}$$

in which E, when expressed in terms of the coordinates and their derivatives with respect to time up to $2\nu - 1$, *shall be called the energy supply of the system* then it will follow from (3) that:

$$\frac{dE}{dt} = \sum_{s=1}^{\mu} P_s \, p_s' .$$

Therefore, the energy supply of the system will continually decrease or increase in measure according to whether the forces P_s *do negative or positive work, resp., such that when external forces are zero, one will have:*

$$E = h ,$$

i.e., the energy will be constant.

If H depends upon only the coordinates and their first derivatives then equation (3) will go to:

$$H - \sum_{s=1}^{\mu} p_s' \frac{\partial H}{\partial p_s'} = \sum_{s=1}^{\mu} \int P_s \, p_s' \, dt + h ,$$

and if H is an entire function of degree m in the p_s' that reads:

$$H = H_0 + H_1 + H_2 + \ldots + H_m$$

when it is arranged into homogeneous functions whose degree is given by the index then one will get:

$$H_0 - H_2 - 2\,H_3 - 3\,H_4 - \ldots - (m-1)\,H_m = \sum_{s=1}^{\mu} \int P_s\, p_s'\, dt + h\,.$$

If all external forces P_s are zero then the principle of the conservation of *vis viva* will assume the form:

$$H_0 - H_2 - 2\,H_3 - 3\,H_4 - \ldots - (m-1)\,H_m = h\,.$$

If:

$$H = -T - U\,,$$

and the force function U, which depends upon the coordinates and their first derivatives, has degree only two in its derivatives, and in addition, the *vis viva* T is a homogeneous function of degree two in the first derivatives (which is always the case when the constraint conditions are independent of time) then if U is represented in the form:

$$U = U_0 + U_1 + U_2\,,$$

the energy principle can be expressed by the equation:

$$T - U_0 + U_2 = h\,.$$

Therefore, for the **Weber** force function, which satisfies the condition that was imposed, it will go to:

$$T - W_0 + W_2 = h\,,$$

in which:

$$W_0 = \frac{m\,m_1}{r}\,, \qquad W_2 = \frac{m\,m_1}{k^2\,r}\, r'^2\,.$$

Under the assumption that H does not include time explicitly, the principle of the conservation of *vis viva* can be derived from the **Lagrange** equations in their second form, so the former is a necessary consequence of those equations for holonomic systems.

In order to exhibit the law of conservation of energy for non-holonomic systems, as well, under the assumption that H does not include time t explicitly, we multiply equations (6) of **§ 4** by x_i', y_i', z_i', in succession, and add all of those equations. Under the assumption that the functions f_{ki}, φ_{ki}, ψ_{ki} do not include time explicitly, the actual displacements will also become virtual ones, so equations (2) of **§ 4** will be satisfied when the increments are set to dx_i, dy_i, dz_i, instead of δx_i, δy_i, δz_i, and the relation will then give:

$$\sum_{i=1}^{n} x_i' \left(\frac{\partial H}{\partial x_i} - \frac{d}{dt}\frac{\partial H}{\partial x_i'} + \cdots + (-1)^\nu \frac{d^\nu}{dt^\nu}\frac{\partial H}{\partial x_i^{(\nu)}} \right)$$
$$+ \sum_{i=1}^{n} y_i' \left(\frac{\partial H}{\partial y_i} - \frac{d}{dt}\frac{\partial H}{\partial y_i'} + \cdots + (-1)^\nu \frac{d^\nu}{dt^\nu}\frac{\partial H}{\partial y_i^{(\nu)}} \right)$$
$$+ \sum_{i=1}^{n} z_i' \left(\frac{\partial H}{\partial z_i} - \frac{d}{dt}\frac{\partial H}{\partial z_i'} + \cdots + (-1)^\nu \frac{d^\nu}{dt^\nu}\frac{\partial H}{\partial z_i^{(\nu)}} \right)$$
$$= \sum_{i=1}^{n} (x_i' Q_i + y_i' R_i + z_i' S_i) .$$

Therefore, in analogy with the transformation that was performed on (1), when:

$$H - \sum_{i=1}^{n} x_i' \left(\frac{\partial H}{\partial x_i'} - \frac{d}{dt}\frac{\partial H}{\partial x_i''} + \cdots \right)$$
$$- \sum_{i=1}^{n} y_i' \left(\frac{\partial H}{\partial y_i'} - \frac{d}{dt}\frac{\partial H}{\partial y_i''} + \cdots \right)$$
$$- \sum_{i=1}^{n} z_i' \left(\frac{\partial H}{\partial z_i'} - \frac{d}{dt}\frac{\partial H}{\partial z_i''} + \cdots \right)$$
$$- \sum_{i=1}^{n} x_i'' \left(\frac{\partial H}{\partial x_i''} - \frac{d}{dt}\frac{\partial H}{\partial x_i'''} + \cdots \right)$$
$$- \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$- \sum_{i=1}^{n} x_i^{(\nu)} \frac{\partial H}{\partial x_i^{(\nu)}} - \sum_{i=1}^{n} y_i^{(\nu)} \frac{\partial H}{\partial y_i^{(\nu)}} - \sum_{i=1}^{n} z_i^{(\nu)} \frac{\partial H}{\partial z_i^{(\nu)}} = E ,$$

in which *E* once more means the energy supply that was referred to above, one will have:

$$\frac{dE}{dt} = \sum_{i=1}^{n} (x_i' Q_i + y_i' R_i + z_i' S_i) .$$

If *H* is once more decomposed into its components, so one sets:

$$H = - T - U ,$$

then:

$$- T + \sum_{s=1}^{\mu} p_s' \left(\frac{\partial T}{\partial p_s'} - \frac{d}{dt}\frac{\partial T}{\partial p_s''} + \cdots \right) + \sum_{s=1}^{\mu} p_s'' \left(\frac{\partial T}{\partial p_s''} - \cdots \right) + \cdots + \sum_{s=1}^{\mu} p_s^{(\nu)} \frac{\partial T}{\partial p_s^{(\nu)}} = E_a$$

shall be called the *actual energy*, and:

$$-U+\sum_{s=1}^{\mu} p'_s\left(\frac{\partial U}{\partial p'_s}-\frac{d}{dt}\frac{\partial U}{\partial p''_s}+\cdots\right)+\sum_{s=1}^{\mu} p''_s\left(\frac{\partial U}{\partial p''_s}-\cdots\right)+\cdots+\sum_{s=1}^{\mu} p_s^{(\nu)}\frac{\partial U}{\partial p_s^{(\nu)}}=E_p$$

shall be called the *potential energy*. When no external forces are present, one will then have:

$$E_a+E_p=E=h\,,$$

so *the sum of the actual and potential energy will be constant*.

If the force system has a force function, so one is dealing with the problem of a kinetic potential, then from Lemma 4, we know that infinitely-many kinetic potentials will exist, but they will all differ by functions that are complete differential quotients with respect to t of a function of the coordinates and their derivatives up to order $\nu-1$. If one now denotes the energy values that belong to two such values H_1 and H_2 of the kinetic potential by E_1 and E_2, resp., then from (4), one will have:

$$E_1=H_1-\sum_{s=1}^{\mu} p'_s\left(\frac{\partial H_1}{\partial p'_s}-\frac{d}{dt}\frac{\partial H_1}{\partial p''_s}+\cdots\right)-\sum_{s=1}^{\mu} p''_s\left(\frac{\partial H_1}{\partial p''_s}-\cdots\right)-\cdots-\sum_{s=1}^{\mu} p_s^{(\nu)}\frac{\partial H_1}{\partial p_s^{(\nu)}}\,,$$

$$E_2=H_2-\sum_{s=1}^{\mu} p'_s\left(\frac{\partial H_2}{\partial p'_s}-\frac{d}{dt}\frac{\partial H_2}{\partial p''_s}+\cdots\right)-\sum_{s=1}^{\mu} p''_s\left(\frac{\partial H_2}{\partial p''_s}-\cdots\right)-\cdots-\sum_{s=1}^{\mu} p_s^{(\nu)}\frac{\partial H_2}{\partial p_s^{(\nu)}}\,.$$

When one sets:

$$H_1-H_2=K\,,$$

and subtracts those two equations, one will get:

$$E_1-E_2=K-\sum_{s=1}^{\mu} p'_s\left(\frac{\partial K}{\partial p'_s}-\frac{d}{dt}\frac{\partial K}{\partial p''_s}+\cdots\right)-\sum_{s=1}^{\mu} p''_s\left(\frac{\partial K}{\partial p''_s}-\cdots\right)-\cdots-\sum_{s=1}^{\mu} p_s^{(\nu)}\frac{\partial K}{\partial p_s^{(\nu)}}\,.$$

Therefore, since H_1 and H_2 (so K, as well) cannot include time t explicitly if the energy principle is to be valid, one will have:

$$\begin{aligned}
\frac{d\,(E_1-E_2)}{dt} &= \sum_{s=1}^{\mu}\frac{\partial K}{\partial p_s}p'_s+\sum_{s=1}^{\mu}\frac{\partial K}{\partial p'_s}p''_s+\cdots+\sum_{s=1}^{\mu}\frac{\partial K}{\partial p_s^{(\nu)}}p_s^{(\nu+1)}\\
&\quad-\sum_{s=1}^{\mu} p''_s\left(\frac{\partial K}{\partial p'_s}-\frac{d}{dt}\frac{\partial K}{\partial p''_s}+\cdots\right)\\
&\quad-\sum_{s=1}^{\mu} p'_s\left(\frac{d}{dt}\frac{\partial K}{\partial p'_s}-\frac{d^2}{dt^2}\frac{\partial K}{\partial p''_s}+\cdots\right)\\
&\quad-\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&= \sum_{s=1}^{\mu} p'_s\left(\frac{\partial K}{\partial p_s}-\frac{d}{dt}\frac{\partial K}{\partial p'_s}+\frac{d^2}{dt^2}\frac{\partial K}{\partial p'_s}+-\cdots+(-1)^{\nu}\frac{d^{\nu}}{dt^{\nu}}\frac{\partial K}{\partial p_s^{(\nu)}}\right).
\end{aligned}$$

However, since K must be a complete differential quotient of a function of t, the coordinates, and their derivatives up to order $\nu - 1$, from Lemma 3, the parentheses on the right-hand side will vanish identically for every value of s, and therefore one will have:

$$\frac{d\,(E_1 - E_2)}{dt} = 0 \qquad \text{or} \qquad E_1 - E_2 = c\,,$$

in which c means a constant, which can also be inferred immediately from the energy principle.

All of the problems that belong to the same kinetic potential will the differ by differential quotients of a function of time, the coordinates, and their derivatives up to order $\nu - 1$, *and when the kinetic potential is free of time, the infinitude of associated energy values will be constants when they are regarded as functions of the coordinates and their derivatives.*

Whereas it should be clear from (4) that the energy supply of the system will be determined uniquely when the kinetic potential H is given as a function of $p_1, \ldots, p_\mu,\ p_1', \ldots,\ p_\mu', \ldots,\ p_1^{(\nu)}, \ldots,\ p_\mu^{(\nu)}$, and indeed as a function of the coordinates and their derivatives up to order $2\nu - 1$, one also has that conversely when E is given, the kinetic potential will be the solution to a partial differential equation, and in addition, it will be subject to the condition that it should include the derivatives of the coordinates only up to order ν. However, one can already see from this that the energy supply of a system cannot be given as an arbitrary function of the coordinates and their derivatives but must be determined by the fact that some conditions that are easy to exhibit must be satisfied, from the foregoing.

Namely, if one sets:

$$\frac{\partial H}{\partial p_s} - \frac{d}{dt}\frac{\partial H}{\partial p_s'} + \frac{d^2}{dt^2}\frac{\partial H}{\partial p_s'} + - \cdots + (-1)^\nu \frac{d^\nu}{dt^\nu}\frac{\partial H}{\partial p_s^{(\nu)}} = K_s$$

then it will follow with the help of the transformation that was applied to equation (1) that:

$$\sum_{s=1}^{\mu} K_s\, p_s' = \frac{dE}{dt}\,. \tag{5}$$

Therefore, one will get the necessary and sufficient conditions for K_s from the necessary and sufficient conditions that were developed in Lemma 4 for the quantities N_k to be representable in terms of a quantity M in the manner that was given there when one exchanges N_k with K_s and M with H. One can then derive the conditions on E that must be fulfilled identically by eliminating K_s from them and (5). If we assume, e.g., that there is only one variable p and that $\nu = 1$ then the only condition equation that will exist will read:

$$\frac{\partial K}{\partial p'} - \frac{d}{dt}\frac{\partial K}{\partial p''} = 0\,.$$

Eliminating K from that equation and (5) with the use of the formulas in Lemma 1 will yield an identity. *The expression for the energy supply, which depends upon the one coordinate and its first derivative, is then subject to no other condition* than that it must always be finite in the course of motion, and furthermore, since:

$$\frac{\partial E}{\partial p'} = - p' \frac{\partial^2 H}{\partial p'^2} ,$$

under the assumption that the values $\frac{\partial H}{\partial p'}$ and $\frac{\partial^2 H}{\partial p'^2}$ will always remain finite for the values that come into question, we must have:

$$\left(\frac{\partial E}{\partial p'}\right)_{p'=0} = 0 .$$

Similarly, for $\nu = 1$ *and an arbitrary number of variables, E can be given as a function of* $p_1, \ldots, p_\mu, p'_1, \ldots, p'_\mu$ *that is constrained by the same restrictions and is arbitrary, moreover.* In that case, the determination of the kinetic potential by means of (4) will lead to the linear partial differential equation:

$$p'_1 \frac{\partial H}{\partial p'_1} + p'_2 \frac{\partial H}{\partial p'_2} + \cdots + p'_\mu \frac{\partial H}{\partial p'_\mu} = H - E ,$$

whose general integral will produce the value for the kinetic potential:

$$H = - p'_1 \int \frac{(E)}{p_1'^2} dp'_1 + p'_1 \varphi\left(\frac{p'_2}{p'_1}, \frac{p'_3}{p'_1}, \ldots, \frac{p'_\mu}{p'_1}\right) ,$$

in which (E) means the expression for energy that was given above when one sets:

$$p'_2 = \alpha_2 \, p'_1 , \quad p'_3 = \alpha_3 \, p'_1 , \quad \ldots, \quad p'_\mu = \alpha_\mu \, p'_1$$

in it. After the integration, one once more substitutes the quotients of the p' for the quantities α, while φ means an arbitrary function, and the quadrature that appears in the expression H will be finite as a result of the assumption that was made. When one recalls the single-valuedness, finitude, and continuity of the kinetic potential, H can be put into the form:

$$H = - p'_1 \int \frac{(E)}{p_1'^2} dp_1 + A_1 \, p'_1 + A_2 \, p'_2 + \cdots + A_\mu \, p'_\mu ,$$

in which $A_1, \ldots, A_\mu$ are arbitrary functions of the $p_1, \ldots, p_\mu$. *That will then determine the energy supply of a system for* $\nu = 1$ *when its kinetic potential is a general linear function in the first derivatives of the coordinates*, which we will come back to later.

For $\mu = 1$ and $\nu = 2$, from Lemma 4, the necessary and sufficient conditions for K read:

$$\frac{\partial K}{\partial p'} - \frac{d}{dt}\frac{\partial K}{\partial p''} + \frac{d^2}{dt^2}\frac{\partial K}{\partial p'''} - \frac{d^3}{dt^3}\frac{\partial K}{\partial p^{IV}} = 0 ,$$

$$\frac{\partial K}{\partial p'''} - 2\frac{d}{dt}\frac{\partial K}{\partial p^{IV}} = 0 .$$

It is easy to see that eliminating K from (5) and those two equations will produce only one necessary and sufficient condition equation that must be satisfied identically by the energy supply E that depends upon p and p':

$$p'\frac{\partial E}{\partial p''} + 2p''\frac{\partial E}{\partial p'''} - p'\frac{d}{dt}\frac{\partial E}{\partial p'''} = 0 ,$$

since the substitution of K in the first of the two equations will lead to an identity due to the condition on E that was found above, or when one sets:

$$E = p'^2 E_1 ,$$

one will get:

$$\frac{\partial E_1}{\partial p''} - \frac{d}{dt}\frac{\partial E_1}{\partial p'''} = 0 .$$

Similarly, the conditions that the energy supply must be subject to when its coordinate derivatives go up to only order $2\nu - 1$ will follow for every value of ν from the necessary and sufficient conditions for K that were exhibited above when it includes the derivatives of the coordinates up to order 2ν.

Finally, in order to examine the energy principle in the case where time t does not enter explicitly into the kinetic potential H or the constraint equations, but the latter includes the coordinates and their derivatives up to order μ, so they are given by:

$$F_1(x_i, y_i, z_i, x_i', y_i', z_i', \ldots, x_i^{(\mu)}, y_i^{(\mu)}, z_i^{(\mu)}) = 0 , \ \ldots, \ F_m(x_i, y_i, z_i, x_i', y_i', z_i', \ldots, x_i^{(\mu)}, y_i^{(\mu)}, z_i^{(\mu)}) = 0 ,$$

one starts from the equations of motion that were described above:

$$\frac{\partial H}{\partial x_i} - \frac{d}{dt}\frac{\partial H}{\partial x_i'} + \cdots + (-1)^\nu \frac{d^\nu}{dt^\nu}\frac{\partial H}{\partial x_s^{(\nu)}} = Q_i + \sum_{r=1}^{m} \lambda_r \left(\frac{\partial F_r}{\partial x_i} - \frac{d}{dt}\frac{\partial F_r}{\partial x_i'} + \cdots + (-1)^\mu \frac{d^\mu}{dt^\mu}\frac{\partial F_r}{\partial x_s^{(\mu)}} \right) ,$$

$$\frac{\partial H}{\partial y_i} - \frac{d}{dt}\frac{\partial H}{\partial y_i'} + \cdots + (-1)^\nu \frac{d^\nu}{dt^\nu}\frac{\partial H}{\partial y_s^{(\nu)}} = R_i + \sum_{r=1}^m \lambda_r \left(\frac{\partial F_r}{\partial y_i} - \frac{d}{dt}\frac{\partial F_r}{\partial y_i'} + \cdots + (-1)^\mu \frac{d^\mu}{dt^\mu}\frac{\partial F_r}{\partial y_s^{(\mu)}} \right),$$

$$\frac{\partial H}{\partial z_i} - \frac{d}{dt}\frac{\partial H}{\partial z_i'} + \cdots + (-1)^\nu \frac{d^\nu}{dt^\nu}\frac{\partial H}{\partial y_s^{(\nu)}} = R_i + \sum_{r=1}^m \lambda_r \left(\frac{\partial F_r}{\partial z_i} - \frac{d}{dt}\frac{\partial F_r}{\partial z_i'} + \cdots + (-1)^\mu \frac{d^\mu}{dt^\mu}\frac{\partial F_r}{\partial z_s^{(\mu)}} \right),$$

multiplies them by x_i', y_i', z_i', and takes the sum over i from 1 to n. Now, since the coefficients of the λ have precisely the form of the left-hand sides of the equations of motion, according to the previous development, when one recalls that $F_k = 0$ and sets:

$$-\sum_{i=1}^n x_i' \left(\frac{\partial F_r}{\partial x_i'} - \frac{d}{dt}\frac{\partial F_r}{\partial x_i''} + \cdots + (-1)^{\mu-1} \frac{d^{\mu-1}}{dt^{\mu-1}}\frac{\partial F_r}{\partial x_i^{(\mu)}} \right) - \cdots - \sum_{i=1}^n z_i' \left(\frac{\partial F_r}{\partial z_i'} - \frac{d}{dt}\frac{\partial F_r}{\partial z_i''} + \cdots + (-1)^{\mu-1} \frac{d^{\mu-1}}{dt^{\mu-1}}\frac{\partial F_r}{\partial z_i^{(\mu)}} \right)$$

$$-\sum_{i=1}^n x_i'' \left(\frac{\partial F_r}{\partial x_i''} - \frac{d}{dt}\frac{\partial F_r}{\partial x_i'''} + \cdots \right) - \cdots - \sum_{i=1}^n z_i'' \left(\frac{\partial F_r}{\partial z_i''} - \frac{d}{dt}\frac{\partial F_r}{\partial z_i'''} + \cdots \right)$$

$$- \cdots\cdots\cdots\cdots$$

$$-\sum_{i=1}^n x_i^{(\mu)} \frac{\partial F_r}{\partial x_i^{(\mu)}} - \sum_{i=1}^n y_i^{(\mu)} \frac{\partial F_r}{\partial y_i^{(\mu)}} - \sum_{i=1}^n z_i^{(\mu)} \frac{\partial F_r}{\partial z_i^{(\mu)}} = \mathfrak{E}_k,$$

one will get the relation:

$$\frac{dE}{dt} = \sum_{i=1}^n (Q_i\, x_i' + R_i\, y_i' + S_i\, z_i') + \lambda_1 \frac{d\mathfrak{E}_1}{dt} + \cdots + \lambda_m \frac{d\mathfrak{E}_m}{dt},$$

or when the external forces are zero:

$$\frac{dE}{dt} = \lambda_1 \frac{d\mathfrak{E}_1}{dt} + \cdots + \lambda_m \frac{d\mathfrak{E}_m}{dt}.$$

Now, since $E = h$ will prove to be independent of the values of λ when:

$$\frac{d\mathfrak{E}_1}{dt} = 0, \qquad \frac{d\mathfrak{E}_2}{dt} = 0, \qquad \ldots, \qquad \frac{d\mathfrak{E}_m}{dt} = 0,$$

it will then follow that:

When the equations of constraint are free of t, but depend upon the coordinates and their derivatives up to an arbitrary order, the principle of the conservation of vis viva will be valid when the quantities $\mathfrak{E}_1$, $\mathfrak{E}_2$, …, $\mathfrak{E}_m$ that were defined above go to constants because of the equations of constraint and their differential quotients with respect to t.

Thus, for $\mu = 1$, the series of constraint equations:

$$F_1(x_i, y_i, z_i, x_i', y_i', z_i') = 0 \, , \, \ldots, \, F_m(x_i, y_i, z_i, x_i', y_i', z_i') = 0$$

will require that in order for the energy principle to be valid, those equations must make the expressions:

$$\sum_{i=1}^{n} \left\{ x_i' \frac{\partial F_k}{\partial x_i'} + y_i' \frac{\partial F_k}{\partial y_i'} + z_i' \frac{\partial F_k}{\partial z_i'} \right\}$$

into constants, which will be the case when, e.g., the equations of constraint represent homogeneous function relative to x_i', y_i', z_i'.

If the equations of constraint are free of t and given as linear homogeneous equations in the variations of the coordinates and their derivatives, so for non-holonomic systems, in the form:

$$\sum_{i=1}^{n} \{ f_{ki}\,\delta x_i + \varphi_{ki}\,\delta y_i + \psi_{ki}\,\delta z_i + f_{ki}^{(1)}\,\delta x_i' + \varphi_{ki}^{(1)}\,\delta y_i' + \psi_{ki}^{(1)}\,\delta z_i' + \cdots$$
$$+ f_{ki}^{(\mu)}\,\delta x_i^{(\mu)} + \varphi_{ki}^{(\mu)}\,\delta y_i^{(\mu)} + \psi_{ki}^{(\mu)}\,\delta z_i^{(\mu)} \} = 0$$

$$(k = 1, 2, \ldots, m),$$

in which the coefficients of the variations mean given functions of the coordinates and their derivatives up to order μ, then the actual variations will once more become virtual ones. Therefore, those conditions will go to the differential equations:

$$\sum_{i=1}^{n} \{ x_i' f_{ki} + y_i' \varphi_{ki} + z_i' \psi_{ki} + x_i'' f_{ki}^{(1)} + y_i'' \varphi_{ki}^{(1)} + z_i' \psi_{ki}^{(1)} + \cdots$$
$$+ x_i^{(\mu+1)} f_{ki}^{(\mu)} + y_i^{(\mu+1)} \varphi_{ki}^{(\mu)} + z_i^{(\mu+1)} \psi_{ki}^{(\mu)} \} = 0$$

$$(k = 1, 2, \ldots, m),$$

which will now enter in place of the previous constraint equations $F_k = 0$.

§ 8. – The extension of Gauss's principle of least constraint

If one forms the expression:

$$M = \sum_{i=1}^{n} \left\{ A_i \left(\frac{\partial H}{\partial x_i} - \frac{d}{dt}\frac{\partial H}{\partial x_i'} + \cdots + (-1)^\nu \frac{d^\nu}{dt^\nu}\frac{\partial H}{\partial x_i^{(\nu)}} - Q_i \right)^2 \right. \tag{1}$$
$$+ B_i \left(\frac{\partial H}{\partial y_i} - \frac{d}{dt}\frac{\partial H}{\partial y_i'} + \cdots + (-1)^\nu \frac{d^\nu}{dt^\nu}\frac{\partial H}{\partial y_i^{(\nu)}} - R_i \right)^2$$
$$\left. + C_i \left(\frac{\partial H}{\partial z_i} - \frac{d}{dt}\frac{\partial H}{\partial z_i'} + \cdots + (-1)^\nu \frac{d^\nu}{dt^\nu}\frac{\partial H}{\partial z_i^{(\nu)}} - S_i \right)^2 \right\},$$

in which A_i, B_i, C_i shall be arbitrary functions of t, the coordinates, and their derivatives up to order $2\nu - 1$, while the Q_i, R_i, S_i depend upon those same quantities in a given way, and one seeks the minimum of M when one considers that quantity to be a function of the $x_i^{(2\nu)}$, $y_i^{(2\nu)}$, $z_i^{(2\nu)}$, while fixing x_i, y_i, z_i, x_i', y_i', z_i', …, $x_i^{(2\nu-1)}$, $y_i^{(2\nu-1)}$, $z_i^{(2\nu-1)}$ then that will give the necessary condition that:

$$\tfrac{1}{2}\delta M = \sum_{r=1}^{n}\sum_{i=1}^{n} \left\{ (-1)^\nu A_i \left(\frac{\partial H}{\partial x_i} - \frac{d}{dt}\frac{\partial H}{\partial x_i'} + \cdots + (-1)^\nu \frac{d^\nu}{dt^\nu}\frac{\partial H}{\partial x_i^{(\nu)}} - Q_i \right) \right. \tag{2}$$
$$\left(\frac{\partial^2 H}{\partial x_i^{(\nu)}\,\partial x_r^{(\nu)}}\delta x_r^{(2\nu)} + \frac{\partial^2 H}{\partial x_i^{(\nu)}\,\partial y_r^{(\nu)}}\delta y_r^{(2\nu)} + \frac{\partial^2 H}{\partial x_i^{(\nu)}\,\partial z_r^{(\nu)}}\delta z_r^{(2\nu)} \right)$$
$$+ (-1)^\nu B_i \left(\frac{\partial H}{\partial y_i} - \frac{d}{dt}\frac{\partial H}{\partial y_i'} + \cdots + (-1)^\nu \frac{d^\nu}{dt^\nu}\frac{\partial H}{\partial y_i^{(\nu)}} - R_i \right)$$
$$\left(\frac{\partial^2 H}{\partial y_i^{(\nu)}\,\partial x_r^{(\nu)}}\delta x_r^{(2\nu)} + \frac{\partial^2 H}{\partial y_i^{(\nu)}\,\partial y_r^{(\nu)}}\delta y_r^{(2\nu)} + \frac{\partial^2 H}{\partial y_i^{(\nu)}\,\partial z_r^{(\nu)}}\delta z_r^{(2\nu)} \right)$$
$$+ (-1)^\nu C_i \left(\frac{\partial H}{\partial z_i} - \frac{d}{dt}\frac{\partial H}{\partial z_i'} + \cdots + (-1)^\nu \frac{d^\nu}{dt^\nu}\frac{\partial H}{\partial z_i^{(\nu)}} - S_i \right)$$
$$\left. \left(\frac{\partial^2 H}{\partial z_i^{(\nu)}\,\partial x_r^{(\nu)}}\delta x_r^{(2\nu)} + \frac{\partial^2 H}{\partial z_i^{(\nu)}\,\partial y_r^{(\nu)}}\delta y_r^{(2\nu)} + \frac{\partial^2 H}{\partial z_i^{(\nu)}\,\partial z_r^{(\nu)}}\delta z_r^{(2\nu)} \right) \right\} = 0,$$

since $x_i^{(2\nu)}$, $y_i^{(2\nu)}$, $z_i^{(2\nu)}$ occur only in the terms:

$$\frac{d^\nu}{dt^\nu}\frac{\partial H}{\partial x_i^{(\nu)}}, \quad \frac{d^\nu}{dt^\nu}\frac{\partial H}{\partial y_i^{(\nu)}}, \quad \frac{d^\nu}{dt^\nu}\frac{\partial H}{\partial z_i^{(\nu)}}.$$

Indeed, they are endowed with the coefficients:

$$\frac{\partial^2 H}{\partial x_i^{(\nu)}\,\partial x_r^{(\nu)}},\ \frac{\partial^2 H}{\partial x_i^{(\nu)}\,\partial y_r^{(\nu)}},\ \frac{\partial^2 H}{\partial x_i^{(\nu)}\,\partial z_r^{(\nu)}},\ \frac{\partial^2 H}{\partial y_i^{(\nu)}\,\partial x_r^{(\nu)}},\ \ldots,\ \frac{\partial^2 H}{\partial z_i^{(\nu)}\,\partial x_r^{(\nu)}},\ \ldots$$

Now, if that equation is to go to the equations of motion (5) in **§ 4** then one would need to have:

(3)
$$\frac{\partial^2 H}{\partial x_i^{(\nu)}\,\partial x_r^{(\nu)}} = \frac{\partial^2 H}{\partial y_i^{(\nu)}\,\partial y_r^{(\nu)}} = \frac{\partial^2 H}{\partial z_i^{(\nu)}\,\partial z_r^{(\nu)}} = 0$$

when i is different from r, and furthermore that:

(4)
$$\frac{\partial^2 H}{\partial x_i^{(\nu)}\,\partial y_r^{(\nu)}} = \frac{\partial^2 H}{\partial x_i^{(\nu)}\,\partial z_r^{(\nu)}} = \frac{\partial^2 H}{\partial y_i^{(\nu)}\,\partial z_r^{(\nu)}} = 0$$

when i and r are either equal or unequal, and finally that:

$$A_i = (-1)^\nu \cdot \frac{1}{\dfrac{\partial^2 H}{\partial x_i^{(\nu)2}}}, \qquad B_i = (-1)^\nu \cdot \frac{1}{\dfrac{\partial^2 H}{\partial y_i^{(\nu)2}}}, \qquad C_i = (-1)^\nu \cdot \frac{1}{\dfrac{\partial^2 H}{\partial z_i^{(\nu)2}}}.$$

In order for the expression:

(5)
$$\begin{aligned} M = \sum_{i=1}^{n} \Bigg\{ & \left(\frac{\partial H}{\partial x_i} - \frac{d}{dt}\frac{\partial H}{\partial x_i'} + \cdots + (-1)^\nu \frac{d^\nu}{dt^\nu}\frac{\partial H}{\partial x_i^{(\nu)}} - Q_i\right)^2 \frac{(-1)^\nu}{\dfrac{\partial^2 H}{\partial x_i^{(\nu)2}}} \\ & + \left(\frac{\partial H}{\partial y_i} - \frac{d}{dt}\frac{\partial H}{\partial y_i'} + \cdots + (-1)^\nu \frac{d^\nu}{dt^\nu}\frac{\partial H}{\partial y_i^{(\nu)}} - Q_i\right)^2 \frac{(-1)^\nu}{\dfrac{\partial^2 H}{\partial y_i^{(\nu)2}}} \\ & + \left(\frac{\partial H}{\partial z_i} - \frac{d}{dt}\frac{\partial H}{\partial z_i'} + \cdots + (-1)^\nu \frac{d^\nu}{dt^\nu}\frac{\partial H}{\partial z_i^{(\nu)}} - Q_i\right)^2 \frac{(-1)^\nu}{\dfrac{\partial^2 H}{\partial z_i^{(\nu)2}}} \end{aligned}$$

to be a minimum, the following relation must be true:

$$\begin{aligned} \tfrac{1}{2}\,\delta M = \sum_{i=1}^{n} \Bigg\{ & \left(\frac{\partial H}{\partial x_i} - \frac{d}{dt}\frac{\partial H}{\partial x_i'} + \cdots - Q_i\right)\delta x_i^{(2\nu)} \\ & + \left(\frac{\partial H}{\partial y_i} - \frac{d}{dt}\frac{\partial H}{\partial y_i'} + \cdots - R_i\right)\delta y_i^{(2\nu)} \end{aligned}$$

$$+\left(\frac{\partial H}{\partial z_i}-\frac{d}{dt}\frac{\partial H}{\partial z_i'}+\cdots-S_i\right)\delta z_i^{(2\nu)}\Bigg\} = 0\,,$$

which agrees with the equations of motion that were referred to above, since when one fixes the values of x_i, y_i, z_i, and their derivatives up to order $2\nu - 1$, the variations $\delta x_i^{(2\nu)}$, $\delta y_i^{(2\nu)}$, $\delta z_i^{(2\nu)}$ will likewise satisfy the equations of motion (2) in **§ 4**, and can therefore be considered to be virtual displacements. However, in order to see whether the expression M does, in fact, experience a maximum or minimum, one forms:

$$\begin{aligned}\tfrac{1}{2}\delta^2 M \;=\; \sum_{i=1}^{n}\Bigg\{&\left(\frac{\partial H}{\partial x_i}-\frac{d}{dt}\frac{\partial H}{\partial x_i'}+\cdots-Q_i\right)\delta^2 x_i^{(2\nu)}\\ &+\left(\frac{\partial H}{\partial y_i}-\frac{d}{dt}\frac{\partial H}{\partial y_i'}+\cdots-R_i\right)\delta^2 y_i^{(2\nu)}\\ &+\left(\frac{\partial H}{\partial z_i}-\frac{d}{dt}\frac{\partial H}{\partial z_i'}+\cdots-S_i\right)\delta^2 z_i^{(2\nu)}\Bigg\}\\ +\sum_{i=1}^{n}\Bigg\{&(-1)^\nu\frac{\partial^2 H}{\partial x_i^{(\nu)2}}\left(\delta^2 x_i^{(2\nu)}\right)^2+(-1)^\nu\frac{\partial^2 H}{\partial y_i^{(\nu)2}}\left(\delta^2 y_i^{(2\nu)}\right)^2+(-1)^\nu\frac{\partial^2 H}{\partial z_i^{(\nu)2}}\left(\delta^2 z_i^{(2\nu)}\right)^2\Bigg\},\end{aligned}$$

under the assumption that equations (3) and (4) are true. That will then imply that when $\delta M = 0$, so during the course of the motion:

$$(\alpha)\qquad (-1)^\nu\frac{\partial^2 H}{\partial x_i^{(\nu)2}}\,,\quad (-1)^\nu\frac{\partial^2 H}{\partial y_i^{(\nu)2}}\,,\quad (-1)^\nu\frac{\partial^2 H}{\partial z_i^{(\nu)2}}$$

will always be positive quantities. One will then have that $\delta^2 M > 0$ and M is positive, and when those quantities are negative, one will have that $\delta^2 M < 0$, but M is negative, so the absolute value of M will always experience a minimum. It will then follow that:

The absolute value of the expression M in equation (5) *will assume a minimum value over all values of* $x_i^{(2\nu)}$, $y_i^{(2\nu)}$, $z_i^{(2\nu)}$ *with fixed values of* x_i, y_i, z_i, x_i', y_i', z_i', …, $x_i^{(2\nu-1)}$, $y_i^{(2\nu-1)}$, $z_i^{(2\nu-1)}$ *that satisfy the **Lagrange** equations of motion when the kinetic potential satisfies the conditions* (3) *and* (4) *(which would be the case when it were an entire function of* $x_i^{(\nu)}$, $y_i^{(\nu)}$, $z_i^{(\nu)}$ *in which only powers of the individual quantities appeared, but not products of them), and the quantities* (α) *always keep the same common sign during the course of motion. Among those conditions, one will then find the equivalence of the principle of least constraint with the **extended d'Alembert** principle, so with **Lagrange**'s equations in the first form.*

For $\nu = 1$, when the actual energy is separated from the potential energy in the kinetic potential:

$$H = -\tfrac{1}{2}\sum_{i=1}^{n} m_i \left(x_i'^2 + y_i'^2 + z_i'^2\right) - U \,,$$

in which U depends upon only the coordinates, so the conditions (3) and (4) are satisfied, and the quantities (α) all assume the value m_i, such that in the mechanics of ponderable masses, the positive expression:

$$M = \sum_{i=1}^{n} \frac{1}{m_i}\left\{\left(m_i\, x_i'' - \frac{\partial U}{\partial x_i} - Q_i\right)^2 + \left(m_i\, y_i'' - \frac{\partial U}{\partial y_i} - R_i\right)^2 + \left(m_i\, z_i'' - \frac{\partial U}{\partial z_i} - S_i\right)^2\right\}$$

will assume a minimum for those values of that are defined by **d'Alembert**'s principle:

$$\sum_{i=1}^{n} \left\{\left(m_i\, x_i'' - \frac{\partial U}{\partial x_i} - Q_i\right)\delta x_i + \left(m_i\, y_i'' - \frac{\partial U}{\partial y_i} - R_i\right)\delta y_i + \left(m_i\, z_i'' - \frac{\partial U}{\partial z_i} - S_i\right)\delta z_i\right\} = 0$$

when the values of x_i, y_i, z_i, x_i', y_i', z_i' are preserved for all value systems being compared.

If the kinetic potential is once more given as a function of the μ free coordinates $p_1, p_2, \ldots, p_\mu$, and one forms:

$$M = \sum_{s=1}^{\mu} \left\{\frac{\partial H}{\partial p_s} - \frac{d}{dt}\frac{\partial H}{\partial p_s'} + \cdots + (-1)^{\nu}\frac{d^{\nu}}{dt^{\nu}}\frac{\partial H}{\partial p_s^{(\nu)}} - P_s\right\}^2$$

then it will become immediately obvious that the **Lagrange** equations in their second form will assign the value zero to the positive quantity M .

If the kinetic potential that is exerted upon a moving point with the origin as its center is:

$$H = -\,T + W\,(r, r', r'', \ldots, r^{(\nu)})\,,$$

in which the *vis viva* T is defined in complete generality, as in **§ 3** by:

$$T = -\tfrac{1}{2}\left\{(-1)^{\nu} A_0\, x^{(\nu)^2} + (-1)^{\nu-1} A_2\, x^{(\nu-1)^2} + \cdots + (-1)^{\frac{\nu+1}{2}} A_{\nu-1}\, x^{\left(\frac{\nu+1}{2}\right)^2}\right\} + \cdots$$

for odd ν, and:

$$T = -\tfrac{1}{2}\left\{(-1)^{\nu} A_0\, x^{(\nu)^2} + (-1)^{\nu-1} A_2\, x^{(\nu-1)^2} + \cdots + (-1)^{\frac{\nu}{2}+1} A_{\nu-2}\, x^{\left(\frac{\nu}{2}+1\right)^2}\right\} + \cdots$$

for even ν, then from (2) in **§ 2**, one will have:

$$\frac{\partial H}{\partial x^{(\nu)}} = (-1)^{\nu} A_0\, x^{(\nu)} + \frac{\partial W}{\partial r^{(\nu)}} \frac{x}{r} \;,$$
$$\frac{\partial H}{\partial y^{(\nu)}} = (-1)^{\nu} A_0\, y^{(\nu)} + \frac{\partial W}{\partial r^{(\nu)}} \frac{y}{r} \;,$$
$$\frac{\partial H}{\partial z^{(\nu)}} = (-1)^{\nu} A_0\, z^{(\nu)} + \frac{\partial W}{\partial r^{(\nu)}} \frac{z}{r} \;,$$

and therefore:

$$\frac{\partial^2 H}{\partial x^{(\nu)}\, \partial y^{(\nu)}} = \frac{\partial^2 W}{\partial r^{(\nu)^2}} \frac{x\,y}{r^2}, \qquad \frac{\partial^2 H}{\partial x^{(\nu)}\, \partial z^{(\nu)}} = \frac{\partial^2 W}{\partial r^{(\nu)^2}} \frac{x\,z}{r^2}, \qquad \frac{\partial^2 H}{\partial y^{(\nu)}\, \partial z^{(\nu)}} = \frac{\partial^2 W}{\partial r^{(\nu)^2}} \frac{y\,z}{r^2} .$$

Now, should the conditions that are expressed by equations (4) be fulfilled, so one would need to have $\frac{\partial^2 W}{\partial r^{(\nu)^2}} = 0$, then one would have:

$$W = \varphi_0\,(r, r', r'', \ldots, r^{(\nu-1)}) + \varphi_1\,(r, r', r'', \ldots, r^{(\nu-1)})\, r^{(\nu)} ,$$

and since the quantities (α) would then assume the constant value A_0, *in this case,* ***Gauss****'s principle of least constraint would exist for:*

$$M = \frac{1}{A_0} \left\{ \left(\frac{\partial H}{\partial x} - \frac{d}{dt}\frac{\partial H}{\partial x'} + \cdots + (-1)^{\nu} \frac{d^{\nu}}{dt^{\nu}} \frac{\partial H}{\partial x^{(\nu)}} \right)^2 \right.$$
$$+ \left(\frac{\partial H}{\partial y} - \frac{d}{dt}\frac{\partial H}{\partial y'} + \cdots + (-1)^{\nu} \frac{d^{\nu}}{dt^{\nu}} \frac{\partial H}{\partial y^{(\nu)}} \right)^2$$
$$\left. + \left(\frac{\partial H}{\partial z} - \frac{d}{dt}\frac{\partial H}{\partial z'} + \cdots + (-1)^{\nu} \frac{d^{\nu}}{dt^{\nu}} \frac{\partial H}{\partial z^{(\nu)}} \right)^2 \right\}$$

for a free or constrained point.

Since the kinetic potential in **Weber**'s law has degree two relative to r', *the extended* ***Gauss*** *principle of least constraint will not be valid under the conditions that were imposed for* ***Weber****'s law when the attracted point is subject to constraints.*

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§ 9. – The extended principle of least action.

Under the assumption that the kinetic potential H can also include time t explicitly, when the initial and final values of the coordinates can vary, along with their derivatives up to order $\nu - 1$, as well as the time duration for the comparison motion, a known formula from the calculus of variations will imply the relation:

$$
\begin{aligned}
(1) \qquad \delta \int_{t_0}^{t_1} H\,dt = & \int_{t_0}^{t_1} \sum_{s=1}^{\mu} \left\{ \left(\frac{\partial H}{\partial p_s} - \frac{d}{dt}\frac{\partial H}{\partial p_s'} + \cdots + (-1)^{\nu} \frac{d^{\nu}}{dt^{\nu}} \frac{\partial H}{\partial p_s^{(\nu)}} \right) (\delta p_s - p_s' \, \delta t) \right\} dt + [H\,dt]_{t_0}^{t} \\
& + \sum_{s=1}^{\mu} \left[\left(\frac{\partial H}{\partial p_s'} - \frac{d}{dt}\frac{\partial H}{\partial p_s''} + \cdots + (-1)^{\nu-1} \frac{d^{\nu-1}}{dt^{\nu-1}} \frac{\partial H}{\partial p_s^{(\nu)}} \right) (\delta p_s - p_s' \, \delta t) \right]_{t_0}^{t} \\
& + \sum_{s=1}^{\mu} \left[\left(\frac{\partial H}{\partial p_s''} - \frac{d}{dt}\frac{\partial H}{\partial p_s'''} + \cdots + (-1)^{\nu-2} \frac{d^{\nu-2}}{dt^{\nu-2}} \frac{\partial H}{\partial p_s^{(\nu)}} \right) (\delta p_s' - p_s'' \, \delta t) \right]_{t_0}^{t} + \cdots \\
& + \sum_{s=1}^{\mu} \left[\frac{\partial H}{\partial p_s^{(\nu)}} (\delta p_s^{(\nu-1)} - p_s^{(\nu)} \, \delta t) \right]_{t_0}^{t} .
\end{aligned}
$$

When one lets the variations of the coordinates and their derivatives up to order $\nu - 1$ vanish and sets $\delta t = \delta t_0 = 0$, so the comparison motions will again require the same time duration, for the case in which the **Lagrange** equations:

$$\frac{\partial H}{\partial p_s} - \frac{d}{dt}\frac{\partial H}{\partial p_s'} + \cdots + (-1)^{\nu} \frac{d^{\nu}}{dt^{\nu}} \frac{\partial H}{\partial p_s^{(\nu)}} = P_s$$

are fulfilled, one will once more get:

$$\delta \int_{t_0}^{t_1} H\,dt = \int_{t_0}^{t_1} \sum_{s=1}^{\mu} P_s \, \delta p_s \, dt .$$

Under the assumption that the external forces P_s are functions of only time, but not the coordinates, one will once more get the previous form of **Hamilton**'s principle:

$$\delta \int_{t_0}^{t_1} \left(H - \sum_{s=1}^{\mu} P_s \, p_s \right) dt = 0 .$$

If one uses the expression (4) of **§ 7** for the energy supply E to put equation (1) into the form:

$$(2) \qquad \delta\int_{t_0}^{t_1} H\,dt = \int_{t_0}^{t_1}\sum_{s=1}^{\mu}\left\{\left(\frac{\partial H}{\partial p_s} - \frac{d}{dt}\frac{\partial H}{\partial p'_s} + \cdots + (-1)^\nu \frac{d^\nu}{dt^\nu}\frac{\partial H}{\partial p_s^{(\nu)}}\right)(\delta p_s - p'_s\,\delta t)\right\}dt + [E\,dt]_{t_0}^{t}$$

$$+ \sum_{s=1}^{\mu}\left[\left(\frac{\partial H}{\partial p'_s} - \frac{d}{dt}\frac{\partial H}{\partial p''_s} + \cdots\right)\delta p_s\right]_{t_0}^{t} + \sum_{s=1}^{\mu}\left[\left(\frac{\partial H}{\partial p''_s} - \frac{d}{dt}\frac{\partial H}{\partial p'''_s} + \cdots\right)\delta p'_s\right]_{t_0}^{t} + \cdots + \sum_{s=1}^{\mu}\left[\frac{\partial H}{\partial p_s^{(\nu)}}\,\delta p_s^{(\nu-1)}\right]_{t_0}^{t}$$

then for the functions $p_1, p_2, \ldots, p_\mu$ of t that correspond to the actual motion, so they fulfill equations that take the form:

$$\frac{\partial H}{\partial p_s} - \frac{d}{dt}\frac{\partial H}{\partial p'_s} + \cdots + (-1)^\nu \frac{d^\nu}{dt^\nu}\frac{\partial H}{\partial p_s^{(\nu)}} = P_s$$

(in other words, when H does not include time t explicitly), from the energy principle, one will have the equation:

$$E = h + \sum_{s=1}^{\mu}\int P_s\, p'_s\, dt\ ,$$

in which h is a constant during the normal motion. Equation (2) will then go to:

$$(3) \qquad \delta\int_{t_0}^{t_1} H\,dt = \int_{t_0}^{t_1}\sum_{s=1}^{\mu} P_s\,(\delta p_s - p'_s\,\delta t)\,dt + \left[h + \int\sum_{s=1}^{\mu} P_s\, p'_s\, dt\right]_{t_0}^{t}$$

$$+ \sum_{s=1}^{\mu}\left[\left(\frac{\partial H}{\partial p'_s} - \frac{d}{dt}\frac{\partial H}{\partial p'_s} + \cdots\right)\delta p_s\right]_{t_0}^{t} + \sum_{s=1}^{\mu}\left[\left(\frac{\partial H}{\partial p''_s} - \frac{d}{dt}\frac{\partial H}{\partial p'''_s} + \cdots\right)\delta p'_s\right]_{t_0}^{t} + \cdots + \sum_{s=1}^{\mu}\left[\frac{\partial H}{\partial p_s^{(\nu)}}\,\delta p_s^{(\nu-1)}\right]_{t_0}^{t}$$

$$= \int_{t_0}^{t_1}\sum_{s=1}^{\mu} P_s\,\delta p_s\,dt + h(\delta t - \delta t_0) - \int_{t_0}^{t_1}\sum_{s=1}^{\mu} P_s\, p'_s\,\delta t\,dt + \left[\delta t\int\sum_{s=1}^{\mu} P_s\, p'_s\,\delta t\right]_{t_0}^{t}$$

$$+ \sum_{s=1}^{\mu}\left[\left(\frac{\partial H}{\partial p'_s} - \frac{d}{dt}\frac{\partial H}{\partial p'_s} + \cdots\right)\delta p_s\right]_{t_0}^{t} + \sum_{s=1}^{\mu}\left[\left(\frac{\partial H}{\partial p''_s} - \frac{d}{dt}\frac{\partial H}{\partial p'''_s} + \cdots\right)\delta p'_s\right]_{t_0}^{t} + \cdots + \sum_{s=1}^{\mu}\left[\frac{\partial H}{\partial p_s^{(\nu)}}\,\delta p_s^{(\nu-1)}\right]_{t_0}^{t}\ .$$

If we consider the variation:

$$\delta\int_{t_0}^{t_1} E\,dt$$

under the assumption that the energy principle is valid for not only the normal motion, but also for the motions that it is compared to, then the variation of time must not be drawn into consideration on the basis of that fact, because that would demand the preservation of not merely the *principle* of energy, but also the *constant* of the energy as a result, and for that reason, when the arbitrariness of the variation of the coordinates is also supposed to remain true, due to the appearance of a new equation between those variations, we must appeal to the variation of time, such that the time duration will not be the same for the normal and comparison motions, as it is in **Hamilton**'s principle, but different.

Now, if one regards all quantities as functions of a non-varying quantity u, in the known way, then when one sets $dt / du = t'$, one will get:

$$\delta\int_{t_0}^{t} E\,dt = \delta\int_{u_0}^{u} t'\,E\,du = \int_{u_0}^{u} \delta(t'\,E)\,du = \int_{u_0}^{u} t'\,\delta E\,du + \int_{u_0}^{u} E\,\delta t'\,du\,,$$

or from the energy principle:

$$\begin{aligned}\delta\int_{t_0}^{t} E\,dt &= \int_{t_0}^{t} \delta E\cdot dt + \int_{u_0}^{u}\left(h + \sum_{s=1}^{\mu}\int (P_s\,p_s'\,dt)\right)\delta t'\,du \\ &= \int_{t_0}^{t} \delta E\cdot dt + h(\delta t - \delta t_0) + \sum_{s=1}^{\mu}\int_{u_0}^{u}\left(\int P_s\,p_s'\,dt\right)\delta t'\,du\,.\end{aligned}$$

However, since:

$$\int_{u_0}^{u}\left(\int P_s\,p_s'\,dt\right)\delta t'\,du = \left[\int P_s\,p_s'\,dt\cdot\delta t\right]_{t_0}^{t} - \int_{u_0}^{u}\frac{d}{du}\left(\int P_s\,p_s'\,dt\right)\delta t'\,du = \left[\int P_s\,p_s'\,dt\cdot\delta t\right]_{t_0}^{t} - \int_{t_0}^{t} P_s\,p_s'\,\delta t\cdot dt\,,$$

one will get:

$$\delta\int_{t_0}^{t} E\,dt = \int_{t_0}^{t}\delta E\cdot dt + h(\delta t - \delta t_0) + \left[\sum_{s=1}^{\mu}\int P_s\,p_s'\,dt\cdot\delta t\right]_{t_0}^{t} - \sum_{s=1}^{\mu}\int_{t_0}^{t} P_s\,p_s'\,\delta t\cdot dt\,.$$

If one subtracts that equation from (3) then that will give:

$$(4)\qquad \delta\int_{t_0}^{t}(H - E)\,dt = -\int_{t_0}^{t}\delta E\cdot dt + \int_{t_0}^{t}\sum_{s=1}^{\mu} P_s\,\delta p_s'\,dt$$

$$+\sum_{s=1}^{\mu}\left[\left(\frac{\partial H}{\partial p_s'} - \frac{d}{dt}\frac{\partial H}{\partial p_s''} + \cdots\right)\delta p_s\right]_{t_0}^{t} + \sum_{s=1}^{\mu}\left[\left(\frac{\partial H}{\partial p_s''} - \frac{d}{dt}\frac{\partial H}{\partial p_s'''} + \cdots\right)\delta p_s'\right]_{t_0}^{t} + \cdots + \sum_{s=1}^{\mu}\left[\frac{\partial H}{\partial p_s^{(\nu)}}\,\delta p_s^{(\nu-1)}\right]_{t_0}^{t},$$

and *that equation represents the principle of least action.*

Should those coordinates and the derivatives of those quantities up to order $\nu - 1$ *remain the same for* t_0 *and* t *under the normal and comparison motion, then the principle of least action would go to:*

$$\delta \int_{t_0}^{t} (H - E)\, dt = - \delta \int_{t_0}^{t} E\, dt + \int_{t_0}^{t} \sum_{s=1}^{\mu} P_s\, \delta p_s\, dt .$$

When no external forces are present, so the energy principle is represented by $E = h$ for the normal motion, but the comparison motions, in turn, satisfy the energy principle, but with a different energy constant, by assumption, it will go to:

$$\delta \int_{t_0}^{t} (H - E)\, dt = -(t - t_0)\, dh .$$

Finally, if one establishes that the energy constant should be the same for the normal and comparison motions, so $\delta h = 0$ (which is permissible, since we also vary the time duration), *then the principle of least action, in its simplest form, will imply that:*

$$\delta \int_{t_0}^{t} (H - E)\, dt = 0 ,$$

in which, here, as well as in the previous equation, one understands $H - E$ to mean the expression in the coordinates and their derivatives that was defined above:

$$\begin{aligned} H - E = {} & \sum_{s=1}^{\mu} p_s' \left\{ \frac{\partial H}{\partial p_s'} - \frac{d}{dt} \frac{\partial H}{\partial p_s''} + \cdots + (-1)^{\nu-1} \frac{d^{\nu-1}}{dt^{\nu-1}} \frac{\partial H}{\partial p_s^{(\nu)}} \right\} \\ & + \sum_{s=1}^{\mu} p_s'' \left\{ \frac{\partial H}{\partial p_s''} - \frac{d}{dt} \frac{\partial H}{\partial p_s'''} + \cdots + (-1)^{\nu-2} \frac{d^{\nu-2}}{dt^{\nu-2}} \frac{\partial H}{\partial p_s^{(\nu)}} \right\} \\ & + \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ & + \sum_{s=1}^{\mu} p_s^{(\nu)} \frac{\partial H}{\partial p_s^{(\nu)}} , \end{aligned}$$

and one regards the equation:

$$E = h$$

as the constraint equation for the variations.

For the mechanics of ponderable masses, one has from the definition of E that:

$$H - E = \sum_{s=1}^{\mu} p_s' \frac{\partial H}{\partial p_s'} ,$$

since:

$$H = - T - U ,$$

in which:

$$T = \tfrac{1}{2}\sum_{i=1}^{n} m_i\, v_i^2 \,,$$

when v_i means the velocity of the i^{th} point. For holonomic constraints that do not include time t explicitly, that will be a homogeneous function of degree two in $p'_1,\ p'_2, \ldots,\ p'_\mu$ with coefficients that are functions of $p_1, p_2, \ldots, p_\mu$, like U, and one will then have:

$$(H - E)\, dt = \sum_{s=1}^{\mu} p'_s \frac{\partial H}{\partial p'_s}\, dt = -\sum_{s=1}^{\mu} p'_s \frac{\partial T}{\partial p'_s}\, dt = -\,2T\, dt = -\sum_{i=1}^{n} m_i\, v_i\, d\sigma_i \,,$$

when $d\sigma_i$ means the path element of the i^{th} point. *The principle of least action* (4) *can then be represented by the equation:*

(5)
$$\delta \int_{t_0}^{t} \sum_{i=1}^{n} m_i\, v_i\, d\sigma_i = \int_{t_0}^{t} \delta E\, dt - \int_{t_0}^{t} \sum_{s=1}^{\mu} P_s\, \delta p_s\, dt + \left[\sum_{s=1}^{\mu} \frac{\partial T}{\partial p'_s}\, \delta p_s\right]_{t_0}^{t} ,$$

which will go to:

$$\delta \int_{t_0}^{t} \sum_{i=1}^{n} m_i\, v_i\, d\sigma_i = (t - t_0)\, \delta h + \left[\sum_{s=1}^{\mu} \frac{\partial T}{\partial p'_s}\, \delta p_s\right]_{t_0}^{t}$$

when the external forces are all zero. If we now establish that the coordinates of the system suffer no variations at the beginning t_0 *of the motion and the arbitrarily-chosen end time t, and the variation of h vanishes, moreover (which is, as is immediately obvious, identical to saying that the comparison motions possess the same vis viva as the normal one at the beginning* t_0*) then the* ***Lagrange*** *equations will be equivalent to the principle of least action that is expressed by the equation:*

(6)
$$\delta \int_{t_0}^{t} \sum_{i=1}^{n} m_i\, v_i\, d\sigma_i = 0 \,.$$

However, if we drop the assumption that the external forces all vanish then, according to equation (5), under the assumption that the coordinates suffer no variations at the beginning and end, that the energy supply of the system can change from the normal motion to the comparison motions, moreover, and that its measure will decrease or increase according to whether the forces P_s do negative or positive work under the displacement δp_s. Then the principle of least action will, in turn, be represented by equation (5), since time does not need to be varied, due to the change in energy supply with time.

However, it is still essential for us to explain how the variation that is included in the principle of least action:

(7) $$\delta \int_{t_0}^{t} (H - E)\, dt = 0 ,$$

under which the time t is also varied, due to the conservation of the energy constant, must be carried out in order to lead to the **Lagrange** equations, which are equivalent to that principle. Indeed, that is again indicated by the **Weber** force function:

$$W = \frac{m\,m_1}{r}\left(1 + \frac{r'^2}{k^2}\right),$$

whose kinetic potential is:

$$H = -\tfrac{1}{2} m (x'^2 + y'^2 + z'^2) - \frac{m\,m_1}{r}\left(1 + \frac{r'^2}{k^2}\right) ,$$

and whose energy is:

$$E = \tfrac{1}{2} m (x'^2 + y'^2 + z'^2) - \frac{m\,m_1}{r} + \frac{m\,m_1}{r}\frac{r'^2}{k^2} ,$$

as was shown above, such that equation (7) will go to:

(8) $$\delta \int_{t_0}^{t} \left(\tfrac{1}{2} m (x'^2 + y'^2 + z'^2) + \frac{m\,m_1}{r}\frac{r'^2}{k^2}\right) dt = 0 .$$

The variation is performed in such a way that *vis viva* constant does not change, so the constraint equation will exist:

(9) $$\tfrac{1}{2} m (x'^2 + y'^2 + z'^2) - \frac{m\,m_1}{r} + \frac{m\,m_1}{r}\frac{r'^2}{k^2} = h .$$

It we regard all variables as functions of one variable u and set:

$$x = x_1 , \quad y = y_1 , \quad z = z_1 , \quad \frac{dx}{du} = x_1' , \quad \frac{dy}{du} = y_1' , \quad \frac{dz}{du} = z_1' , \quad r = r_1 , \quad \frac{dr}{du} = r_1'$$

then when we set:

(10) $$\frac{m\,m_1}{r} + h = M , \qquad \tfrac{1}{2} m (x'^2 + y'^2 + z'^2) + \frac{m\,m_1}{r}\frac{r'^2}{k^2} = N ,$$

the two equations (8) and (9) will go to:

(11) $$\delta \int_{u_0}^{u} N \cdot \frac{du}{dt}\, du = 0$$

and

$$\sqrt{N}\,du = \sqrt{M}\,dt\,. \tag{12}$$

Eliminating *dt* from (11) and (12) will produce the variation that must now be performed:

$$\delta\int_{u_0}^{u}\sqrt{M}\cdot\sqrt{N}\,du = 0\,. \tag{13}$$

Now since the known transformation of the calculus of variations will make that equation assume the form:

$$\begin{aligned}\delta\int_{u_0}^{u}\Bigg\{&\left(\frac{\partial\sqrt{M\,N}}{\partial x_1}-\frac{d}{du}\frac{\partial\sqrt{M\,N}}{\partial x_1'}\right)\delta x_1+\left(\frac{\partial\sqrt{M\,N}}{\partial y_1}-\frac{d}{du}\frac{\partial\sqrt{M\,N}}{\partial y_1'}\right)\delta y_1\,du\\ &+\left(\frac{\partial\sqrt{M\,N}}{\partial z_1}-\frac{d}{du}\frac{\partial\sqrt{M\,N}}{\partial z_1'}\right)\delta z_1\Bigg\}\,du = 0\,,\end{aligned} \tag{14}$$

one will have the following three equations:

$$\left\{\begin{aligned}&\frac{\partial\sqrt{M\,N}}{\partial x_1}-\frac{d}{du}\frac{\partial\sqrt{M\,N}}{\partial x_1'}=0\,,\\ &\frac{\partial\sqrt{M\,N}}{\partial y_1}-\frac{d}{du}\frac{\partial\sqrt{M\,N}}{\partial y_1'}=0\,,\\ &\frac{\partial\sqrt{M\,N}}{\partial z_1}-\frac{d}{du}\frac{\partial\sqrt{M\,N}}{\partial z_1'}=0\,,\end{aligned}\right. \tag{15}$$

since the point is free.

However, equations (10) will imply the expressions:

$$\frac{\partial\sqrt{M\,N}}{\partial x_1} = \frac{\sqrt{M}}{2\sqrt{N}}\left\{-\frac{3\,m\,m_1}{k^2\,r_1^3}\,x_1\,r_1'^2+\frac{2\,m\,m_1}{k^2\,r_1^2}\,x_1'\,r_1'\right\}-\frac{\sqrt{N}}{2\sqrt{M}}\,\frac{m\,m_1}{r_1^3}\,x_1\;,$$

$$\frac{\partial\sqrt{M\,N}}{\partial x_1'} = \frac{\sqrt{M}}{2\sqrt{N}}\left\{m\,x_1'+\frac{2\,m\,m_1}{k^2\,r_1^2}\,x_1\,r_1'\right\},$$

or with the help of (12):

$$\frac{\partial\sqrt{M\,N}}{\partial x_1} = \frac{\sqrt{N}}{2\sqrt{M}}\left\{-\frac{3\,m\,m_1}{k^2\,r^3}\,x\,r'^{\,2} + \frac{2\,m\,m_1}{k^2\,r^2}\,x'\,r' - \frac{m\,m_1}{r^3}\,x\right\},$$

$$\frac{\partial\sqrt{M\,N}}{\partial x_1'} = \frac{1}{2}\left\{m\,x' + \frac{2\,m\,m_1}{k^2\,r^2}\,x\,r'\right\}.$$

Since:

$$\frac{d}{du}\frac{\partial\sqrt{M\,N}}{\partial x_1'} = \frac{dt}{du}\frac{d}{dt}\frac{\partial\sqrt{M\,N}}{\partial x_1'} = \frac{\sqrt{N}}{\sqrt{M}}\frac{d}{dt}\frac{\partial\sqrt{M\,N}}{\partial x_1'},$$

the first of equations (15) will go to:

$$m\,x'' = -\frac{m\,m_1}{r^3}\,x + \frac{m\,m_1}{k^2\,r^3}\,x\,r'^{\,2} - \frac{2\,m\,m_1}{k^2\,r^2}\,x\,r'',$$

and those three equations will assume the necessary form of the **Lagrange** equations of motion:

$$m\,x'' = \left(\frac{\partial W}{\partial r} - \frac{d}{dt}\frac{\partial W}{\partial r'}\right)\frac{x}{r},$$

$$m\,y'' = \left(\frac{\partial W}{\partial r} - \frac{d}{dt}\frac{\partial W}{\partial r'}\right)\frac{y}{r},$$

$$m\,z'' = \left(\frac{\partial W}{\partial r} - \frac{d}{dt}\frac{\partial W}{\partial r'}\right)\frac{z}{r}.$$

____

§ 10. – The extended principle of the conservation of areas

If one starts from the **Lagrange** equations of motion in the first form and assumes, e.g., that the ones that belong to the *x* and *y* coordinates are:

(1) $$\frac{\partial H}{\partial x_i} - \frac{d}{dt}\frac{\partial H}{\partial x_i'} + \cdots + (-1)^\nu \frac{d^\nu}{dt^\nu}\frac{\partial H}{\partial x_i^{(\nu)}} - Q_i - \lambda_1 f_{1i} - \cdots - \lambda_m f_{mi} = 0 ,$$

(2) $$\frac{\partial H}{\partial y_i} - \frac{d}{dt}\frac{\partial H}{\partial y_i'} + \cdots + (-1)^\nu \frac{d^\nu}{dt^\nu}\frac{\partial H}{\partial y_i^{(\nu)}} - Q_i - \lambda_1 \varphi_{1i} - \cdots - \lambda_m \varphi_{mi} = 0 ,$$

multiplies them by y_i and x_i, resp., and subtracts them from each other then it will follow that:

(3) $$x_i \frac{\partial H}{\partial y_i} - y_i \frac{\partial H}{\partial x_i} - \left(x_i \frac{d}{dt}\frac{\partial H}{\partial y_i'} - y_i \frac{d}{dt}\frac{\partial H}{\partial x_i'} \right) + \cdots + (-1)^\nu \left(x_i \frac{d^\nu}{dt^\nu}\frac{\partial H}{\partial y_i^{(\nu)}} - y_i \frac{d^\nu}{dt^\nu}\frac{\partial H}{\partial x_i^{(\nu)}} \right)$$
$$- (x_i R_i - y_i Q_i) - \lambda_1 (x_i \varphi_{1i} - y_i f_{1i}) - \cdots - \lambda_m (x_i \varphi_{mi} - y_i f_{mi}) = 0 .$$

However, one now has:

(4) $$x_i \frac{d^r}{dt^r}\frac{\partial H}{\partial y_i^{(r)}} - y_i \frac{d^r}{dt^r}\frac{\partial H}{\partial x_i^{(r)}}$$

$$= \frac{d}{dt}\left\{ \left(x_i \frac{d^{r-1}}{dt^{r-1}}\frac{\partial H}{\partial y_i^{(r)}} - y_i \frac{d^{r-1}}{dt^{r-1}}\frac{\partial H}{\partial x_i^{(r)}} \right) - \left(x_i' \frac{d^{r-2}}{dt^{r-2}}\frac{\partial H}{\partial y_i^{(r)}} - y_i' \frac{d^{r-2}}{dt^{r-2}}\frac{\partial H}{\partial x_i^{(r)}} \right) + \cdots \right.$$
$$\left. + (-1)^{r-1} \left(x_i^{(r-1)} \frac{\partial H}{\partial y_i^{(r)}} - y_i^{(r-1)} \frac{\partial H}{\partial x_i^{(r)}} \right) \right\} + (-1)^r \left(x_i^{(r)} \frac{\partial H}{\partial y_i^{(r)}} - y_i^{(r)} \frac{\partial H}{\partial x_i^{(r)}} \right)$$

$$= \frac{d}{dt}\sum_{\lambda=1}^{r} (-1)^{\lambda-1} \left(x_i^{(\lambda-1)} \frac{d^{r-\lambda}}{dt^{r-\lambda}}\frac{\partial H}{\partial y_i^{(r)}} - y_i^{(\lambda-1)} \frac{d^{r-\lambda}}{dt^{r-\lambda}}\frac{\partial H}{\partial x_i^{(r)}} \right) + (-1)^r \left(x_i^{(r)} \frac{\partial H}{\partial y_i^{(r)}} - y_i^{(r)} \frac{\partial H}{\partial x_i^{(r)}} \right) ,$$

and equation (3) will then go to:

(5) $$\frac{d}{dt}\sum_{r=1}^{\nu} (-1)^r \sum_{\lambda=1}^{r} (-1)^{\lambda-1} \left(x_i^{(\lambda-1)} \frac{d^{r-\lambda}}{dt^{r-\lambda}}\frac{\partial H}{\partial y_i^{(r)}} - y_i^{(\lambda-1)} \frac{d^{r-\lambda}}{dt^{r-\lambda}}\frac{\partial H}{\partial x_i^{(r)}} \right) + \sum_{r=0}^{\nu} \left(x_i^{(r)} \frac{\partial H}{\partial y_i^{(r)}} - y_i^{(r)} \frac{\partial H}{\partial x_i^{(r)}} \right)$$
$$- (x_i R_i - y_i Q_i) - \lambda_1 (x_i \varphi_{1i} - y_i f_{1i}) - \cdots - \lambda_m (x_i \varphi_{mi} - y_i f_{mi}) = 0 .$$

Finally, when one sums over i from 1 to n and further *assumes that:*

(6) $$\sum_{i=1}^{n}\sum_{r=0}^{\nu}\left(x_i^{(r)}\frac{\partial H}{\partial y_i^{(r)}} - y_i^{(r)}\frac{\partial H}{\partial x_i^{(r)}}\right) = 0 ,$$

(7) $$\sum_{i=1}^{n}(x_i\,R_i - y_i\,Q_i) = 0 ,$$

(8) $$\sum_{i=1}^{n}(x_i\,\varphi_{\rho i} - y_i\,f_{\rho i}) = 0 \qquad (\rho = 1, 2, \ldots, m)$$

upon integrating over t, one will get the principle of areas:

(9) $$\sum_{i=1}^{n}\sum_{r=0}^{\nu}(-1)^r\sum_{\lambda=1}^{r}(-1)^{\lambda-1}\left\{x_i^{(\lambda-1)}\frac{d^{r-\lambda}}{dt^{r-\lambda}}\frac{\partial H}{\partial y_i^{(r)}} - y_i^{(\lambda-1)}\frac{d^{r-\lambda}}{dt^{r-\lambda}}\frac{\partial H}{\partial x_i^{(r)}}\right\} = c ,$$

in which c is a constant, and the left-hand side of that equation is a differential expression of order $2\nu - 1$.

For $\nu = 1$ and $H = - T - U$, that equation will go to:

$$\sum_{i=1}^{n} m_i\,(x_i\,y_i' - y_i\,x_i') = c ,$$

and the geometric interpretation of that equation is what gives the principle its name.

However, the form of H that is required by equation (6) is easy to find, since that partial differential equation leads to the total differential equation:

$$\frac{dy_1^{(r)}}{x_1^{(r)}} = \ldots = \frac{dy_n^{(r)}}{x_n^{(r)}} = -\frac{dx_n^{(r)}}{y_n^{(r)}} = \ldots = -\frac{dx_n^{(r)}}{y_n^{(r)}}$$
$$= \frac{dy_1^{(s)}}{x_1^{(s)}} = \ldots = \frac{dy_n^{(s)}}{x_n^{(s)}} = -\frac{dx_n^{(s)}}{y_n^{(s)}} = \ldots = -\frac{dx_n^{(s)}}{y_n^{(s)}} ,$$

whose integral functions can be represented by:

$$(x_i^{(r)})^2 + (y_i^{(r)})^2 , \qquad x_i^{(r)}x_{i_1}^{(r)} + y_i^{(r)}y_{i_1}^{(r)} , \qquad x_i^{(r)}x_i^{(s)} + y_i^{(r)}y_i^{(s)}$$

$$(i = 1, 2, \ldots, n ; \quad r, s = 0, 1, 2, \ldots, \nu),$$

such that all of the forms for the kinetic potential that satisfy the principle of areas for the *x* and *y* coordinates are represented by:

$$(10) \qquad H = F((x_i^{(r)})^2 + (y_i^{(r)})^2, x_i^{(r)} x_{i_1}^{(r)} + y_i^{(r)} y_{i_1}^{(r)}, x_i^{(r)} x_i^{(s)} + y_i^{(r)} y_i^{(s)}, t)$$

$$(i = 1, 2, \ldots, n\,;\ r, s = 0, 1, 2, \ldots, \nu)\,.$$

We then find that:

For a kinetic potential of the form (10), *the principle of the conservation of areas is given in the form that is represented in equation* (9) *when equations* (7) *and* (8) *are valid identically.*

Since the kinetic potential of **Weber**'s law is given by the expression:

$$H = -\tfrac{1}{2} m (x'^2 + y'^2 + z'^2) - \frac{m m_1}{\sqrt{x^2 + y^2 + z^2}} \left(1 + \frac{(x x' + y y' + z z')^2}{k^2 (x^2 + y^2 + z^2)} \right),$$

the law of areas that was defined above will then be true for the three coordinate planes.

In order to derive the principle of areas for the second form of **Lagrange**'s equations, one must multiply:

$$\frac{\partial H}{\partial p_\kappa} - \frac{d}{dt}\frac{\partial H}{\partial p'_\kappa} + \cdots + (-1)^\nu \frac{d^\nu}{dt^\nu}\frac{\partial H}{\partial p_\kappa^{(\nu)}} = P_\kappa\,,$$

$$\frac{\partial H}{\partial p_\lambda} - \frac{d}{dt}\frac{\partial H}{\partial p'_\lambda} + \cdots + (-1)^\nu \frac{d^\nu}{dt^\nu}\frac{\partial H}{\partial p_\lambda^{(\nu)}} = P_\lambda$$

by p_λ (p_κ, resp.) and subtract them from each other, which will give:

$$p_\kappa \frac{\partial H}{\partial p_\lambda} - p_\lambda \frac{\partial H}{\partial p_\kappa} - \left(p_\kappa \frac{d}{dt}\frac{\partial H}{\partial p'_\lambda} - p_\lambda \frac{d}{dt}\frac{\partial H}{\partial p'_\kappa} \right) + \cdots + (-1)^\nu \left(p_\kappa \frac{d^\nu}{dt^\nu}\frac{\partial H}{\partial p_\lambda^{(\nu)}} - p_\lambda \frac{d^\nu}{dt^\nu}\frac{\partial H}{\partial p_\kappa^{(\nu)}} \right)$$

$$= p_\kappa P_\lambda - p_\lambda P_\kappa\,,$$

or, in turn, when one applies the relation (4) that was developed above, when the expression for the kinetic potential satisfies the equation:

$$\sum_{\kappa,\lambda=1,2,\ldots,\mu} \sum_{s=0}^{\nu} \left(p_\kappa^{(\nu)} \frac{\partial H}{\partial p_\lambda^{(\nu)}} - p_\lambda^{(\nu)} \frac{\partial H}{\partial p_\kappa^{(\nu)}} \right) = 0$$

identically, and the external forces are subject to the condition that:

$$\sum_{\kappa,\lambda=1,2,\ldots,\mu} (p_\kappa P_\lambda - p_\lambda P_\kappa) = 0 ,$$

in which the sums over κ and λ extend over all different combinations of values for the numbers 1, 2, …, μ, one will get the *principle of areas in the form:*

$$\sum_{\kappa,\lambda=1,2,\ldots,\mu} \sum_{s=1}^{\nu} (-1)^s \sum_{\rho=1}^{s} (-1)^{\rho-1} \left\{ p_\kappa^{(\rho-1)} \frac{d^{s-\rho}}{dt^{s-\rho}} \frac{\partial H}{\partial p_\lambda^{(s)}} - p_\lambda^{(\rho-1)} \frac{d^{s-\rho}}{dt^{s-\rho}} \frac{\partial H}{\partial p_\kappa^{(s)}} \right\} = c .$$

If one multiplies equations (1) and (2), in which the index i should be replaced with κ, by $y_\kappa^{(\rho)}$ and $x_\kappa^{(\rho)}$, resp., subtracts them from each other, and remarks that:

$$x_\kappa^{(\rho)} \frac{d^r}{dt^r} \frac{\partial H}{\partial p_\kappa^{(s)}} - y_\kappa^{(\rho)} \frac{d^r}{dt^r} \frac{\partial H}{\partial p_\kappa^{(r)}}$$

$$= \frac{d}{dt}\left\{\left(x_\kappa^{(\rho)} \frac{d^{r-1}}{dt^{r-1}} \frac{\partial H}{\partial y_\kappa^{(r)}} - y_\kappa^{(\rho)} \frac{d^{r-1}}{dt^{r-1}} \frac{\partial H}{\partial x_\kappa^{(r)}} \right) - \left(x_\kappa^{(\rho+1)} \frac{d^{r-2}}{dt^{r-2}} \frac{\partial H}{\partial y_\kappa^{(r)}} - y_\kappa^{(\rho+1)} \frac{d^{r-2}}{dt^{r-2}} \frac{\partial H}{\partial x_\kappa^{(r)}} \right) + \cdots \right.$$

$$\left. + (-1)^{r-1} \left(x_\kappa^{(\rho+r-1)} \frac{\partial H}{\partial y_\kappa^{(r)}} - y_\kappa^{(\rho+r-1)} \frac{\partial H}{\partial x_\kappa^{(r)}} \right) \right\} + (-1)^r \left(x_\kappa^{(\rho+r)} \frac{\partial H}{\partial y_\kappa^{(r)}} - y_\kappa^{(\rho+r)} \frac{\partial H}{\partial x_\kappa^{(r)}} \right)$$

$$= \frac{d}{dt} \sum_{\lambda=1}^{r} (-1)^{\lambda-1} \left(x_\kappa^{(\rho+\lambda-1)} \frac{d^{r-\lambda}}{dt^{r-\lambda}} \frac{\partial H}{\partial y_\kappa^{(r)}} - y_\kappa^{(\rho+\lambda-1)} \frac{d^{r-\lambda}}{dt^{r-\lambda}} \frac{\partial H}{\partial x_\kappa^{(r)}} \right) + (-1)^2 \left(x_\kappa^{(\rho+r)} \frac{\partial H}{\partial y_\kappa^{(r)}} - y_\kappa^{(\rho+r)} \frac{\partial H}{\partial x_\kappa^{(r)}} \right)$$

then that will give:

$$(11) \quad \frac{d}{dt} \sum_{r=1}^{\nu} (-1)^r \sum_{\lambda=1}^{r} (-1)^{\lambda-1} \left\{ x_\kappa^{(\rho+\lambda-1)} \frac{d^{r-\lambda}}{dt^{r-\lambda}} \frac{\partial H}{\partial y_\kappa^{(r)}} - y_\kappa^{(\rho+\lambda-1)} \frac{d^{r-\lambda}}{dt^{r-\lambda}} \frac{\partial H}{\partial x_\kappa^{(r)}} \right\} + \sum_{r=0}^{\nu} \left(x_\kappa^{(\rho+r)} \frac{\partial H}{\partial y_\kappa^{(r)}} - y_\kappa^{(\rho+r)} \frac{\partial H}{\partial x_\kappa^{(r)}} \right)$$

$$- (x_\kappa^{(\rho)} R_\kappa - y_\kappa^{(\rho)} Q_\kappa) - \lambda_1 (x_\kappa^{(\rho)} \varphi_{1\kappa} - y_\kappa^{(\rho)} f_{1\kappa}) - \cdots - \lambda_m (x_\kappa^{(\rho)} \varphi_{m\kappa} - y_\kappa^{(\rho)} f_{m\kappa}) = 0 .$$

If one now sums over κ from 1 to n and once more makes the assumption that:

$$\sum_{\kappa=1}^{n} (x_\kappa^{(\rho)} R_\kappa - y_\kappa^{(\rho)} Q_\kappa) = 0 ,$$

$$\sum_{\kappa=1}^{n} (x_\kappa^{(\rho)} \varphi_{s\kappa} - y_\kappa^{(\rho)} f_{s\kappa}) = 0 \qquad (s = 1, 2, \ldots, \mu),$$

such that the functions $\varphi_{s\kappa}$ and $f_{s\kappa}$ must vanish since they should not include the derivatives of the coordinates, then under the assumption that the second term in the equation (11), thus-converted, will be a complete differential quotient with respect to t, or that:

(12)
$$\sum_{\kappa=1}^{n}\sum_{r=0}^{\nu}\left(x_\kappa^{(\rho+r)}\frac{\partial H}{\partial y_\lambda^{(r)}}-y_\lambda^{(\rho+r)}\frac{\partial H}{\partial x_\kappa^{(r)}}\right)$$
$$=\frac{d}{dt}F(t,x_s,x_s',\dots,x_s^{(\rho+\nu-1)},y_s,y_s',\dots,y_s^{(\rho+\nu-1)},z_s,z_s',\dots,z_s^{(\nu-1)}),$$

that will give an integral equation of the form:

(13)
$$\sum_{\kappa=1}^{n}\sum_{r=1}^{\nu}(-1)^r\sum_{\lambda=1}^{r}(-1)^{\lambda-1}\left\{x_\kappa^{(\rho+\lambda-1)}\frac{d^{r-\lambda}}{dt^{r-\lambda}}\frac{\partial H}{\partial y_\kappa^{(r)}}-y_\kappa^{(\rho+\lambda-1)}\frac{d^{r-\lambda}}{dt^{r-\lambda}}\frac{\partial H}{\partial x_\kappa^{(r)}}\right\}$$
$$+\;F(t,x_s,\dots,x_s^{(\rho+\nu-1)},y_s,\dots,y_s^{(\rho+\nu-1)},z_s,\dots,z_s^{(\nu-1)})\;=C,$$

in which $\rho\le\nu$. That can then be regarded as a further generalization of the principle of areas.

We can next ignore the case of $\rho = 0$, since equation (12) would require an identity of the form:

$$\sum_{\kappa=1}^{n}\sum_{r=0}^{\nu}\left(x_\kappa^{(r)}\frac{\partial H}{\partial y_\lambda^{(r)}}-y_\lambda^{(r)}\frac{\partial H}{\partial x_\kappa^{(r)}}\right)$$
$$=\frac{d}{dt}F(t,x_s,x_s',\dots,x_s^{(\nu-1)},y_s,y_s',\dots,y_s^{(\nu-1)},z_s,z_s',\dots,z_s^{(\nu-1)})$$
$$=\frac{\partial F}{\partial t}+\sum_{\kappa=1}^{n}\frac{\partial F}{\partial x_\kappa}x_\kappa'+\dots+\sum_{\kappa=1}^{n}\frac{\partial F}{\partial x_\kappa^{(\nu-1)}}x_\kappa^{(\nu)}+\sum_{\kappa=1}^{n}\frac{\partial F}{\partial y_\kappa}y_\kappa'+\cdots$$

In the mechanics of ponderable masses, so for $\nu = 1$, when one sets:

$$H=-\tfrac{1}{2}\sum_{\kappa=1}^{n}m_\kappa(x_\kappa'^2+y_\kappa'^2+z_\kappa'^2)-U,$$

that will go to:

$$-\sum_{\kappa=1}^{n}\left(x_\kappa\frac{\partial H}{\partial y_\kappa}-y_\kappa\frac{\partial H}{\partial x_\kappa}\right)=\frac{\partial F}{\partial t}+\sum_{\kappa=1}^{n}\frac{\partial F}{\partial x_\kappa}x_\kappa'+\sum_{\kappa=1}^{n}\frac{\partial F}{\partial y_\kappa}y_\kappa'+\sum_{\kappa=1}^{n}\frac{\partial F}{\partial z_\kappa}z_\kappa'\;.$$

Therefore, it cannot be satisfied identically when F is not a constant, such that this equation will go to the previous condition equation (6). If we would then like to look for an extension of the principle of areas that was developed above that could also produce integrals for the mechanics of ponderable masses then we would have to set $\rho\ge 1$.

For $\rho = 1$, the question would arise of when equation (12) could be satisfied identically, as a result of the relation:

$$\sum_{\kappa=1}^{n}\sum_{r=0}^{\nu}\left(x_\kappa^{(r+1)}\frac{\partial H}{\partial y_\lambda^{(r)}} - y_\lambda^{(r+1)}\frac{\partial H}{\partial x_\kappa^{(r)}}\right)$$

$$= \frac{d}{dt}F(t, x_s, x_s', \ldots, x_s^{(\nu)}, y_s, y_s', \ldots, y_s^{(\nu)}, z_s, z_s', \ldots, z_s^{(\nu-1)})$$

or

(14) $$\sum_{\kappa=1}^{n}\left\{\left(x_\kappa'\frac{\partial H}{\partial y_\kappa} - y_\kappa'\frac{\partial H}{\partial x_\kappa}\right) + \left(x_\kappa''\frac{\partial H}{\partial y_\kappa'} - y_\kappa''\frac{\partial H}{\partial x_\kappa'}\right) + \cdots + \left(x_\kappa^{(\nu+1)}\frac{\partial H}{\partial y_\kappa^{(\nu)}} - y_\kappa^{(\nu+1)}\frac{\partial H}{\partial x_\kappa^{(\nu)}}\right)\right\}$$

$$\begin{aligned} = \frac{\partial F}{\partial t} + \sum_{\kappa=1}^{n}\Bigg\{&\frac{\partial F}{\partial x_\kappa}x_\kappa' + \frac{\partial F}{\partial x_\kappa'}x_\kappa'' + \cdots + \frac{\partial F}{\partial x_\kappa^{(\nu)}}x_\kappa^{(\nu+1)} \\ &+ \sum_{\kappa=1}^{n}\frac{\partial F}{\partial y_\kappa}y_\kappa' + \frac{\partial F}{\partial y_\kappa'}y_\kappa'' + \cdots + \frac{\partial F}{\partial y_\kappa^{(\nu)}}y_\kappa^{(\nu+1)} \\ &+ \frac{\partial F}{\partial z_\kappa}z_\kappa' + \frac{\partial F}{\partial z_\kappa'}z_\kappa'' + \cdots + \frac{\partial F}{\partial z_\kappa^{(\nu-1)}}z_\kappa^{(\nu)}\Bigg\}. \end{aligned}$$

Since H includes the derivatives of the coordinates only up to order ν, (14) can be fulfilled identically only when:

$$\frac{\partial F}{\partial x_\kappa^{(\nu)}} = \frac{\partial H}{\partial y_\kappa^{(\nu)}} \qquad \text{and} \qquad \frac{\partial F}{\partial y_\kappa^{(\nu)}} = -\frac{\partial H}{\partial x_\kappa^{(\nu)}}.$$

So F is then the real part of a function of $x_\kappa^{(\nu)} + i\,y_\kappa^{(\nu)}$, and H is the imaginary part, without the i, and F must subject to the further condition that it must satisfy equation (14), once the terms that are linear in $x_\kappa^{(\nu+1)}$ and $y_\kappa^{(\nu+1)}$ are dropped from both sides of them. In order to satisfy that equation, it is obviously sufficient that one has:

(15) $$\frac{\partial F}{\partial x_\kappa^{(r)}} = \frac{\partial H}{\partial y_\kappa^{(r)}} \qquad \text{and} \qquad \frac{\partial F}{\partial y_\kappa^{(r)}} = -\frac{\partial H}{\partial x_\kappa^{(r)}}$$

and

(16) $$\frac{\partial F}{\partial t} + \sum_{\kappa=1}^{n}\left\{\frac{\partial F}{\partial z_\kappa}z_\kappa' + \frac{\partial F}{\partial z_\kappa'}z_\kappa'' + \cdots + \frac{\partial F}{\partial z_\kappa^{(\nu-1)}}z_\kappa^{(\nu)}\right\} = 0,$$

in general, for $r = 0, 1, 2, \ldots, \nu$ and $\kappa = 1, 2, \ldots, n$. However, since F includes the derivatives of z_κ only up to order $\nu - 1$, equation (16) demands that one must have $\frac{\partial F}{\partial z_\kappa^{(\nu-1)}} = 0$, and then once more,

$\frac{\partial F}{\partial z_\kappa^{(\nu-2)}} = \ldots = \frac{\partial F}{\partial z_\kappa} = \frac{\partial F}{\partial t} = 0$, so F must be independent of t, z_κ, z'_κ, …, $z_\kappa^{(\nu-1)}$. Therefore, from equations (15), the same thing must be true of the partial differential quotients of H with respect to the x_κ, y_κ, and their derivatives, such that under the assumption that the sufficient conditions (15) are verified (which will, in turn, be satisfied by functions of complex variables), that will yield the following forms for F and H :

$$F = \frac{1}{2}\Big\{ f(x_\kappa + i\, y_\kappa, x'_\kappa + i\, y'_\kappa, \ldots, x_\kappa^{(\nu)} + i\, y_\kappa^{(\nu)}) + f(x_\kappa - i\, y_\kappa, x'_\kappa - i\, y'_\kappa, \ldots, x_\kappa^{(\nu)} - i\, y_\kappa^{(\nu)})\Big\},$$

(17)
$$H = \frac{1}{2i}\Big\{ f(x_\kappa + i\, y_\kappa, x'_\kappa + i\, y'_\kappa, \ldots, x_\kappa^{(\nu)} + i\, y_\kappa^{(\nu)}) - f(x_\kappa - i\, y_\kappa, x'_\kappa - i\, y'_\kappa, \ldots, x_\kappa^{(\nu)} - i\, y_\kappa^{(\nu)})\Big\} + f_1(t, z_\kappa, z'_\kappa, \ldots, z_\kappa^{(\nu)}),$$

in which f and f_1 mean arbitrary real functions of their arguments.

For the form of the kinetic potential that is represented by (17), *the extension of the principle of areas is given by the equation* (*):

(*) Thus, if one has, e.g.:

$$H = \frac{1}{2i}\{(x_1 + i\,y_1)^2 (x''_2 + i\,y''_2) - (x'_3 + i\,y'_3)(x''_2 + i\,y''_2)^2 - (x_1 - i\,y_1)^2 (x''_2 - i\,y''_2) + (x'_3 - i\,y'_3)(x''_2 - i\,y''_2)^2\} + t^2 z_1^2 z_2''^3 + z''_3 z'_1 z'_2$$
$$= (x_1^2 - y_1^2)\, y''_2 + 2x_1 y_1 x''_2 - y'_3(x_2''^2 - y_2''^2) + 2x'_3 x''_2 y''_2 + t^2 z_1^2 z_2''^3 + z''_3 z'_1 z'_2,$$

so one sets:

$$F = \frac{1}{2}\{(x_1 + i\,y_1)^2 (x''_2 + i\,y''_2) - (x'_3 + i\,y'_3)(x''_2 + i\,y''_2)^2 + (x_1 - i\,y_1)^2 (x''_2 - i\,y''_2) + (x'_3 - i\,y'_3)(x''_2 - i\,y''_2)^2\}$$
$$= (x_1^2 - y_1^2)\, x''_2 + 2x_1 y_1 y''_2 - x'_3(x_2''^2 - y_2''^2) + 2y'_3 x''_2 y''_2,$$

then, as one can see immediately, the equation:

$$\sum_{\kappa=0}^{3}\sum_{r=0}^{2}\left(x_\kappa^{(r+1)} \frac{\partial H}{\partial y_\kappa^{(r)}} - y_\kappa^{(r+1)} \frac{\partial H}{\partial x_\kappa^{(r)}} \right) = \frac{dF}{dt}$$

will be satisfied identically, and the integral equation (18) will then read:

$$2x'''_2(y'_2\, y'_3 - x'_2\, x'_3) + 2y'''_2(x'_2\, y'_3 + y'_2\, x'_3) + 2y''_3(x'_2\, y''_2 - y'_2\, x''_3)$$
$$+\ 2x''_3(y''_2\, y''_2 - x'_3\, x''_2) + 2x'_3(x_2''^2 - y_2''^2) - 4y'_3\, x''_2\, y''_2 + 2x'_2(x_1 x'_1 - y_1 y'_1) - 2y'_2(x_1 y'_1 + y_1 x'_1) = C.$$

(18) $$\sum_{\kappa=1}^{n}\sum_{r=1}^{\nu}(-1)^r\sum_{\lambda=1}^{r}(-1)^{\lambda-1}\left\{x_\kappa^{(\lambda)}\frac{d^{r-\lambda}}{dt^{r-\lambda}}\frac{\partial H}{\partial y_\kappa^{(r)}}-y_\kappa^{(\lambda)}\frac{d^{r-\lambda}}{dt^{r-\lambda}}\frac{\partial H}{\partial x_\kappa^{(r)}}\right\}+F=C\,.$$

One can investigate the conditions for the kinetic potential when one assumes that $\rho = 2, 3, \ldots, \nu$ similarly. It is easy to see that the first form that was found (9) for the principle of areas cannot be included in the integral equations (13) that were just presented.

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§ 11. – The extended principle of the conservation of the motion of the center of mass

Assume that when the constraint conditions of the problem are given in finite form, they shall depend upon only the difference of coordinates of the same type, so:

$$\delta x_1 = \delta x_2 = \ldots = \delta x_n = p\,, \qquad \delta y_1 = \delta y_2 = \ldots = \delta y_n = q\,, \qquad \delta z_1 = \delta z_2 = \ldots = \delta z_n = r\,,$$

in which p, q, r mean arbitrary quantities, so they can be considered to be virtual displacements, or that one has:

$$\sum_{i=1}^{n} f_{ki} = 0\,, \qquad \sum_{i=1}^{n} \varphi_{ki} = 0\,, \qquad \sum_{i=1}^{n} \psi_{ki} = 0 \qquad (k = 1, 2, \ldots, m)$$

in the **Lagrange** equations in the first form. Equations (6) of **§ 4** will then emerge from the relations:

$$\text{(1)} \qquad \begin{cases} \displaystyle\sum_{i=1}^{n} \frac{\partial H}{\partial x_i} - \frac{d}{dt}\sum_{i=1}^{n} \frac{\partial H}{\partial x_i'} + \cdots + (-1)^{\nu} \frac{d^{\nu}}{dt^{\nu}} \sum_{i=1}^{n} \frac{\partial H}{\partial x_i^{(\nu)}} = \sum_{i=1}^{n} Q_i\,, \\ \displaystyle\sum_{i=1}^{n} \frac{\partial H}{\partial y_i} - \frac{d}{dt}\sum_{i=1}^{n} \frac{\partial H}{\partial y_i'} + \cdots + (-1)^{\nu} \frac{d^{\nu}}{dt^{\nu}} \sum_{i=1}^{n} \frac{\partial H}{\partial y_i^{(\nu)}} = \sum_{i=1}^{n} R_i\,, \\ \displaystyle\sum_{i=1}^{n} \frac{\partial H}{\partial z_i} - \frac{d}{dt}\sum_{i=1}^{n} \frac{\partial H}{\partial z_i'} + \cdots + (-1)^{\nu} \frac{d^{\nu}}{dt^{\nu}} \sum_{i=1}^{n} \frac{\partial H}{\partial z_i^{(\nu)}} = \sum_{i=1}^{n} S_i\,. \end{cases}$$

Now let H_1 be an arbitrary function of t, ξ, ξ', …, $\xi^{(\nu)}$, η, η', …, $\eta^{(\nu)}$, ζ, ζ', …, $\zeta^{(\nu)}$, in which ξ, η, ζ shall initially be functions of $x_1, \ldots, x_h$; $y_1, \ldots, y_h$; $z_1, \ldots, z_h$ that are still arbitrary. It will then follow from the equation in Lemma 2 that:

$$\frac{\partial H_1}{\partial x_i} - \frac{d}{dt}\frac{\partial H_1}{\partial x_i'} + \cdots + (-1)^{\nu} \frac{d^{\nu}}{dt^{\nu}} \frac{\partial H_1}{\partial x_i^{(\nu)}} = \left(\frac{\partial H_1}{\partial \xi} - \frac{d}{dt}\frac{\partial H_1}{\partial \xi'} + \cdots + (-1)^{\nu} \frac{d^{\nu}}{dt^{\nu}} \frac{\partial H_1}{\partial \xi^{(\nu)}} \right) \frac{\partial \xi}{\partial x_i}\,.$$

There will be two corresponding equations for the other coordinates, such that when ξ, η, ζ are subject to the conditions that:

$$\text{(2)} \qquad \sum_{i=1}^{n} \frac{\partial \xi}{\partial x_i} = 1\,, \qquad \sum_{i=1}^{n} \frac{\partial \eta}{\partial y_i} = 1\,, \qquad \sum_{i=1}^{n} \frac{\partial \zeta}{\partial z_i} = 1\,,$$

or

$$\xi = x_\lambda + \omega_1 (x_1 - x_\lambda, \ldots, x_n - x_\lambda)\,, \qquad \eta = y_\lambda + \omega_2 (y_1 - y_\lambda, \ldots, y_n - y_\lambda)\,,$$
$$\zeta = z_\lambda + \omega_3 (z_1 - z_\lambda, \ldots, z_n - z_\lambda)\,,$$

in which λ means any of the indices 1, 2, …, n, and ω_1, ω_2, ω_3 mean arbitrary functions, that will give the relations:

$$
(3)\qquad \left\{
\begin{aligned}
&\frac{\partial H_1}{\partial \xi} - \frac{d}{dt}\frac{\partial H_1}{\partial \xi'} + \cdots + (-1)^\nu \frac{d^\nu}{dt^\nu}\frac{\partial H_1}{\partial \xi^{(\nu)}} \\
&\qquad = \sum_{i=1}^n \frac{\partial H_1}{\partial x_i} - \frac{d}{dt}\sum_{i=1}^n \frac{\partial H_1}{\partial x_i'} + \cdots + (-1)^\nu \frac{d^\nu}{dt^\nu}\sum_{i=1}^n \frac{\partial H_1}{\partial x_i^{(\nu)}}, \\
&\frac{\partial H_1}{\partial \eta} - \frac{d}{dt}\frac{\partial H_1}{\partial \eta'} + \cdots + (-1)^\nu \frac{d^\nu}{dt^\nu}\frac{\partial H_1}{\partial \eta^{(\nu)}} \\
&\qquad = \sum_{i=1}^n \frac{\partial H_1}{\partial y_i} - \frac{d}{dt}\sum_{i=1}^n \frac{\partial H_1}{\partial y_i'} + \cdots + (-1)^\nu \frac{d^\nu}{dt^\nu}\sum_{i=1}^n \frac{\partial H_1}{\partial y_i^{(\nu)}}, \\
&\frac{\partial H_1}{\partial \zeta} - \frac{d}{dt}\frac{\partial H_1}{\partial \zeta'} + \cdots + (-1)^\nu \frac{d^\nu}{dt^\nu}\frac{\partial H_1}{\partial \zeta^{(\nu)}} \\
&\qquad = \sum_{i=1}^n \frac{\partial H_1}{\partial z_i} - \frac{d}{dt}\sum_{i=1}^n \frac{\partial H_1}{\partial z_i'} + \cdots + (-1)^\nu \frac{d^\nu}{dt^\nu}\sum_{i=1}^n \frac{\partial H_1}{\partial z_i^{(\nu)}}.
\end{aligned}\right.
$$

Now if the kinetic potential H has the form:

$$
(4)\qquad
\begin{aligned}
H = H_2\,(t,\ & x_\lambda + \omega_1\,(x_1 - x_\lambda, \ldots, x_n - x_\lambda), \\
& y_\lambda + \omega_2\,(y_1 - y_\lambda, \ldots, y_n - y_\lambda), \\
& z_\lambda + \omega_3\,(z_1 - z_\lambda, \ldots, z_n - z_\lambda), \\
& x_\lambda' + \omega_1',\ y_\lambda' + \omega_2',\ z_\lambda' + \omega_3', \ldots, \\
& x_\lambda^{(\nu)} + \omega_1^{(\nu)},\ y_\lambda^{(\nu)} + \omega_2^{(\nu)},\ z_\lambda^{(\nu)} + \omega_3^{(\nu)}) \\[1ex]
+ H_3\,(t,\ & x_r - x_s,\ x_r' - x_s', \ldots, x_r^{(\nu)} - x_s^{(\nu)}, \\
& y_r - y_s,\ y_r' - y_s', \ldots, y_r^{(\nu)} - y_s^{(\nu)}, \\
& z_r - z_s,\ z_r' - z_s', \ldots, z_r^{(\nu)} - z_s^{(\nu)}),
\end{aligned}
$$

in which H_2 and H_3 are arbitrary functions of their arguments, then on the one hand, from (3), one will have:

$$
(5)\qquad
\begin{aligned}
&\sum_{i=1}^n \frac{\partial H_2}{\partial x_i} - \frac{d}{dt}\sum_{i=1}^n \frac{\partial H_2}{\partial x_i'} + \cdots + (-1)^\nu \frac{d^\nu}{dt^\nu}\sum_{i=1}^n \frac{\partial H_2}{\partial x_i^{(\nu)}} \\
&\qquad = \sum_{i=1}^n \frac{\partial H_2}{\partial \xi} - \frac{d}{dt}\sum_{i=1}^n \frac{\partial H_2}{\partial \xi'} + \cdots + (-1)^\nu \frac{d^\nu}{dt^\nu}\sum_{i=1}^n \frac{\partial H_2}{\partial \xi^{(\nu)}},
\end{aligned}
$$

along with corresponding equations for the y_i, z_i (η, ζ, resp.). On the other hand, since it is known that:

$$\sum_{i=1}^{n} \frac{\partial H_3}{\partial x_i^{(\rho)}} = 0 , \qquad \sum_{i=1}^{n} \frac{\partial H_3}{\partial y_i^{(\rho)}} = 0 , \quad \sum_{i=1}^{n} \frac{\partial H_3}{\partial z_i^{(\rho)}} = 0 \qquad (\rho = 0, 1, 2, \ldots, \nu),$$

(6)
$$\sum_{i=1}^{n} \frac{\partial H_3}{\partial x_i} - \frac{d}{dt} \sum_{i=1}^{n} \frac{\partial H_3}{\partial x_i'} + \cdots + (-1)^{\nu} \frac{d^{\nu}}{dt^{\nu}} \sum_{i=1}^{n} \frac{\partial H_3}{\partial x_i^{(\nu)}} = 0$$

with corresponding equations for the other coordinates, such that when one adds equations (5) and (6), one will get:

$$\sum_{i=1}^{n} \frac{\partial H}{\partial x_i} - \frac{d}{dt} \sum_{i=1}^{n} \frac{\partial H}{\partial x_i'} + \cdots + (-1)^{\nu} \frac{d^{\nu}}{dt^{\nu}} \sum_{i=1}^{n} \frac{\partial H}{\partial x_i^{(\nu)}} = \frac{\partial H_2}{\partial \xi} - \frac{d}{dt} \frac{\partial H_2}{\partial \xi'} + \cdots + (-1)^{\nu} \frac{d^{\nu}}{dt^{\nu}} \frac{\partial H_2}{\partial \xi^{(\nu)}} .$$

Therefore from (1), that will give the relations:

(7)
$$\left\{ \begin{aligned} \frac{\partial H_2}{\partial \xi} - \frac{d}{dt} \frac{\partial H_2}{\partial \xi'} + \cdots + (-1)^{\nu} \frac{d^{\nu}}{dt^{\nu}} \frac{\partial H_2}{\partial \xi^{(\nu)}} &= \sum_{i=1}^{n} Q_i , \\ \frac{\partial H_2}{\partial \eta} - \frac{d}{dt} \frac{\partial H_2}{\partial \eta'} + \cdots + (-1)^{\nu} \frac{d^{\nu}}{dt^{\nu}} \frac{\partial H_2}{\partial \eta^{(\nu)}} &= \sum_{i=1}^{n} R_i , \\ \frac{\partial H_2}{\partial \zeta} - \frac{d}{dt} \frac{\partial H_2}{\partial \zeta'} + \cdots + (-1)^{\nu} \frac{d^{\nu}}{dt^{\nu}} \frac{\partial H_2}{\partial \zeta^{(\nu)}} &= \sum_{i=1}^{n} S_i . \end{aligned} \right.$$

The relations that they express, which will then imply equations (7), *when H is given by equation* (4) *as:*

$$H = H_2 + H_3 ,$$

and ξ, η, ζ *are determined by the expressions:*

$$\xi = x_\lambda + \omega_1 (x_1 - x_\lambda, \ldots, x_n - x_\lambda) , \qquad \eta = y_\lambda + \omega_2 (y_1 - y_\lambda, \ldots, y_n - y_\lambda) ,$$
$$\zeta = z_\lambda + \omega_3 (z_1 - z_\lambda, \ldots, z_n - z_\lambda) ,$$

shall represent the extended principle of the conservation of the motion of the center of mass.

In the mechanics of ponderable masses, when $\sum m_i = M$, if one sets

$$T = \tfrac{1}{2} \sum_{i=1}^{n} m_i (x_i'^2 + y_i'^2 + z_i'^{2)})$$

$$= \frac{1}{2M} (m_1 x_1' + m_2 x_1' + \cdots + m_2 x_1')^2 + \sum \frac{m_i m_k}{2M} (x_i' - x_k')^2 + \cdots$$

$$= \frac{M}{2}(x_1' + \frac{m_2}{M}(x_2' - x_1') + \cdots + \frac{m_n}{M}(x_n' - x_1')) + \sum \frac{m_i\, m_k}{2M}(x_i' - x_k')^2 + \cdots$$

then if U depends upon only the differences of the coordinates, the kinetic potential $H = -T - U$ will have the form (4), in which:

$$H_2 = -\frac{1}{2M}\{(m_1\, x_1' + m_2\, x_2' + \cdots + m_n\, x_n')^2 + (m_1\, y_1' + m_2\, y_2' + \cdots + m_n\, y_n')^2 + (m_1\, z_1' + m_2\, z_2' + \cdots + m_n\, z_n')^2\}$$

and

$$\xi = \frac{1}{M}(m_1\, x_1 + \ldots + m_n\, x_n)\,, \quad \eta = \frac{1}{M}(m_1\, y_1 + \ldots + m_n\, y_n)\,, \quad \zeta = \frac{1}{M}(m_1\, z_1 + \ldots + m_n\, z_n)\,.$$

However, ξ, η, ζ are the coordinates of the center of mass then, and from (7), since one has:

$$H_2 = -\tfrac{1}{2} M(\xi'^2 + \eta'^2 + \zeta'^2)\,,$$

its motion will be subject to the known equations:

$$M\frac{d^2\xi}{dt^2} = \sum_{i=1}^{n} Q_i\,, \qquad M\frac{d^2\eta}{dt^2} = \sum_{i=1}^{n} R_i\,, \qquad M\frac{d^2\zeta}{dt^2} = \sum_{i=1}^{n} S_i\,.$$

In order to give a simple application of the mechanical principles that were developed up to now, we would next like to treat the motion of a point that is attracted to a fixed center according to **Weber**'s law. When the center possesses a mass m_2 and the coordinates x_2, y_2, while the coordinates of the moving point of mass m_1 are denoted by x_1, y_1, that motion will be described by the differential equations:

$$m_1\, x_1'' = \left(\frac{\partial W}{\partial r} - \frac{d}{dt}\frac{\partial W}{\partial r'}\right)\frac{x_1 - x_2}{r}\,,$$

$$m_1\, y_1'' = \left(\frac{\partial W}{\partial r} - \frac{d}{dt}\frac{\partial W}{\partial r'}\right)\frac{y_1 - y_2}{r}\,,$$

in which the xy-plane is laid through the center and the direction of the initial velocity, and we set:

$$r^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2\,.$$

Since the law of conservation of energy:

$$\tfrac{1}{2} m_1 (x_1'^2 + y_1'^2) - \frac{m_1\, m_2}{r}\left(1 - \frac{r'^2}{k^2}\right) = h$$

further implies the principle of areas:

$$m_1\{(x_1 - x_2)\,y_1' - (y_1 - y_2)\,x_1'\} = \alpha ,$$

upon substituting:

$$x_1 - x_2 = r \cos \vartheta , \qquad y_1 - y_2 = r \sin \vartheta$$

in that, it will follow, as we can see immediately, that:

$$t + c = \int \frac{\sqrt{r}\,\sqrt{r + \dfrac{2m_2}{k^2}}}{\sqrt{2r\left(\dfrac{h}{m_1}r + m_2\right) - \dfrac{\alpha^2}{m_1^2}}}\,dr$$

and

$$\vartheta + c_1 = \frac{1}{\alpha}\int \frac{\sqrt{r + \dfrac{2m_2}{k^2}}}{\sqrt{r^3}\,\sqrt{2r\left(\dfrac{h}{m_1}r + m_2\right) - \dfrac{\alpha^2}{m_1^2}}}\,dr .$$

Therefore, *we have been led back to a problem that involves elliptic integrals.*

However, if we assume that the attracting center is also moving, so we investigate the motion of two free points that are attracted to each other according to **Weber**'s law, and whose initial velocities might be assumed to lie in a plane, for the sake of brevity, so the entire motion will then take place in that plane, then, from the above, the principle of the center of mass will be valid for the four equations of motion:

$$m_1\, x_1'' = \left(\frac{\partial W}{\partial r} - \frac{d}{dt}\frac{\partial W}{\partial r'}\right)\frac{x_1 - x_2}{r} , \qquad m_1\, y_1'' = \left(\frac{\partial W}{\partial r} - \frac{d}{dt}\frac{\partial W}{\partial r'}\right)\frac{y_1 - y_2}{r} ,$$
$$m_2\, x_2'' = \left(\frac{\partial W}{\partial r} - \frac{d}{dt}\frac{\partial W}{\partial r'}\right)\frac{x_2 - x_1}{r} , \qquad m_2\, y_2'' = \left(\frac{\partial W}{\partial r} - \frac{d}{dt}\frac{\partial W}{\partial r'}\right)\frac{y_2 - y_1}{r} .$$

Thus, for the coordinates that are defined by the equations:

$$(m_1 + m_2)\ \xi = m_1\, x_1 + m_2\, x_2 , \qquad (m_1 + m_2)\ \eta = m_1\, y_1 + m_2\, y_2 ,$$

one will have the equations:

$$\frac{d^2\xi}{dt^2} = 0 , \qquad \frac{d^2\eta}{dt^2} = 0 .$$

Therefore, the center of mass will advance along a straight line with constant velocity. If one now denotes the relative coordinates of the two points relative to the center of mass by ξ_1, η_1, ξ_2, η_2 such that:

$$\xi_1 = x_1 - \xi\,, \qquad \eta_1 = y_1 - \eta\,, \qquad \xi_2 = x_2 - \xi\,, \qquad \eta_2 = y_1 - \eta\,,$$

then when one sets:

$$\xi_1^2 + \eta_1^2 = \rho_1^2\,, \qquad \xi_2^2 + \eta_2^2 = \rho_2^2\,,$$

so:

$$r = \frac{m_1 + m_2}{m_2}\,\rho_1 = \frac{m_1 + m_2}{m_2}\,\rho_2\,,$$

and

$$\frac{W\,m_2}{m_1 + m_2} = W_1\,, \qquad \frac{W\,m_1}{m_1 + m_2} = W_2\,,$$

the equations will go to:

$$m_1\,\xi_1'' = \left(\frac{\partial W_1}{\partial \rho_1} - \frac{d}{dt}\frac{\partial W_1}{\partial \rho_1'}\right)\frac{\xi_1}{\rho_1}\,, \qquad m_1\,\eta_1'' = \left(\frac{\partial W_1}{\partial \rho_1} - \frac{d}{dt}\frac{\partial W_1}{\partial \rho_1'}\right)\frac{\eta_1}{\rho_1}\,,$$

$$m_2\,\xi_2'' = \left(\frac{\partial W_2}{\partial \rho_2} - \frac{d}{dt}\frac{\partial W_2}{\partial \rho_2'}\right)\frac{\xi_2}{\rho_2}\,, \qquad m_2\,\eta_2'' = \left(\frac{\partial W_2}{\partial \rho_2} - \frac{d}{dt}\frac{\partial W_2}{\partial \rho_2'}\right)\frac{\eta_2}{\rho_2}\,.$$

When one sets:

$$\frac{m_2\,k}{m_1 + m_2} = k_1\,, \quad \frac{m_1\,k}{m_1 + m_2} = k_2\,,$$

one will then find that:

$$W_1 = m_1 \cdot \frac{m_2^3}{(m_1 + m_2)^2}\frac{1}{\rho_1}\left(1 + \frac{\rho_1'^{\,2}}{k_1^2}\right),$$

$$W_2 = m_2 \cdot \frac{m_1^3}{(m_1 + m_2)^2}\frac{1}{\rho_2}\left(1 + \frac{\rho_2'^{\,2}}{k_2^2}\right),$$

such that:

The motion of two points that are attracted to each other by ***Weber****'s law will be around a center of mass that advances along a straight line with constant velocity in such a way that it is as if the masses were found to be:*

$$\frac{m_2^3}{(m_1 + m_2)^2} \quad \text{and} \quad \frac{m_1^3}{(m_1 + m_2)^2}\text{, resp.,}$$

which attract the two mass-points m_1 (m_2, *resp.*) *according to* ***Weber****'s law with the constants:*

$$k_1 = \frac{m_2\,k}{m_1 + m_2} \quad \text{and} \quad k_2 = \frac{m_1\,k}{m_1 + m_2}\text{, resp.}$$

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§ 12. – Transforming the extended Lagrange equations into Hamilton's system of total differential equations

If the external forces P_s are all zero then the second form of the **Lagrange** equations of motion will read:

$$\text{(1)} \qquad \frac{\partial H}{\partial p_s} - \frac{d}{dt}\frac{\partial H}{\partial p_s'} + \cdots + (-1)^\nu \frac{d^\nu}{dt^\nu}\frac{\partial H}{\partial p_s^{(\nu)}} = 0 \qquad (s = 1, 2, \ldots, \mu),$$

while the energy E will be defined by the expression:

$$\text{(2)} \qquad E = H - \sum_{\rho=1}^{\mu} p_\rho' \left(\frac{\partial H}{\partial p_\rho'} - \frac{d}{dt}\frac{\partial H}{\partial p_\rho''} + \cdots \right) - \sum_{\rho=1}^{\mu} p_\rho'' \left(\frac{\partial H}{\partial p_\rho''} - \frac{d}{dt}\frac{\partial H}{\partial p_\rho'''} + \cdots \right) - \cdots - \sum_{\rho=1}^{\mu} p_\rho^{(\nu)} \frac{\partial H}{\partial p_\rho^{(\nu)}} .$$

If one now sets:

$$\text{(3)} \qquad \left\{ \begin{aligned} & \frac{\partial H}{\partial p_\rho'} - \frac{d}{dt}\frac{\partial H}{\partial p_\rho''} + \cdots + (-1)^{\nu-1} \frac{d^{\nu-1}}{dt^{\nu-1}}\frac{\partial H}{\partial p_\rho^{(\nu)}} = q_{\rho 0} , \\ & \frac{\partial H}{\partial p_\rho''} - \frac{d}{dt}\frac{\partial H}{\partial p_\rho'''} + \cdots + (-1)^{\nu-2} \frac{d^{\nu-2}}{dt^{\nu-2}}\frac{\partial H}{\partial p_\rho^{(\nu)}} = q_{\rho 1} , \\ & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ & \frac{\partial H}{\partial p_\rho^{(\nu)}} = q_{\rho,\nu-1} , \end{aligned} \right.$$

and calculates the $\nu\mu$ quantities $p_\rho^{(\nu)}$, $p_\rho^{(\nu+1)}$, …, $p_\rho^{(2\nu-1)}$ as functions of t, p_s, p_s', …, $p_s^{(\nu+1)}$, q_{s0}, q_{s1}, …, $q_{s,\nu-1}$ using those $\nu\mu$ equations then the $p_\rho^{(\nu)}$ will be given as functions of only p_s, p_s', …, $p_s^{(\nu-1)}$, and $q_{s,\nu-1}$ by the last μ of equations (3), while the other equations, which are linear in the quantities $p_\rho^{(\nu+1)}$, $p_\rho^{(\nu+2)}$, …, $p_\rho^{(2\nu-1)}$, resp., will give their values as functions of all of the quantities that were just referred to. The energy, which assumes the following form in terms of the quantities q that were introduced in (3):

$$E = H - \sum_{\rho=1}^{\mu} p_\rho' \, q_{\rho 0} - \sum_{\rho=1}^{\mu} p_\rho'' \, q_{\rho 1} - \cdots - \sum_{\rho=1}^{\mu} p_\rho^{(\nu)} \, q_{\rho,\nu-1} ,$$

when we let (E), (H), $(p_\rho^{(\nu)})$ denote the values that the quantities E, H, and $p_\rho^{(\nu)}$ go to after substituting them in the expressions that are calculated from the system of equations (3), will yield the relation:

(4) $$(E) = (H) - \sum_{\rho=1}^{\mu} p'_\rho \, q_{\rho 0} - \sum_{\rho=1}^{\mu} p''_\rho \, q_{\rho 1} - \cdots - \sum_{\rho=1}^{\mu} (p_\rho^{(\nu)}) \, q_{\rho,\nu-1} \ .$$

If one partially differentiates that equation with respect to $p_\rho^{(\nu)}$, in which $\lambda = 1, 2, \ldots, \nu - 1$, then one will get:

$$\frac{\partial (E)}{\partial p_s^{(\lambda)}} = \frac{\partial (H)}{\partial p_s^{(\lambda)}} - q_{s,\lambda-1} - \sum_{\rho=1}^{\mu} \frac{\partial (p_\rho^{(\nu)})}{\partial p_s^{(\lambda)}} \, q_{\rho,\lambda-1} ,$$

and since one has:

$$\frac{\partial (H)}{\partial p_s^{(\lambda)}} = \left(\frac{\partial H}{\partial p_s^{(\lambda)}}\right) + \sum_{\rho=1}^{\mu} \left(\frac{\partial H}{\partial p_\rho^{(\nu)}}\right) \frac{\partial (p_\rho^{(\nu)})}{\partial p_s^{(\lambda)}} = \left(\frac{\partial H}{\partial p_s^{(\lambda)}}\right) + \sum_{\rho=1}^{\mu} \frac{\partial (p_\rho^{(\nu)})}{\partial p_s^{(\lambda)}} \, q_{\rho,\nu-1} ,$$

in which the parentheses shall always denote the value of the bracketed expression after one applies the substitutions that were given above, one will get the relation:

$$\frac{\partial (E)}{\partial p_s^{(\lambda)}} = \left(\frac{\partial H}{\partial p_s^{(\lambda)}}\right) - q_{s,\lambda-1} ,$$

or since:

$$\frac{\partial H}{\partial p_s^{(\lambda)}} - \frac{d}{dt} \frac{\partial H}{\partial p_s^{(\lambda+1)}} + \cdots + (-1)^{\nu-\lambda} \frac{d^{\nu-\lambda}}{dt^{\nu-\lambda}} \frac{\partial H}{\partial p_s^{(\nu)}} = q_{s,\lambda-1} ,$$

and therefore:

$$\left(\frac{\partial H}{\partial p_s^{(\lambda)}}\right) = q_{s,\lambda-1} + \left(\frac{d}{dt}\left[\frac{\partial H}{\partial p_s^{(\lambda+1)}} - \frac{d}{dt} \frac{\partial H}{\partial p_s^{(\lambda+2)}} + \cdots + (-1)^{\nu-\lambda-1} \frac{d^{\nu-\lambda-1}}{dt^{\nu-\lambda-1}} \frac{\partial H}{\partial p_s^{(\nu)}}\right]\right) = q_{s,\lambda-1} + \frac{dq_{s\lambda}}{dt} ,$$

one will get the relation:

$$\frac{\partial (E)}{\partial p_s^{(\lambda)}} = \frac{dq_{s\lambda}}{dt} \qquad (\lambda = 1, 2, \ldots, \nu - 1).$$

By contrast, when equation (4) is differentiated with respect to p_s, only the terms in $q_{s,\lambda-1}$ will drop out of the previous equations, and the relation:

$$\frac{\partial (E)}{\partial p_s} = \left(\frac{\partial H}{\partial p_s}\right)$$

will go to

$$\frac{\partial (E)}{\partial p_s} = \frac{dq_s}{dt}$$

as a result of the **Lagrange** equations (1), which will then yield ν equations of the form:

$$\frac{\partial(E)}{\partial p_s^{(\lambda)}} = \frac{dq_{s\lambda}}{dt} \qquad (\lambda = 0, 1, 2, \ldots, \nu - 1). \tag{5}$$

Furthermore, since (4) implies that:

$$\frac{\partial(E)}{\partial q_{s\lambda}} = \frac{\partial(H)}{\partial q_{s\lambda}} - p_s^{(\lambda+1)} - \sum_{\rho=1}^{\mu} \frac{\partial\left(p_\rho^{(\nu)}\right)}{\partial q_{s\lambda}} q_{\rho,\nu-1} = \sum_{\rho=1}^{\mu}\left(\frac{\partial(H)}{\partial p_\rho^{(\nu)}}\right)\frac{\partial\left(p_\rho^{(\nu)}\right)}{\partial q_{s\lambda}} - p_s^{(\lambda+1)} - \sum_{\rho=1}^{\mu} \frac{\partial\left(p_\rho^{(\nu)}\right)}{\partial q_{s\lambda}} q_{\rho,\nu-1} ,$$

due to the identity of the first and third terms on the right-hand side, that will give:

$$\frac{\partial(E)}{\partial q_{s\lambda}} = -\frac{dp_s^{(\lambda)}}{dt}. \tag{6}$$

One will then get from (5) and (6) that:

***Hamilton**'s system of total differential equations, which consists of $\nu \cdot \mu$ first-order differential equations that is equivalent to the **Lagrange** equations of motion in the case where the external forces are all zero is expressed in the form:*

$$\frac{dq_{s\lambda}}{dt} = \frac{\partial(E)}{\partial p_s^{(\lambda)}}, \qquad \frac{dp_s^{(\lambda)}}{dt} = -\frac{\partial(E)}{\partial q_{s\lambda}} \tag{7}$$

$$(\lambda = 0, 1, 2, \ldots, \nu - 1 \,;\, s = 1, 2, \ldots, \mu) ,$$

in which (E) *means the value of energy that is defined by* (2) *when the values of* $p_\rho^{(\lambda)}$, $p_\rho^{(\lambda+1)}$, …, $p_\rho^{(2\nu-1)}$ *as functions of* t, p_s, p_s', …, $p_s^{(\nu-1)}$, q_{s0}, q_{s1}, …, $q_{s,\nu-1}$ *that are inferred from equations* (3) *are substituted in them.*

For the case in which the kinetic potential includes only the first derivatives of the coordinates, as one can see immediately, **Hamilton**'s differential equations go to the system of 2μ simultaneous differential equations:

$$\frac{dq_{s0}}{dt} = \frac{\partial(E)}{\partial p_s}, \qquad \frac{dp_s}{dt} = -\frac{\partial(E)}{\partial q_{s0}},$$

in which (E) represents the expression:

$$H - \sum_{\lambda=1}^{\mu} p_\lambda' \frac{\partial H}{\partial p_\lambda'},$$

when the p'_λ in them are replaced with their values as functions of t, p_s, q_{s0} that one infers from the μ equations:

$$\frac{\partial H}{\partial p'_\rho} = q_{\rho\,0}\ .$$

If the actual energy is, in turn, separated from the potential energy, so $H = -T - U$, then the last equation will go to the μ equations:

$$-\frac{\partial T}{\partial p'_\rho} = q_{\rho\,0}\ ,$$

which are linear in p'_1, p'_2, …, p'_μ, and in which:

$$E = T - U\ .$$

From the form of the differential equations (7), it is immediately clear that:

The multiplier in the extended ***Hamiltonian*** *system is also a constant, and therefore the known theorem of* ***Jacobi*** *on the last multiplier will retain its validity*.

Similarly, **Poisson**'s theorem for the differential system (7) shows that:

If:

$$\varphi(p_s, p'_s, \ldots, p_s^{(\nu-1)}, q_{s0}, q_{s1}, \ldots, q_{s,\nu-1}, t)$$

and

$$\psi(p_s, p'_s, \ldots, p_s^{(\nu-1)}, q_{s0}, q_{s1}, \ldots, q_{s,\nu-1}, t)$$

are two integral functions of system of the differential equations then the expression:

$$\sum_{\sigma=1}^{\mu}\sum_{\rho=1}^{\nu-1}\left\{\frac{\partial\varphi}{\partial p_\sigma^{(\rho)}}\frac{\partial\psi}{\partial q_{\sigma\rho}} - \frac{\partial\varphi}{\partial q_{\sigma\rho}}\frac{\partial\psi}{\partial p_\sigma^{(\rho)}}\right\}$$

will likewise represent an integral function of that differential equation.

Finally, we might add a theorem on the nature of the integrals of the extended **Hamilton**ian equations of motion.

Let the kinetic potential H_1 be an algebraic function of t, p_s, p'_s, …, $p_s^{(\nu)}$ that might satisfy the equation:

(8) $$H^\delta + r_1(t, p_s, p'_s, \ldots, p_s^{(\nu)})\, H^{\delta-1} + \cdots + r_\delta(t, p_s, p'_s, \ldots, p_s^{(\nu)}) = 0\ ,$$

in which $r_1, r_2, \ldots, r_\delta$ mean rational functions of the quantities that are included in it. The derivatives of H_1 of arbitrary order with respect to the coordinates and their derivatives with respect to time will then be rational functions of just those quantities and H_1 itself, and equations (3) can then be put into the form:

$$(9)\qquad \begin{cases} R_{\rho 0}\quad (t, H_1, p_s, p_s'', \ldots, p_s^{(2\nu-1)}) = q_{\rho 0}\,, \\ R_{\rho 1}\quad (t, H_1, p_s, p_s'', \ldots, p_s^{(2\nu-2)}) = q_{\rho 1}\,, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ R_{\rho,\nu-1}(t, H_1, p_s, p_s'', \ldots, p_s^{(\nu)}) \quad = q_{\rho,\nu-1}\,, \end{cases}$$

in which $R_{\rho 0}, \ldots, R_{\rho,\nu-1}$ are rational functions of the quantities that they include, the first of which is linear in $p_s^{(2\nu-1)}$, the second of which is linear in $p_s^{(2\nu-2)}$, …, and last of which is linear in $p_s^{(\nu)}$. Eliminating the $\mu + 1$ quantities $p_s^{(\nu)}$ and H_1 from the $\mu + 2$ equations (8) for $H = H_1$, the μ equations:

$$R_{\rho,\nu-1}(t, H_1, p_s, p_s'', \ldots, p_s^{(\nu)}) = q_{\rho,\nu-1}\,,$$

and the equation:

$$(10)\qquad E_1 = H_1 - \sum_{\rho=1}^{\mu} p_\rho'\, q_{\rho 0} - \sum_{\rho=1}^{\mu} p_\rho''\, q_{\rho 1} - \cdots - \sum_{\rho=1}^{\mu} p_\rho^{(\nu)}\, q_{\rho,\nu-1}\,,$$

in which E_1 denotes the unique value E that corresponds to H_1 by way of equation (2), will yield (E_1) as the solution to the algebraic equation:

$$(11)\qquad \begin{aligned} &(E)^\eta + \mathfrak{r}_1(t, p_s, p_s', \ldots, p_s^{(\nu-1)}, q_{s0}, q_{s1}, \ldots, q_{s,\nu-1})(E)^{\eta-1} + \cdots \\ &\quad + \mathfrak{r}_\eta(t, p_s, p_s', \ldots, p_s^{(\nu-1)}, q_{s0}, q_{s1}, \ldots, q_{s,\nu-1}) = 0\,, \end{aligned}$$

in which $\mathfrak{r}_1, \ldots, \mathfrak{r}_\eta$ mean rational functions of the quantities that they include. Now, equation (11) is assumed to be irreducible when one adjoins the quantities $t, p_s, p_s', \ldots, p_s^{(\nu-1)}, q_{s0}, q_{s1}, \ldots, q_{s,\nu-1}$ [because otherwise we would have to substitute those irreducible factors in it that possess the solution (E_1), instead of equation (11)]. If one assumes that **Hamilton**'s system of differential equations (7), in which E is replaced with E_1, possesses an algebraic integral:

$$\omega_1(t, p_s, p_s', \ldots, p_s^{(\nu-1)}, q_{s0}, q_{s1}, \ldots, q_{s,\nu-1}) = \alpha\,,$$

in which α means an arbitrary constant, and the integral function ω_1 might be the solution of an irreducible equation with the adjunction of (E_1), $t, p_s, p_s', \ldots, p_s^{(\nu-1)}, q_{s0}, q_{s1}, \ldots, q_{s,\nu-1}$:

$$(12)\qquad \begin{aligned} &\omega^m + f_1((E_1), t, p_s, p_s', \ldots, p_s^{(\nu-1)}, q_{s0}, q_{s1}, \ldots, q_{s,\nu-1})\,\omega^{m-1} + \cdots \\ &\quad + f_m((E_1), t, p_s, p_s', \ldots, p_s^{(\nu-1)}, q_{s0}, q_{s1}, \ldots, q_{s,\nu-1}) = 0\,, \end{aligned}$$

in which $f_1, \ldots, f_m$ represent rational functions of the quantities that they include.

Now since, by definition, an integral function must satisfy the equation:

$$(13)\qquad \frac{\partial\omega_1}{\partial t} - \sum_{s=1}^{\mu} \frac{\partial\omega_1}{\partial p_s}\frac{\partial(E_1)}{\partial q_{s0}} - \sum_{s=1}^{\mu} \frac{\partial\omega_1}{\partial p'_s}\frac{\partial(E_1)}{\partial q_{s1}} - \cdots - \sum_{s=1}^{\mu} \frac{\partial\omega_1}{\partial p_s^{(\nu-1)}}\frac{\partial(E_1)}{\partial q_{s,\nu-1}}$$
$$+ \sum_{s=1}^{\mu} \frac{\partial\omega_1}{\partial q_{s0}}\frac{\partial(E_1)}{\partial p_s} + \sum_{s=1}^{\mu} \frac{\partial\omega_1}{\partial q_{s1}}\frac{\partial(E_1)}{\partial p'_s} + \cdots + \sum_{s=1}^{\mu} \frac{\partial\omega_1}{\partial q_{s,\nu-1}}\frac{\partial(E_1)}{\partial p_s^{(\nu-1)}} = 0$$

identically by means of (7), but from equation (11), the partial differential quotients of (E_1) can be expressed rationally in terms of (E_1) and the other quantities that enter into the coefficients of that equation, and therefore, from (12), the partial differential quotients of ω_1 will also be rational functions of:

$$\omega_1,\ (E_1),\ t,\ p_s,\ p'_s,\ \ldots,\ p_s^{(\nu-1)},\ q_{s0},\ q_{s1},\ \ldots,\ q_{s,\nu-1}$$

then (13) will go to an equation in ω of the same type as equation (12) and will likewise be satisfied by ω_1. Therefore, it will follow from the irreducibility of (12) that all solutions of the latter also satisfy equation (13), so they will also be integrals of **Hamilton**'s system of differential equations. However, it will follow from this that since:

$$\frac{d\omega_1}{dt},\ \frac{d\omega_2}{dt},\ \ldots,\ \frac{d\omega_m}{dt}$$

must then vanish identically with the use of those differential equations, either $f_1, f_2, \ldots, f_m$ themselves must be integral functions of the system or we must have $m = 1$. We will then find that *an algebraic integral function of Hamilton's system of differential equations will either be itself composed rationally from* $(E_1), t, p_s, p'_s, \ldots, p_s^{(\nu-1)}, q_{s0}, q_{s1}, \ldots, q_{s,\nu-1}$ *or it will define an algebraic combination of such rational integral functions.*

If one now replaces the quantities $q_{s0}, q_{s1}, \ldots, q_{s,\nu-1}$ in those rational integral functions with rational functions that are given by equations (9) then (E_1) will go to E_1, which will, in turn, be expressible rationally in terms of $H_1, t, p_s, p'_s, \ldots, p_s^{(\nu-1)}$, and that will give the following theorem:

If the kinetic potential is an algebraic function of time, the coordinates, and their derivatives with respect to time up to order ν*, and if* ***Hamilton****'s system of differential equations possesses an algebraic integral function then it will either be itself a rational function of the kinetic potential, time, the coordinates, and their derivatives with respect to time up to order* $2\nu - 1$ *or an algebraic combination of such rational functions*.

Similarly, as one can easily show, there are corresponding theorems when the kinetic potential includes logarithms with algebraic logarithmands or **Abel**ian integrals of the logarithmands and the upper limits of the integrals whose upper limits are algebraic functions of the quantities that were just referred to.

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§ 13. – Extension of Hamilton's partial differential equation.

If we assume, in turn, that the external forces all vanish then from a known theorem of **Jacobi** regarding the complete integrals of the first-order differential equation that one obtains when one substitutes the partial differential quotients $\partial V / \partial p_s^{(k)}$ for the quantities q_{sk} in the expression for (E) that represents it as a function of t, p_s, p_s', …, $p_s^{(\nu-1)}$, q_{s0}, q_{s1}, …, $q_{s,\nu-1}$, the general integral of the total system of differential equations:

$$\frac{dq_{s0}}{dt} = \frac{\partial (E)}{\partial p_s}, \qquad \frac{dq_{s1}}{dt} = \frac{\partial (E)}{\partial p_s'}, \qquad \dots, \qquad \frac{dq_{s,\nu-1}}{dt} = \frac{\partial (E)}{\partial p_s^{(\nu-1)}},$$

$$\frac{dp_s}{dt} = -\frac{\partial (E)}{\partial q_{s0}}, \qquad \frac{dp_s'}{dt} = -\frac{\partial (E)}{\partial q_{s1}}, \qquad \dots, \qquad \frac{dp_s^{(\nu-1)}}{dt} = -\frac{\partial (E)}{\partial q_{s,\nu-1}}$$

will imply that:

$$\text{(1)} \qquad \frac{\partial V}{\partial t} + \left(E\left(p_s, p_s', \dots, p_s^{(\nu-1)}, \frac{\partial V}{\partial p_s}, \frac{\partial V}{\partial p_s'}, ,\dots, \frac{\partial V}{\partial p_s^{(\nu-1)}} \right)\right) = 0 .$$

However, if the values of $p_s^{(\nu)}$ that one gets from the last of equations (3) in **§ 12** are denoted by:

$$p_s^{(\nu)} = \omega_\rho (t, p_s, p_s', \dots, p_s^{(\nu-1)}, q_{s,\nu-1})$$

then since equation (4) of **§ 12** says that one has:

$$(E) = (H) - \sum_{\rho=1}^{\mu} p_\rho' \, q_{\rho 0} - \sum_{\rho=1}^{\mu} p_\rho'' \, q_{\rho 1} - \dots - \sum_{\rho=1}^{\mu} p_\rho^{(\nu-1)} \, q_{\rho,\nu-2} - \sum_{\rho=1}^{\mu} \omega_\rho (t, p_s, p_\rho', \dots, p_\rho^{(\nu-1)}, q_{\rho,\nu-1}) \, q_{\rho,\nu-1},$$

the *partial differential equation* (1) *will go to:*

$$\text{(2)} \qquad \frac{\partial V}{\partial t} + \left(E\left(p_s, p_s', \dots, p_s^{(\nu-1)}, \frac{\partial V}{\partial p_s}, \frac{\partial V}{\partial p_s'}, ,\dots, \frac{\partial V}{\partial p_s^{(\nu-1)}} \right)\right)$$

$$- \sum_{\rho=1}^{\mu} p_\rho' \frac{\partial V}{\partial p_\rho} - \sum_{\rho=1}^{\mu} p_\rho'' \frac{\partial V}{\partial p_\rho'} - \dots - \sum_{\rho=1}^{\mu} p_\rho^{(\nu-1)} \frac{\partial V}{\partial p_\rho^{(\nu-2)}} - \sum_{\rho=1}^{\mu} \omega_\rho \left(t, p_s, p_\rho', \dots, p_\rho^{(\nu-1)}, \frac{\partial V}{\partial p_\rho^{(\nu-1)}} \right) \frac{\partial V}{\partial p_\rho^{(\nu-1)}} = 0 ,$$

with the dependent variable V and the $\mu\,\nu + 1$ *independent variables:*

$$t, p_1, p'_1, \ldots, p_1^{(\nu-1)}, p_2, p'_2, \ldots, p_2^{(\nu-1)}, q_{s0}, q_{s1}, \ldots, q_{s,\nu-1},$$

which will be called ***Hamilton****'s partial equation.*

If one knows the complete integral of that partial differential equation, which includes $\mu\,\nu + 1$ arbitrary constants, one of which is an additive one, while the others might be denoted by α_1, α_2, …, $\alpha_{\mu\nu}$, *then it is known that one will obtain the solutions of* ***Hamilton****'s system of total differential equations when one uses the equations:*

$$\frac{\partial V}{\partial \alpha_1} = \beta_1, \qquad \frac{\partial V}{\partial \alpha_2} = \beta_2, \qquad \ldots, \qquad \frac{\partial V}{\partial \alpha_{\mu\nu}} = \beta_{\mu\nu},$$

in which β_1, β_2, …, $\beta_{\mu\nu}$, *in turn, mean a system of* $\mu\nu$ *arbitrary constants, to represent:*

$$p_1, p_2, \ldots, p_\mu, p'_1, \ldots, p'_\mu, \ldots, p_1^{(\nu-1)}, \ldots, p_\mu^{(\nu-1)}$$

as functions of t *and the* $2\mu\nu$ *constants.*

For the case in which the kinetic potential includes only the first derivatives of the coordinates, the partial differential equation will go to:

$$\frac{\partial V}{\partial t} + H\left(t, p_s, \frac{\partial V}{\partial p_s}\right) - \sum_{\rho=1}^{\mu} \omega_\rho\left(t, p_s, \frac{\partial V}{\partial p_s}\right)\frac{\partial V}{\partial p_s} = 0,$$

when the solutions of the equations:

$$\frac{\partial H}{\partial p'_\rho} = q_{\rho 0} \qquad (\rho = 1, 2, \ldots, \mu)$$

are denoted by:

$$p'_\rho = \omega_\rho\,(t, p_s, q_{s0}).$$

Now consider the case of the mechanics of ponderable masses with a separation of actual and potential energy:

$$H = -\,T - U$$

and assume that the constraint equations do not include time t *explicitly*, such that T will be a homogeneous function of degree two in p'_1, p'_2, …, p'_μ whose coefficients are functions of the coordinates themselves. When the values of p'_ρ that are obtained from the equation:

$$- \frac{\partial T}{\partial p'_\rho} = q_{\rho 0}$$

are represented by:

$$p'_\rho = \omega_\rho (p_s, q_{s0}) = B_{\rho 1}\, q_{10} + B_{\rho 2}\, q_{20} + \ldots + B_{\rho\mu}\, q_{\mu 0} ,$$

in which $B_{\rho 1}, \ldots, B_{\rho\mu}$ are functions of $p_1, \ldots, p_\mu$, *the partial differential equation will assume the form:*

$$\frac{\partial V}{\partial t} - \tfrac{1}{2}\sum_{\rho=1}^{\mu} \frac{\partial V}{\partial p_s}\left(B_{\rho 1} \frac{\partial V}{\partial p_1} + B_{\rho 2} \frac{\partial V}{\partial p_2} + \cdots + B_{\rho\mu} \frac{\partial V}{\partial p_\mu} \right) - U = 0 ,$$

as is easy to see.

If the vis viva includes only the square of the first derivatives of the coordinates, so:

$$T = \tfrac{1}{2}\sum_{\rho=1}^{\mu} a_\rho\, p'^{2}_\rho ,$$

then **Hamilton***'s partial differential equation will go to:*

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sum_{\rho=1}^{\mu} \frac{1}{a_\rho}\left(\frac{\partial V}{\partial p_s} \right)^2 - U = 0 .$$

---

§ 14. – Helmholtz's principle of hidden motion in the mechanics of ponderable masses and its application to the motion of three points.

In his paper "Über die physikalische Bedeutung des Princips der kleinsten Wirkung," **Helmholtz** highlighted two cases of equations of motion in which an essential reduction in the number of coordinates could be achieved, and indeed not by restricting the freedom of motion of the system by fixed constraints that are expressed by equations in the coordinates, as is customary, but by a special property of the kinetic potential and the nature of **Lagrange**'s equations of motion. He initially assumed that the kinetic potential:

$$H = -T - U,$$

in which T means the *vis viva*, and U means the force function, was free of a number of the mutually-independent coordinates $p_1, p_2, \ldots, p_\mu$, such that when they are denoted by $p_1, p_2, \ldots, p_\rho$ (to use the same notations here that shall be preserved in what follows), the associated **Lagrange** equations:

$$\frac{\partial H}{\partial p_r} - \frac{d}{dt}\frac{\partial H}{\partial p_r'} = P_r$$

will go to:

$$\frac{d}{dt}\frac{\partial H}{\partial p_r'} = 0 \qquad (r = 1, 2, \ldots, \rho)$$

when one assumes that $P_r = 0$, in addition, while the other equations of motion will be represented in the form:

$$\frac{\partial H}{\partial \mathfrak{p}_r} - \frac{d}{dt}\frac{\partial H}{\partial \mathfrak{p}_r'} = \mathfrak{P}_s \qquad (s = 1, 2, \ldots, \sigma)$$

when one has:

$$\mathfrak{p}_s = p_{\rho+s}, \qquad \mathfrak{P}_s = P_{\rho+s}, \qquad \rho + \sigma = \mu.$$

Now when $p_1', p_2', \ldots, p_\rho'$ are expressed in terms of $\mathfrak{p}_1, \ldots, \mathfrak{p}_\sigma$ using the ρ equations:

$$\frac{\partial H}{\partial p_r'} = c_r,$$

in which the quantities c_r mean integration constants, and one substitutes them in the other equations of motion, it will then follow that since:

$$\frac{\partial (H)}{\partial \mathfrak{p}_s} = \left(\frac{\partial H}{\partial \mathfrak{p}_s}\right) + \left(\frac{\partial H}{\partial \mathfrak{p}'_1}\right)\frac{\partial (p'_1)}{\partial \mathfrak{p}_s} + \cdots + \left(\frac{\partial H}{\partial \mathfrak{p}'_\rho}\right)\frac{\partial (p'_\rho)}{\partial \mathfrak{p}_s},$$

$$\frac{\partial (H)}{\partial \mathfrak{p}'_s} = \left(\frac{\partial H}{\partial \mathfrak{p}'_s}\right) + \left(\frac{\partial H}{\partial \mathfrak{p}'_1}\right)\frac{\partial (p'_1)}{\partial \mathfrak{p}'_s} + \cdots + \left(\frac{\partial H}{\partial \mathfrak{p}'_\rho}\right)\frac{\partial (p'_\rho)}{\partial \mathfrak{p}'_s}$$

(in which the previous notation is preserved), when one sets:

$$\mathfrak{H} = (H) - c_1 (p'_1) - c_2 (p'_2) - \cdots - c_\rho (p'_\rho),$$

the latter σ equations, which are free of p_r and p'_r (in which the quantities in parentheses shall once more denote the values after completing the substitution), will go to:

$$\frac{\partial \mathfrak{H}}{\partial \mathfrak{p}_r} - \frac{d}{dt}\frac{\partial \mathfrak{H}}{\partial \mathfrak{p}'_r} = \mathfrak{P}_s \qquad (s = 1, 2, \ldots, \sigma).$$

Thus, one has the **Lagrange** form for a first-order kinetic potential, and therefore **Hamilton**'s principle will still remain valid. The fact that the energy does not change in this case is self-explanatory in the mechanics of ponderable masses, but one can also see it as analytically immediate for the general first-order kinetic potential. Namely, since one has:

$$\mathfrak{E} = \mathfrak{H} - \sum_{\lambda=1}^{\sigma} \mathfrak{p}'_\lambda \frac{\partial \mathfrak{H}}{\partial \mathfrak{p}'_\lambda}$$

and

$$\frac{\partial \mathfrak{H}}{\partial \mathfrak{p}'_\lambda} = \frac{\partial (H)}{\partial \mathfrak{p}'_\lambda} - c_1 \frac{\partial (\mathfrak{p}'_1)}{\partial \mathfrak{p}'_\lambda} - c_2 \frac{\partial (\mathfrak{p}'_2)}{\partial \mathfrak{p}'_\lambda} - \cdots - c_\rho \frac{\partial (\mathfrak{p}'_\rho)}{\partial \mathfrak{p}'_\lambda}$$

$$= \left(\frac{\partial H}{\partial \mathfrak{p}'_\lambda}\right) + \left(\frac{\partial H}{\partial p'_1}\right)\frac{\partial (p'_1)}{\partial \mathfrak{p}'_\lambda} + \left(\frac{\partial H}{\partial p'_2}\right)\frac{\partial (p'_2)}{\partial \mathfrak{p}'_\lambda} + \cdots - c_1 \frac{\partial (p'_1)}{\partial \mathfrak{p}'_\lambda} - c_1 \frac{\partial (p'_1)}{\partial \mathfrak{p}'_\lambda} - \cdots = \left(\frac{\partial H}{\partial \mathfrak{p}'_\lambda}\right),$$

it will follow that:

$$\mathfrak{E} = (H) - c_1 (p'_1) - c_2 (p'_2) - \cdots - c_\rho (p'_\rho) - \sum_{\lambda=1}^{\sigma} \mathfrak{p}'_\lambda \left(\frac{\partial H}{\partial \mathfrak{p}'_\lambda}\right),$$

while it will follow from:

$$E = H - p'_1 \frac{\partial H}{\partial p'_1} - \cdots - p'_\rho \frac{\partial H}{\partial p'_\rho} - \mathfrak{p}'_1 \frac{\partial H}{\partial \mathfrak{p}'_1} - \cdots - \mathfrak{p}'_\sigma \frac{\partial H}{\partial \mathfrak{p}'_\sigma},$$

after a subsequent substitution, that:

$$(E) = (H) - c_1\,(p_1') - c_2\,(p_2') - \cdots - c_\rho\,(p_\rho') - \sum_{\lambda=1}^{\sigma} \mathfrak{p}_\lambda' \left(\frac{\partial H}{\partial \mathfrak{p}_\lambda'}\right) ,$$

which will give:

$$\mathfrak{E} = (E) .$$

However, in the mechanics of ponderable masses, as well, since the substitution equations include the quantities p_r' and $\mathfrak{p}_s'$ to the first degree, the kinetic potential $\mathfrak{H}$ will now include not only terms of dimension two in the $\mathfrak{p}_s'$, but also terms of degree one. Because of that analogy with the mechanics of ponderable masses, **Helmholtz** called other cases of physical processes in which the kinetic potential also included terms that are linear in the velocities *cases with hidden motion*, in order to suggest that those physical processes can take place as motions of ponderable masses, some of which are not visible, and whose influence corresponds to the algebraic elimination process.

Now, before this principle is extended to the general first-order kinetic potential and the path to the extension of that theory to potentials of arbitrary order is stated in advance, the application and meaning of it shall first be discussed for the simplest case of hidden motion, and indeed for *one* point that is constrained to remain on a surface.

Let the equation of the surface be:

$$z = F\,(x, y) ,$$

so the kinetic potential:

$$H = -\tfrac{1}{2} m (x'^2 + y'^2 + z'^2) - U\,(x, y, z)$$

will assume the form:

$$H = -\tfrac{1}{2} m\, x'^2 \left(1 + \left(\frac{\partial F}{\partial x}\right)^2\right) - \tfrac{1}{2} m\, y'^2 \left(1 + \left(\frac{\partial F}{\partial y}\right)^2\right) + m \frac{\partial F}{\partial x} \frac{\partial F}{\partial y}\, x'\, y' - U\,(x, y, F)$$

in the free coordinates x and y. When one makes the assumption that H does not include the variable x, one must have:

$$\frac{\partial F}{\partial x} = F_1\,(y) , \quad \frac{\partial F}{\partial y} = F_2\,(y) , \quad U\,(x, y, F) = V\,(y) .$$

Therefore, since:

$$\frac{\partial^2 F}{\partial x\, \partial y} = F_1'(y) = \frac{\partial F_2(y)}{\partial x} = 0 , \qquad F_1\,(y) = \kappa ,$$

in which κ means a constant, the required form for the kinetic potential must be:

$$H = -\tfrac{1}{2} m\, x'^2\,(1 + \kappa^2) - \tfrac{1}{2} m\, y'^2\,(1 + F_2(y)^2) - m \kappa F_2(y)\, x'\, y' - V(y) ,$$

while the equation of the surface will be given by:

$$z = \kappa\, x + \int F_2(y)\, dy\,.$$

It will then define a cylinder surface whose generating line is in the XZ-plane and parallel to the line $x = -(1/\kappa)\, z$ through the origin, and U and V are given by the expressions:

$$U = \left(z - \kappa\, x - \int F_2(y)\, dy\right)\omega_1(x, y, z) + \omega_2(z - \kappa\, x, y)\,,$$

$$V(y) = \omega_2\left(\int F_2(y)\, dy, y\right)\,,$$

in which ω_1 and ω_2 mean arbitrary functions.

Now since the **Lagrange** equation that belongs to the variable x:

$$\frac{\partial H}{\partial x} - \frac{d}{dt}\frac{\partial H}{\partial x'} = 0$$

will imply the relation:

$$x'(1+\kappa^2) + \kappa\, y'\, F_2(y) = c \qquad \text{or} \qquad x' = \frac{1}{1+\kappa^2}(c - \kappa\, y'\, F_2(y))\,,$$

since one has $\partial H / \partial x = 0$, the substitution of that value of x' in the second **Lagrange** equation will again produce one of the form:

$$\frac{\partial \mathfrak{H}}{\partial y} - \frac{d}{dt}\frac{\partial \mathfrak{H}}{\partial y'} = \mathfrak{P}\,,$$

in which the kinetic potential is given by:

$$\mathfrak{H} = \frac{m}{2(1+\kappa^2)}\{y'^2(1+F_2(y)^2 + \kappa^2 + 2\kappa\, c\, y' F_2(y) + 3c^2\}\,.$$

The image of the moving point on the Y-axis will then move as if it were driven autonomously by the kinetic potential $\mathfrak{H}$, which depends upon only y and y', but also includes y' to the first power.

Now in order to make the meaning of **Helmholtz**'s principle emerge even clearer, we would like to consider the motion of three material points with masses m_1, m_2, m_3 whose coordinates are subject to the constraint equation:

(1) $$z_1 = f(x_1, y_1, x_2, y_2, z_2, x_3, y_3, z_3)\,,$$

and its internal forces might be given by a force function:

$$U\,(x_1, y_1, x_2, y_2, z_2, x_3, y_3, z_3)\ .$$

Since the kinetic potential:

$$H = -\,T - U = -\tfrac{1}{2}m_1\,(x_1'^2 + y_1'^2 + z_1'^2) - \tfrac{1}{2}m_2\,(x_2'^2 + y_2'^2 + z_2'^2) - \tfrac{1}{2}m_3\,(x_3'^2 + y_3'^2 + z_3'^2) - U$$

assumes the form:

$$(2)\qquad \begin{cases} H = & -\tfrac{1}{2}m_1\left(1+\left(\dfrac{\partial f}{\partial x_1}\right)^2\right)x_1'^2 - \tfrac{1}{2}m_1\left(1+\left(\dfrac{\partial f}{\partial y_1}\right)^2\right)y_1'^2 \\ & -\tfrac{1}{2}\left(m_2 + m_1\left(\dfrac{\partial f}{\partial x_2}\right)^2\right)x_2'^2 - \tfrac{1}{2}\left(m_2 + m_1\left(\dfrac{\partial f}{\partial y_2}\right)^2\right)y_2'^2 \\ & -\tfrac{1}{2}\left(m_2 + m_1\left(\dfrac{\partial f}{\partial z_2}\right)^2\right)z_2'^2 - \tfrac{1}{2}\left(m_3 + m_1\left(\dfrac{\partial f}{\partial x_3}\right)^2\right)x_3'^2 \\ & -\tfrac{1}{2}\left(m_3 + m_1\left(\dfrac{\partial f}{\partial y_2}\right)^2\right)y_3'^2 - \tfrac{1}{2}\left(m_3 + m_1\left(\dfrac{\partial f}{\partial z_3}\right)^2\right)z_3'^2 \\ & -\,m_1\dfrac{\partial f}{\partial x_1}\dfrac{\partial f}{\partial y_1}x_1'\,y_1' - m_1\dfrac{\partial f}{\partial x_1}\dfrac{\partial f}{\partial x_2}x_1'\,x_2' - \cdots \\ & -\,U(x_1, y_1, f, x_2, y_2, z_2, x_3, y_3, z_3) \end{cases}$$

as a result of the relation (1), it will follow that when it is supposed to be independent of the coordinates x_1 and y_1, that must also be the case for the coefficients of all derivatives of the coordinates. Therefore, when $\partial f / \partial x_1$ and $\partial f / \partial y_1$ are independent of x_1 and y_1 :

$$f = x_1\ \varphi\,(x_2, y_2, z_2, x_2, y_3, z_3) + y_1\ \psi\,(x_2, y_2, z_2, x_2, y_3, z_3) + \omega\,(x_2, y_2, z_2, x_2, y_3, z_3)\ .$$

Hence, since $\dfrac{\partial f}{\partial x_2}$, $\dfrac{\partial f}{\partial x_3}$, $\dfrac{\partial f}{\partial y_3}$, … must also be free of x_1 and y_1, the function f, and thus the constraint equation (1), will have the form:

$$(3)\qquad z_1 = a\,x_1 + b\,y_1 + \omega\,(x_2, y_2, z_2, x_2, y_3, z_3)\ ,$$

in which a and b mean constants, and the kinetic potential will be:

$$
(4)\quad \left\{
\begin{aligned}
H = \; & -\tfrac{1}{2} m_1 \left(1+a^2\right) x_1'^2 - \tfrac{1}{2} m_1 \left(1+b^2\right) y_1'^2 \\
& -\tfrac{1}{2}\left(m_2 + m_1 \left(\frac{\partial \omega}{\partial x_2}\right)^2 \right) x_2'^2 - \tfrac{1}{2}\left(m_2 + m_1 \left(\frac{\partial \omega}{\partial y_2}\right)^2 \right) y_2'^2 - \tfrac{1}{2}\left(m_2 + m_1 \left(\frac{\partial \omega}{\partial z_2}\right)^2 \right) z_2'^2 \\
& -\tfrac{1}{2}\left(m_3 + m_1 \left(\frac{\partial \omega}{\partial x_2}\right)^2 \right) x_3'^2 - \tfrac{1}{2}\left(m_3 + m_1 \left(\frac{\partial \omega}{\partial y_3}\right)^2 \right) y_3'^2 - \tfrac{1}{2}\left(m_3 + m_1 \left(\frac{\partial \omega}{\partial z_3}\right)^2 \right) z_3'^2 \\
& - m_1 \, a \, b \, x_1' \, y_1' - m_1 \, a \frac{\partial \omega}{\partial x_2} x_1' \, x_2' - m_1 \, a \frac{\partial \omega}{\partial y_2} x_1' \, y_2' + \cdots \\
& - m_1 \, b \frac{\partial \omega}{\partial x_1} y_1' \, x_2' - m_1 \, b \frac{\partial \omega}{\partial y_2} y_1' \, y_2' - \cdots - m_1 \frac{\partial \omega}{\partial x_2} \frac{\partial \omega}{\partial y_2} x_2' \, y_2' \\
& - U(x_1, y_1, f, x_2, y_2, z_2, x_3, y_3, z_3) \,.
\end{aligned}
\right.
$$

However, since the last part of U is supposed to be free of x_1 and y_1, it is clear from (3) that U must have the form:

$$
(5)\quad \left\{
\begin{aligned}
U \; & = (z_1 - a \, x_1 - b \, y_1 - \omega) \, F_1 \, (x_1, y_1, z_1, x_2, y_2, z_2, x_3, y_3, z_3) \\
& + F \, (x_1, y_1, z_1, x_2, y_2, z_2, x_3, y_3, z_3).
\end{aligned}
\right.
$$

The **Lagrange** equations that belong to the coordinates x_1 and y_1 will then assume the form:

$$
\frac{d}{dt}\frac{\partial H}{\partial x_1'} = 0 \qquad \text{and} \qquad \frac{d}{dt}\frac{\partial H}{\partial y_1'} = 0 \,,
$$

or as is easy to see:

$$
\begin{aligned}
(1+a^2) \, x_1' + a \, b \, y_1' &= c_1 - a \frac{d\omega}{dt} \,, \\
a \, b \, x_1' + (1+b^2) \, y_1' &= c_2 - b \frac{d\omega}{dt} \,,
\end{aligned}
$$

when c_1 and c_2 mean integration constants, which will then yield:

$$
(6)\quad \left\{
\begin{aligned}
(1+a^2+b^2) \, x_1' &= \quad c_1 (1+b^2) - c_2 \, a \, b - a \frac{d\omega}{dt} , \\
(1+a^2+b^2) \, y_1' &= - c_1 \, a \, b + c_2 (1+b^2) - b \frac{d\omega}{dt} .
\end{aligned}
\right.
$$

If one substitutes the values of x_1' and y_1' that follow from that in the other six equations of motion, in which all of the external forces are assumed to be zero, then with the previously-emphasized meaning of the expressions in parentheses, e.g., the first one of them will give:

$$\left(\frac{\partial H}{\partial x_2}\right) - \frac{d}{dt}\left(\frac{\partial H}{\partial x_2'}\right) = 0 \,,$$

or since one has:

$$\frac{\partial (H)}{\partial x_2} = \left(\frac{\partial H}{\partial x_2}\right) + \left(\frac{\partial H}{\partial x_1'}\right)\frac{\partial x_1'}{\partial x_2} + \left(\frac{\partial H}{\partial y_1'}\right)\frac{\partial y_1'}{\partial x_2} = \left(\frac{\partial H}{\partial x_2}\right) + c_1\, m_1 \frac{\partial x_1'}{\partial x_2} + c_2\, m_1 \frac{\partial y_1'}{\partial x_2} \,,$$

$$\frac{\partial (H)}{\partial x_2'} = \left(\frac{\partial H}{\partial x_2'}\right) + \left(\frac{\partial H}{\partial x_1'}\right)\frac{\partial x_1'}{\partial x_2'} + \left(\frac{\partial H}{\partial y_1'}\right)\frac{\partial y_1'}{\partial x_2'} = \left(\frac{\partial H}{\partial x_2'}\right) + c_1\, m_1 \frac{\partial x_1'}{\partial x_2'} + c_2\, m_1 \frac{\partial y_1'}{\partial x_2'} \,,$$

when one sets:

(7) $$\mathfrak{H} = (H) - c_1\, m_1\, x_1' - c_2\, m_1\, y_1' \,,$$

one will again get the **Lagrange** form:

(8) $$\frac{\partial \mathfrak{H}}{\partial x_2} - \frac{d}{dt}\frac{\partial \mathfrak{H}}{\partial x_2'} = 0 \,.$$

The other five equations of motion for that same kinetic potential $\mathfrak{H}$ will be similar.

When we substitute the values of x_1' and y_1' from (6) in (7), an easy calculation will give the following simple form to the kinetic potential that now arises:

(9) $$\left\{\begin{aligned} \mathfrak{H} = \; & -\frac{m_1}{2(1+a^2+b^2)}\left(\frac{d\omega}{dt}\right)^2 + m_1 \frac{a\,c_1 + b\,c_2}{1+a^2+b^2}\frac{d\omega}{dt} \\ & -\frac{3}{2} m_1 \frac{c_1^2(1+b^2) - 2\,a\,b\,c_1\,c_2 + c_2^2(1+a^2)}{1+a^2+b^2} - \tfrac{1}{2} m_2 (x_2'^2 + y_2'^2 + z_2'^2) - \tfrac{1}{2} m_3 (x_3'^2 + y_3'^2 + z_3'^2) \\ & -F(\omega, x_2, y_2, z_2, x_3, y_3, z_3) \,, \end{aligned}\right.$$

and we will then find that:

The necessary and sufficient condition for the motion of three material points whose coordinates are subject to ***one*** *constraint equation to go from eight equations of motion to two equations in complete differential quotients with respect to time, or equivalently, that the kinetic potential must be independent of two of the eight coordinates, is that the constraint equation must have the form:*

$$z_1 = a\, x_1 + b\, y_1 + \omega \,,$$

in which a and b mean constants, and ω depends upon only x_2, y_2, z_2, x_3, y_3, z_3, *while the force function possesses the form:*

$$U = (z_1 - a\,x_1 - b\,y_1 - \omega)\,F_1\,(x_1, y_1, z_1, x_2, y_2, z_2, x_2, y_3, z_3) + F_2\,(z_1 - a\,x_1 - b\,y_1, x_2, y_2, z_2, x_2, y_3, z_3).$$

In that case, the six equations of motion for the coordinates x_2, y_2, z_2, x_2, y_3, z_3 *will once more assume the* ***Lagrangian*** *form for the kinetic potential* $\mathfrak{H}$ *that is given by equation* (9).

When one assumes that $F_1 = 0$, the form for the force function that was found above will demand that since:

$$X_1 = -\frac{\partial F}{\partial(z_1 - a\,x_1 - b\,y_1)}\,a\,, \quad Y_1 = -\frac{\partial F}{\partial(z_1 - a\,x_1 - b\,y_1)}\,b\,, \quad Z_1 = -\frac{\partial F}{\partial(z_1 - a\,x_1 - b\,y_1)}\,c\,,$$

the direction of the force that acts upon the point m_1 must be constant.

We would now like to raise the question of the necessary and sufficient conditions for the kinetic potential to have the form:

$$\mathfrak{H} = -\tfrac{1}{2}\,m_2\,(x_2'^2 + y_2'^2 + z_2'^2) - \tfrac{1}{2}\,m_3\,(x_3'^2 + y_3'^2 + z_3'^2) - W(r, r')\,, \tag{10}$$

in which W is an arbitrarily-given function of r and r', and:

$$r^2 = (x_2 - x_3)^2 + (y_2 - y_3)^2 + (z_2 - z_3)^2\,.$$

One next infers from the values (9) that $W(r, r')$ can only be an entire function of degree two in r' with the form:

$$W = \varphi_0(r) + \varphi_1(r)\frac{dr}{dt} + \varphi_2(r)\left(\frac{dr}{dt}\right)^2, \tag{11}$$

and therefore:

$$\varphi_2(r)\left(\frac{dr}{dt}\right)^2 = \frac{m_1}{2\,(1 + a^2 + b^2)}\left(\frac{d\omega}{dt}\right)^2 \tag{12}$$

and

$$\varphi_1(r)\left(\frac{dr}{dt}\right)^2 = -\,m_1\,\frac{a\,c_1 + b\,c_2}{1 + a^2 + b^2}\,\frac{d\omega}{dt}\,. \tag{13}$$

However, it will now follow from equation (12) that:

$$\omega = \sqrt{\frac{2\,(1 + a^2 + b^2)}{m_1}}\int\sqrt{\varphi_2(r)}\,dr\,,$$

such that for an arbitrary choice of $\varphi_2(r)$, one can determine $\varphi_1(r)$ from (13) as:

$$\varphi_1(r) = -(a\,c_1 + b\,c_2)\sqrt{\frac{2m_1}{1+a^2+b^2}}\sqrt{\varphi_2(r)}\,,$$

while $\varphi_0(r)$ is characterized by the expression:

$$\varphi_0(r) = \frac{3}{2}\,m_1\,\frac{c_1^{\,2}(1+b^2) - 2a\,b\,c_1\,c_2 + c_2^{\,2}(1+a^2)}{1+a^2+b^2} + F(\omega, x_2, y_2, z_3, x_3, y_3, z_3)\ .$$

From that, one can get:

$$F\left(\sqrt{\frac{2(1+a^2+b^2)}{m_1}}\int\sqrt{\varphi_2(r)}\,dr, x_2, y_2, z_3, x_3, y_3, z_3\right)$$

as a function of r, and therefore, from (5), one can get the force function in the form:

$$U = -\left(z_1 - a\,x_1 - b\,y_1 - \sqrt{\frac{2(1+a^2+b^2)}{m_1}}\int\sqrt{\varphi_2(r)}\,dr\right)F_1(x_1, y_1, z_1, x_2, y_2, z_2, x_3, y_3, z_3)$$
$$+ F(z_1 - a\,x_1 - b\,y_1, r)\,,$$

while the coordinates are subject to the condition:

$$z_1 = a\,x_1 + b\,y_1 + \sqrt{\frac{2(1+a^2+b^2)}{m_1}}\int\sqrt{\varphi_2(r)}\,dr\,.$$

We then find that:

The necessary and sufficient condition for the motion of three material points m_1, m_2, m_3 whose coordinates are subject to ***one*** *condition to go from the eight equations of motion to two equations in complete differential quotients with respect to time, while the remaining six again assume the* ***Lagrangian*** *form after eliminating the coordinates of one point as a result of that, and for the kinetic potential of the latter equations to further be composed of the negative sum of the vis viva of the two other points and a function of the distance r between them and its derivatives with respect to time is that the condition that exists between the coordinates of the three points must have the form:*

$$z_1 = a\,x_1 + b\,y_1 + \sqrt{\frac{2(1+a^2+b^2)}{m_1}}\int\sqrt{\varphi_2(r)}\,dr\,,$$

in which a and b mean arbitrary constants, and φ_2 *means an arbitrary function, and that the force function must be defined by an expression of the form:*

$$U = -\left(z_1 - a\,x_1 - b\,y_1 - \sqrt{\frac{2(1+a^2+b^2)}{m_1}} \int \sqrt{\varphi_2(r)}\;dr \right) F_1\,(x_1, y_1, z_1, x_2, y_2, z_2, x_3, y_3, z_3)$$
$$+ F\,(z_1 - a\,x_1 - b\,y_1, r)\,,$$

moreover, in which F_1 *and* F *are also arbitrary. The kinetic potential for the motion of the two points will then read:*

$$\begin{aligned} \mathfrak{H} = &- \tfrac{1}{2} m_2\,(x_2'^2 + y_2'^2 + z_2'^2) - \tfrac{1}{2} m_3\,(x_3'^2 + y_3'^2 + z_3'^2) - \varphi_2(r)\,r'^2 \\ &- (a\,c_1 + b\,c_1)\sqrt{\frac{2m_1}{1+a^2+b^2}}\sqrt{\varphi_2(r)} \cdot r' - \tfrac{3}{2} m_1 \frac{c_1^2\,(1+b^2) - 2\,a\,b\,c_1\,c_2 + c_2^2\,(1+a^2)}{1+a^2+b^2} \\ &- F\left(\sqrt{\frac{2(1+a^2+b^2)}{m_1}} \int \sqrt{\varphi_2(r)}\;dr \right), \end{aligned}$$

and from (6), *the coordinates* x_1, y_1, z_1 *will be given by the expressions:*

$$\begin{aligned} x_1 &= \frac{c_1^2\,(1+b^2) - c_2\,a\,b}{1+a^2+b^2}\,t - a\sqrt{\frac{2}{m_1(1+a^2+b^2)}} \int \sqrt{\varphi_2(r)}\,dr\,, \\ y_1 &= \frac{-c_1^2\,(1+b^2) - c_2\,a\,b}{1+a^2+b^2}\,t - b\sqrt{\frac{2}{m_1(1+a^2+b^2)}} \int \sqrt{\varphi_2(r)}\,dr\,, \\ z_1 &= \frac{a\,c_1 + c\,b_2}{1+a^2+b^2}\,t + \sqrt{\frac{2}{m_1(1+a^2+b^2)}} \int \sqrt{\varphi_2(r)}\,dr\,. \end{aligned}$$

If we choose the constants a, b, c_1, c_2 such that:

$$a\,c_1 + b\,c_2 = 0\,,$$

and further choose:

$$\varphi_2\,(r) = \frac{m_2\,m_3}{k^2}\frac{1}{r}\,, \qquad \text{so} \qquad \int \sqrt{\varphi_2(r)}\,dr = 2\sqrt{\frac{m_2\,m_3}{k^2}}\,r^{1/2}\,,$$

along with:

$$U = F\left(\frac{2}{k}\sqrt{\frac{2\,m_2\,m_3(1+a^2+b^2)}{m_1}}\,r^{1/2}, r \right) = \frac{m_2\,m_3}{r} - \tfrac{3}{2} m_1\,(c_1^2 + c_2^2)\,,$$

then the expression:

$$\mathfrak{H} = -\tfrac{1}{2} m_2\,(x_2'^2 + y_2'^2 + z_2'^2) - \tfrac{1}{2} m_3\,(x_3'^2 + y_3'^2 + z_3'^2) - \frac{m_2\,m_3}{r}\left(1 + \frac{r'^2}{k^2}\right)$$

will yield the kinetic potential of **Weber**'s law, while the coordinates x_1, y_1, z_1 are given by:

$$x_1 = c_1 t - \frac{2a}{k}\sqrt{\frac{2m_2\,m_3}{m_1(1+a^2+b^2)}}\,r^{1/2},$$
$$y_1 = c_2 t - \frac{2b}{k}\sqrt{\frac{2m_2\,m_3}{m_1(1+a^2+b^2)}}\,r^{1/2},$$
$$z_1 = \frac{2}{k}\sqrt{\frac{2m_2\,m_3}{m_1(1+a^2+b^2)}}\,r^{1/2},$$

in which one can also choose $a = 0$, $b = 0$ such that one will have:

$$x_1 = c_1\,t\,, \qquad y_1 = c_2\,t\,, \qquad z_1 = \frac{2}{k}\sqrt{\frac{2\,m_2\,m_3}{m_1}}\,r^{1/2}.$$

Therefore, if one of three mass-points is coupled with the other two by the condition that its distance from a fixed plane is always proportional to the square root of the distance between the other two mass-points, while the latter are attracted according to **Newton***'s law, then its motion will take place according to* **Weber***'s law.*

Finally, if we consider the case in which two constraint equations exist between the coordinates the three mass-points, which might take the form:

$$y_1 = f\,(x_1, y_1, z_1, x_2, y_2, z_2, x_2, y_3, z_3)\,, \quad z_1 = \varphi\,(x_1, y_1, z_1, x_2, y_2, z_2, x_2, y_3, z_3)\,,$$

then the kinetic potential will go to:

$$(14)\qquad \left\{\begin{aligned} H = \; & -\tfrac{1}{2}m_1\left(1+\left(\frac{\partial f}{\partial x_1}\right)^2+\left(\frac{\partial \varphi}{\partial x_1}\right)^2\right)x_1'^{\,2} \\ & -\tfrac{1}{2}\left(m_2+m_1\left(\frac{\partial f}{\partial x_1}\right)^2+m_1\left(\frac{\partial \varphi}{\partial x_1}\right)^2\right)x_2'^{\,2} \\ & -\tfrac{1}{2}\left(m_2+m_1\left(\frac{\partial f}{\partial x_1}\right)^2+m_1\left(\frac{\partial \varphi}{\partial x_1}\right)^2\right)x_2'^{\,2} - \cdots \\ & -m_1\left(\frac{\partial f}{\partial x_1}\frac{\partial f}{\partial x_2}+\frac{\partial \varphi}{\partial x_1}\frac{\partial \varphi}{\partial x_2}\right)x_1'\,x_2' \\ & -m_1\left(\frac{\partial f}{\partial x_1}\frac{\partial f}{\partial y_2}+\frac{\partial \varphi}{\partial x_1}\frac{\partial \varphi}{\partial y_2}\right)x_1'\,y_2' + \cdots \\ & -U(x_1, f, \varphi, x_2, y_2, z_2, x_3, y_3, z_3). \end{aligned}\right.$$

If that should be independent of x_1 then that would also have to be true for the coefficients of the derivatives of the coordinates, and therefore the expressions:

$$\left(\frac{\partial f}{\partial x_1}\right)^2+\left(\frac{\partial \varphi}{\partial x_1}\right)^2, \qquad \left(\frac{\partial f}{\partial x_2}\right)^2+\left(\frac{\partial \varphi}{\partial x_2}\right)^2, \qquad \left(\frac{\partial f}{\partial y_2}\right)^2+\left(\frac{\partial \varphi}{\partial y_2}\right)^2, \qquad \ldots,$$

$$\frac{\partial f}{\partial x_1}\frac{\partial f}{\partial x_2}+\frac{\partial \varphi}{\partial x_1}\frac{\partial \varphi}{\partial x_2}, \qquad \frac{\partial f}{\partial x_1}\frac{\partial f}{\partial y_2}+\frac{\partial \varphi}{\partial x_1}\frac{\partial \varphi}{\partial y_2}, \qquad \ldots$$

would be functions of only x_2, y_2, z_2, x_2, y_3, z_3 .

If one sets:

$$(15) \qquad \left(\frac{\partial f}{\partial x_1}\right)^2+\left(\frac{\partial \varphi}{\partial x_1}\right)^2=\omega_1, \qquad \left(\frac{\partial f}{\partial x_2}\right)^2+\left(\frac{\partial \varphi}{\partial x_2}\right)^2=\omega_2, \qquad \frac{\partial f}{\partial x_1}\frac{\partial f}{\partial x_2}+\frac{\partial \varphi}{\partial x_1}\frac{\partial \varphi}{\partial x_2}=\Omega,$$

in which ω_1, ω_2, Ω mean the coordinates of the second and third mass-points, then by simply combining them, that will imply that the function φ must satisfy the partial differential equation:

$$(16) \qquad \omega_2\left(\frac{\partial \varphi}{\partial x_1}\right)^2+\omega_1\left(\frac{\partial \varphi}{\partial x_1}\right)^2-2\Omega\frac{\partial \varphi}{\partial x_1}\frac{\partial \varphi}{\partial x_2}=\omega_1\,\omega_2-\Omega^2,$$

while f and φ are coupled by the relations:

$$(17) \qquad \begin{cases} \sqrt{\omega_1\,\omega_2-\Omega^2}\,\dfrac{\partial f}{\partial x_1}=-\Omega\,\dfrac{\partial \varphi}{\partial x_1}+\omega_1\,\dfrac{\partial \varphi}{\partial x_1}, \\ \sqrt{\omega_1\,\omega_2-\Omega^2}\,\dfrac{\partial f}{\partial x_2}=-\omega_2\,\dfrac{\partial \varphi}{\partial x_1}+\Omega\,\dfrac{\partial \varphi}{\partial x_1}, \end{cases}$$

and, in turn, upon differentiating the first of those equations by x_2 and the second one by x_1, one will get a further second-order partial differential equation for φ :

$$(18) \qquad \begin{cases} \omega_2\,\dfrac{\partial^2\varphi}{\partial x_1^2}+\omega_1\,\dfrac{\partial^2\varphi}{\partial x_2^2}-2\Omega\,\dfrac{\partial^2\varphi}{\partial x_1\,\partial x_2} \\ \qquad +\dfrac{\partial \varphi}{\partial x_1}\left[\dfrac{1}{2}\dfrac{\Omega}{\omega_1\,\omega_2-\Omega^2}\dfrac{\partial}{\partial x_2}(\omega_1\,\omega_2-\Omega^2)-\dfrac{\partial \varphi}{\partial x_2}\right] \\ \qquad +\dfrac{\partial \varphi}{\partial x_2}\left[\dfrac{\partial \omega_1}{\partial x_2}-\dfrac{1}{2}\dfrac{\omega_1}{\omega_1\,\omega_2-\Omega^2}\dfrac{\partial}{\partial x_2}(\omega_1\,\omega_2-\Omega^2)\right]=0. \end{cases}$$

If one now partially differentiates equation (16) with respect to x_1 and x_2, and then the first equation thus-obtained with respect to x_1 and x_2 and the second one with respect to x_2 then one will get five equations in the differential quotients that have orders two and three in the function φ. If one differentiates equation (18) with respect to x_1 and x_2, along with those three equations in the same quantities, then that will give eight equations (which one easily sees to be independent of each other) in the seven quantities:

$$\frac{\partial^2\varphi}{\partial x_1^2}, \quad \frac{\partial^2\varphi}{\partial x_1\,\partial x_2}, \quad \frac{\partial^2\varphi}{\partial x_2^2}, \quad \frac{\partial^3\varphi}{\partial x_1^3}, \quad \frac{\partial^3\varphi}{\partial x_1^2\,\partial x_2}, \quad \frac{\partial^3\varphi}{\partial x_1\,\partial x_2^2}, \quad \frac{\partial^3\varphi}{\partial x_2^3}.$$

When one eliminates them, that will yield a relation of the form:

$$F_1\left(x_2, \frac{\partial\varphi}{\partial x_1}, \frac{\partial\varphi}{\partial x_2}\right) = 0, \tag{19}$$

in which the variable x_1 does not occur. The two equations (16) and (19) now demand that $\frac{\partial\varphi}{\partial x_1}$ and $\frac{\partial\varphi}{\partial x_2}$ must be independent of x_1, and therefore since the same thing must be true for the other variables, one will have:

$$\varphi = b\,x_1 + \Phi\,(x_2, y_2, z_2, x_3, y_3, z_3), \tag{20}$$

in which b means a constant. Therefore, from (17), $\frac{\partial f}{\partial x_1}$ and $\frac{\partial f}{\partial x_2}$ must also be independent of x_1, and f can be represented in the form:

$$f = a\,x_1 + F\,(x_2, y_2, z_2, x_3, y_3, z_3), \tag{21}$$

in which a, in turn, represents a constant. As one can see immediately, the forms for f and φ that were just found are not only the necessary, but also sufficient, conditions for the kinetic potential H to be independent of x_1 when the force function is subject to a condition of the form:

$$\left\{\begin{aligned} U = \quad & (y_1 - a\,x_1 - F)\,\chi_1\,(x_1, y_1, z_1, x_2, y_2, z_2, x_3, y_3, z_3) \\ & + (z_1 - b\,x_1 - \Phi)\,\chi_2\,(x_1, y_1, z_1, x_2, y_2, z_2, x_3, y_3, z_3) \\ & + \chi\,(y_1 - a\,x_1, z_1 - b\,x_1, x_2, y_2, z_2, x_3, y_3, z_3). \end{aligned}\right. \tag{22}$$

Now, since the form of the kinetic potential:

$$
(23)\qquad \left\{
\begin{aligned}
H = \; & -\tfrac{1}{2} m_1 (1 + a^2 + b^2)\, x_1'^{\,2} \\
& -\tfrac{1}{2}\left(m_2 + m_1 \left(\frac{\partial F}{\partial x_2}\right)^2 + m_1 \left(\frac{\partial \Phi}{\partial x_2}\right)^2 \right) x_2'^{\,2} \\
& -\tfrac{1}{2}\left(m_2 + m_1 \left(\frac{\partial F}{\partial y_2}\right)^2 + m_1 \left(\frac{\partial \Phi}{\partial y_2}\right)^2 \right) y_2'^{\,2} + \cdots \\
& - m_1 \left(a\,\frac{\partial F}{\partial x_2} + b\,\frac{\partial \Phi}{\partial x_2} \right) x_1'\, x_2' \\
& - m_1 \left(a\,\frac{\partial F}{\partial y_2} + b\,\frac{\partial \Phi}{\partial y_2} \right) x_1'\, y_2' - \cdots \\
& - m_1 \left(\frac{\partial F}{\partial x_2}\,\frac{\partial F}{\partial y_2} + \frac{\partial \Phi}{\partial x_2}\,\frac{\partial \Phi}{\partial y_2} \right) x_2'\, y_2' - \cdots \\
& - \chi\,(F, \Phi, x_2, y_2, z_2, x_3, y_2, z_3)
\end{aligned}
\right.
$$

implies that the **Lagrange** equation that belongs to x_1 :

$$\frac{\partial H}{\partial x_1'} = c\, m_1 ,$$

in which c means an integration constant, will assume the form:

$$(24)\qquad (1 + a^2 + b^2)\, x_1' = -\, c - a\,\frac{dF}{dt} - b\,\frac{d\Phi}{dt} ,$$

upon substituting the value of x_1' in the six following equations, e.g., the first of them will give:

$$\left(\frac{\partial H}{\partial x_2}\right) - \frac{d}{dt}\left(\frac{\partial H}{\partial x_2'}\right) = 0 ,$$

when all external forces are assumed to be equal to zero. Since:

$$
\begin{aligned}
\frac{\partial (H)}{\partial x_2} &= \left(\frac{\partial H}{\partial x_2}\right) + \left(\frac{\partial H}{\partial x_1'}\right)\frac{\partial x_1'}{\partial x_2} = \left(\frac{\partial H}{\partial x_2}\right) + c\, m_1\,\frac{\partial x_1'}{\partial x_2} , \\
\frac{\partial (H)}{\partial x_2'} &= \left(\frac{\partial H}{\partial x_2'}\right) + \left(\frac{\partial H}{\partial x_1'}\right)\frac{\partial x_1'}{\partial x_2'} = \left(\frac{\partial H}{\partial x_2'}\right) + c\, m_1\,\frac{\partial x_1'}{\partial x_2'} ,
\end{aligned}
$$

when one sets:

(25) $$\mathfrak{H} = (H) - c\, m_1\, x_1' ,$$

the **Lagrange** equation will become:

$$\frac{\partial \mathfrak{H}}{\partial x_2} - \frac{d}{dt}\frac{\partial \mathfrak{H}}{\partial x_2'} = 0 ,$$

and the same thing will be true for the other five equations of motion for the same kinetic potential $\mathfrak{H}$.

If one now substitutes the value of x_1' from equation (24) in (25) then that will give the kinetic potential in the form:

(26) $$\begin{cases} \mathfrak{H} = & -\tfrac{1}{2} m_1 \dfrac{1+b^2}{1+a^2+b^2}\left(\dfrac{dF}{dt}\right)^2 - \tfrac{1}{2} m_1 \dfrac{1+a^2}{1+a^2+b^2}\left(\dfrac{d\Phi}{dt}\right)^2 + m_1 \dfrac{1+a^2}{1+a^2+b^2}\dfrac{dF}{dt}\dfrac{d\Phi}{dt} \\ & + m_1 \dfrac{1+a^2}{1+a^2+b^2}\dfrac{dF}{dt} + m_1 \dfrac{1+b^2}{1+a^2+b^2}\dfrac{d\Phi}{dt} + m_1 \dfrac{c^2}{1+a^2+b^2} \\ & -\tfrac{1}{2} m_2 (x_2'^2 + y_2'^2 + z_2'^2) - \tfrac{1}{2} m_3 (x_3'^2 + y_3'^2 + z_3'^2) \\ & -\chi (F, \Phi, x_2, y_2, z_2, x_3, y_3, z_3), \end{cases}$$

and one will then find that:

The necessary and sufficient condition for the motion of three material points whose coordinates are subject to two constraint equations to go from seven equations of motion to one equation in complete differential quotients with respect to time, or equivalently, for the kinetic potential to be independent of one of the seven coordinates, is that the constraint equations must have the form:

$$y_1 = a\, x_1 + F , \qquad z_1 = b\, x_1 + \Phi ,$$

in which a and b are constants, and F, as well as Φ, must depend upon only:

$$x_2 , y_2 , z_2 , x_3 , y_3 , z_3 ,$$

and the force function must possess the form:

$$\begin{aligned} U = {} & (y_1 - a\, x_1 - F)\, \chi_1 (x_1 , y_1 , z_1 , x_2 , y_2 , z_2 , x_3 , y_3 , z_3) \\ & + (z_1 - b\, x_1 - \Phi)\, \chi_2 (x_1 , y_1 , z_1 , x_2 , y_2 , z_2 , x_3 , y_3 , z_3) \\ & + \chi (y_1 - a\, x_1 , z_1 - b\, x_1 , x_2 , y_2 , z_2 , x_3 , y_3 , z_3) . \end{aligned}$$

*In that case, the six equations of motion for the coordinates $x_2 , y_2 , z_2 , x_3 , y_3 , z_3$ will once more assume the **Lagrang**ian form for the kinetic potential $\mathfrak{H}$ that is given by equation* (26).

The form for the force function that was found above demands that the force that acts upon the first point must satisfy the relation:

$$X_1 = - a\, Y_1 - b\, Z_1 ,$$

when $\chi_1 = 0$, $\chi_2 = 0$.

Should the kinetic potential, in turn, have the form (10), so W is determined by the expression (11), then one would need to have:

$$\varphi_2(r)\left(\frac{dr}{dt}\right)^2 = \tfrac{1}{2} m_1 \frac{1+b^2}{1+a^2+b^2}\left(\frac{dF}{dt}\right)^2 + \tfrac{1}{2} m_1 \frac{1+a^2}{1+a^2+b^2}\left(\frac{d\Phi}{dt}\right)^2 - m_1 \frac{a\,b}{1+a^2+b^2}\frac{dF}{dt}\frac{d\Phi}{dt},$$

$$\varphi_1(r)\frac{dr}{dt} = - m_1 \frac{a\,c}{1+a^2+b^2}\frac{dF}{dt} - m_1 \frac{b\,c}{1+a^2+b^2}\frac{d\Phi}{dt},$$

$$\varphi_0(r) = - \tfrac{1}{2} m_1 \frac{c^2}{1+a^2+b^2} + \chi(F, \Phi, x_2, y_2, z_2, x_3, y_3, z_3),$$

from which it would easily follow that:

$$\frac{dF}{dt} = \frac{-a(1+a^2+b^2)\,\varphi_1(r) + b\sqrt{2m_1 c^2 (a^2+b^2)\,\varphi_2(r) - (1+a^2+b^2)\,\varphi_1(r)^2}}{m_1 c\,(a^2+b^2)} \cdot \frac{dr}{dt},$$

$$\frac{d\Phi}{dt} = \frac{-b(1+a^2+b^2)\,\varphi_1(r) - a\sqrt{2m_1 c^2 (a^2+b^2)\,\varphi_2(r) - (1+a^2+b^2)\,\varphi_1(r)^2}}{m_1 c\,(a^2+b^2)} \cdot \frac{dr}{dt},$$

and therefore:

(27)
$$F = \int \frac{-a(1+a^2+b^2)\,\varphi_1(r) + b\sqrt{2m_1 c^2 (a^2+b^2)\,\varphi_2(r) - (1+a^2+b^2)\,\varphi_1(r)^2}}{m_1 c\,(a^2+b^2)}\, dr$$

and

(28)
$$\Phi = \int \frac{-b(1+a^2+b^2)\,\varphi_1(r) - a\sqrt{2m_1 c^2 (a^2+b^2)\,\varphi_2(r) - (1+a^2+b^2)\,\varphi_1(r)^2}}{m_1 c\,(a^2+b^2)}\, dr$$

are functions of only r. In addition:

(29)
$$\varphi_0(r) = - \tfrac{1}{2} m_1 \frac{c^2}{1+a^2+b^2} + \chi(F(r), \Phi(r), x_2, y_2, z_2, x_3, y_3, z_3),$$

so

$$\text{(30)} \qquad \begin{cases} U = (y_1 - a\,x_1 - F(r))\,\chi_1(x_1, y_1, z_1, x_2, y_2, z_2, x_3, y_3, z_3) \\ \quad + (z_1 - b\,x_1 - \Phi(r))\,\chi_2(x_1, y_1, z_1, x_2, y_2, z_2, x_3, y_3, z_3) \\ \quad + \chi(y_1 - a\,x_1, z_1 - b\,x_1), \end{cases}$$

and the constraint equations between the coordinates will become:

$$\text{(31)} \qquad y_1 = a\,x_1 + F(r), \qquad z_1 = b\,x_1 + \Phi(r).$$

We then find that:

The necessary and sufficient condition for the motion of three material points m_1, m_2, m_3 *whose coordinates are subject to two conditions to go from seven equations of motion to one equation in complete differential quotients with respect to time, while the remaining six again assume the* ***Lagrange*** *form as a result of eliminating the coordinates of one point, and for the kinetic potential of the latter to be composed of the negative sum of the vis viva of the other two points and a function of the distance r between them and its derivatives with respect to time, moreover, is that the two constraint equations between the coordinates must have the form* (31), *in which* $F(r)$ *and* $\Phi(r)$ *are determined by the expressions* (27) *and* (28), *in which* $\varphi_1(r)$ and $\varphi_2(r)$ *mean arbitrary functions of r, and furthermore that the force function must be represented by an expression of the form* (30). *The kinetic potential for the motion of the two points will then read:*

$$\text{(32)} \qquad \begin{cases} \mathfrak{H} = -\tfrac{1}{2} m_2 (x_2'^2 + y_2'^2 + z_2'^2) - \tfrac{1}{2} m_3 (x_3'^2 + y_3'^2 + z_3'^2) \\ \quad - \varphi_2(r) \left(\dfrac{dr}{dt} \right)^2 - \varphi_1(r) \dfrac{dr}{dt} + \tfrac{1}{2} m_1 \dfrac{c^2}{1 + a^2 + b^2} - \chi(F, \Phi, r), \end{cases}$$

and from (24), *the coordinates* x_1, y_1, z_1 *will be given by the expressions:*

$$x_1 = -\frac{c^2}{1 + a^2 + b^2}\,t + \frac{1}{m_1 c} \int \varphi_1(r)\,dr,$$

$$y_1 = -\frac{c^2}{1 + a^2 + b^2}\,t + \int \frac{-a(1 + a^2 + b^2)\,\varphi_1(r) + b\sqrt{2 m_1 c^2 (a^2 + b^2)\,\varphi_2(r) - (1 + a^2 + b^2)\,\varphi_1(r)^2}}{m_1 c\,(a^2 + b^2)}\,dr,$$

$$z_1 = -\frac{b\,c}{1 + a^2 + b^2}\,t + \int \frac{-b(1 + a^2 + b^2)\,\varphi_1(r) - a\sqrt{2 m_1 c^2 (a^2 + b^2)\,\varphi_2(r) - (1 + a^2 + b^2)\,\varphi_1(r)^2}}{m_1 c\,(a^2 + b^2)}\,dr.$$

If one would like to be led to **Weber**'s law then one must, in turn, set:

$$\varphi_1(r) = 0\,, \qquad \varphi_2(r) = \frac{m_2\,m_3}{k^2}\,\frac{1}{r}\,,$$

$$U = \chi(F, \Phi, r) = \frac{m_2\,m_3}{r} + \tfrac{1}{2}\,m_1\,\frac{c^2}{1+a^2+b^2}\,,$$

for which the kinetic potential (32) will assume the form:

$$\mathfrak{H} = \tfrac{1}{2}\,m_2\,(x_2'^2 + y_2'^2 + z_2'^2) - \tfrac{1}{2}\,m_3\,(x_3'^2 + y_3'^2 + z_3'^2) - \frac{m_2\,m_3}{r}\left(1 + \frac{r'^2}{k^2}\right),$$

while from (27) and (28) the constraint equations between the coordinates will go to:

$$y_1 = a\,x_1 + \frac{2b}{k}\sqrt{\frac{2\,m_2\,m_3}{m_1(a^2+b^2)}}\;r^{1/2}\,,$$

$$z_1 = b\,x_1 - \frac{2a}{k}\sqrt{\frac{2\,m_2\,m_3}{m_1(a^2+b^2)}}\;r^{1/2}\,,$$

and the coordinates x_1, y_1, z_1 will be given by the expressions:

$$x_1 = -\,\frac{c}{1+a^2+b^2}\,t\,,$$

$$y_1 = -\,\frac{a\,c}{1+a^2+b^2}\,t + \frac{2b}{k}\sqrt{\frac{2\,m_2\,m_3}{m_1(a^2+b^2)}}\;r^{1/2}\,,$$

$$z_1 = -\,\frac{b\,c}{1+a^2+b^2}\,t - \frac{2a}{k}\sqrt{\frac{2\,m_2\,m_3}{m_1(a^2+b^2)}}\;r^{1/2}\,.$$

If one takes $b = a$ then the constraint equations will go to:

$$y_1 = a\,x_1 + \frac{2}{k}\sqrt{\frac{2\,m_2\,m_3}{m_1(a^2+b^2)}}\;r^{1/2}\,,$$

$$z_1 = a\,x_1 - \frac{2}{k}\sqrt{\frac{2\,m_2\,m_3}{m_1(a^2+b^2)}}\;r^{1/2}\,,$$

and if one finally sets $a = 0$ then it will follow that:

If one of the three mass-points is subject to the condition that its distance from two mutually-perpendicular planes always remains proportional to the square root of the distance between the *other two mass-points, but with opposite signs on the factor, while the latter are attracted according to* **Newton***'s law, then its motion will take place according to* **Weber***'s law.*

The force function will then have the form:

$$U = \frac{m_2\, m_3}{r} + \tfrac{1}{2} m_1\, c^2 ,$$

in which c means the x-component of the initial velocity of the first point.

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§ 15. – Extension of Helmholtz's principle of hidden motion for the general first-order kinetic potential

The idea that led **Helmholtz** to introduce his principle into the mechanics of ponderable masses was that for physical processes that could be described by **Lagrange**'s equations when the kinetic potential is not clearly resolvable into current and potential energy and possesses not only terms that are quadratic in the derivatives of the coordinates, but also ones that are linear in them, one might introduce a large number of ponderable material points that belong to a kinetic potential in the conventional sense and whose **Lagrange** equations would lead back to the **Lagrange** equations for the original physical process upon eliminating the coordinates of the newly-introduced points.

In what follows, we would now like to take up the problem of eliminating the coordinates between the **Lagrange** equations of motion in full generality for the extended **Lagrange** equations, as well, but for the sake of simplicity of representation, we shall first make the assumption that the kinetic potential H depends upon only the coordinates and their first derivatives, so it is one of first order, but other than that, it is an arbitrary function of those quantities that is free of time. The extension to the case in which the kinetic potential includes the derivatives of the coordinates to arbitrarily-high order will be immediately clear (*).

The problem that will be solved in what follows will then read:

*Under what conditions will the elimination of a number of coordinates in a system of **Lagrange** equations with a first-order kinetic potential lead, in turn, to **Lagrange** equations with a first-order potential?*

If we first look for the necessary and sufficient conditions for the left-hand sides of the first ρ **Lagrange** equations of motion to be representable as complete differential quotients with respect to time of a function of the coordinates and their derivatives, such that we will have:

$$\frac{\partial H}{\partial p_r} - \frac{d}{dt}\frac{\partial H}{\partial p'_r} = \frac{dK_r}{dt} \qquad \text{or} \qquad \frac{\partial H}{\partial p_r} = \frac{d}{dt}\left(K_r + \frac{\partial H}{\partial p'_r}\right),$$

then if we preserve the previously-used notations, we will have:

$$K_r + \frac{\partial H}{\partial p'_r} = \omega_r\,(p_1, \ldots, p_\rho, \mathfrak{p}_1, \ldots, \mathfrak{p}_s)\,.$$

Since $\partial H / \partial p_r$ does not include the second derivatives of the coordinates, it must be free of their first derivatives, and therefore $\partial H / \partial p_r$, as the total differential quotient of a function of just the coordinates, must be a linear function of their first derivatives in the form:

(*) How that same problem can be treated as a problem in the calculus of variations on the basis of **Hamilton**'s principle will be shown in a later investigation.

(1)
$$\frac{\partial H}{\partial p_r} = \frac{d\omega_r}{dt}.$$

When r_1 and r_2 mean two numbers from the sequence 1, 2, …, ρ, that will immediately imply that:

(2)
$$\frac{d}{dt}\frac{\partial \omega_{r_1}}{\partial p_{r_2}} = \frac{d}{dt}\frac{\partial \omega_{r_2}}{\partial p_{r_1}},$$

so

(3)
$$\frac{\partial \omega_{r_1}}{\partial p_{r_2}} = \frac{\partial \omega_{r_2}}{\partial p_{r_1}} + c_{r_1 r_2},$$

in which $c_{r_1 r_2}$ mean constants for which one has:

$$c_{r_1 r_2} = - c_{r_2 r_1}.$$

If one now multiplies equation (1) by dp_r and adds the equations thus-obtained for $r = 1, 2, \ldots, \rho$ then it will follow that:

$$\frac{\partial H}{\partial p_1} dp_1 + \frac{\partial H}{\partial p_2} dp_2 + \cdots + \frac{\partial H}{\partial p_\rho} dp_\rho$$

$$\begin{aligned} = \; & p_1'\left(\frac{\partial \omega_1}{\partial p_1} dp_1 + \cdots + \frac{\partial \omega_\rho}{\partial p_1} dp_\rho\right) + \cdots + p_\rho'\left(\frac{\partial \omega_1}{\partial p_\rho} dp_1 + \cdots + \frac{\partial \omega_\rho}{\partial p_\rho} dp_\rho\right) \\ + \; & \mathfrak{p}_1'\left(\frac{\partial \omega_1}{\partial \mathfrak{p}_1} dp_1 + \cdots + \frac{\partial \omega_\rho}{\partial \mathfrak{p}_1} dp_\rho\right) + \cdots + \mathfrak{p}_\rho'\left(\frac{\partial \omega_1}{\partial \mathfrak{p}_\rho} dp_1 + \cdots + \frac{\partial \omega_\rho}{\partial \mathfrak{p}_\rho} dp_\rho\right), \end{aligned}$$

or by means of (3), upon integration:

$$\begin{aligned} H = \; & p_1'(\omega_1 + c_{21}\, p_2 + \cdots + c_{\rho 1}\, p_\rho) + \cdots + p_\rho'(\omega_\rho + c_{1\rho}\, p_1 + \cdots + c_{\rho-1,\rho}\, p_{\rho-1}) \\ + \; & \mathfrak{p}_1' \int \left(\frac{\partial \omega_1}{\partial \mathfrak{p}_1} dp_1 + \cdots + \frac{\partial \omega_\rho}{\partial \mathfrak{p}_1} dp_\rho\right) + \cdots + \mathfrak{p}_\rho' \int \left(\frac{\partial \omega_1}{\partial \mathfrak{p}_\rho} dp_1 + \cdots + \frac{\partial \omega_\rho}{\partial \mathfrak{p}_\rho} dp_\rho\right) \\ + \; & \Omega(\mathfrak{p}_1, \ldots, \mathfrak{p}_\sigma, \mathfrak{p}_1', \ldots, \mathfrak{p}_\sigma', p_1', \ldots, p_\rho'), \end{aligned}$$

in which Ω means an arbitrary function of the included quantities, and the coefficients of $\mathfrak{p}_1'$, …, $\mathfrak{p}_\sigma'$ are represented as complete differential expressions in the variables p_1, …, p_r by means of equations (3). Finally, if one sets:

$$\omega_r - \tfrac{1}{2}\, c_{r1}\, p_1 - \cdots - \tfrac{1}{2}\, c_{r,r-1}\, p_{r-1} - \tfrac{1}{2} c_{r,r+1}\, p_{r+1} - \cdots - \tfrac{1}{2}\, c_{r\rho}\, p_\rho = \bar{\omega}_r$$

then $\bar{\omega}_r$ *will mean functions of* $p_1, \ldots, p_\rho, \mathfrak{p}_1, \ldots, \mathfrak{p}_s$ *that satisfy the condition:*

(4)
$$\frac{\partial \bar{\omega}_{r_1}}{\partial p_{r_2}} = \frac{\partial \bar{\omega}_{r_2}}{\partial p_{r_1}}$$

according to (3). *That will then imply that the necessary condition for the first* ρ *Lagrange equations to go to complete differential quotients with respect to time is that the kinetic potential H must take the form:*

$$H = p_1' \bar{\omega}_1 + \cdots + p_\rho' \bar{\omega}_\rho) + \mathfrak{p}_1' \int \left(\frac{\partial \bar{\omega}_1}{\partial \mathfrak{p}_1} dp_1 + \cdots + \frac{\partial \bar{\omega}_1}{\partial \mathfrak{p}_\rho} dp_\rho \right) + \cdots + \mathfrak{p}_\sigma' \int \left(\frac{\partial \bar{\omega}_1}{\partial \mathfrak{p}_\sigma} dp_1 + \cdots + \frac{\partial \bar{\omega}_\rho}{\partial \mathfrak{p}_\sigma} dp_\rho \right)$$
$$+ \tfrac{1}{2} \sum_{r=1}^{\rho} p_r' (c_{1r}\, p_1 + c_{2r}\, p_2 + \cdots + c_{\rho r}\, p_\rho) + \Omega(\mathfrak{p}_1, \ldots, \mathfrak{p}_\sigma, \mathfrak{p}_1', \ldots, \mathfrak{p}_\sigma', p_1', \ldots, p_\rho'),$$

or since Lemma 3 says that one can add the differential quotient of an arbitrary function of t, p_1, …, p_ρ, $\mathfrak{p}_1$, …, $\mathfrak{p}_s$ with respect to t to any first-order kinetic potential, and:

$$p_1' \bar{\omega}_1 + \cdots + p_\rho' \bar{\omega}_\rho) + \mathfrak{p}_1' \int \left(\frac{\partial \bar{\omega}_1}{\partial \mathfrak{p}_1} dp_1 + \cdots + \frac{\partial \bar{\omega}_1}{\partial \mathfrak{p}_\rho} dp_\rho \right) + \cdots + \mathfrak{p}_\sigma' \int \left(\frac{\partial \bar{\omega}_1}{\partial \mathfrak{p}_\sigma} dp_1 + \cdots + \frac{\partial \bar{\omega}_\rho}{\partial \mathfrak{p}_\sigma} dp_\rho \right)$$

is the differential quotient of:

$$\int (\bar{\omega}_1\, dp_1 + \bar{\omega}_2\, dp_2 + \cdots + \bar{\omega}_\rho\, dp_\rho)$$

with respect to t, *it must take the form:*

(5)
$$\sum_{r=1}^{\rho} p_r' (C_{r,r+1}\, p_{r+1} + C_{r,r+2}\, p_{r+2} + \cdots + C_{r\rho}\, p_\rho) + \Omega(\mathfrak{p}_1, \ldots, \mathfrak{p}_\sigma, \mathfrak{p}_1', \ldots, \mathfrak{p}_\sigma', p_1', \ldots, p_\rho'),$$

in which $C_{r,r+1}$, …, $C_{r\rho}$ *mean arbitrary constants. However, that form is also sufficient.*

That is because it follows immediately from this that in that form, the system of the first ρ **Lagrange** equations will be:

$$C_{1r}\, p_1' - C_{2r}\, p_2' - \cdots - C_{r-1,r}\, p_{r-1}' + C_{r,r+1}\, p_{r+1}' + C_{r,r+2}\, p_{r+2}' + \cdots + C_{r\rho}\, p_\rho' + \frac{d}{dt} \frac{\partial \Omega}{\partial p_r'} = -P_r,$$

or when integrated over t :

(6) $$C_{1r}\,p_1 - C_{2r}\,p_2 - \cdots - C_{r-1,r}\,p_{r-1} + C_{r,r+1}\,p_{r+1} + C_{r,r+2}\,p_{r+2} + \cdots + C_{r\rho}\,p_\rho + \frac{\partial\Omega}{\partial p'_r} = h_r - \int P_r\,dt\,,$$

in which h_r means an integration constant, and the external forces might be given functions of time.

Now, should the ρ coordinates $p_1, \ldots, p_\rho$, and their first derivatives be eliminated from the σ equations of motion:

(7) $$\frac{\partial H}{\partial \mathfrak{p}_\sigma} - \frac{d}{dt}\frac{\partial H}{\partial \mathfrak{p}'_\sigma} = \mathfrak{P}_s$$

with the help of (6) then two and only two assumptions are permissible:

a) If we assume that equations (6) are independent of the coordinates $p_1, \ldots, p_\rho$, and develop the values of $p'_1, \ldots, p'_\rho$ from those equations then since those ρ coordinates are not included in Ω, we must have:

$$C_{1r} = C_{2r} = \ldots = C_{r-1,r} = C_{r,r+1} = C_{r,r+2} = \ldots = C_{r\rho} = 0\,.$$

The values of $p'_1, \ldots, p'_\rho$ that are obtained from the equations:

(8) $$\frac{\partial\Omega}{\partial p'_r} = h_r - \int P_r\,dt\,,$$

whose determination assumes that the determinant of the partial differential quotients of Ω with respect to the derivatives of the coordinates $p_1, \ldots, p_\rho$ is non-zero, will then be represented by:

(9) $$\begin{cases} p'_1 = \Omega_1(t, \mathfrak{p}_1, \ldots, \mathfrak{p}_\sigma, \mathfrak{p}'_1, \ldots, \mathfrak{p}'_\sigma)\,, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ p'_\rho = \Omega_\rho(t, \mathfrak{p}_1, \ldots, \mathfrak{p}_\sigma, \mathfrak{p}'_1, \ldots, \mathfrak{p}'_\sigma)\,, \end{cases}$$

into which t does not enter explicitly when the external forces $P_1, \ldots, P_\rho$ are zero. If one now forms the system of σ **Lagrange** equations (7) from (5) then that will give them in the form:

$$\frac{\partial\Omega}{\partial \mathfrak{p}_s} - \frac{d}{dt}\frac{\partial\Omega}{\partial \mathfrak{p}'_s} = \mathfrak{P}_s\,,$$

in which Ω is a function of $\mathfrak{p}_1, \ldots, \mathfrak{p}_\sigma$, $\mathfrak{p}'_1, \ldots, \mathfrak{p}'_\sigma$, $p'_1, \ldots, p'_\rho$, or after substituting the values (9), in the form:

(10) $$\left(\frac{\partial \Omega}{\partial \mathfrak{p}_s}\right) - \frac{d}{dt}\left(\frac{\partial \Omega}{\partial \mathfrak{p}'_s}\right) = \mathfrak{P}_s \,,$$

with the notation that we have always used.

However, since (8) implies that:

$$\frac{\partial (\Omega)}{\partial \mathfrak{p}_s} = \left(\frac{\partial \Omega}{\partial \mathfrak{p}_s}\right) + \left(h_1 - \int P_1\, dt\right)\frac{\partial \Omega_1}{\partial \mathfrak{p}_s} + \cdots + \left(h_\rho - \int P_\rho\, dt\right)\frac{\partial \Omega_\rho}{\partial \mathfrak{p}_s} \,,$$
$$\frac{\partial (\Omega)}{\partial \mathfrak{p}'_s} = \left(\frac{\partial \Omega}{\partial \mathfrak{p}'_s}\right) + \left(h_1 - \int P_1\, dt\right)\frac{\partial \Omega_1}{\partial \mathfrak{p}'_s} + \cdots + \left(h_\rho - \int P_\rho\, dt\right)\frac{\partial \Omega_\rho}{\partial \mathfrak{p}'_s} \,,$$

when one sets:

$$\mathfrak{H} = (\Omega) + \left(h_1 - \int P_1\, dt\right)\Omega_1 + \cdots + \left(h_\rho - \int P_\rho\, dt\right)\Omega_\rho \,,$$

in which $\mathfrak{H}$ means a function of t, $\mathfrak{p}_1, \ldots, \mathfrak{p}_\sigma$, $\mathfrak{p}'_1, \ldots, \mathfrak{p}'_\sigma$, equations (10) will then go to:

$$\frac{\partial \mathfrak{H}}{\partial \mathfrak{p}_s} - \frac{d}{dt}\frac{\partial \mathfrak{H}}{\partial \mathfrak{p}'_s} = \mathfrak{P}_s \,.$$

The **Lagrange** equations of motion will then keep their original form, while the kinetic potential $\mathfrak{H}$ in them will be a function of the coordinates $\mathfrak{p}_1, \ldots, \mathfrak{p}_\sigma$, and their first derivatives that will have a completely-different form from the given potential H.

If we next summarize the first part of the theorem that this gives then we will find that:

The necessary and sufficient condition for the left-hand sides of the ***Lagrange*** *equations of motion that correspond to the coordinates $p_1, \ldots, p_\rho$ for a kinetic potential H that is free of t to be completed by the time derivatives of functions of all coordinates and their first derivatives, which do not, however, include the coordinates $p_1, \ldots, p_\rho$ themselves, is that the kinetic potential must have the form:*

(11) $$H = \Omega(t, \mathfrak{p}_1, \ldots, \mathfrak{p}_\sigma, \mathfrak{p}'_1, \ldots, \mathfrak{p}'_\sigma, p'_1, \ldots, p'_\rho) \,,$$

in which Ω is an arbitrary function of the quantities in parentheses.

When the first ρ of the equations of motion, which assume the form:

$$\frac{\partial \Omega}{\partial p'_1} = h_1 - \int P_1\, dt \,, \qquad \ldots, \qquad \frac{\partial \Omega}{\partial p'_\rho} = h_\rho - \int P_\rho\, dt \,,$$

determine the quantities $p'_1, \ldots, p'_\rho$ *as functions of t,* $\mathfrak{p}_1, \ldots, \mathfrak{p}_\sigma, \mathfrak{p}'_1, \ldots, \mathfrak{p}'_\sigma$ *, under the assumption that the determinant of the second differential quotients of* Ω *with respect to* $p'_1, \ldots, p'_\rho$ *does not vanish identically, and one then substitutes them and sets:*

$$\mathfrak{H} = (\Omega) - \left(h_1 - \int P_1\, dt\right)(p'_1) - \cdots - \left(h_\rho - \int P_\rho\, dt\right)(p'_\rho),$$

the other ρ *equations of motion will, in turn, go to the* ***Lagrang****ian form:*

$$\frac{\partial \mathfrak{H}}{\partial \mathfrak{p}_s} - \frac{d}{dt}\frac{\partial \mathfrak{H}}{\partial \mathfrak{p}'_s} = \mathfrak{P}_s ,$$

in which $\mathfrak{H}$ *is, in turn, a first-order kinetic potential.*

The form (11) of the kinetic potential is independent of $p_1, \ldots, p_\rho$, so:

$$\frac{\partial H}{\partial p_1} = 0 , \qquad \ldots, \qquad \frac{\partial H}{\partial p_\rho} = 0 ,$$

and the left-hand sides of the first ρ **Lagrange** equations will then become complete differential quotients:

$$\frac{d}{dt}\frac{\partial H}{\partial p'_1}, \qquad \ldots, \qquad \frac{d}{dt}\frac{\partial H}{\partial p'_\rho},$$

which is the case that **Helmholtz** considered for the kinetic potential in the mechanics of ponderable masses:

$$H = -T - U ,$$

in which the derivatives of the coordinates enter in only degree two, since the equations of motion do not include time explicitly, and equations (6) will then be linear in them. It will then be clear that:

For kinetic potentials in the mechanics of ponderable masses, the case of hidden motion that ***Helmholtz*** *emphasized, for which the kinetic potential should be independent of some of the coordinates, will be the only case in which the associated* ***Lagrange*** *equations will go to complete differential quotients with respect to time (which is always the case then) and admit eliminations of the coordinates that make the resulting equations of motion, in turn, assume the* ***Lagrang****ian form for a first-order kinetic potential.*

The theorem that was developed above for an arbitrary first-order kinetic potential then yields a generalization of the principle of hidden motion in one direction.

We now go on to the second part of the investigation, in which we, in turn, establish the necessary and sufficient form (5) for the kinetic potential H to represent the first ρ **Lagrange** equations as complete differential quotients with respect to time, and assume:

b) Equations (6) are independent of the $p_1', \ldots, p_\rho'$. We will then be dealing with the calculation of the coordinates $p_1, \ldots, p_\rho$ from those equations and their substitution in the other σ equations of motion. It will follow immediately from the assumption that was made that:

$$\Omega = p_1' \varphi_1 (\mathfrak{p}_1, \ldots, \mathfrak{p}_\sigma, \mathfrak{p}_1', \ldots, \mathfrak{p}_\sigma') + \cdots + p_\rho' \varphi_\rho (\mathfrak{p}_1, \ldots, \mathfrak{p}_\sigma, \mathfrak{p}_1', \ldots, \mathfrak{p}_\sigma') + \varphi(\mathfrak{p}_1, \ldots, \mathfrak{p}_\sigma, \mathfrak{p}_1', \ldots, \mathfrak{p}_\sigma'), \tag{12}$$

and from (5), the kinetic potential will then assume the form:

$$\begin{aligned} H = \sum_{r=1}^{\rho} \{ p_r \, (C_{r,r+1} \, p_{p+1} + C_{r,r+2} \, p_{p+2} + \cdots + C_{r\rho} \, p_\rho \\ + \varphi_r \, (\mathfrak{p}_1, \ldots, \mathfrak{p}_\sigma, \mathfrak{p}_1', \ldots, \mathfrak{p}_\sigma')\} \\ + \varphi(\mathfrak{p}_1, \ldots, \mathfrak{p}_\sigma, \mathfrak{p}_1', \ldots, \mathfrak{p}_\sigma') . \end{aligned} \tag{13}$$

The fact that this case is excluded in the mechanics of ponderable masses emerges from the fact that the kinetic potential includes $p_1', \ldots, p_\rho'$ only linearly. If one again forms the expression:

$$\frac{\partial H}{\partial \mathfrak{p}_\sigma} - \frac{d}{dt} \frac{\partial H}{\partial \mathfrak{p}_\sigma'}$$

then, as is easy to see, when one sets:

$$p_1' \varphi_1 + \cdots + p_\rho' \varphi_\rho + \varphi = \Phi , \tag{14}$$

in which Φ is independent $p_1, \ldots, p_\rho$, that will give the second set of σ **Lagrange** equations in the form:

$$\frac{\partial \mathfrak{H}}{\partial \mathfrak{p}_s} - \frac{d}{dt} \frac{\partial \mathfrak{H}}{\partial \mathfrak{p}_s'} = \mathfrak{P}_s . \tag{15}$$

The question will now arise of whether, for arbitrary functions $\varphi_1, \varphi_2, \ldots, \varphi_\rho$ of $\mathfrak{p}_1, \ldots, \mathfrak{p}_\sigma, \mathfrak{p}_1', \ldots, \mathfrak{p}_\sigma'$, one can use the first ρ **Lagrange** equations, which assume the form:

$$\begin{aligned} - C_{1r} \, p_1 - C_{2r} \, p_2 - \ldots - C_{r-1,r} \, p_{r-1} + C_{r,r+2} \, p_{r+2} + \ldots + C_{r\rho} \, p_\rho \\ = h_r - \int P_r \, dt - \varphi_r \, (\mathfrak{p}_1, \ldots, \mathfrak{p}_\sigma, \mathfrak{p}_1', \ldots, \mathfrak{p}_\sigma') \qquad (r = 1, 2, \ldots, \rho), \end{aligned} \tag{16}$$

from (6) and (12), in order to express the quantities $p_1, \ldots, p_\rho$ in terms of $t, \mathfrak{p}_1, \ldots, \mathfrak{p}_\sigma, \mathfrak{p}'_1, \ldots, \mathfrak{p}'_\sigma$, so $p'_1, \ldots, p'_\rho$ are expressed in terms of $t, \mathfrak{p}_1, \ldots, \mathfrak{p}_\sigma, \mathfrak{p}'_1, \ldots, \mathfrak{p}'_\sigma, \mathfrak{p}''_1, \ldots, \mathfrak{p}''_\sigma$, such that substituting that in (15) will make the resulting equation:

$$\left(\frac{\partial \Phi}{\partial \mathfrak{p}_s}\right) - \frac{d}{dt}\left(\frac{\partial \Phi}{\partial \mathfrak{p}'_s}\right) = \mathfrak{P}_s , \tag{17}$$

in turn, assume the **Lagrang**ian form.

The investigation has now become essentially more complicated, and as we will see, that fact is closely connected to some later general considerations that will also allow us to treat the question here without the help of the transformation of **Hamilton**'s principle.

Since one has:

$$\frac{\partial \Phi}{\partial \mathfrak{p}_s} = \sum_{r=1}^{\rho} p'_r \frac{\partial \varphi_r}{\partial \mathfrak{p}_s} + \frac{\partial \varphi}{\partial \mathfrak{p}_s} , \qquad \frac{\partial \Phi}{\partial \mathfrak{p}'_s} = \sum_{r=1}^{\rho} p'_r \frac{\partial \varphi_r}{\partial \mathfrak{p}'_s} + \frac{\partial \varphi}{\partial \mathfrak{p}'_s} ,$$

from (14), equation (17) will go to:

$$\sum_{r=1}^{\rho} (p'_r) \frac{\partial \varphi_r}{\partial \mathfrak{p}_s} - \frac{d}{dt} \sum_{r=1}^{\rho} (p'_r) \frac{\partial \varphi_r}{\partial \mathfrak{p}'_s} + \frac{\partial \varphi}{\partial \mathfrak{p}_s} - \frac{d}{dt}\frac{\partial \varphi}{\partial \mathfrak{p}'_s} = \mathfrak{P}_s . \tag{18}$$

If one is to investigate the possibility of converting the left-hand side of that equation into **Lagrang**ian form then it will suffice to ignore the last two terms, since they already appear in the desired form.

If one differentiates equation (16) with respect to t, multiplies it by (p'_r), and sums over r from 1 to ρ then that will yield:

$$\mathfrak{p}'_1 \sum_{r=1}^{\rho} (p'_r) \frac{\partial \varphi_r}{\partial \mathfrak{p}_s} + \cdots + \mathfrak{p}'_\sigma \sum_{r=1}^{\rho} (p'_r) \frac{\partial \varphi_r}{\partial \mathfrak{p}'_\sigma} + \mathfrak{p}''_1 \sum_{r=1}^{\rho} (p'_r) \frac{\partial \varphi_r}{\partial \mathfrak{p}'_1} + \cdots + \mathfrak{p}''_\sigma \sum_{r=1}^{\rho} (p'_r) \frac{\partial \varphi_r}{\partial \mathfrak{p}'_\sigma} = - \sum_{r=1}^{\rho} P_r \, (p'_r) . \tag{19}$$

If one differentiates that with respect to t, makes a subsequent substitution, differentiates the identity (16) with respect to $\mathfrak{p}''_s$, multiplies it by $\dfrac{\partial (p'_r)}{\partial \mathfrak{p}''_s}$, and in turn sums over r from 1 to ρ then it will follow that:

$$\sum_{r=1}^{\rho} \frac{\partial \varphi_r}{\partial \mathfrak{p}'_s} \frac{\partial (p'_r)}{\partial \mathfrak{p}''_s} = 0 \qquad (s = 1, 2, \ldots, \sigma). \tag{20}$$

If we now set:

$$\sum_{r=1}^{\rho} (p'_r) \frac{\partial \varphi_r}{\partial \mathfrak{p}_s} - \frac{d}{dt} \sum_{r=1}^{\rho} (p'_r) \frac{\partial \varphi_r}{\partial \mathfrak{p}'_s} = Q_s$$

in equation (18), or:

$$(21)\qquad Q_s = \sum_{r=1}^{\rho}(p'_r)\frac{\partial\varphi_r}{\partial\mathfrak{p}_s}$$
$$- \mathfrak{p}'_1\sum_{r=1}^{\rho}\left((p'_r)\frac{\partial^2\varphi_r}{\partial\mathfrak{p}'_s\,\partial\mathfrak{p}_1}+(p'_r)\frac{\partial\varphi_r}{\partial\mathfrak{p}'_s}\frac{\partial(p'_r)}{\partial\mathfrak{p}_1}\right)-\cdots-\mathfrak{p}'_\sigma\sum_{r=1}^{\rho}\left((p'_r)\frac{\partial^2\varphi_r}{\partial\mathfrak{p}'_s\,\partial\mathfrak{p}_\sigma}+(p'_r)\frac{\partial\varphi_r}{\partial\mathfrak{p}'_s}\frac{\partial(p'_r)}{\partial\mathfrak{p}_\sigma}\right)$$
$$- \mathfrak{p}''_1\sum_{r=1}^{\rho}\left((p'_r)\frac{\partial^2\varphi_r}{\partial\mathfrak{p}'_s\,\partial\mathfrak{p}'_1}+(p'_r)\frac{\partial\varphi_r}{\partial\mathfrak{p}'_s}\frac{\partial(p'_r)}{\partial\mathfrak{p}'_1}\right)-\cdots-\mathfrak{p}''_\sigma\sum_{r=1}^{\rho}\left((p'_r)\frac{\partial^2\varphi_r}{\partial\mathfrak{p}'_s\,\partial\mathfrak{p}'_\sigma}+(p'_r)\frac{\partial\varphi_r}{\partial\mathfrak{p}'_s}\frac{\partial(p'_r)}{\partial\mathfrak{p}'_\sigma}\right)$$
$$- \mathfrak{p}'''_1\sum_{r=1}^{\rho}\frac{\partial\varphi_r}{\partial\mathfrak{p}'_s}\frac{\partial(p'_r)}{\partial\mathfrak{p}''_1}-\cdots-\mathfrak{p}'''_\sigma\sum_{r=1}^{\rho}\frac{\partial\varphi_r}{\partial\mathfrak{p}'_s}\frac{\partial(p'_r)}{\partial\mathfrak{p}''_\sigma},$$

then we will see from (20) that the coefficient of $\mathfrak{p}'''_\lambda$ in Q_s will generally be zero only when we have $\lambda = s$, or that Q_s is a linear function of:

$$\mathfrak{p}'''_1,\ \ldots,\ \mathfrak{p}'''_{s-1},\ \mathfrak{p}'''_{s+1},\ \ldots,\ \mathfrak{p}'''_\sigma .$$

However, it already follows from this that Q_s cannot be put into the **Lagrang**ian form with a first-order kinetic potential, as it should, since it cannot include third-order derivatives. [Whether kinetic potentials of order higher than one can occur will first be implied by a later investigation (*).]

(*) In order to examine whether Q_s possesses a *second*-order kinetic potential, so it can be represented in the form:

$$Q_s = \frac{\partial\mathfrak{H}_1}{\partial\mathfrak{p}_s}-\frac{d}{dt}\frac{\partial\mathfrak{H}_1}{\partial\mathfrak{p}'_s}+\frac{d^2}{dt^2}\frac{\partial\mathfrak{H}_1}{\partial\mathfrak{p}''_s},$$

it should be remarked that since Q_s is linear in $\mathfrak{p}'''_1,\ \ldots,\ \mathfrak{p}'''_\sigma$ from (21), $\mathfrak{H}_1$ can also include the derivatives $\mathfrak{p}''_1,\ \ldots,\ \mathfrak{p}''_\sigma$ only linearly. Now in order to be able to satisfy that equation, from Lemma 4, the quantities Q_s, which do not include $\mathfrak{p}''''_1,\ \ldots,\ \mathfrak{p}''''_s$, must satisfy the equations:

$$(\alpha)\qquad \frac{\partial Q_\kappa}{\partial p_\lambda^{(\tau)}}-(\tau+1)_1\frac{d}{dt}\frac{\partial Q_\kappa}{\partial p_\lambda^{(\tau+1)}}+(\tau+2)_2\frac{d^2}{dt^2}\frac{\partial Q_\kappa}{\partial p_\lambda^{(\tau+2)}}+(-1)^{3-\tau}(3-\tau)_3\frac{d^{3-\tau}}{dt^{3-\tau}}\frac{\partial Q_\kappa}{\partial p'''_\kappa} = (-1)^\tau\frac{\partial Q_\lambda}{\partial p_\kappa^{(\tau)}}$$

identically, in which $\tau = 0, 1, 2, 3$, and κ, λ assume the values $1, 2, \ldots, \sigma$.

For $\kappa = s$, $\lambda = s$, $\tau = 0$, that equation will go to:

$$\frac{d}{dt}\left(\frac{\partial Q_s}{\partial\mathfrak{p}'_s}-\frac{d}{dt}\frac{\partial Q_s}{\partial\mathfrak{p}''_s}+\frac{d^2}{dt^2}\frac{\partial Q_s}{\partial\mathfrak{p}'''_s}\right)=0 .$$

However, since:

$$Q_s = \sum_{r=1}^{\rho}(p'_r)\frac{\partial\varphi_r}{\partial\mathfrak{p}_s}-\frac{d}{dt}\sum_{r=1}^{\rho}(p'_r)\frac{\partial\varphi_r}{\partial\mathfrak{p}'_s},$$

Under the assumption that the functions $\varphi_1, \ldots, \varphi_r, \varphi$ that are included in the kinetic potential (13) are subject to no restrictions, it is only in the case when $\sigma = 1$ (so *only one* **Lagrange** equation to be transformed comes under consideration) that equation (21) can go to an expression that is free of third derivatives, in which one can now drop the index s :

$$Q = \sum_{r=1}^{\rho}(p_r')\frac{\partial \varphi_r}{\partial \mathfrak{p}} - \mathfrak{p}'\sum_{r=1}^{\rho}\left((p_r')\frac{\partial^2\varphi_r}{\partial\mathfrak{p}\,\partial\mathfrak{p}'} + \frac{\partial\varphi_r}{\partial\mathfrak{p}'}\frac{\partial(p_r')}{\partial\mathfrak{p}}\right) - \mathfrak{p}''\sum_{r=1}^{\rho}\left((p_r')\frac{\partial^2\varphi_r}{\partial\mathfrak{p}'^2} + \frac{\partial\varphi_r}{\partial\mathfrak{p}'}\frac{\partial(p_r')}{\partial\mathfrak{p}'}\right)$$
$$= \sum_{r=1}^{\rho}(p_r')\frac{\partial\varphi_r}{\partial\mathfrak{p}} - \frac{d}{dt}\sum_{r=1}^{\rho}(p_r')\frac{\partial\varphi_r}{\partial\mathfrak{p}'} .$$

The question of whether that expression can be converted into **Lagrang**ian form obviously needs to be addressed for only even ρ, since for odd ρ, the determinant of the left-hand side of equations (16), when they are differentiated with respect to t, is known to vanish.

If one now sets:

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it is easy to see, by means of (2) and (3) in **§ 2** that:

$$\frac{d}{dt}\left(\frac{\partial Q_s}{\partial\mathfrak{p}_s'} - \frac{d}{dt}\frac{\partial Q_s}{\partial\mathfrak{p}_s''} + \frac{d^2}{dt^2}\frac{\partial Q_s}{\partial\mathfrak{p}_s'''}\right)$$

$$= \sum_{r=1}^{\rho}\frac{\partial(p_r')}{\partial\mathfrak{p}_s'}\frac{\partial\varphi_r}{\partial\mathfrak{p}_s} + \sum_{r=1}^{\rho}(p_r')\frac{\partial^2\varphi_r}{\partial\mathfrak{p}_s\,\partial\mathfrak{p}_s'} - \frac{d}{dt}\left\{\sum_{r=1}^{\rho}\frac{\partial(p_r')}{\partial\mathfrak{p}_s'}\frac{\partial\varphi_r}{\partial\mathfrak{p}_s'} + (p_r')\frac{\partial^2\varphi_r}{\partial\mathfrak{p}_s'^2}\right\} - \sum_{r=1}^{\rho}\frac{\partial(p_r')}{\partial\mathfrak{p}_s'}\frac{\partial\varphi_r}{\partial\mathfrak{p}_s'} + (p_r')\frac{\partial^2\varphi_r}{\partial\mathfrak{p}_s\,\partial\mathfrak{p}_s'}$$
$$-\frac{d}{dt}\left\{\sum_{r=1}^{\rho}\frac{\partial(p_r')}{\partial\mathfrak{p}_s''}\frac{\partial\varphi_r}{\partial\mathfrak{p}_s'} - \frac{d}{dt}\sum_{r=1}^{\rho}\frac{\partial(p_r')}{\partial\mathfrak{p}_s''}\frac{\partial\varphi_r}{\partial\mathfrak{p}_s'} - \sum_{r=1}^{\rho}\frac{\partial(p_r')}{\partial\mathfrak{p}_s'}\frac{\partial\varphi_r}{\partial\mathfrak{p}_s'} + (p_r')\frac{\partial^2\varphi_r}{\partial\mathfrak{p}_s'^2}\right\} - \frac{d^2}{dt^2}\sum_{r=1}^{\rho}\frac{\partial(p_r')}{\partial\mathfrak{p}_s''}\frac{\partial\varphi_r}{\partial\mathfrak{p}_s'}$$
$$= \sum_{r=1}^{\rho}\left(\frac{\partial(p_r')}{\partial\mathfrak{p}_s'}\frac{\partial\varphi_r}{\partial\mathfrak{p}_s} - \frac{\partial(p_r')}{\partial\mathfrak{p}_s}\frac{\partial\varphi_r}{\partial\mathfrak{p}_s'}\right) - \frac{d}{dt}\sum_{r=1}^{\rho}\frac{\partial(p_r')}{\partial\mathfrak{p}_s''}\frac{\partial\varphi_r}{\partial\mathfrak{p}_s} .$$

Now, the expressions for $p_1', \ldots, p_\rho'$ that are obtained from the system of equations (16) by differentiating them with respect to t are substituted in that equation in order to see whether that expression will vanish identically. For the sake of simplicity, let $\rho = 2$. That will yield the values:

$$p_1' = \sum_{s=1}^{\sigma}\left(\frac{\partial\varphi_2}{\partial\mathfrak{p}_s}\mathfrak{p}_s' + \frac{\partial\varphi_2}{\partial\mathfrak{p}_s'}\mathfrak{p}_s''\right), \qquad p_2' = -\sum_{s=1}^{\sigma}\left(\frac{\partial\varphi_1}{\partial\mathfrak{p}_s}\mathfrak{p}_s' + \frac{\partial\varphi_1}{\partial\mathfrak{p}_s'}\mathfrak{p}_s''\right),$$

and upon substituting that in the last expression:

$$\frac{\partial Q_s}{\partial\mathfrak{p}_s'} - \frac{d}{dt}\frac{\partial Q_s}{\partial\mathfrak{p}_s''} + \frac{d^2}{dt^2}\frac{\partial Q_s}{\partial\mathfrak{p}_s'''} = \frac{\partial\varphi_1}{\partial\mathfrak{p}_s}\left(\frac{\partial\varphi_2}{\partial\mathfrak{p}_s} + \mathfrak{p}_s'\frac{\partial^2\varphi_2}{\partial\mathfrak{p}_s\,\partial\mathfrak{p}_s'} + \mathfrak{p}_s''\frac{\partial^2\varphi_2}{\partial\mathfrak{p}_s'^2}\right) - \frac{\partial\varphi_2}{\partial\mathfrak{p}_s}\left(\frac{\partial\varphi_1}{\partial\mathfrak{p}_s} + \mathfrak{p}_s'\frac{\partial^2\varphi_1}{\partial\mathfrak{p}_s\,\partial\mathfrak{p}_s'} + \mathfrak{p}_s''\frac{\partial^2\varphi_1}{\partial\mathfrak{p}_s'^2}\right) - \frac{d}{dt}\left(\frac{\partial\varphi_1}{\partial\mathfrak{p}_s}\frac{\partial\varphi_2}{\partial\mathfrak{p}_s'} - \frac{\partial\varphi_2}{\partial\mathfrak{p}_s}\frac{\partial\varphi_1}{\partial\mathfrak{p}_s'}\right) = 0 .$$

The identity is proved similarly for the other equations that are included in (α), which we will come back to later.

$$\sum_{r=1}^{\rho}(p'_r)\frac{\partial\varphi_r}{\partial\mathfrak{p}} = K\,, \qquad \sum_{r=1}^{\rho}(p'_r)\frac{\partial\varphi_r}{\partial\mathfrak{p}'} = L\,,$$

such that one will have:

(22) $$Q = K - \frac{dL}{dt}\,,$$

then since Q does not include the third-order derivative of $\mathfrak{p}$, L must also be free of $\mathfrak{p}''$, since the (p'_r) depend upon only $\mathfrak{p}$, $\mathfrak{p}'$, and $\mathfrak{p}''$, and equation (19) will assume the form:

(23) $$\mathfrak{p}'\,K + \mathfrak{p}''L = -P_1(p'_1) - \cdots - P_\rho(p'_\rho)\,,$$

in the present case.

However, from Lemma 4, the necessary and sufficient condition for a function Q that depends upon t, $\mathfrak{p}$, $\mathfrak{p}'$, $\mathfrak{p}''$ to possess a first-order kinetic potential is that:

(24) $$\frac{\partial Q}{\partial\mathfrak{p}'} - \frac{d}{dt}\frac{\partial Q}{\partial\mathfrak{p}''} = 0\,,$$

and when one considers the fact that L does not include $\mathfrak{p}''$, it will then follow from equation (22) that:

(25) $$\frac{\partial Q}{\partial\mathfrak{p}'} - \frac{d}{dt}\frac{\partial Q}{\partial\mathfrak{p}''} = \frac{\partial K}{\partial\mathfrak{p}'} - \frac{d}{dt}\frac{\partial K}{\partial\mathfrak{p}''} - \frac{\partial L}{\partial\mathfrak{p}}\,.$$

Furthermore, when one sets:

(26) $$\frac{1}{\mathfrak{p}'}(P_1(p'_1) + \cdots + P_\rho(p'_\rho)) = N\,,$$

in which N depends upon $\mathfrak{p}$, $\mathfrak{p}'$, $\mathfrak{p}''$, and indeed linearly in the last quantity, equation (23) will yield the value:

$$K = -\frac{\mathfrak{p}''}{\mathfrak{p}'}L - N\,,$$

so the substitution of that expression in (25) will give:

(27) $$\frac{\partial Q}{\partial\mathfrak{p}'} - \frac{d}{dt}\frac{\partial Q}{\partial\mathfrak{p}''} = -\frac{\partial N}{\partial\mathfrak{p}'} + \frac{d}{dt}\frac{\partial N}{\partial\mathfrak{p}''}\,.$$

Equation (24) will then be fulfilled, so Q will possess a first-order kinetic potential when the external forces P_1, …, P_r are all zero, since it will also follow from (26) that $N = 0$ in this case. However, if the external forces do not vanish then it will suffice to examine the case of $\rho = 2$, in which differentiating equations (16) with respect to t will yield:

$$p'_1 = \frac{\partial \varphi_2}{\partial \mathfrak{p}}\mathfrak{p}' + \frac{\partial \varphi_2}{\partial \mathfrak{p}'}\mathfrak{p}'' + P_2, \qquad p'_2 = -\frac{\partial \varphi_1}{\partial \mathfrak{p}}\mathfrak{p}' - \frac{\partial \varphi_1}{\partial \mathfrak{p}'}\mathfrak{p}'' - P_1,$$

and (26) will give:

$$N = \frac{P_1}{\mathfrak{p}'}\frac{d\varphi_2}{dt} - \frac{P_2}{\mathfrak{p}'}\frac{d\varphi_1}{dt}.$$

From that, as one can see immediately, it will follow by means of (2) and (3) in **§ 2.** that:

$$-\frac{\partial N}{\partial \mathfrak{p}'} + \frac{d}{dt}\frac{\partial N}{\partial \mathfrak{p}''} = \frac{1}{\mathfrak{p}'}\left(P'_1\frac{\partial \varphi_2}{\partial \mathfrak{p}'} - P'_2\frac{\partial \varphi_1}{\partial \mathfrak{p}'}\right),$$

and it can then be concluded from (27) that equation (24) can only be fulfilled when the external forces are all constant.

If we summarize the results that we have obtained then that will give the following theorem:

*The necessary and sufficient condition for the ρ **Lagrange** equations:*

$$\frac{\partial H}{\partial \mathfrak{p}_r} - \frac{d}{dt}\frac{\partial H}{\partial \mathfrak{p}'_r} = P_r \qquad (r = 1, 2, \ldots, \rho)$$

to go to complete differential quotients with respect to time of functions of the coordinates $p_1, \ldots, p_\rho$, $\mathfrak{p}_1, \ldots, \mathfrak{p}_\sigma$, and their derivatives (which are, however, independent of $p'_1, \ldots, p'_\rho$) is that the kinetic potential must have the form:

$$H = \sum_{r=1}^{\rho}\left\{p'_r\,(C_{r,r+1}\,p_{r+1} + C_{r,r+2}\,p_{r+2} + \cdots + C_{r\rho}\,p_\rho + \varphi_r\right\} + \varphi,$$

*in which $C_{r,r+1}, \ldots, C_{r\rho}$ are arbitrary constants, and $\varphi_1, \ldots, \varphi_\rho$, φ represent arbitrary functions of $\mathfrak{p}_1, \ldots, \mathfrak{p}_\sigma$, $\mathfrak{p}'_1, \ldots, \mathfrak{p}'_\sigma$. In that case, if the functions that enter into H are subject to no further conditions then eliminating the coordinates $p_1, \ldots, p_\rho$, and their derivatives from the ρ equations and the σ **Lagrange** equations*

$$\frac{\partial H}{\partial \mathfrak{p}_s} - \frac{d}{dt}\frac{\partial H}{\partial \mathfrak{p}'_s} = \mathfrak{P}_s \qquad (s = 1, 2, \ldots, \sigma)$$

*will again lead to **Lagrange** equations of the form:*

$$\frac{\partial \mathfrak{H}}{\partial \mathfrak{p}_s} - \frac{d}{dt}\frac{\partial \mathfrak{H}}{\partial \mathfrak{p}'_s} = \mathfrak{P}_s \qquad (s = 1, 2, \ldots, \sigma),$$

in which $\mathfrak{H}$ *represents a first-order kinetic potential in* $\mathfrak{p}_1, \ldots, \mathfrak{p}_\sigma, \mathfrak{p}'_1, \ldots, \mathfrak{p}'_\sigma$, *if and only if* $\sigma = 1$. *Thus,* $\rho + 1$ *equations of motion will be included in the system, in total, and the external forces* $P_1, \ldots, P_\rho$ *will be either zero or constant.*

If $\sigma > 1$ then equations (20) must be fulfilled, and when one sets:

$$\sum_{r=1}^{\rho} (p'_r) \frac{\partial \varphi_r}{\partial \mathfrak{p}_s} = K_s , \qquad \sum_{r=1}^{\rho} (p'_r) \frac{\partial \varphi_r}{\partial \mathfrak{p}'_s} = L_s ,$$

the quantity:

(28) $$Q_s = K_s - \frac{dL_s}{dt}$$

will not include the third derivatives of $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$, so L_s must be independent of the second derivatives of those quantities, while K_s must include the second derivatives linearly. Now, if one combines that equation with the relations that (19) will imply under the assumption that the forces $P_1, \ldots, P_\rho$ are zero, namely:

$$\sum_{s=1}^{\sigma} \mathfrak{p}'_s K_s + \sum_{s=1}^{\sigma} \mathfrak{p}''_s L_s = 0 ,$$

then when one multiplies the equation:

$$Q_s = K_s - \frac{dL_s}{dt} = \frac{\partial \mathfrak{H}_1}{\partial \mathfrak{p}_s} - \frac{d}{dt} \frac{\partial \mathfrak{H}_1}{\partial \mathfrak{p}'_s}$$

by $\mathfrak{p}'_s$ and sums over s from 1 to σ, the demand that it must be true will lead to:

$$\sum_{s=1}^{\sigma} \mathfrak{p}'_s \frac{\partial \mathfrak{H}_1}{\partial \mathfrak{p}_s} - \sum_{s=1}^{\sigma} \mathfrak{p}'_s \frac{d}{dt} \frac{\partial \mathfrak{H}_1}{\partial \mathfrak{p}'_s} = \sum_{s=1}^{\sigma} \mathfrak{p}'_s K_s + \sum_{s=1}^{\sigma} \mathfrak{p}'_s \frac{dL_s}{dt} = - \frac{d}{dt} \sum_{s=1}^{\sigma} \mathfrak{p}'_s L_s ,$$

or since:

$$\sum_{s=1}^{\sigma} \mathfrak{p}'_s \frac{\partial \mathfrak{H}_1}{\partial \mathfrak{p}_s} - \sum_{s=1}^{\sigma} \mathfrak{p}'_s \frac{d}{dt} \frac{\partial \mathfrak{H}_1}{\partial \mathfrak{p}'_s} = \frac{d}{dt} \left(\mathfrak{H}_1 - \sum_{s=1}^{\sigma} \mathfrak{p}'_s \frac{\partial \mathfrak{H}_1}{\partial \mathfrak{p}'_s} \right) ,$$

that will lead to:

(29) $$\sum_{s=1}^{\sigma} \mathfrak{p}'_s \frac{\partial \mathfrak{H}_1}{\partial \mathfrak{p}_s} = \mathfrak{H}_1 + \sum_{s=1}^{\sigma} \mathfrak{p}'_s L_s$$

upon integration. Integrating that partial differential equation will yield $\mathfrak{H}$ as a function of $\mathfrak{p}_s$ and $\mathfrak{p}'_s$, since L does not include the second derivatives. The details of the conditions that the $\varphi_1, \ldots,$

φ_ρ must fulfill in order for equations (20) to be satisfied identically are not very interesting, but for $\rho = 2$, it might suffice to give the condition in the form:

$$\frac{\partial \varphi_1}{\partial \mathfrak{p}_1'} \frac{\partial \varphi_2}{\partial \mathfrak{p}_2'} - \frac{\partial \varphi_1}{\partial \mathfrak{p}_2'} \frac{\partial \varphi_2}{\partial \mathfrak{p}_1'} = 0 .$$

If we set $\sigma = 1$ in that then, from (29), the first-order kinetic potential that was proved to always exist in the theorem that was stated above will be given by the equation:

$$\mathfrak{p}' \frac{\partial \mathfrak{H}_1}{\partial \mathfrak{p}} - \mathfrak{H}_1 = \mathfrak{p}' L \qquad \text{or} \qquad \mathfrak{H}_1 = \mathfrak{p}' \int \frac{L}{\mathfrak{p}'} d\mathfrak{p}' + \psi(\mathfrak{p})\,\mathfrak{p}' ,$$

in which $\psi(\mathfrak{p})$ means an arbitrary function of $\mathfrak{p}$, so, e.g., for $\rho = 2$, since in that case, it would emerge immediately upon differentiating equations (16) with respect to t that:

$$L = \mathfrak{p}' \left(\frac{\partial \varphi_1}{\partial \mathfrak{p}'} \frac{\partial \varphi_2}{\partial \mathfrak{p}} - \frac{\partial \varphi_1}{\partial \mathfrak{p}'} \frac{\partial \varphi_2}{\partial \mathfrak{p}} \right) .$$

Thus, for the case in which $P_1 = P_2 = 0$, the kinetic potential $\mathfrak{H}$ will be given by:

$$\mathfrak{H} = \mathfrak{p}' \int \left(\frac{\partial \varphi_1}{\partial \mathfrak{p}'} \frac{\partial \varphi_2}{\partial \mathfrak{p}} - \frac{\partial \varphi_1}{\partial \mathfrak{p}'} \frac{\partial \varphi_2}{\partial \mathfrak{p}} \right) d\mathfrak{p}' + \psi(\mathfrak{p})\,\mathfrak{p}' + \varphi(\mathfrak{p}, \mathfrak{p}') ,$$

according to equation (18).

Once the assumption has been made that the first ρ **Lagrange** equations are represented as complete differential quotients with respect to time, the only two cases in which an elimination of the coordinates is still possible that shall be brought under consideration are defined by either the case in which the quantities $p_1, \ldots, p_\rho, p_1', \ldots, p_\rho'$ are not included explicitly in the first ρ equations. One then determines the values of $p_1'', \ldots, p_\rho''$ from them and substitutes the latter in the other σ equations of motion, or the case in which the second derivatives of $p_1, \ldots, p_\rho$ do not enter into the first ρ equations, in which one agrees that the new kinetic potential should once more be of order one, while the elimination of an arbitrary number of coordinates (which is always possible in the absence of conditions) will lead to completely-different forms for the results of the elimination, as we will see later on. Thus:

1) If the equations:

$$\frac{\partial H}{\partial p_r} - \frac{d}{dt} \frac{\partial H}{\partial p_r'} = P_r ,$$

in which P_r are given functions of t, or:

(30) $$\frac{\partial H}{\partial p_r} - \sum_{\delta=1}^{\rho} \frac{\partial^2 H}{\partial p'_r\, \partial p_\delta}\, p'_\delta - \sum_{\delta=1}^{\rho} \frac{\partial^2 H}{\partial p'_r\, \partial p'_\delta}\, p''_\delta - \sum_{s=1}^{\sigma} \frac{\partial^2 H}{\partial p'_r\, \partial \mathfrak{p}_s}\, \mathfrak{p}'_s - \sum_{s=1}^{\sigma} \frac{\partial^2 H}{\partial p'_r\, \partial \mathfrak{p}'_s}\, \mathfrak{p}''_s = P_r$$

are free of the coordinates $p_1, \ldots, p_\rho$, and their first derivatives then obviously the same thing must be true of the quantities:

$$\frac{\partial^2 H}{\partial p'_r\, \partial p'_\delta} \qquad \text{and} \qquad \frac{\partial^2 H}{\partial p'_r\, \partial \mathfrak{p}'_s}\,.$$

They must then have the forms:

$$\frac{\partial^2 H}{\partial p'_r\, \partial p'_\delta} = \omega_{r\delta}\,(\mathfrak{p}_1, \ldots, \mathfrak{p}_\sigma, \mathfrak{p}'_1, \ldots, \mathfrak{p}'_\sigma)\ ,$$

$$\frac{\partial^2 H}{\partial p'_r\, \partial \mathfrak{p}'_s} = \Omega_{rs}\,(\mathfrak{p}_1, \ldots, \mathfrak{p}_\sigma, \mathfrak{p}'_1, \ldots, \mathfrak{p}'_\sigma)\,.$$

Thus, since:

$$\frac{\partial^3 H}{\partial p'_r\, \partial p'_\delta\, \partial \mathfrak{p}'_s} = \frac{\partial \omega_{r\delta}}{\partial \mathfrak{p}'_s} = \frac{\partial\, \Omega_{rs}}{\partial p'_\delta} = 0\ ,$$

$\omega_{r\delta}$ will be independent of $\mathfrak{p}'_s$, and therefore:

(31) $$\frac{\partial^2 H}{\partial p'_r\, \partial p'_\delta} = \omega_{r\delta}\,(\mathfrak{p}_1, \ldots, \mathfrak{p}_\sigma)\,.$$

However, since the remaining part of (30) must be free of the derivatives of the ρ coordinates, so for $\lambda = 1, 2, \ldots, \rho$, the differential quotient with respect to p'_λ:

$$\frac{\partial^2 H}{\partial p_r\, \partial p'_\lambda} - \frac{\partial^2 H}{\partial p'_r\, \partial p_\lambda} - \sum_{\delta=1}^{\rho} \frac{\partial^3 H}{\partial p'_r\, \partial p'_\delta\, \partial \mathfrak{p}_s}\, p'_\delta - \sum_{s=1}^{\sigma} \frac{\partial^3 H}{\partial p'_r\, \partial \mathfrak{p}'_s\, \partial \mathfrak{p}'_\lambda}\, \mathfrak{p}'_s = 0$$

must be satisfied identically, it will follow that:

$$\frac{\partial^2 H}{\partial p_r\, \partial p'_\lambda} - \frac{\partial^2 H}{\partial p'_r\, \partial p_\lambda} - \sum_{s=1}^{\sigma} \frac{\partial \omega_{r\lambda}}{\partial \mathfrak{p}_s}\, \mathfrak{p}'_s = 0\,.$$

From that, one will have $\omega_{r\lambda} = 2\, c_{r\lambda}$, in which $c_{r\lambda} = c_{\lambda r}$ means a constant, such that (31) will give the kinetic potential as:

(33) $$H = \sum_{r,\delta=1}^{\rho} c_{r\delta}\, p'_r\, p'_\delta + \sum_{\delta=1}^{\rho} p'_\delta\, \nu_\delta(p_1, \ldots, \mathfrak{p}_1, \ldots, \mathfrak{p}'_1, \ldots) + N(p_1, \ldots, \mathfrak{p}_1, \ldots, \mathfrak{p}'_1, \ldots)\,,$$

in which the functions ν satisfy the condition:

$$\frac{\partial \nu_\delta}{\partial p_r} = \frac{\partial \nu_r}{\partial p_\delta}, \tag{34}$$

according to the first of equations (32), and the first ρ equations of motion will assume the form:

$$-2\sum_{\delta=1}^{\rho} c_{r\delta}\, p''_\delta + \frac{\partial N}{\partial p_r} - \sum_{\eta=1}^{\sigma} \frac{\partial \nu_r}{\partial \mathfrak{p}_\eta}\mathfrak{p}'_\eta - \sum_{\eta=1}^{\sigma} \frac{\partial \nu_r}{\partial \mathfrak{p}'_\eta}\mathfrak{p}''_\eta = P_r\,, \tag{35}$$

but they are still not free of $p_1, \ldots, p_\rho$. In order for that to be the case, the coefficients of $\mathfrak{p}''_\eta$ would have to be independent of those quantities, and therefore:

$$\nu_r = R_r\,(\mathfrak{p}_1, \ldots, \mathfrak{p}'_1, \ldots) + Q_r\,(p_1, \ldots, \mathfrak{p}_1, \ldots)\,, \tag{36}$$

in which the functions Q_r are subject to the condition:

$$\frac{\partial Q_{r_1}}{\partial p_{r_2}} = \frac{\partial Q_{r_2}}{\partial p_{r_1}}, \tag{37}$$

as a result of equation (34). If that is fulfilled then the middle two terms in equation (35) must be free of $p_1, \ldots, p_\rho$, and therefore with the use of (36):

$$\frac{\partial N}{\partial p_r} - \sum_{\eta=1}^{\sigma} \frac{\partial Q_r}{\partial \mathfrak{p}_\eta}\mathfrak{p}'_\eta = T_r\,(\mathfrak{p}_1, \ldots, \mathfrak{p}'_1, \ldots)$$

or

$$N = \sum_{\eta=1}^{\sigma} \mathfrak{p}'_\eta \int \left(\frac{\partial Q_1}{\partial \mathfrak{p}_\eta} dp_1 + \cdots + \frac{\partial Q_\rho}{\partial \mathfrak{p}_\eta} dp_\rho \right) + p_1 T_1 + \cdots + p_\rho T_\rho + U(\mathfrak{p}_1, \ldots, \mathfrak{p}'_1, \ldots)\,, \tag{38}$$

in which a complete differential in the variables $p_1, \ldots, p_\rho$ is under the integral, due to (37). If we substitute the values of ν_r and N from (36) and (38) in (33) then that will give the necessary condition for the first ρ **Lagrange** equations to be free of the quantities $p_1, \ldots, p'_1, \ldots$ in the form of saying that the kinetic potential must have the following form:

$$H = \sum_{r,\delta=1}^{\rho} c_{r\delta}\, p'_r\, p'_\delta + \sum_{\delta=1}^{\rho} p'_\delta (R_\delta + Q_\delta) + \sum_{\eta=1}^{\sigma} p'_\eta \int \left(\frac{\partial Q_1}{\partial \mathfrak{p}_\eta} dp_1 + \cdots + \frac{\partial Q_\rho}{\partial \mathfrak{p}_\eta} dp_\rho \right) + \sum_{r=1}^{\rho} p_r\, T_r + U\,,$$

in which all functions are arbitrary, but subject to only the condition (37), or since one again has that:

$$\sum_{\delta=1}^{\rho} p'_\delta Q_\delta + \sum_{\eta=1}^{\sigma} p'_\eta \int \left(\frac{\partial Q_1}{\partial \mathfrak{p}_\eta} dp_1 + \cdots + \frac{\partial Q_\rho}{\partial \mathfrak{p}_\eta} dp_\rho \right)$$

is a complete differential quotient with respect to t:

$$H = \sum_{r,\delta=1}^{\rho} c_{r\delta} p'_r \, p'_\delta + \sum_{\delta=1}^{\rho} p'_\delta R_\delta + \sum_{r=1}^{\rho} p_r T_r + U \; . \tag{39}$$

Conversely, as is immediately obvious from (39), the first ρ equations of motion will assume the form:

$$- \sum_{r,\delta=1}^{\rho} c_{r\delta} p''_\delta - \sum_{s=1}^{\sigma} \frac{\partial R_r}{\partial \mathfrak{p}_s} \mathfrak{p}'_s + \sum_{s=1}^{\sigma} \frac{\partial R_r}{\partial \mathfrak{p}'_s} \mathfrak{p}''_s + T_r = P_r \, , \tag{40}$$

from which $p_1, \ldots, p'_1, \ldots$ are missing, and we will then find that:

The necessary and sufficient condition for the first ρ ***Lagrange*** *equations to be free of the coordinates $p_1, \ldots, p_\rho$, and their first derivatives is that the kinetic potential must have the form that is represented in* (39).

If one substitutes the form that was found for H in the second set of σ **Lagrange** equations then that will give immediately:

$$- \sum_{\delta=1}^{\rho} p''_\delta \frac{\partial R_r}{\partial \mathfrak{p}_s} + \sum_{\delta=1}^{\rho} p''_\delta \left(\frac{\partial R_\delta}{\partial \mathfrak{p}_s} - \frac{d}{dt} \frac{\partial R_\delta}{\partial \mathfrak{p}'_s} \right) + \sum_{r=1}^{\rho} p'_r \frac{\partial T_r}{\partial \mathfrak{p}'_s} + \sum_{r=1}^{\rho} p_r \left(\frac{\partial T_r}{\partial \mathfrak{p}_s} - \frac{d}{dt} \frac{\partial T_r}{\partial \mathfrak{p}'_s} \right) + \frac{\partial U}{\partial \mathfrak{p}_s} - \frac{d}{dt} \frac{\partial U}{\partial \mathfrak{p}'_s} = \mathfrak{P}_s \, , \tag{41}$$

in which the values of $p''_1, \ldots, p''_\rho$ that are obtained from (40) have been substituted. However, since that substitution does not once more introduce the quantities p and p', and the second set of **Lagrange** equations should include only the coordinates $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$, and their derivatives, the quantities $p_1, \ldots, p'_1, \ldots$ will not occur in (41) at all, and therefore the relations:

$$\frac{\partial T_r}{\partial \mathfrak{p}_s} - \frac{d}{dt} \frac{\partial T_r}{\partial \mathfrak{p}'_s} = 0 \tag{42}$$

and

$$\frac{\partial R_\delta}{\partial \mathfrak{p}_s} - \frac{d}{dt} \frac{\partial R_\delta}{\partial \mathfrak{p}'_s} - \frac{\partial T_r}{\partial \mathfrak{p}'_s} = 0 \tag{43}$$

must be satisfied identically. However, from Lemma 3, equation (42) demands that T_r must have the form:

$$(44) \qquad -T_r = T_{1r}(\mathfrak{p}_1, \ldots)\,\mathfrak{p}_1' + \cdots + T_{\sigma r}(\mathfrak{p}_1, \ldots)\,\mathfrak{p}_\sigma' + c_r\,,$$

in which c_r is a constant, and:

$$\frac{\partial T_{s_1 r}}{\partial \mathfrak{p}_s} = \frac{\partial T_{sr}}{\partial \mathfrak{p}_{s_1}},$$

from which, (43) will go to:

$$\frac{d}{dt}\frac{\partial R_r}{\partial \mathfrak{p}_s'} = \frac{\partial R_r}{\partial \mathfrak{p}_s} + T_{sr}\,,$$

which will therefore imply that:

$$(45) \qquad R_r = R_{1r}(\mathfrak{p}_1, \ldots)\,\mathfrak{p}_1' + \cdots + R_{\sigma r}(\mathfrak{p}_1, \ldots)\,\mathfrak{p}_\sigma' + \bar{R}_r(\mathfrak{p}_1, \ldots)\,,$$

in which:

$$(46) \qquad \frac{\partial R_{s_1 r}}{\partial \mathfrak{p}_s} = \frac{\partial R_{sr}}{\partial \mathfrak{p}_{s_1}}, \qquad T_{sr} + \frac{\partial \bar{R}_r}{\partial \mathfrak{p}_s} = 0\,.$$

The equations of motion (41) will then assume the form:

$$(47) \qquad -\sum_{\delta=1}^{\rho} p_\delta''\,R_{s\delta} + \frac{\partial U}{\partial \mathfrak{p}_s} - \frac{d}{dt}\frac{\partial U}{\partial \mathfrak{p}_s'} = \mathfrak{P}_s\,,$$

while the kinetic potential (39) will go to:

$$H = \sum_{r,\delta=1}^{\rho} c_{r\delta}\,p_r'\,p_\delta' + \sum_{\delta=1}^{\rho} p_\delta'(R_{1\delta}\,\mathfrak{p}_1' + \ldots + R_{\sigma r}\,\mathfrak{p}_\sigma' + \bar{R}_\delta) + \sum_{r=1}^{\rho} p_r\left(\frac{\partial \bar{R}_r}{\partial \mathfrak{p}_1}\mathfrak{p}_1' + \cdots + \frac{\partial \bar{R}_r}{\partial \mathfrak{p}_\sigma}\mathfrak{p}_\varsigma' + c_r\right) + U\,,$$

or once more, after substituting the complete differential quotient with respect to t :

$$\frac{d}{dt}\sum_{r=1}^{\rho} p_r\,\bar{R}_r\,,$$

it will go to:

$$H = \sum_{r,\delta=1}^{\rho} c_{r\delta}\,p_r'\,p_\delta' + \sum_{r=1}^{\rho} c_r\,p_r + \sum_{r=1}^{\rho} p_r'(R_{1r}\,\mathfrak{p}_1' + \ldots + R_{\sigma r}\,\mathfrak{p}_\sigma') + U\,.$$

Finally, if one substitutes in equations (47) the values of p_1'', …, p_ρ'' that are obtained from equation (40), when it is converted with the aid of (44), (45), (46):

$$- 2 \sum_{\delta=1}^{\rho} c_{r\delta} \, p''_\delta - \frac{d}{dt} \sum_{s=1}^{\rho} R_{sr} \, \mathfrak{p}'_s - c_r = P_r \,,$$

then an easy calculation will show that when one sets:

$$W = \tfrac{1}{2} \sum_{\nu,\mu=1}^{\sigma} \mathfrak{p}'_\nu \, \mathfrak{p}'_\mu \{(C_{11} R_{\mu 1} + \cdots + C_{\rho 1} R_{\mu\rho}) R_{\nu 1} + \cdots + (C_{1\rho} R_{\mu 1} + \cdots + C_{\rho\rho} R_{\mu\rho}) R_{\nu\rho}\}$$
$$- \sum_{\mu=1}^{\rho} \left(C_\mu + \sum_{s=1}^{\rho} C_{\mu s} P_s \right) \int (R_{\mu 1} \, d\mathfrak{p}_1 + \cdots + R_{\sigma\mu} d\mathfrak{p}_\sigma) \,,$$

in which the quantities $C_{rs} = C_{sr}$ are all constants, and complete differential expressions are under the integral sign due to (46), one will have:

$$- \sum_{\delta=1}^{\rho} p''_\delta \, R_{s\delta} = \frac{\partial W}{\partial \mathfrak{p}_s} - \frac{d}{dt} \frac{\partial W}{\partial \mathfrak{p}'_s} \,.$$

When one sets:

$$U + W = \mathfrak{H} \,,$$

the second set of **Lagrange** equations will then go to:

$$\frac{\partial \mathfrak{H}}{\partial \mathfrak{p}_s} - \frac{d}{dt} \frac{\partial \mathfrak{H}}{\partial \mathfrak{p}'_s} = \mathfrak{P}_s \qquad (s = 1, 2, \ldots, \sigma).$$

If we combine the results that we obtained then that will give the following theorem:

*The necessary and sufficient condition for the first ρ **Lagrange** equations to be free of the coordinates $p_1, \ldots, p_\rho$, and their first derivatives is that the kinetic potential must have the form:*

$$H = \sum_{r,\delta=1}^{\rho} c_{r\delta} \, p'_r \, p'_\delta + \sum_{r=1}^{\rho} p'_r \, R_r + \sum_{r=1}^{\rho} p_r \, T_r + U \,,$$

in which R, T, U are arbitrary functions of $\mathfrak{p}_1, \ldots, \mathfrak{p}'_1, \ldots$, and $c_{r\delta} = c_{\delta r}$.

*If one further adds the requirement that the other **Lagrange** equations must also be independent of $p_1, \ldots, p_\rho, p'_1, \ldots, p'_\rho$ then the necessary and sufficient condition for that is that the kinetic potential must take the form:*

$$H = \sum_{r,\delta=1}^{\rho} c_{r\delta} \, p'_r \, p'_\delta + \sum_{r=1}^{\rho} c_r \, p_r + \sum_{r=1}^{\rho} p'_r \, (R_{1r} \, \mathfrak{p}'_1 + \cdots + R_{\sigma r} \, \mathfrak{p}'_\sigma) + U \,,$$

in which c_r means constants, R_{sr} mean functions of $\mathfrak{p}_1, \dots, \mathfrak{p}_s$ that are subject to only the condition that:

$$\frac{\partial R_{s_1 r}}{\partial \mathfrak{p}_s} = \frac{\partial R_{sr}}{\partial \mathfrak{p}_{s_1}},$$

and U represents an arbitrary function of $\mathfrak{p}_1, \dots, \mathfrak{p}'_1, \dots$ In that case, eliminating the quantities from all of the equations of motion will yield the last σ of those equations in the coordinates p''_1, …, p''_ρ, and the derivatives, which will be, in turn, equations in the ***Lagrang****ian form:*

$$\frac{\partial \mathfrak{H}}{\partial \mathfrak{p}_s} - \frac{d}{dt}\frac{\partial \mathfrak{H}}{\partial \mathfrak{p}'_s} = \mathfrak{P}_s \qquad (s = 1, 2, \dots, \sigma),$$

in which the kinetic potential has the form:

$$\mathfrak{H} = U + \tfrac{1}{2}\sum_{\nu,\mu=1}^{\sigma} \mathfrak{p}'_\nu\, \mathfrak{p}'_\mu \{(C_{11} R_{\mu 1} + \cdots + C_{\rho 1} R_{\mu\rho}) R_{\nu 1} + \cdots + (C_{1\rho} R_{\mu 1} + \cdots + C_{\rho\rho} R_{\mu\rho}) R_{\nu\rho}\}$$
$$- \sum_{\mu=1}^{\rho} \left(C_\mu + \sum_{\varepsilon=1}^{\rho} C_{\mu\varepsilon} P_\varepsilon \right) \int (R_{1\mu}\, d\mathfrak{p}_1 + \cdots + R_{\sigma\mu}\, d\mathfrak{p}_\sigma),$$

in which $C_{mn} = C_{nm}$ and C_μ represent arbitrary constants.

In so doing, it is essential to point out that this case also finds application in the mechanics of ponderable masses, since H includes terms of degree two in the derivatives $p'_1, \dots, p'_\rho$, and will therefore be either itself the kinetic potential of a problem in mechanics when the linear terms in those quantities do not occur in it, since the equations of motion should not include time explicitly, or when they do occur, it can be the kinetic potential of a problem with hidden motion, in the **Helmholtz** sense, when the coefficients of the linear terms are constants (*).

(*) For the sake of the later investigations, one might put forth an example of the case that was treated above that takes the form of a system of only two **Lagrange** equations of motion in p and $\mathfrak{p}$ whose kinetic potential is then represented in the form:

$$H = c\, p'^2 + R_1\, p' \mathfrak{p}' + k\, p + U,$$

which was found to be necessary and sufficient above, and in which c and k are arbitrary constants, R_1 is an arbitrary function of $\mathfrak{p}$, and U an arbitrary function of $\mathfrak{p}$ and $\mathfrak{p}'$. The elimination of p, p', p'' from the two equations of motion will, in turn, produce a **Lagrange** equation whose potential has order one, and indeed when P is zero, if and only if that potential is represented by:

$$\mathfrak{H} = U - \frac{1}{4c} R_1^2\, \mathfrak{p}'^2 - \frac{k}{2c} \int R_1\, d\mathfrak{p} .$$

2) If the first ρ **Lagrange** equations are free of the quantities $p_1'', \ldots, p_\rho''$ then under the assumption that (with the exception of the integrability of those equations) the only allowable assumption is that those equations are independent of either the ρ coordinates or their first derivatives, in addition.

The independence of the ρ equations of motion of the second derivatives next demands that since the second partial derivatives of the kinetic potential with respect to $p_1', \ldots, p_\rho'$ must vanish, it must have the form:

$$H = \varphi_1(p_1,\ldots,\mathfrak{p}_1,\ldots,\mathfrak{p}_1',\ldots)\, p_1' + \cdots + \varphi_\rho(p_1,\ldots,\mathfrak{p}_1,\ldots,\mathfrak{p}_1',\ldots)\, p_\rho' + \varphi(p_1,\ldots,\mathfrak{p}_1,\ldots,\mathfrak{p}_1',\ldots)\,, \tag{48}$$

and that will take the first ρ equations of motion to:

$$\left(\frac{\partial\varphi_1}{\partial p_r} - \frac{\partial\varphi_r}{\partial p_1}\right) p_1' + \cdots + \left(\frac{\partial\varphi_\rho}{\partial p_r} - \frac{\partial\varphi_r}{\partial p_\rho}\right) p_\rho' + \frac{\partial\varphi}{\partial p_r} - \frac{\partial\varphi_r}{\partial\mathfrak{p}_1}\mathfrak{p}_1' - \cdots - \frac{\partial\varphi_r}{\partial\mathfrak{p}_1'}\mathfrak{p}_1'' - \cdots = P_r\,. \tag{49}$$

For the mechanics of ponderable masses, H must include only terms of degree two in p' and $\mathfrak{p}'$, and when one sets:

$$U = U_0(\mathfrak{p}) + U_1(\mathfrak{p})\,\mathfrak{p}' + U_2(\mathfrak{p})\,\mathfrak{p}'^2,$$

one must have $U_1(\mathfrak{p}) = 0$. The original and transformed kinetic potentials will then assume the forms:

$$H = c\,p'^2 + R_1(\mathfrak{p})\,p'\mathfrak{p}' + \mathfrak{p}'^2 U_2(\mathfrak{p}) + k\,p + U_0(\mathfrak{p})$$

and

$$\mathfrak{H} = \mathfrak{p}'^2\left(U_2(\mathfrak{p}) - \frac{1}{4c}R_1^2(\mathfrak{p})\right) + U_0(\mathfrak{p}) - \frac{k}{2c}\int R_1(\mathfrak{p})\,d\mathfrak{p}\,,$$

resp. It will then follow that this case does not represent a hidden motion in the **Helmholtz** sense, since $\mathfrak{H}$ does not include terms that are linear in $\mathfrak{p}'$. For the value of H that was found, the two equations of motion will read:

$$2c\,p'' + R_1(\mathfrak{p})\,\mathfrak{p}'' + R_1'(\mathfrak{p})\,\mathfrak{p}'^2 - k = 0$$

and

$$R_1(\mathfrak{p})\,\mathfrak{p}'' + 2\,\mathfrak{p}''U_2(\mathfrak{p}) + \mathfrak{p}'^2 U_2'(\mathfrak{p}) - U_0'(\mathfrak{p}) = \mathfrak{P},$$

resp. Eliminating p'' will produce the equation:

$$\mathfrak{p}''\left(2U_2(\mathfrak{p}) - \frac{1}{2c}R_1(\mathfrak{p})^2\right) + \mathfrak{p}'^2\left(U_2'(\mathfrak{p}) - \frac{1}{2c}R_1(\mathfrak{p})\,R_1'(\mathfrak{p})\right) + \frac{1}{2c}R_1(\mathfrak{p}) - U_0'(\mathfrak{p}) = \mathfrak{P}\,,$$

which can also be represented in the form:

$$\frac{\partial\mathfrak{H}}{\partial\mathfrak{p}} - \frac{d}{dt}\frac{\partial\mathfrak{H}}{\partial\mathfrak{p}'} = \mathfrak{P}\,.$$

Now, should those equations:

a) be independent of the coordinates $p_1, \ldots, p_\rho$, then that would also have to be true for the coefficients of $\mathfrak{p}_s''$, and φ_r would then have the form:

(50) $$\varphi_r = \omega_r(\mathfrak{p}_1, \ldots, \mathfrak{p}_1', \ldots) + \Omega_r(p_1, \ldots, \mathfrak{p}_1, \ldots) .$$

The equations of motion (49) would go to:

$$\left(\frac{\partial \Omega_1}{\partial p_r} - \frac{\partial \Omega_r}{\partial p_1}\right) p_1' + \cdots + \left(\frac{\partial \Omega_\rho}{\partial p_r} - \frac{\partial \Omega_r}{\partial p_\rho}\right) p_\rho' + \frac{\partial \varphi}{\partial p_r} - \frac{\partial \omega_r}{\partial \mathfrak{p}_1} \mathfrak{p}_1' - \cdots - \frac{\partial \Omega_r}{\partial \mathfrak{p}_1} \mathfrak{p}_1' - \cdots - \frac{\partial \varphi_r}{\partial \mathfrak{p}_1'} \mathfrak{p}_1'' - \cdots = P_r .$$

However, since the coefficients of $p_1', \ldots, p_\rho'$ must be independent of the first ρ coordinates, just like the remaining part of the equation, when one sets:

(51) $$\frac{\partial \Omega_{r_1}}{\partial p_r} - \frac{\partial \Omega_r}{\partial p_{r_1}} = \omega_{r r_1}(\mathfrak{p}_1, \ldots, \mathfrak{p}_\sigma)$$

and

(52) $$\frac{\partial \varphi}{\partial p_r} - \frac{\partial \Omega_r}{\partial \mathfrak{p}_1} \mathfrak{p}_1' - \cdots - \frac{\partial \Omega_r}{\partial \mathfrak{p}_\sigma} \mathfrak{p}_\sigma' = \bar{\Omega}_r(\mathfrak{p}_1, \ldots, \mathfrak{p}_1', \ldots),$$

so:

(53) $$\varphi = \mathfrak{p}_1' \int \left(\frac{\partial \Omega_r}{\partial \mathfrak{p}_1} dp_1 + \cdots + \frac{\partial \Omega_\rho}{\partial \mathfrak{p}_1} dp_\rho\right) + \cdots + \mathfrak{p}_\sigma' \int \left(\frac{\partial \Omega_r}{\partial \mathfrak{p}_\sigma} dp_1 + \cdots + \frac{\partial \Omega_\rho}{\partial \mathfrak{p}_\sigma} dp_\rho\right)$$
$$+ \bar{\Omega}_1 p_1 + \cdots + \bar{\Omega}_\rho p_\rho + \psi(\mathfrak{p}_1, \ldots, \mathfrak{p}_1', \ldots),$$

and simultaneously considers the fact that from (52):

$$\frac{\partial^2 \Omega_r}{\partial \mathfrak{p}_s \, \partial \mathfrak{p}_{r_1}} = \frac{\partial^2 \Omega_{r_1}}{\partial \mathfrak{p}_s \, \partial \mathfrak{p}_r},$$

so from (51):

$$\omega_{r r_1} = c_{r r_1},$$

in which $c_{r r_1} = -c_{r_1 r}$ means a constant, and $c_{rr} = 0$, the equation above will go to:

(54) $$c_{r1} p_1' + \cdots + c_{r\rho} p_\rho' + \bar{\Omega}_r - \frac{\partial \omega_r}{\partial \mathfrak{p}_1} \mathfrak{p}_1' - \cdots - \frac{\partial \omega_r}{\partial \mathfrak{p}_1'} \mathfrak{p}_1'' - \cdots = P_r ,$$

which is now, in fact, free of $p_1, \ldots, p_\rho$.

Now, since the introduction of just those quantities will make the kinetic potential assume the form:

$$H = (\omega_1 + \Omega_1)\, p_1' + \cdots + (\omega_\rho + \Omega_\rho)\, p_\rho' + \sum_{\delta=1}^{\sigma} \mathfrak{p}_\delta' \int \left(\frac{\partial \Omega_1}{\partial \mathfrak{p}_\delta} dp_1 + \cdots + \frac{\partial \Omega_\rho}{\partial \mathfrak{p}_\delta} dp_\rho \right) + \bar{\Omega}_1\, p_1 + \cdots + \bar{\Omega}_\rho\, p_\rho + \psi$$

by means of (48), or since $\omega_{r r_1} = c_{r r_1}$, by means of equation (51), based upon the argument that was given in the derivation of (5), it will assume the form:

$$(55) \qquad H = \sum_{r=1}^{\rho} p_r' \{C_{r,r+1}\, p_{r+1} + C_{r,r+2}\, p_{r+2} + \cdots + C_{r\rho}\, p_\rho + \omega_r\} + \bar{\Omega}_1\, p_1 + \cdots + \bar{\Omega}_\rho\, p_\rho + \psi\,,$$

in which $C_{r,r+1}, \ldots, C_{r\rho}$ mean arbitrary constants. As one sees immediately, *that is not only the necessary, but also the sufficient, condition for the first* ρ ***Lagrange*** *equations to be independent of the coordinates* $p_1, \ldots, p_\rho$, *and their second derivatives*. The other σ equations of motion will go to:

$$(56) \quad \sum_{r=1}^{\rho} p_r' \left(\frac{\partial \omega_r}{\partial \mathfrak{p}_s} - \frac{d}{dt} \frac{\partial \omega_r}{\partial \mathfrak{p}_s'} \right) - \sum_{r=1}^{\rho} p_r'' \frac{\partial \omega_r}{\partial \mathfrak{p}_s'} + \sum_{r=1}^{\rho} p_r \left(\frac{\partial \bar{\Omega}_r}{\partial \mathfrak{p}_s} - \frac{d}{dt} \frac{\partial \bar{\Omega}_r}{\partial \mathfrak{p}_s'} \right) - \sum_{r=1}^{\rho} p_r' \frac{\partial \bar{\Omega}_r}{\partial \mathfrak{p}_s'} + \frac{\partial \psi}{\partial \mathfrak{p}_s} - \frac{d}{dt} \frac{\partial \psi}{\partial \mathfrak{p}_s'} = \mathfrak{P}_s\,.$$

Now, should it be possible to eliminate the quantities $p_1, \ldots, p_\rho$, and their derivatives from (54) then (56) would itself have to be free of $p_1, \ldots, p_\rho$, so:

$$\frac{\partial \bar{\Omega}_r}{\partial \mathfrak{p}_s} - \frac{d}{dt} \frac{\partial \bar{\Omega}_r}{\partial \mathfrak{p}_s'} = 0$$

would be fulfilled identically. Thus, it would follow from Lemma 3 that:

$$\bar{\Omega}_r = \Phi_{1r}\, (\mathfrak{p}_1, \ldots) \mathfrak{p}_1' + \cdots + \Phi_{\sigma r}\, (\mathfrak{p}_1, \ldots) \mathfrak{p}_\sigma' + C_r\,,$$

in which C_r is a constant, while:

$$(57) \qquad \frac{\partial \Phi_{sr}}{\partial \mathfrak{p}_{s_1}} = \frac{\partial \Phi_{s_1 r}}{\partial \mathfrak{p}_s}\,,$$

and the two equations (54) and (56) would go to:

$$(58) \qquad C_{1r}\, p_1' + C_{2r}\, p_2' + \cdots + C_{r-1,r}\, p_{r-1}' - C_{r,r+1}\, p_{r+1}' - C_{r,r+2}\, p_{r+2}' - \cdots - C_{r\rho}\, p_\rho' + C_r$$

$$- \mathfrak{p}_1' \left(\frac{\partial \omega_r}{\partial \mathfrak{p}_1} - \Phi_{1r} \right) - \cdots - \mathfrak{p}_\sigma' \left(\frac{\partial \omega_r}{\partial \mathfrak{p}_\sigma} - \Phi_{\sigma r} \right) - \frac{\partial \omega_r}{\partial \mathfrak{p}_1'} \mathfrak{p}_1'' - \cdots - \frac{\partial \omega_r}{\partial \mathfrak{p}_\sigma'} \mathfrak{p}_\sigma'' = P_r$$

and

(59) $$- \frac{\partial \omega_1}{\partial \mathfrak{p}_s'} \mathfrak{p}_1'' - \cdots - \frac{\partial \omega_\rho}{\partial \mathfrak{p}_s'} \mathfrak{p}_\rho'' + \mathfrak{p}_1' \left(\frac{\partial \omega_1}{\partial \mathfrak{p}_s} - \frac{d}{dt} \frac{\partial \omega_1}{\partial \mathfrak{p}_s'} \right) + \cdots + \mathfrak{p}_\rho' \left(\frac{\partial \omega}{\partial \mathfrak{p}_s} - \frac{d}{dt} \frac{\partial \omega_\rho}{\partial \mathfrak{p}_s'} \right) + p_1' \Phi_{s1} + \cdots + p_\rho' \Phi_{s\rho}$$
$$+ \left(\frac{\partial \psi}{\partial \mathfrak{p}_s} - \frac{d}{dt} \frac{\partial \psi}{\partial \mathfrak{p}_s'} \right) = \mathfrak{P}_s ,$$

while, from (55), the kinetic potential H would assume the form:

(60) $$H = \sum_{r=1}^{\rho} p_r' \{ C_{r,r+1} \, p_{r+1} + C_{r,r+2} \, p_{r+2}' + \cdots + C_{r\rho} \, p_\rho + \omega_r \}$$
$$+ \; p_1 (\Phi_{11} \, \mathfrak{p}_1' + \cdots + \Phi_{\sigma 1} \, \mathfrak{p}_\sigma' + C_1) + \cdots + p_\rho (\Phi_{1\rho} \, \mathfrak{p}_1' + \cdots + \Phi_{\sigma\rho} \, \mathfrak{p}_\sigma' + C_\rho) + \psi .$$

If one now substitutes the values of $p_1', \ldots, p_\rho'$, and the values of $p_1'', \ldots, p_\rho''$ that one derives from them using (58) in the σ equations (59) then one must examine whether those equations once more assume the **Lagrang**ian form, and what their kinetic potential would be. In order to get the lowest number of arbitrary functions in the last three equations, one sets:

$$\int (\Phi_{1r} \, d\mathfrak{p}_1 + \cdots + \Phi_{\sigma r} \, d\mathfrak{p}_\sigma) = \Phi_r (\mathfrak{p}_1, \ldots, \mathfrak{p}_s) ,$$

which is allowed, as a result of equation (57), and the ρ equations (58) will go to:

(61) $$C_{1r} \, p_1' + C_{2r} \, p_2' + \cdots + C_{r-1,r} \, p_{r-1}' - C_{r,r+1} \, p_{r+1}' - C_{r,r+2} \, p_{r+2}' - \cdots - C_{r\rho} \, p_\rho' + C_r$$
$$= \frac{d(\omega_r - \Phi_r)}{dt} + P_r .$$

Moreover, equations (59) will go to:

(62) $$\frac{\partial (p_1' \, \omega_1 + \cdots + p_\rho' \, \omega_\rho)}{\partial \mathfrak{p}_s} - \cdots - \frac{d}{dt} \frac{\partial (p_1' \, \omega_1 + \cdots + p_\rho' \, \omega_\rho)}{\partial \mathfrak{p}_s'} + p_1' \frac{\partial \Phi_1}{\partial \mathfrak{p}_s} + \cdots + p_\rho' \frac{\partial \Phi_\rho}{\partial \mathfrak{p}_s} + \frac{\partial \psi}{\partial \mathfrak{p}_s} - \frac{d}{dt} \frac{\partial \psi}{\partial \mathfrak{p}_s'}$$
$$= \mathfrak{P}_s ,$$

while the kinetic potential will assume the form:

(63) $$H = \sum_{r=1}^{\rho} p_r' \{ C_{r,r+1} \, p_{r+1} + C_{r,r+2} \, p_{r+2} + \cdots + C_{r\rho} \, p_\rho + \omega_r \} + p_1 \left(\frac{d\Phi_1}{dt} + C_1 \right) + \cdots + p_\rho \left(\frac{d\Phi_\rho}{dt} + C_\rho \right) + \psi .$$

However, since equation (61) can be put into the form:

$$\frac{d}{dt}\{C_{1r}\, p_1' + C_{2r}\, p_2' + \cdots + C_{r-1,r}\, p_{r-1}' - C_{r,r+1}\, p_{r+1}' - C_{r,r+2}\, p_{r+2}' - \cdots - C_{r\rho}\, p_\rho' - \omega_r + \Phi_r\} = P_r - C_r\,,$$

in which Φ_r is a function of $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$, and ω_r is a function of $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$, $\mathfrak{p}_1', \ldots, \mathfrak{p}_\sigma'$, we will be led to the case that was treated above in which the left-hand sides of the first ρ **Lagrange** equations can be represented as complete differential quotients with respect to t, in which the basis for the differentials is independent of $p_1', \ldots, p_\rho'$. It was shown that for those forms, for arbitrary ρ and $\sigma = 1$, when the external forces P are zero or constant, the last equation of motion will again assume the **Lagrang**ian form, while that is not permissible for $\sigma > 1$, in general, since that would require certain relations to exist between the functions that are included in the kinetic potential.

We will then find that:

*The necessary and sufficient condition for the ρ **Lagrange** equations to be independent of the second derivatives of the coordinates $p_1, \ldots, p_\rho$, as well as the coordinates themselves, is that none of the other equations of motion can include the coordinates $p_1, \ldots, p_\rho$, either. That is, the kinetic potential must possess the form:*

$$H = \sum_{r=1}^{\rho} p_r'\{C_{r,r+1}\, p_{r+1} + C_{r,r+2}\, p_{r+2} + \cdots + C_{r\rho}\, p_\rho + \omega_r\} + p_1\left(\frac{d\Phi_1}{dt} + C_1\right) + \cdots + p_\rho\left(\frac{d\Phi_\rho}{dt} + C_\rho\right) + \psi\,,$$

in which $C_1, \ldots, C_\rho$, $C_{r,r+1}, \ldots, C_{r\rho}$ are constants, $\omega_1, \ldots, \omega_\rho$, and ψ are functions of $\mathfrak{p}_1, \ldots, \mathfrak{p}_1', \ldots$, and $\Phi_1, \ldots, \Phi_\rho$ mean functions of $\mathfrak{p}_1, \ldots, \mathfrak{p}_\sigma$,

or, as is obvious when one adds a complete differential quotient with respect to t,

the equivalent form:

$$H = \sum_{r=1}^{\rho} p_r'\{C_{r,r+1}\, p_{r+1} + C_{r,r+2}\, p_{r+2} + \cdots + C_{r\rho}\, p_\rho + \Psi_r\} + \sum_{r=1}^{\rho} C_r\, p_r + \psi\,,$$

*in which Ψ_r and ψ represent arbitrary functions of $\mathfrak{p}_1, \ldots, \mathfrak{p}_\sigma$, $\mathfrak{p}_1', \ldots, \mathfrak{p}_\sigma'$. In that case, the first ρ **Lagrange** equations will go to the complete differential expressions:*

$$\frac{d}{dt}[C_{1r}\, p_1 + C_{2r}\, p_2 + \cdots + C_{r-1,r}\, p_{r-1} - C_{r,r+1}\, p_{r+1} - C_{r,r+2}\, p_{r+2} - \cdots - C_{r\rho}\, p_\rho - \Psi_r] = P_r + C_r\,,$$

while the following set of equations of motion will assume the form:

$$\sum_{r=1}^{\rho} p_r' \left(\frac{\partial \Psi_r}{\partial \mathfrak{p}_s} - \frac{d}{dt} \frac{\partial \Psi_r}{\partial \mathfrak{p}_s'} \right) - \sum_{r=1}^{\rho} p_r'' \frac{\partial \Psi_r}{\partial \mathfrak{p}_s''} + \frac{\partial \psi}{\partial \mathfrak{p}_s} - \frac{d}{dt} \frac{\partial \psi}{\partial \mathfrak{p}_s'} = \mathfrak{P}_s .$$

*For arbitrary ρ and $\sigma = 1$, when $P_1, \ldots, P_\rho$ are zero or constant, those equations can be reduced to the **Lagrang**ian normal form:*

$$\frac{\partial \mathfrak{H}}{\partial \mathfrak{p}_1} - \frac{d}{dt} \frac{\partial \mathfrak{H}}{\partial \mathfrak{p}_1'} = \mathfrak{P}_1 ,$$

with the first-order potential $\mathfrak{H}$, while for $\sigma > 1$, conditions must exist between the functions that are included in the kinetic potential.

Once more, that case is therefore excluded from the mechanics of ponderable masses, as would emerge from the form of the kinetical potential H.

The independence of the ρ **Lagrange** functions of the quantities $p_1'', \ldots, p_\rho''$ requires the form (48) for the kinetic potential, so the only case that remains to be considered is the one for which:

b) those very equations are independent of the quantities $p_1', \ldots, p_\rho'$, and the elimination of the values of the coordinates $p_1, \ldots, p_\rho$, and the derivatives that emerge from the following set of equations of motion can be performed.

It follows from the form (49) of the first ρ equations of motion that their independence from the quantities $p_1', \ldots, p_\rho'$ will imply the conditions:

(64) $$\frac{\partial \varphi_r}{\partial p_1} = \frac{\partial \varphi_1}{\partial p_r}, \quad \ldots, \quad \frac{\partial \varphi_r}{\partial p_\rho} = \frac{\partial \varphi_\rho}{\partial p_r} .$$

Therefore, those equations themselves will assume the form:

(65) $$\frac{\partial \varphi}{\partial p_r} - \frac{\partial \varphi_r}{\partial \mathfrak{p}_1} \mathfrak{p}_1' - \cdots - \frac{\partial \varphi_r}{\partial \mathfrak{p}_\sigma} \mathfrak{p}_\sigma' - \frac{\partial \varphi_r}{\partial \mathfrak{p}_1'} \mathfrak{p}_1'' - \cdots - \frac{\partial \varphi_r}{\partial \mathfrak{p}_\sigma'} \mathfrak{p}_\sigma'' = P_r ,$$

in which $\varphi_1, \ldots, \varphi_\rho, \varphi$ depend upon $p_1, \ldots, \mathfrak{p}_1, \ldots, \mathfrak{p}_1', \ldots$, and the kinetic potential will possess the form:

(66) $$H = \varphi_1 p_1' + \cdots + \varphi_\rho p_\rho' + \varphi .$$

Now, should the ρ equations (65) give the values:

(67) $$p_r = \omega_r\,(t, \mathfrak{p}', \ldots, \mathfrak{p}_1', \ldots, \mathfrak{p}_1'', \ldots)\,,$$

then it will follow that:

(68) $$p_r' = \frac{\partial\omega_r}{\partial t} + \sum_{\lambda=1}^{\sigma} \frac{\partial\omega_r}{\partial\mathfrak{p}_\lambda}\,\mathfrak{p}_\lambda' + \sum_{\lambda=1}^{\sigma} \frac{\partial\omega_r}{\partial\mathfrak{p}_\lambda'}\,\mathfrak{p}_\lambda'' + \sum_{\lambda=1}^{\sigma} \frac{\partial\omega_r}{\partial\mathfrak{p}_\lambda''}\,\mathfrak{p}_\lambda''',$$

(69) $$\begin{aligned} p_r'' = {} & \frac{\partial^2\omega_r}{\partial t^2} + 2\sum_{\lambda=1}^{\sigma}\left(\frac{\partial^2\omega_r}{\partial t\,\partial\mathfrak{p}_\lambda}\,\mathfrak{p}_\lambda' + \frac{\partial^2\omega_r}{\partial t\,\partial\mathfrak{p}_\lambda'}\,\mathfrak{p}_\lambda'' + \frac{\partial^2\omega_r}{\partial t\,\partial\mathfrak{p}_\lambda''}\,\mathfrak{p}_\lambda'''\right) \\ & + \sum_{\lambda=1}^{\sigma}\mathfrak{p}_\lambda''\left(\frac{\partial^2\omega_r}{\partial\mathfrak{p}_\lambda\,\partial\mathfrak{p}_1}\,\mathfrak{p}_1' + \cdots + \frac{\partial^2\omega_r}{\partial\mathfrak{p}_\lambda\,\partial\mathfrak{p}_\lambda'}\,\mathfrak{p}_\lambda'' + \cdots + \frac{\partial^2\omega_r}{\partial\mathfrak{p}_\lambda\,\partial\mathfrak{p}_\lambda''}\,\mathfrak{p}_\lambda''' + \cdots\right) \\ & + \sum_{\lambda=1}^{\sigma}\frac{\partial\omega_r}{\partial\mathfrak{p}_\lambda}\,\mathfrak{p}_\lambda'' + \sum_{\lambda=1}^{\sigma}\mathfrak{p}_\lambda''\left(\frac{\partial^2\omega_r}{\partial\mathfrak{p}_\lambda'\,\partial\mathfrak{p}_1}\,\mathfrak{p}_1' + \cdots + \frac{\partial^2\omega_r}{\partial\mathfrak{p}_\lambda'\,\partial\mathfrak{p}_\lambda'}\,\mathfrak{p}_\lambda'' + \cdots + \frac{\partial^2\omega_r}{\partial\mathfrak{p}_\lambda'\,\partial\mathfrak{p}_\lambda''}\,\mathfrak{p}_\lambda''' + \cdots\right) \\ & + \sum_{\lambda=1}^{\sigma}\frac{\partial\omega_r}{\partial\mathfrak{p}_\lambda'}\,\mathfrak{p}_\lambda''' + \sum_{\lambda=1}^{\sigma}\mathfrak{p}_\lambda'''\left(\frac{\partial^2\omega_r}{\partial\mathfrak{p}_\lambda''\,\partial\mathfrak{p}_1}\,\mathfrak{p}_1' + \cdots + \frac{\partial^2\omega_r}{\partial\mathfrak{p}_\lambda''\,\partial\mathfrak{p}_\lambda'}\,\mathfrak{p}_\lambda'' + \cdots + \frac{\partial^2\omega_r}{\partial\mathfrak{p}_\lambda''\,\partial\mathfrak{p}_\lambda''}\,\mathfrak{p}_\lambda''' + \cdots\right) + \sum_{\lambda=1}^{\sigma}\frac{\partial\omega_r}{\partial\mathfrak{p}_\lambda''}\,\mathfrak{p}_\lambda''''\,. \end{aligned}$$

If one substitutes those values in the system of equations:

$$\frac{\partial H}{\partial\mathfrak{p}_s} - \frac{d}{dt}\frac{\partial H}{\partial\mathfrak{p}_s'} = \mathfrak{P}_s$$

or

(70) $$\begin{aligned} & p_1'\,\frac{\partial\varphi_1}{\partial\mathfrak{p}_s} + \cdots + \frac{\partial\varphi}{\partial\mathfrak{p}_s} - \frac{\partial\varphi_1}{\partial\mathfrak{p}_s'}\,p_1'' - \cdots - \frac{d}{dt}\frac{\partial\varphi}{\partial\mathfrak{p}_s'} \\ & \quad - p_1'\left(\frac{\partial^2\varphi_1}{\partial\mathfrak{p}_s'\,\partial p_1}\,p_1' + \cdots + \frac{\partial^2\varphi_1}{\partial\mathfrak{p}_s'\,\partial\mathfrak{p}_1}\,\mathfrak{p}_1' + \cdots + \frac{\partial^2\varphi_1}{\partial\mathfrak{p}_s'\,\partial\mathfrak{p}_1'}\,\mathfrak{p}_1'' + \cdots\right) \\ & \quad - p_2'\left(\frac{\partial^2\varphi_2}{\partial\mathfrak{p}_s'\,\partial p_1}\,p_1' + \cdots + \frac{\partial^2\varphi_2}{\partial\mathfrak{p}_s'\,\partial\mathfrak{p}_1}\,\mathfrak{p}_1' + \cdots + \frac{\partial^2\varphi_2}{\partial\mathfrak{p}_s'\,\partial\mathfrak{p}_1'}\,\mathfrak{p}_1'' + \cdots\right) - \cdots = \mathfrak{P}_s\,, \end{aligned}$$

then under the assumption that the σ equations should, in turn, have a first-order kinetic potential, it will then follow that the coefficients of $\mathfrak{p}''''$ and $\mathfrak{p}'''$ must vanish. However, as is easy to see, that will then imply that the equations:

(71) $$\left(\frac{\partial\varphi_1}{\partial\mathfrak{p}_s'}\right)\frac{\partial\varphi_1}{\partial\mathfrak{p}_\lambda''} + \left(\frac{\partial\varphi_2}{\partial\mathfrak{p}_s'}\right)\frac{\partial\varphi_2}{\partial\mathfrak{p}_\lambda''} + \cdots + \left(\frac{\partial\varphi_\rho}{\partial\mathfrak{p}_s'}\right)\frac{\partial\varphi_\rho}{\partial\mathfrak{p}_\lambda''} = 0 \qquad (s, \lambda = 1, 2, \ldots, \sigma)$$

must be true identically, in which the quantities in parentheses should once more denote their values after the substitution. Therefore, when we again exclude the cases in which special relations are required between the functions $\varphi_1, \ldots, \varphi_\rho$, we will have:

$$\text{(72)} \qquad \left(\frac{\partial \varphi_1}{\partial \mathfrak{p}_s'}\right) = 0 , \quad \left(\frac{\partial \varphi_2}{\partial \mathfrak{p}_s'}\right) = 0 , \quad \ldots, \quad \left(\frac{\partial \varphi_\rho}{\partial \mathfrak{p}_s'}\right) = 0 .$$

However, it will follow from the identity (65):

$$\text{(73)} \qquad \left(\frac{\partial \varphi}{\partial p_r}\right) - \left(\frac{\partial \varphi_r}{\partial \mathfrak{p}_1}\right)\mathfrak{p}_1' - \cdots - \left(\frac{\partial \varphi_r}{\partial \mathfrak{p}_\sigma}\right)\mathfrak{p}_\sigma' = P_r$$

that the p_r cannot include the $\mathfrak{p}_s''$, such that one will have:

$$\text{(74)} \qquad p_r = \omega_r\,(t, \mathfrak{p}', \ldots, \mathfrak{p}_1', \ldots) \qquad \text{or} \qquad \left(\frac{\partial \omega_r}{\partial \mathfrak{p}_s''}\right) = 0 .$$

Now since the parentheses that are multiplied by p_λ''' in the expression (69) vanishes, after one substitutes the values (67), (68), (69) in (70), since p_r' will also no longer include p_λ''', due to (74), the coefficient of p_λ''' will be zero because of (72), so neither $\mathfrak{p}_\lambda''''$ nor p_λ''' will be included in the σ **Lagrange** equations.

However, if we preserve the meaning of the parentheses, then:

$$\frac{\partial \left(\dfrac{\partial \varphi_r}{\partial \mathfrak{p}_s'}\right)}{\partial \mathfrak{p}_s} = 0 = \left(\frac{\partial^2 \varphi_r}{\partial \mathfrak{p}_s' \, \partial \mathfrak{p}_\lambda}\right) + \left(\frac{\partial^2 \varphi_r}{\partial \mathfrak{p}_s' \, \partial p_1}\right)\frac{\partial \omega_1}{\partial \mathfrak{p}_\lambda} + \cdots,$$

$$\frac{\partial \left(\dfrac{\partial \varphi_r}{\partial \mathfrak{p}_s'}\right)}{\partial \mathfrak{p}_s'} = 0 = \left(\frac{\partial^2 \varphi_r}{\partial \mathfrak{p}_s' \, \partial \mathfrak{p}_\lambda'}\right) + \left(\frac{\partial^2 \varphi_r}{\partial \mathfrak{p}_s' \, \partial p_1}\right)\frac{\partial \omega_1}{\partial \mathfrak{p}_\lambda'} + \cdots$$

When we assume, for brevity, that the ω_r are independent of t, so the P_r are zero or constant, multiplying the latter equations by $\mathfrak{p}_\lambda'$, $\mathfrak{p}_\lambda''$, adding them, and summing over λ will give:

$$\sum_{\lambda=1}^{\sigma} \left\{ \left(\frac{\partial^2 \varphi_r}{\partial \mathfrak{p}_s' \, \partial \mathfrak{p}_\lambda}\right)\mathfrak{p}_\lambda' + \left(\frac{\partial^2 \varphi_r}{\partial \mathfrak{p}_s' \, \partial \mathfrak{p}_\lambda'}\right)\mathfrak{p}_\lambda'' \right\}$$

$$= -\left(\frac{\partial^2\varphi_r}{\partial\mathfrak{p}_s'\,\partial p_1}\right)\sum_{\lambda=1}^{\sigma}\left(\frac{\partial\omega_1}{\partial\mathfrak{p}_\lambda}\mathfrak{p}_\lambda' + \frac{\partial\omega_1}{\partial\mathfrak{p}_\lambda'}\mathfrak{p}_\lambda''\right) - \left(\frac{\partial^2\varphi_r}{\partial\mathfrak{p}_s'\,\partial p_2}\right)\sum_{\lambda=1}^{\sigma}\left(\frac{\partial\omega_2}{\partial\mathfrak{p}_\lambda}\mathfrak{p}_\lambda' + \frac{\partial\omega_2}{\partial\mathfrak{p}_\lambda'}\mathfrak{p}_\lambda''\right) - \cdots$$

$$= -\left(\frac{\partial^2\varphi_r}{\partial\mathfrak{p}_s'\,\partial p_1}\right)p_1' - \left(\frac{\partial^2\varphi_r}{\partial\mathfrak{p}_s'\,\partial p_2}\right)p_2' - \cdots,$$

so the parentheses in equation (70) that are multiplied by p_1', p_2', … will vanish, and after the substitution, the σ **Lagrange** equations will then assume the form:

$$(75) \qquad (p_1')\left(\frac{\partial\varphi_1}{\partial\mathfrak{p}_s}\right) + \cdots + (p_\rho')\left(\frac{\partial\varphi_\rho}{\partial\mathfrak{p}_s}\right) + \left(\frac{\partial\varphi}{\partial\mathfrak{p}_s}\right) - \frac{d}{dt}\left(\frac{\partial\varphi}{\partial\mathfrak{p}_s'}\right) = \mathfrak{P}_s$$

or

$$\left(\frac{\partial\varphi_1}{\partial\mathfrak{p}_s}\right)\left(\frac{\partial\omega_1}{\partial\mathfrak{p}_1}\mathfrak{p}_1' + \cdots + \frac{\partial\omega_1}{\partial\mathfrak{p}_1'}\mathfrak{p}_1'' + \cdots\right) + \cdots + \left(\frac{\partial\varphi_\rho}{\partial\mathfrak{p}_s}\right)\left(\frac{\partial\omega_\rho}{\partial\mathfrak{p}_1}\mathfrak{p}_1' + \cdots + \frac{\partial\omega_\rho}{\partial\mathfrak{p}_1'}\mathfrak{p}_1'' + \cdots\right) + \left(\frac{\partial\varphi}{\partial\mathfrak{p}_s}\right) - \frac{d}{dt}\left(\frac{\partial\varphi}{\partial\mathfrak{p}_s'}\right)$$

$$= \mathfrak{P}_s\,.$$

However, since:

$$\frac{\partial(\varphi)}{\partial\mathfrak{p}_s} = \left(\frac{\partial\varphi}{\partial\mathfrak{p}_s}\right) + \left(\frac{\partial\varphi}{\partial p_s}\right)\frac{\partial\omega_1}{\partial\mathfrak{p}_s} + \cdots, \qquad \frac{\partial(\varphi)}{\partial\mathfrak{p}_s'} = \left(\frac{\partial\varphi}{\partial\mathfrak{p}_s'}\right) + \left(\frac{\partial\varphi}{\partial p_s}\right)\frac{\partial\omega_1}{\partial\mathfrak{p}_s'} + \cdots,$$

that will make:

$$\left(\frac{\partial\varphi}{\partial\mathfrak{p}_s}\right) - \frac{d}{dt}\left(\frac{\partial\varphi}{\partial\mathfrak{p}_s'}\right) = \frac{\partial(\varphi)}{\partial\mathfrak{p}_s} - \frac{d}{dt}\frac{\partial(\varphi)}{\partial\mathfrak{p}_s'} - \sum_{r=1}^{\rho}\left(\frac{\partial\varphi}{\partial p_r}\right)\frac{\partial\omega_r}{\partial\mathfrak{p}_s} + \frac{d}{dt}\sum_{r=1}^{\rho}\left(\frac{\partial\varphi}{\partial p_r}\right)\frac{\partial\omega_r}{\partial\mathfrak{p}_s'}\,.$$

Therefore, in order to investigate whether equations (75) can once more be put into **Lagrang**ian form, one only needs to establish whether the identity:

$$(76) \qquad \sum_{r=1}^{\rho}\left(\frac{\partial\varphi_r}{\partial\mathfrak{p}_s}\right)\left(\frac{\partial\omega_r}{\partial\mathfrak{p}_1}\mathfrak{p}_1' + \cdots + \frac{\partial\omega_r}{\partial\mathfrak{p}_1'}\mathfrak{p}_1'' + \cdots\right) - \sum_{r=1}^{\rho}\left(\frac{\partial\varphi}{\partial p_r}\right)\frac{\partial\omega_r}{\partial\mathfrak{p}_s} + \frac{d}{dt}\sum_{r=1}^{\rho}\left(\frac{\partial\varphi}{\partial p_r}\right)\frac{\partial\omega_r}{\partial\mathfrak{p}_s'} = \frac{\partial K}{\partial\mathfrak{p}_s} - \frac{d}{dt}\frac{\partial K}{\partial\mathfrak{p}_s'}$$

can be fulfilled with the use of equation (73) when K is supposed to be a function of $\mathfrak{p}_1$, …, $\mathfrak{p}_1'$, … If one now sets:

$$-\frac{\partial K}{\partial\mathfrak{p}_s'} - \sum_{r=1}^{\rho}\frac{\partial\omega_r}{\partial\mathfrak{p}_s'}\left(\frac{\partial\varphi}{\partial p_r}\right) = L\,,$$

such that (76) will go to:

(77) $$\frac{dL}{dt} = \sum_{r=1}^{\rho}\left(\frac{\partial\varphi_r}{\partial\mathfrak{p}_s}\right)\left(\frac{\partial\omega_r}{\partial\mathfrak{p}_1}\mathfrak{p}_1'+\cdots+\frac{\partial\omega_r}{\partial\mathfrak{p}_1'}\mathfrak{p}_1''+\cdots\right)-\sum_{r=1}^{\rho}\left(\frac{\partial\varphi}{\partial p_r}\right)\frac{\partial\omega_r}{\partial\mathfrak{p}_s}-\frac{\partial K}{\partial\mathfrak{p}_s},$$

then when one considers the values of $\left(\frac{\partial\varphi}{\partial p_r}\right)$ that equation (73) implies, it will follow that the coefficients of $\mathfrak{p}_\delta''$ and $\mathfrak{p}_s''$ on the two sides of equation (77) will yield the relations:

$$\frac{\partial L}{\partial\mathfrak{p}_\delta'} = \sum_{r=1}^{\rho}\left(\frac{\partial\varphi_r}{\partial\mathfrak{p}_s}\right)\frac{\partial\omega_r}{\partial\mathfrak{p}_\delta'}, \qquad \frac{\partial L}{\partial\mathfrak{p}_s'} = \sum_{r=1}^{\rho}\left(\frac{\partial\varphi_r}{\partial\mathfrak{p}_s}\right)\frac{\partial\omega_r}{\partial\mathfrak{p}_s'},$$

from which, it will again follow that:

(78) $$\sum_{r=1}^{\rho}\frac{\partial}{\partial\mathfrak{p}_s'}\left[\left(\frac{\partial\varphi_r}{\partial\mathfrak{p}_s}\right)\frac{\partial\omega_r}{\partial\mathfrak{p}_\delta'}\right] = \sum_{r=1}^{\rho}\frac{\partial}{\partial\mathfrak{p}_\delta'}\left[\left(\frac{\partial\varphi_r}{\partial\mathfrak{p}_s}\right)\frac{\partial\omega_r}{\partial\mathfrak{p}_s'}\right].$$

For $\sigma > 1$, that equation will once more require a relation between the functions $\varphi_1, \ldots, \varphi_r$, which was excluded from the investigation from the outset. However, if ρ is arbitrary and $\sigma = 1$ then the left-hand side of equation (76) will go to:

$$\sum_{r=1}^{\rho}\left(\frac{\partial\varphi_r}{\partial\mathfrak{p}}\right)\left(\frac{\partial\omega_r}{\partial\mathfrak{p}}\mathfrak{p}'+\frac{\partial\omega_r}{\partial\mathfrak{p}'}\mathfrak{p}''\right)-\sum_{r=1}^{\rho}\left(P_r+\left(\frac{\partial\varphi_r}{\partial\mathfrak{p}}\right)\mathfrak{p}'\right)\frac{\partial\omega_r}{\partial\mathfrak{p}}+\frac{d}{dt}\sum_{r=1}^{\rho}\frac{\partial\omega_r}{\partial\mathfrak{p}'}\left(P_r+\left(\frac{\partial\varphi_r}{\partial\mathfrak{p}}\right)\mathfrak{p}'\right)$$

by means of (73), or under the assumption that the P_r are zero or constants, when one sets:

(79) $$K = -\,\mathfrak{p}'\int\sum_{r=1}^{\rho}\frac{\partial\omega_r}{\partial\mathfrak{p}'}\left(\frac{\partial\varphi_r}{\partial\mathfrak{p}}\right)d\mathfrak{p}'-\sum_{r=1}^{\rho}P_r\,\omega_r\,,$$

it will go to the desired form (*):

(*) For example, let:

$$H = p'(p^2+\mathfrak{p}^2+\mathfrak{p}'^2\,p+\mathfrak{p}')+2\mathfrak{p}\mathfrak{p}'\,p-\mathfrak{p}'\,p^2-p+1.$$

When the external force is zero, the first **Lagrange** equation will read:

$$2\,\mathfrak{p}'\,p+1=0,$$

so it will be free of p' and p'', and that will yield $p = -\,1/2\mathfrak{p}'$. If one substitutes that value in the second equation of motion:

$$-\,2\,p\,p'\,\mathfrak{p}''-2\mathfrak{p}'\,p'^2-2\,\mathfrak{p}'p\,p''-p''+2\,p\,p' = \mathfrak{P}$$

then that will give:

$$-\frac{\mathfrak{p}''}{2\,\mathfrak{p}'^3} = \mathfrak{P}.$$

$$\frac{\partial K}{\partial \mathfrak{p}} - \frac{d}{dt}\frac{\partial K}{\partial \mathfrak{p}'},$$

and we will then find that:

The necessary and sufficient condition for the first ρ ***Lagrange*** *equations to be independent of the first and second derivatives of the coordinates $p_1, \ldots, p_\rho$ is that the kinetic potential must have the form:*

$$H = \varphi_1(p_1, \ldots, \mathfrak{p}_1, \ldots, \mathfrak{p}'_1, \ldots)\, p'_1 + \cdots + \varphi_\rho(p_1, \ldots, \mathfrak{p}_1, \ldots, \mathfrak{p}'_1, \ldots)\, p'_\rho + \varphi(p_1, \ldots, \mathfrak{p}_1, \ldots, \mathfrak{p}'_1, \ldots) \;.$$

Now should the elimination of $p_1, \ldots, p_\rho$ again yield equations in ***Lagrang****ian form with a first-order kinetic potential, then if no special relations exist between the φ-functions, it would be necessary and sufficient that $\sigma = 1$ and the forces $P_1, \ldots, P_\rho$ are zero or constants. Indeed, that differential equation will then assume the form:*

$$\frac{\partial \mathfrak{H}}{\partial \mathfrak{p}} - \frac{d}{dt}\frac{\partial \mathfrak{H}}{\partial \mathfrak{p}'} = \mathfrak{P} \,,$$

in which the first-order kinetic potential is:

$$\mathfrak{H} = (\varphi) - \mathfrak{p}' \int \sum_{r=1}^{\rho} \frac{\partial \omega_r}{\partial \mathfrak{p}'} \left(\frac{\partial \varphi_r}{\partial \mathfrak{p}}\right) d\mathfrak{p}' - \sum_{r=1}^{\rho} P_r\, \omega_r \,,$$

and the expressions in parentheses mean the values after substituting the expressions for $p_1, \ldots, p_\rho$ as functions of $\mathfrak{p}$ and $\mathfrak{p}'$ that are derived by the first ρ equations of motion.

However, since:

$$\varphi = p\, p' \mathfrak{p}'' - \mathfrak{p}'\, p'^2 - p + 1$$

in the present case, so:

$$(\varphi) = - \mathfrak{p} + \frac{1}{4\mathfrak{p}'} + 1 \,,$$

and furthermore:

$$\left(\frac{\partial \varphi_1}{\partial \mathfrak{p}}\right) = 2\,\mathfrak{p} \,, \qquad \frac{\partial \omega_1}{\partial \mathfrak{p}'} = \frac{1}{2\,\mathfrak{p}'^2} \,,$$

one will then have:

$$\mathfrak{H} = \frac{1}{4\mathfrak{p}'} + 1 \,,$$

for which one will, in fact, have:

$$\frac{\partial \mathfrak{H}}{\partial \mathfrak{p}} - \frac{d}{dt}\frac{\partial \mathfrak{H}}{\partial \mathfrak{p}'} = \mathfrak{P} \,.$$

That case cannot occur in the mechanics of ponderable masses either, which should be clear from the form of H.

With that, all of the cases have been examined in which the behavior of a set of equations of a system with a first-order kinetic potential will permit the elimination of coordinates without performing possible integrations, and in turn lead to **Lagrange** equations with a first-order kinetic potential. Thus, all cases of the extended hidden equations have also been ascertained that assume kinetic potentials that depend upon only the coordinates and their first derivatives, and indeed that will imply the following five forms that are necessary and sufficient for that elimination to be possible:

1) *When the left-hand sides of the **Lagrange** equations of motion that correspond to the coordinates $p_1, \ldots, p_\rho$ for a kinetic potential that is free of t are complete derivatives with respect to time of functions of all coordinates and their first derivatives but does not, however, include the $p_1, \ldots, p_\rho$ themselves:*

$$H = \Omega(\mathfrak{p}_1, \ldots, \mathfrak{p}_\sigma, \mathfrak{p}_1', \ldots, \mathfrak{p}_\sigma', p_1', \ldots, p_\rho') ,$$

in which Ω represents an arbitrary function of the included quantities.

2) *When one makes the same assumption that the left-hand sides of the ρ **Lagrange** equations can be represented as complete differential quotients and assumes that the basic function for the differential quotients is independent of $p_1', \ldots, p_\rho'$, for $\rho + 1$ **Lagrange** equations, one has:*

$$H = \sum_{r=1}^{\rho} p_r' (C_{r,r+1}\, p_{r+1} + C_{r,r+2}\, p_{r+2} + \cdots + C_{r\rho}\, p_\rho + \varphi_r) + \varphi ,$$

in which $C_{r,r+1}, \ldots, C_{r\rho}$ are arbitrary constants, and $\varphi_1, \ldots, \varphi_\rho, \varphi$ mean arbitrary functions of $\mathfrak{p}$ and $\mathfrak{p}'$.

3) *When the first $\rho + \sigma$ **Lagrange** equations are free of $p_1, \ldots, p_\rho, p_1', \ldots, p_\rho'$:*

$$H = \sum_{r,\delta=1}^{\rho} c_{r\delta}\, p_r'\, p_\delta' + \sum_{r=1}^{\rho} p_r' (R_{1r}\, \mathfrak{p}_1' + R_{2r}\, \mathfrak{p}_2' + \cdots + R_{\sigma r}\, \mathfrak{p}_\sigma') + \sum_{r=1}^{\rho} c_r\, p_r + U ,$$

in which c_r and $c_{r\delta} = c_{\delta r}$ mean arbitrary constants, R_{sr} are arbitrary functions of $\mathfrak{p}_1, \ldots, \mathfrak{p}_\sigma$ that are subject to the condition that:

$$\frac{\partial R_{s_1 r}}{\partial \mathfrak{p}_s} = \frac{\partial R_{s r}}{\partial \mathfrak{p}_{s_1}} ,$$

and U represents an arbitrary function of $\mathfrak{p}_1, \ldots, \mathfrak{p}_\sigma, \mathfrak{p}_1', \ldots, \mathfrak{p}_\sigma'$.

4) *When all* $\rho + \sigma$ *equations of motion are free of* $p_1, \ldots, p_\rho$, *and the first* ρ *equations are free of* $p_1'', \ldots, p_\rho''$, *in addition. For* $\sigma = 1$, *one has:*

$$H = \sum_{r=1}^{\rho} p_r' \{C_{r,r+1}\, p_{r+1} + C_{r,r+2}\, p_{r+2} + \cdots + C_{r\rho}\, p_\rho + \Psi_r\} + \sum_{r=1}^{\rho} C_r\, p_r + \psi\,,$$

in which $C_1, \ldots, C_\rho, C_{r,r+1}, \ldots, C_{r\rho}$ *mean arbitrary constants, and* Ψ_r, *as well as* ψ, *are arbitrary functions of* $\mathfrak{p}$ *and* $\mathfrak{p}'$.

5) *When the first* ρ ***Lagrange*** *equations are independent of the first and second derivatives of the coordinates* $p_1, \ldots, p_\rho$, *and only* $\rho + 1$ *equations of motion are present, one will have:*

$$H = \varphi_1(p_1, \ldots, p_\rho, \mathfrak{p}, \mathfrak{p}')\, p_1' + \cdots + \varphi_\rho(p_1, \ldots, p_\rho, \mathfrak{p}, \mathfrak{p}')\, p_\rho' + \varphi(p_1, \ldots, p_\rho, \mathfrak{p}, \mathfrak{p}'),$$

in which $\varphi_1, \ldots, \varphi_\rho, \varphi$ *represent arbitrary functions of their arguments.*

In all of that, one overlooks a complete differential quotient with respect to t of an arbitrary function.

The foregoing treatment of the problem itself shows how the investigation of kinetic potentials must be extended when they include derivatives of the coordinates of arbitrarily-high order, which has been the basis for our investigations up to now.

§ 16. – Helmholtz's case of incomplete problems.

The treatment of incomplete problems that **Helmholtz** gave for the mechanics of ponderable masses can be adapted directly to the case in which the kinetic potential H is an arbitrary function of $t, p_1, \ldots, p_\rho, \mathfrak{p}_1, \ldots, \mathfrak{p}_\sigma$, and their first derivatives.

One assumes that:

$$p_1 = c_1\,, \qquad p_2 = c_2\,, \qquad \ldots, \qquad p_\rho = c_\rho\,,$$

in which the c_r mean constants, are possible solutions of the problem, and one further assumes that the terms in H in which $p'_1, p'_2, \ldots, p'_\rho$ occur linearly possess coefficients that depend upon only $p_1, p_2, \ldots, p_\rho$, but not upon $t, \mathfrak{p}_1, \ldots, \mathfrak{p}_\sigma, \mathfrak{p}'_1, \ldots, \mathfrak{p}'_\sigma$. In the mechanics of ponderable masses, H includes only terms of degree two in $p'_1, \ldots, p'_\rho, \mathfrak{p}'_1, \ldots, \mathfrak{p}'_\sigma$, and one will then need only to add the condition that **Helmholtz** exhibited:

$$\frac{\partial^2 H}{\partial p'_r\,\partial \mathfrak{p}'_s} = 0\,.$$

The first ρ **Lagrange** equations will then become:

$$\frac{\partial H}{\partial p_r} - \frac{d}{dt}\frac{\partial H}{\partial p'_r} = P_r$$

or

$$\frac{\partial H}{\partial p_r} - \frac{\partial^2 H}{\partial p'_r\,\partial t} - \sum_{\lambda=1}^{\rho} \frac{\partial^2 H}{\partial p'_r\,\partial p_\lambda} p'_\lambda - \sum_{\lambda=1}^{\rho} \frac{\partial^2 H}{\partial p'_r\,\partial p'_\lambda} p''_\lambda - \sum_{\lambda=1}^{\rho} \frac{\partial^2 H}{\partial p'_r\,\partial p_\lambda} p'_\lambda - \sum_{s=1}^{\sigma} \frac{\partial^2 H}{\partial p'_r\,\partial \mathfrak{p}_s} \mathfrak{p}'_s - \sum_{s=1}^{\sigma} \frac{\partial^2 H}{\partial p'_r\,\partial \mathfrak{p}'_s} \mathfrak{p}''_s$$
$$= P_r\,,$$

since under the assumption that the values satisfy:

$$p'_1 = p'_2 = \ldots = p'_\rho = 0$$

they will go to:

$$\left(\frac{\partial H}{\partial p_r}\right)_{p'_1=\cdots=p'_\rho=0} = P_r \qquad (r = 1, 2, \ldots, \rho).$$

If one uses those equations to calculate:

$$p_1 = \omega_1(t, \mathfrak{p}_1, \ldots, \mathfrak{p}_\sigma, \mathfrak{p}'_1, \ldots, \mathfrak{p}'_\sigma)\,, \ldots,$$

$$p_\rho = \omega_\rho(t, \mathfrak{p}_1, \dots, \mathfrak{p}_\sigma, \mathfrak{p}'_1, \dots, \mathfrak{p}'_\sigma), \dots,$$

and substitutes those values in the other σ **Lagrange** equations then if one preserves the previous meaning for the parentheses, one will get:

$$(1) \qquad \left(\frac{\partial \mathfrak{H}}{\partial \mathfrak{p}_s}\right) - \frac{d}{dt}\left(\frac{\partial \mathfrak{H}}{\partial \mathfrak{p}'_s}\right) = \mathfrak{P}_s .$$

However, since one further has:

$$\frac{\partial (H)}{\partial \mathfrak{p}_s} = \left(\frac{\partial H}{\partial \mathfrak{p}_s}\right) + \sum_{r=1}^{\rho}\left(\frac{\partial H}{\partial p_r}\right)\frac{\partial \omega_r}{\partial \mathfrak{p}_s} = \left(\frac{\partial H}{\partial \mathfrak{p}_s}\right) + \sum_{r=1}^{\rho} P_r \frac{\partial \omega_r}{\partial \mathfrak{p}_s},$$

$$\frac{\partial (H)}{\partial \mathfrak{p}'_s} = \left(\frac{\partial H}{\partial \mathfrak{p}'_s}\right) + \sum_{r=1}^{\rho}\left(\frac{\partial H}{\partial p_r}\right)\frac{\partial \omega_r}{\partial \mathfrak{p}'_s} = \left(\frac{\partial H}{\partial \mathfrak{p}'_s}\right) + \sum_{r=1}^{\rho} P_r \frac{\partial \omega_r}{\partial \mathfrak{p}'_s},$$

it will follow from (1) that for the case in which the P_r are given as functions of t, one will have:

$$\frac{\partial (H)}{\partial \mathfrak{p}_s} - \frac{d}{dt}\frac{\partial (H)}{\partial \mathfrak{p}'_s} - \left(\frac{\partial}{\partial \mathfrak{p}_s}\sum_{r=1}^{\rho} P_r\,\omega_r - \frac{d}{dt}\frac{\partial}{\partial \mathfrak{p}'_s}\sum_{r=1}^{\rho} P_r\,\omega_r\right) = \mathfrak{P}_s,$$

and when one sets:

$$(H) - \sum_{r=1}^{\rho} P_r\,\omega_r = \mathfrak{H},$$

those σ equations of motion will then have the **Lagrang**ian form:

$$\frac{\partial \mathfrak{H}}{\partial \mathfrak{p}_s} - \frac{d}{dt}\frac{\partial \mathfrak{H}}{\partial \mathfrak{p}'_s} = \mathfrak{P}_s \qquad (s = 1, 2, \dots, \sigma),$$

in which $\mathfrak{H}$ is, in turn, a first-order kinetic potential.

Therefore, if it is possible to have motions in a system that is defined by $\rho + \sigma$ ***Lagrange*** *equations for which $p_1, p_2, \dots, p_\rho$ are constants, and if the kinetic potential H possesses the property that the terms in which $p'_1, \dots, p'_\rho$ occur linearly possess coefficients that depend upon only $p_1, p_2, \dots, p_\rho$, but not upon t, $\mathfrak{p}_1, \dots, \mathfrak{p}_\sigma$, $\mathfrak{p}'_1, \dots, \mathfrak{p}'_\sigma$, then when one sets $p'_1 = p'_2 = \dots = p'_\rho = 0$ in the first ρ equations and expresses the $p_1, \dots, p_\rho$ in terms of t, $\mathfrak{p}_1, \dots, \mathfrak{p}_\sigma$, $\mathfrak{p}'_1, \dots, \mathfrak{p}'_\sigma$ by using those equations and substitutes the values obtained in the other σ equations of motion, one will, in turn, get σ* ***Lagrange*** *equations with a first-order kinetic potential that is represented by:*

$$\mathfrak{H} = (H) - \sum_{r=1}^{\rho} P_r \, \omega_r \, ,$$

when $\omega_1, \ldots, \omega_\rho$ *are the substituted values.*

That case includes the conditions for extended *monocyclic* systems.

We shall not go into a more detailed examination of the other cases in which other possible solutions to the problem are also known at this point.

§ 17. – Lowering the number of coordinates in the Lagrange equations of motion by raising the order of the kinetic potential.

Once the investigation has been carried out of the necessary and sufficient conditions for the form of a first-order kinetic potential in order for the elimination of a number of coordinates from the system of **Lagrange** equations of motion to lead to a smaller number of **Lagrange** equations that will be, in turn, based upon a first-order kinetic potential (which is an investigation that came under consideration for the mechanics of ponderable masses only for the special form of the separation of the kinetic potential into current and potential energy), the essential and interesting question in our investigations that will come to the foreground is the question of whether it is possible to reduce the number of equations in a system of **Lagrange** equations of motion with a kinetic potential or forces of a certain order when the new system, in turn, is to be represented by **Lagrange** equations, but for a kinetic potential or forces of higher order. More briefly, whether some of points in the motion of a system of points under the influence of forces of a certain order can be described by the action of forces of higher order.

We would first like to consider the case of *two* **Lagrange** equations with the coordinates p and $\mathfrak{p}$, and which belong a first-order kinetic potential H that can also include t explicitly:

$$\text{(1)} \qquad \frac{\partial H}{\partial p} - \frac{d}{dt}\frac{\partial H}{\partial p'} = P ,$$

$$\text{(2)} \qquad \frac{\partial H}{\partial \mathfrak{p}} - \frac{d}{dt}\frac{\partial H}{\partial \mathfrak{p}'} = \mathfrak{P} ,$$

and assume that the first of those equations does not include p'', so we will have:

$$\text{(3)} \qquad \frac{\partial^2 H}{\partial p'^2} = 0 .$$

It will then follow immediately that they must also be free of p', since the partial differential quotient with respect to p' of its left-hand side:

$$\frac{\partial^2 H}{\partial p\,\partial p'} - \frac{d}{dt}\frac{\partial^2 H}{\partial p'^2} - \frac{\partial^2 H}{\partial p'\,\partial p}$$

will vanish identically because of (3). H will then have the form:

$$\text{(4)} \qquad H = f(t, p, \mathfrak{p}, \mathfrak{p}')\, p' + f_1(t, p, \mathfrak{p}, \mathfrak{p}') ,$$

which is then necessary and sufficient for the ***Lagrange*** *equation to be independent of* p' *and* p''.

Now in order to eliminate the coordinate p and its derivatives from (1) and (2), it will follow from (1) that:

$$p = \omega(t, \mathfrak{p}, \mathfrak{p}', \mathfrak{p}''), \tag{5}$$

so

$$p' = \frac{\partial \omega}{\partial t} + \frac{\partial \omega}{\partial \mathfrak{p}} \mathfrak{p}' + \frac{\partial \omega}{\partial \mathfrak{p}'} \mathfrak{p}'' + \frac{\partial \omega}{\partial \mathfrak{p}''} \mathfrak{p}''',$$

$$p'' = \frac{\partial^2 \omega}{\partial t^2} + 2\,\frac{\partial^2 \omega}{\partial t\,\partial \mathfrak{p}} \mathfrak{p}' + \cdots + \frac{\partial \omega}{\partial \mathfrak{p}''} \mathfrak{p}'''',$$

and if one substitutes those values for p, p', p'' in the second **Lagrange** equation then one will get:

$$\left(\frac{\partial H}{\partial \mathfrak{p}}\right) - \frac{d}{dt}\left(\frac{\partial H}{\partial \mathfrak{p}'}\right) = \mathfrak{P}. \tag{6}$$

However, when one substitutes those values in H, (H) will become a function of t, $\mathfrak{p}$, $\mathfrak{p}'$, $\mathfrak{p}''$, $\mathfrak{p}'''$ that includes $\mathfrak{p}'''$ only linearly, such that the following relations will exist:

$$\frac{\partial (H)}{\partial \mathfrak{p}} = \left(\frac{\partial H}{\partial \mathfrak{p}}\right) + \left(\frac{\partial H}{\partial p}\right)\frac{\partial p}{\partial \mathfrak{p}} + \left(\frac{\partial H}{\partial p'}\right)\frac{\partial p'}{\partial \mathfrak{p}},$$

$$\frac{\partial (H)}{\partial \mathfrak{p}'} = \left(\frac{\partial H}{\partial \mathfrak{p}'}\right) + \left(\frac{\partial H}{\partial p}\right)\frac{\partial p}{\partial \mathfrak{p}'} + \left(\frac{\partial H}{\partial p'}\right)\frac{\partial p'}{\partial \mathfrak{p}'},$$

$$\frac{\partial (H)}{\partial \mathfrak{p}''} = \left(\frac{\partial H}{\partial p}\right)\frac{\partial p}{\partial \mathfrak{p}''} + \left(\frac{\partial H}{\partial p'}\right)\frac{\partial p'}{\partial \mathfrak{p}''}, \qquad \frac{\partial (H)}{\partial \mathfrak{p}'''} = \left(\frac{\partial H}{\partial p'}\right)\frac{\partial p'}{\partial \mathfrak{p}'''},$$

or since one has:

$$\frac{\partial p'}{\partial \mathfrak{p}} = \frac{d}{dt}\frac{\partial p}{\partial \mathfrak{p}}, \qquad \frac{\partial p'}{\partial \mathfrak{p}'} = \frac{d}{dt}\frac{\partial p}{\partial \mathfrak{p}'} + \frac{\partial p}{\partial \mathfrak{p}},$$

$$\frac{\partial p'}{\partial \mathfrak{p}''} = \frac{d}{dt}\frac{\partial p}{\partial \mathfrak{p}''} + \frac{\partial p}{\partial \mathfrak{p}'}, \qquad \frac{\partial p'}{\partial \mathfrak{p}'''} = \frac{\partial p}{\partial \mathfrak{p}''},$$

from the auxiliary formulas of **§ 2**, that will make:

$$\frac{\partial (H)}{\partial \mathfrak{p}} = \left(\frac{\partial H}{\partial \mathfrak{p}}\right) + \left(\frac{\partial H}{\partial p}\right)\frac{\partial p}{\partial \mathfrak{p}} + \left(\frac{\partial H}{\partial p'}\right)\frac{d}{dt}\frac{\partial p}{\partial \mathfrak{p}},$$

$$\frac{\partial(H)}{\partial\mathfrak{p}'} = \left(\frac{\partial H}{\partial\mathfrak{p}'}\right) + \left(\frac{\partial H}{\partial p}\right)\frac{\partial p}{\partial\mathfrak{p}'} + \left(\frac{\partial H}{\partial p'}\right)\left[\frac{d}{dt}\frac{\partial p}{\partial\mathfrak{p}'} + \frac{\partial p}{\partial\mathfrak{p}}\right],$$

$$\frac{\partial(H)}{\partial\mathfrak{p}''} = \left(\frac{\partial H}{\partial p}\right)\frac{\partial p}{\partial\mathfrak{p}''} + \left(\frac{\partial H}{\partial p'}\right)\left[\frac{d}{dt}\frac{\partial p}{\partial\mathfrak{p}''} + \frac{\partial p}{\partial\mathfrak{p}'}\right], \qquad \frac{\partial(H)}{\partial\mathfrak{p}'''} = \left(\frac{\partial H}{\partial p'}\right)\frac{\partial p}{\partial\mathfrak{p}''}.$$

However, it follows immediately from the identity that:

$$\frac{\partial(H)}{\partial\mathfrak{p}} - \frac{d}{dt}\frac{\partial(H)}{\partial\mathfrak{p}'} + \frac{d^2}{dt^2}\frac{\partial(H)}{\partial\mathfrak{p}''} - \frac{d^3}{dt^3}\frac{\partial(H)}{\partial\mathfrak{p}'''} = \left(\frac{\partial H}{\partial\mathfrak{p}}\right) - \frac{d}{dt}\left(\frac{\partial H}{\partial\mathfrak{p}'}\right),$$

and the second equation of motion (6) will then assume the form:

$$\frac{\partial(H)}{\partial\mathfrak{p}} - \frac{d}{dt}\frac{\partial(H)}{\partial\mathfrak{p}'} + \frac{d^2}{dt^2}\frac{\partial(H)}{\partial\mathfrak{p}''} - \frac{d^3}{dt^3}\frac{\partial(H)}{\partial\mathfrak{p}'''} = \mathfrak{P}. \tag{7}$$

That will then describe the motion of the coordinate $\mathfrak{p}$ with the help of the kinetic potential (H), which has order three, but is linear in $\mathfrak{p}'''$. From **Hamilton**'s principle in **§ 6**, the equation of motion (6), which is a fourth-order differential equation, will be equivalent to the variational equation:

$$\delta\int_{t_0}^{t_1}((H) - \mathfrak{P}\,\mathfrak{p})\,dt = 0, \tag{8}$$

which can be excluded from the outset when we do not give preference to the deduction above, in light of the previous discussion. However, it is also easy to show that a third-order kinetic potential can be replaced with a second-order one, because since (H) is linear in $\mathfrak{p}'''$, so it will have the form:

$$(H) = F(t, \mathfrak{p}, \mathfrak{p}', \mathfrak{p}'')\,\mathfrak{p}''' + F_1(t, \mathfrak{p}, \mathfrak{p}', \mathfrak{p}''), \tag{9}$$

when one sets:

$$\psi = \int F(t, \mathfrak{p}, \mathfrak{p}', \mathfrak{p}'')\,d\,\mathfrak{p}''$$

and determines:

$$K = \frac{d\psi}{dt} = \int\frac{\partial F}{\partial t}\,d\,\mathfrak{p}'' + \mathfrak{p}'\int\frac{\partial F}{\partial\mathfrak{p}}\,d\,\mathfrak{p}'' + \mathfrak{p}''\int\frac{\partial F}{\partial\mathfrak{p}'}\,d\,\mathfrak{p}'' + F(t, \mathfrak{p}, \mathfrak{p}', \mathfrak{p}'')\,\mathfrak{p}''', \tag{10}$$

since K is a complete differential quotient with respect to t, one will have:

$$\delta \int_{t_0}^{t_1} K\,dt = 0\,,$$

from Lemma 3. When that equation is subtracted from (8), one will get:

$$\delta \int_{t_0}^{t_1} ((H) - \mathfrak{P}\,\mathfrak{p})\,dt = \delta \int_{t_0}^{t_1} (\mathfrak{H} - \mathfrak{P}\,\mathfrak{p})\,dt\,,$$

in which:

$$(H) - K = \mathfrak{H}$$

represents a second-order kinetic potential by means of (9) and (10) (*).

(*) It emerges from the presentation above that a kinetic potential in one variable and arbitrary order that includes the highest derivative of the coordinate only linearly can always be replaced with a kinetic potential with order one less, but we can make that remark more generally, such that it yields an extension of Lemma 4, which we added at that point in order to make its applicability emerge more clearly. The necessary and sufficient conditions were presented above for a function N that depends upon $t, p, p', \ldots, p^{(2\nu)}$ to possess a kinetic potential M, or for a function M of $t, p, p', \ldots, p^{(\nu)}$ to exist that has the property that:

$$(\alpha) \qquad \delta \int_{t_0}^{t_1} M\,dt = \int_{t_0}^{t_1} N\,\delta p\,dt\,.$$

However, if one drops the condition that the kinetic potential has order ν and asks what the existence condition would be for a kinetic potential M of higher order ρ, such that:

$$N = \frac{\partial M}{\partial p} - \frac{d}{dt}\frac{\partial M}{\partial p'} + \cdots + (-1)^{\rho-1}\frac{d^{\rho-1}}{dt^{\rho-1}}\frac{\partial M}{\partial p^{(\rho-1)}} + (-1)^{\rho}\frac{d^{\rho}}{dt^{\rho}}\frac{\partial M}{\partial p^{(\rho)}}\,,$$

in which $\rho > \nu$, then it will be easy to see that $\dfrac{\partial M}{\partial p^{(\rho)}}$ must be independent of $p^{(\rho)}$, because otherwise the right-hand side would have to include the derivative $p^{(2\rho)}$, while the left-hand side has order only 2ν. However, if M has the form:

$$M = \varphi(t, p, p', \ldots, p^{(\rho-1)})\,p^{(\rho)} + \psi(t, p, p', \ldots, p^{(\rho-1)})\,,$$

and one defines the function:

$$\omega = \int \varphi(t, p, p', \ldots, p^{(\rho-1)})\,dp^{(\rho-1)}\,,$$

then one will have:

$$K = \frac{d\omega}{dt} = \int \frac{\partial\varphi}{\partial t}dp^{(\rho-1)} + p'\int\frac{\partial\varphi}{\partial p}dp^{(\rho-1)} + \cdots + p^{(\rho-1)}\int\frac{\partial\varphi}{\partial p^{(\rho-2)}}dp^{(\rho-1)} + \varphi\cdot p^{(\rho)}\,.$$

Since K is a complete differential quotient with respect to t, so:

$$(\beta) \qquad \delta \int_{t_0}^{t_1} K\,dt = 0\,,$$

the difference of equations (α) and (β) will yield:

We then find that:

When one of two ***Lagrange*** *equations is independent of the second derivatives of the associated coordinate, the elimination of that coordinate and its derivatives from the two equations will lead to a fourth-order differential equation that possesses a second-order kinetic potential* (*).

$$\delta\int_{t_0}^{t_1}(M-K)\,dt = \delta\int_{t_0}^{t_1} M_1\,dt = \int_{t_0}^{t_1} N\,\delta p\,dt,$$

in which M_1 is thus a kinetic potential for N, but due to its value, $M - K$ will no longer include the derivative $p^{(\rho)}$. On the same grounds, when $\rho - 1 > \nu$, M_1 must be linear in $p^{(\rho-1)}$. As a consequence of that, one can derive a new kinetic potential M_2 that includes only $p^{(\rho-2)}$, etc., such that one will have the theorem:

If a function N that includes the derivatives up to order 2ν *possesses a kinetic potential with* $\rho > \nu$ *then it must include the derivative* $p^{(\rho)}$ *linearly. One frees it of* $p^{(\rho)}$ *in the way that was given, such that the new kinetic potential that arises, which has order* $\rho - 1$, *will includes* $p^{(\rho-1)}$ *only linearly, etc., and one sees from those successive reductions that when a function N of order* 2ν *possesses a kinetic potential of any order at all, it must possess a kinetic potential of order v. The necessary and sufficient conditions for the existence of a kinetic potential that were given above will then characterize a kinetic potential completely.*

The extension of that theorem to kinetic potentials in several variables is immediately clear.

(*) For example, suppose the first-order kinetic potential:

$$H = \mathfrak{p}\mathfrak{p}'\,p' + \frac{1}{2}\,p^2$$

is given, and let $P = 0$, such that the *two* **Lagrange** equations will read:

$$p + \mathfrak{p}'^{\,2} - \mathfrak{p}\mathfrak{p}'' = 0 \qquad \text{and} \qquad -\mathfrak{p}\,p'' = \mathfrak{P}\,.$$

Eliminating p and p'' from the equation will produce:

$$-\mathfrak{p}\,(3\mathfrak{p}''^{\,2} + 4\mathfrak{p}'\mathfrak{p}''' + \mathfrak{p}\mathfrak{p}'''') = \mathfrak{P}\,.$$

However, if one forms:

$$(H) = 4\mathfrak{p}\mathfrak{p}'^{\,2}\,\mathfrak{p}'' + \mathfrak{p}^2\mathfrak{p}'\,\mathfrak{p}''' + \frac{1}{2}\mathfrak{p}'^{\,4} - \frac{1}{2}\,\mathfrak{p}^2\,\mathfrak{p}''^{\,2}$$

then the equation:

$$\frac{\partial(H)}{\partial\mathfrak{p}} - \frac{d}{dt}\frac{\partial(H)}{\partial\mathfrak{p}'} + \frac{d^2}{dt^2}\frac{\partial(H)}{\partial\mathfrak{p}''} - \frac{d^3}{dt^3}\frac{\partial(H)}{\partial\mathfrak{p}'''} = \mathfrak{P}$$

will go to the second transformed equation of motion that was obtained above when one sets:

$$\psi = \mathfrak{p}^2\,\mathfrak{p}'\,\mathfrak{p}'', \qquad \text{so} \qquad K = \mathfrak{p}^2\,\mathfrak{p}'\mathfrak{p}'' + 2\mathfrak{p}\,\mathfrak{p}'^{\,2}\,\mathfrak{p}'' + \mathfrak{p}^2\,\mathfrak{p}''^{\,2}.$$

The kinetic potential (H) can then be replaced with:

$$\mathfrak{H} = (H) - K = 2\mathfrak{p}\mathfrak{p}'^{\,2}\,\mathfrak{p}'' + \frac{1}{2}\mathfrak{p}'^{\,4} - \frac{1}{2}\,\mathfrak{p}^2\,\mathfrak{p}''^{\,2},$$

The motion of the second coordinate can then be described by the effect of a force of the next-higher order.

However, it is also easy to see that *not only the sufficient, but also the necessary, condition for a third-order kinetic potential to lead to a fourth-order differential equation is that it must be linear in the third derivative*.

That is because if the $\mathfrak{p}^{(VI)}$ is to be absent from the expression:

$$\frac{\partial\mathfrak{H}}{\partial\mathfrak{p}} - \frac{d}{dt}\frac{\partial\mathfrak{H}}{\partial\mathfrak{p}'} + \frac{d^2}{dt^2}\frac{\partial\mathfrak{H}}{\partial\mathfrak{p}''} - \frac{d^3}{dt^3}\frac{\partial\mathfrak{H}}{\partial\mathfrak{p}'''}$$

then one must have:

$$\frac{\partial^2\mathfrak{H}}{\partial\mathfrak{p}'''^2} = 0 \qquad \text{or} \qquad \mathfrak{H} = \varphi(t, \mathfrak{p}, \mathfrak{p}', \mathfrak{p}'')\,\mathfrak{p}''' + \varphi_1(t, \mathfrak{p}, \mathfrak{p}', \mathfrak{p}'') .$$

If that is the case then since the coefficient $\mathfrak{p}^{(V)}$ of arises only from the last two terms in the expression, and obviously it will vanish, $\mathfrak{p}^{(V)}$ will not be included in the differential equation either.

However, if p'' is missing from not only the first of the two ***Lagrange*** *equations, but also the second one*, then the following conditions must be fulfilled:

$$\frac{\partial^2 H}{\partial p'^2} = 0 \qquad \text{and} \qquad \frac{\partial^2 H}{\partial p'\,\partial\mathfrak{p}'} = 0 .$$

Therefore, H will have the form:

$$H = f_1(t, \mathfrak{p})\,\mathfrak{p}' + f_2(t, \mathfrak{p}, \mathfrak{p}') ,$$

such that the two differential equations will read:

$$-\ \mathfrak{p}'\frac{\partial f_1}{\partial\mathfrak{p}} - \frac{\partial f_1}{\partial t} + \frac{\partial f_2}{\partial p} = P ,$$

$$-\frac{\partial^2 f_2}{\partial\mathfrak{p}'^2}\mathfrak{p}'' - \frac{\partial^2 f_2}{\partial\mathfrak{p}'\,\partial p}\,p' - \frac{\partial^2 f_2}{\partial\mathfrak{p}'\,\partial\mathfrak{p}}\mathfrak{p}' - \frac{\partial^2 f_2}{\partial\mathfrak{p}'\,\partial t} + \frac{\partial f_2}{\partial\mathfrak{p}} + p'\frac{\partial f_1}{\partial\mathfrak{p}} = \mathfrak{P} .$$

If one now substitutes the value:

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with which the same equation can be represented in the form:

$$\frac{\partial\mathfrak{H}}{\partial\mathfrak{p}} - \frac{d}{dt}\frac{\partial\mathfrak{H}}{\partial\mathfrak{p}'} + \frac{d^2}{dt^2}\frac{\partial\mathfrak{H}}{\partial\mathfrak{p}''} = \mathfrak{P} .$$

$$p = F\,(t, \mathfrak{p}, \mathfrak{p}')$$

that one gets from first of those two equations in the second equation then that will give only one second-order differential equation in $\mathfrak{p}$, and since:

$$H = f_1(t, F, \mathfrak{p})\left(\frac{\partial F}{\partial t} + \frac{\partial F}{\partial \mathfrak{p}}\mathfrak{p}' + \frac{\partial F}{\partial \mathfrak{p}'}\mathfrak{p}''\right) + f_2(t, F, \mathfrak{p}, \mathfrak{p}')$$

is linear in $\mathfrak{p}''$, *there will always be a first-order kinetic potential in this case* (*).

If equation (2) *does not include the derivatives* p'' *and* p' then the following conditions must be fulfilled:

$$\frac{\partial^2 H}{\partial \mathfrak{p}'\,\partial p'} = 0\,, \quad \frac{\partial^2 H}{\partial \mathfrak{p}\,\partial p'} = \frac{\partial^2 H}{\partial \mathfrak{p}'\,\partial p}\,,$$

and therefore:

(11) $$H = \varphi_1(t, p, \mathfrak{p}, p') + \varphi_2(t, p, \mathfrak{p}, \mathfrak{p}')\,,$$

in which:

$$\frac{\partial^2 \varphi_1}{\partial \mathfrak{p}\,\partial p'} = \frac{\partial^2 \varphi_1}{\partial p\,\partial \mathfrak{p}'} = \Omega\,(t, \mathfrak{p}, p)\,,$$

Thus, as is easy to see:

(12) $$H = p'\,\omega_1(t, p, \mathfrak{p}) + \mathfrak{p}'\,\omega_2(t, p, \mathfrak{p}) + \omega_3(t, p, \mathfrak{p}) + \omega_4(t, p, \mathfrak{p}) + \omega_5(t, p, \mathfrak{p})\,,$$

in which on must have:

(13) $$\frac{\partial \omega_1}{\partial \mathfrak{p}} = \frac{\partial \omega_2}{\partial p}\,.$$

However, it will then follow immediately that the two **Lagrange** equations read:

(*) If one sets, e.g.:

$$f_1 = p^2 + p\,\mathfrak{p}\,, \qquad f_2 = p\,\mathfrak{p}\,\mathfrak{p}'^2\,,$$

then the **Lagrange** equations will read:

$$-p + \mathfrak{p}\;\mathfrak{p}' = 0\,, \quad -2\,p\,\mathfrak{p}\,\mathfrak{p}'^2 - 2\,p'\,\mathfrak{p}\,\mathfrak{p}' - p\,\mathfrak{p}'^2 + p\,p' = 0\,,$$

and the result of elimination will be:

$$2\,\mathfrak{p}\,\mathfrak{p}'^3 + 3\,\mathfrak{p}^2\,\mathfrak{p}'\,\mathfrak{p}'' = 0\,.$$

The kinetic potential will then have the form:

$$\mathfrak{H} = -\tfrac{1}{2}\,\mathfrak{p}^2\,\mathfrak{p}'^3\,.$$

$$(14)\qquad \begin{cases} \dfrac{\partial\omega_3}{\partial p}+\dfrac{\partial\omega_5}{\partial p}-\dfrac{\partial\omega_1}{\partial t}-\dfrac{\partial^2\omega_4}{\partial p'\,\partial t}-\dfrac{\partial^2\omega_3}{\partial p'\,\partial p}\,p'-\dfrac{\partial^2\omega_3}{\partial p'^{\,2}}\,p''=P, \\ \dfrac{\partial\omega_4}{\partial \mathfrak{p}}+\dfrac{\partial\omega_5}{\partial \mathfrak{p}}-\dfrac{\partial\omega_2}{\partial t}-\dfrac{\partial^2\omega_4}{\partial \mathfrak{p}'\,\partial t}-\dfrac{\partial^2\omega_4}{\partial \mathfrak{p}'\,\partial \mathfrak{p}}\,\mathfrak{p}'-\dfrac{\partial^2\omega_3}{\partial \mathfrak{p}'^{\,2}}\,\mathfrak{p}''=\mathfrak{P}. \end{cases}$$

Therefore, the first one is independent of $\mathfrak{p}'$ *and* $\mathfrak{p}''$, *while the second one is independent of* p' *and* p''.

If one now infers the value:

$$p = f(t, \mathfrak{p}, \mathfrak{p}', \mathfrak{p}'')$$

from the second equation, in which one considers the external forces P and $\mathfrak{P}$ to be given functions of t, and substitutes those values, along with their derivatives, in the first differential equation then one will get a fourth-order differential equation in $\mathfrak{p}$ for which the kinetic potential H will go to a function that generally includes the quantity $\mathfrak{p}'''$ not just linearly, and for which *a second-order kinetic potential will also not exist, in general*. Now, in order to examine the conditions under which the result of the elimination:

$$Q=\left(\frac{\partial\omega_3(t,p,p')}{\partial p}\right)+\left(\frac{\partial\omega_5(t,p,\mathfrak{p})}{\partial p}\right)-\left(\frac{\partial\omega_3(t,p,\mathfrak{p})}{\partial t}\right)-\left(\frac{\partial^2\omega_3(t,p,p')}{\partial p'\,\partial t}\right)$$
$$-\,(p')\left(\frac{\partial^2\omega_3(t,p,p')}{\partial p\,\partial p'}\right)-(p'')\left(\frac{\partial^2\omega_3(t,p,p')}{\partial p'^2}\right)-P=0$$

will possess a second-order kinetic potential, one forms:

$$\frac{\partial Q}{\partial \mathfrak{p}'''}=-\left(\frac{\partial^3\omega_3}{\partial p'^2\,\partial t}\right)\frac{\partial f}{\partial \mathfrak{p}''}-\frac{df}{dt}\left(\frac{\partial^3\omega_3}{\partial p\,\partial p'^2}\right)\frac{\partial f}{\partial \mathfrak{p}''}-\frac{d^2 f}{dt^2}\left(\frac{\partial^3\omega_3}{\partial p'^3}\right)\frac{\partial f}{\partial \mathfrak{p}''}-\left(\frac{\partial^2\omega_3}{\partial p'^2}\right)\left(2\frac{d}{dt}\frac{\partial f}{\partial \mathfrak{p}''}+\frac{\partial f}{\partial \mathfrak{p}'}\right),$$

$$\frac{\partial Q}{\partial \mathfrak{p}''''}=-\left(\frac{\partial^2\omega_3}{\partial p'^2}\right)\frac{\partial f}{\partial \mathfrak{p}''}.$$

From equation (24) of **§ 3**, one must then have:

$$(15)\qquad \frac{\partial Q}{\partial \mathfrak{p}'''}-2\frac{d}{dt}\frac{\partial Q}{\partial \mathfrak{p}''''}=\left(\frac{\partial^3\omega_3}{\partial p'^2\,\partial t}\right)\frac{\partial f}{\partial \mathfrak{p}''}+\frac{df}{dt}\left(\frac{\partial^3\omega_3}{\partial p\,\partial p'^2}\right)\frac{\partial f}{\partial \mathfrak{p}''}+\frac{d^2 f}{dt^2}\left(\frac{\partial^3\omega_3}{\partial p'^3}\right)\frac{\partial f}{\partial \mathfrak{p}''}-\left(\frac{\partial^2\omega_3}{\partial p'^2}\right)\frac{\partial f}{\partial \mathfrak{p}'}=0.$$

However, it follows from that equation that the coefficient $\mathfrak{p}''''$ must satisfy:

$$\left(\frac{\partial f}{\partial \mathfrak{p}''}\right)\left(\frac{\partial^3 \omega_3}{\partial p'^3}\right) = 0 \,, \quad \text{so} \qquad \frac{\partial^3 \omega_3}{\partial p'^3} = 0$$

must be fulfilled identically, so:

$$\omega_3\,(t, p, p') = \Omega_1(t, p)\,p'^2 + \Omega_2(t, p)\,p + \Omega_3(t, p) \;,$$

while the remaining part of equation (15) prescribes the condition that must be satisfied identically:

$$\frac{\partial \Omega_1}{\partial t}\frac{\partial f}{\partial \mathfrak{p}''} + \frac{df}{dt}\frac{\partial \Omega_1}{\partial p}\frac{\partial f}{\partial \mathfrak{p}''} - \Omega_1\frac{\partial f}{\partial \mathfrak{p}'} = 0 \;.$$

Now, since the coefficient of $\mathfrak{p}'''$:

$$\frac{\partial \Omega_1}{\partial p} = 0 \,, \qquad \text{so} \qquad \Omega_1 = \Omega_1\,(t) \,,$$

and therefore, we must have:

$$\Omega'(t)\,\frac{\partial f}{\partial \mathfrak{p}''} - \Omega(t)\,\frac{\partial f}{\partial \mathfrak{p}'} = 0 \,,$$

when we now assume, for simplicity, that t is not included in the kinetic potential explicitly, it will follow that:

$$\Omega\,(t) = a \qquad \text{and} \qquad \frac{\partial f}{\partial \mathfrak{p}'} = 0 \;.$$

However, if f is not supposed to contain the derivative then the left-hand side of the second **Lagrange** equation (14), which will assume the form:

$$\frac{\partial \omega_3(\mathfrak{p}, \mathfrak{p}')}{\partial \mathfrak{p}} + \frac{\partial \omega_5(p, \mathfrak{p})}{\partial \mathfrak{p}} - \frac{\partial^2 \omega_4(\mathfrak{p}, \mathfrak{p}')}{\partial \mathfrak{p}\,\partial \mathfrak{p}'}\,\mathfrak{p}' - \frac{\partial^2 \omega_4(\mathfrak{p}, \mathfrak{p}')}{\partial \mathfrak{p}'^{\,2}}\,\mathfrak{p}'' = \mathfrak{P} \,,$$

under the assumptions that were just made, will be independent of $\mathfrak{p}'$, then we will have:

$$\mathfrak{p}'\,\frac{\partial^3 \omega_4(\mathfrak{p}, \mathfrak{p}')}{\partial \mathfrak{p}\,\partial \mathfrak{p}'^{\,2}} + \mathfrak{p}''\,\frac{\partial^3 \omega_4(\mathfrak{p}, \mathfrak{p}')}{\partial \mathfrak{p}'^{\,3}} = 0$$

identically. It will then follow from this that:

$$\frac{\partial^2 \omega_4(\mathfrak{p}, \mathfrak{p}')}{\partial \mathfrak{p}'^2} = c \qquad \text{or} \qquad \omega_4(\mathfrak{p}, \mathfrak{p}') = c\,\mathfrak{p}'^2 + \psi_1(\mathfrak{p})\,\mathfrak{p}' + \psi_2(\mathfrak{p})\,.$$

Therefore, from equation (12), the kinetic potential will assume the form:

$$H = p'\,\omega_1(p, \mathfrak{p}) + \mathfrak{p}'\,\omega_2(p, \mathfrak{p}) + a\,p'^2 + \Omega_2(p)\,p' + \Omega_3(p) + c\,\mathfrak{p}'^2 + \psi_1(\mathfrak{p})\,\mathfrak{p}' + \psi_2(\mathfrak{p}) + \omega_5(p, \mathfrak{p})\,,$$

or when one drops the complete differential quotient with respect to t, while recalling the relation (13):

$$H = a\,p'^2 + \Omega_3(p) + c\,\mathfrak{p}'^2 + \psi_2(\mathfrak{p}) + \omega_5(p, \mathfrak{p})\,, \tag{16}$$

from which the **Lagrange** equations will read:

$$\left\{\begin{aligned} -2a\,p'' + \frac{\partial \Omega_3}{\partial \mathfrak{p}} + \frac{\partial \omega_5}{\partial p} &= P, \\ -2c\,p'' + \frac{\partial \psi_3}{\partial \mathfrak{p}} + \frac{\partial \omega_5}{\partial \mathfrak{p}} &= \mathfrak{P}. \end{aligned}\right. \tag{17}$$

However, the result of eliminating Q, in which one substitutes:

$$p = f(t, \mathfrak{p}, \mathfrak{p}'')\,, \tag{18}$$

will go to:

$$Q = \left(\frac{\partial \Omega_3(p)}{\partial p}\right)_{p=f} + \left(\frac{\partial \omega_5(p, \mathfrak{p})}{\partial p}\right)_{p=f} - 2a\,\frac{d^2 f}{dt^2} - P = 0\,,$$

and as one can see immediately, from the relations (2) and (3) of **§ 2**, one will have

$$\frac{\partial Q}{\partial \mathfrak{p}'} = -\,4a\,\frac{d}{dt}\frac{\partial f}{\partial \mathfrak{p}}\,,$$

$$\frac{\partial Q}{\partial \mathfrak{p}''} = \left(\frac{\partial^2 \Omega_3}{\partial p^2}\right)\frac{\partial f}{\partial \mathfrak{p}''} + \left(\frac{\partial^2 \omega_5}{\partial p^2}\right)\frac{\partial f}{\partial \mathfrak{p}''} - 2a\left[\frac{d^2}{dt^2}\frac{\partial f}{\partial \mathfrak{p}''} + \frac{\partial f}{\partial \mathfrak{p}}\right],$$

$$\frac{\partial Q}{\partial \mathfrak{p}'''} = -\,4a\,\frac{d}{dt}\frac{\partial f}{\partial \mathfrak{p}''}\,, \qquad \frac{\partial Q}{\partial \mathfrak{p}''''} = -\,2a\,\frac{\partial f}{\partial \mathfrak{p}''}\,.$$

The condition equation (23) of **§ 3** that must be fulfilled in order for a second-order kinetic potential to exist will imply that the equation to be satisfied is:

$$2a\,\frac{d}{dt}\frac{\partial f}{\partial \mathfrak{p}} + \frac{d}{dt}\left\{\frac{\partial f}{\partial \mathfrak{p}''}\left[\left(\frac{\partial^2 \Omega_3}{\partial p^2}\right) + \left(\frac{\partial^2 \omega_5}{\partial p^2}\right)\right]\right\} = 0$$

or

$$(19) \qquad 2a\frac{\partial f}{\partial \mathfrak{p}} + \frac{\partial f}{\partial \mathfrak{p}''}\left[\left(\frac{\partial^2 \Omega_3}{\partial p^2}\right) + \left(\frac{\partial^2 \omega_5}{\partial p^2}\right)\right] = c\,,$$

in which c means an arbitrary constant. However, since the expression (18) must satisfy the second of equations (17) identically, one must have:

$$(20) \qquad \frac{\partial \psi_2(\mathfrak{p})}{\partial \mathfrak{p}} + \left(\frac{\partial \omega_5(p, \mathfrak{p})}{\partial \mathfrak{p}}\right) - 2c\,\mathfrak{p}'' = 0\,.$$

Differentiating the identity (20) with respect to $\mathfrak{p}$ and $\mathfrak{p}''$ will give:

$$\left(\frac{\partial^2 \omega_5}{\partial \mathfrak{p}^2}\right) + \left(\frac{\partial^2 \omega_5}{\partial \mathfrak{p}\,\partial p}\right)\frac{\partial f}{\partial \mathfrak{p}} + \frac{\partial^2 \psi_2(\mathfrak{p})}{\partial \mathfrak{p}^2} = 0\,, \qquad \left(\frac{\partial^2 \omega_5}{\partial \mathfrak{p}\,\partial p}\right)\frac{\partial f}{\partial \mathfrak{p}''} - 2c = 0\,.$$

Upon substituting the values of $\frac{\partial f}{\partial \mathfrak{p}}$ and $\frac{\partial f}{\partial \mathfrak{p}''}$ that follow from that in (19), the equation will emerge:

$$(21) \qquad -\,a\left[\frac{\partial^2 \psi_2(\mathfrak{p})}{\partial \mathfrak{p}^2} + \frac{\partial^2 \omega_5(\mathfrak{p})}{\partial \mathfrak{p}^2}\right] + c\left[\frac{\partial^2 \Omega_2(p)}{\partial p^2} + \frac{\partial^2 \omega_5(p, \mathfrak{p})}{\partial p^2}\right] = C\,\frac{\partial^2 \omega_5(p, \mathfrak{p})}{\partial p\,\partial \mathfrak{p}}\,,$$

which must be fulfilled identically by $\mathfrak{p}$ and $\mathfrak{p}''$, so also $\mathfrak{p}$ and p.

If one sets:

$$(22) \qquad \omega_5(p, \mathfrak{p}) + \psi_2(\mathfrak{p}) + \Omega_3(p) = F(p, \mathfrak{p})$$

in that then the equation (21) to be satisfied identically will go to:

$$a\frac{\partial^2 F}{\partial \mathfrak{p}^2} + C\,\frac{\partial^2 F}{\partial \mathfrak{p}\,\partial p} - c\,\frac{\partial^2 F}{\partial p^2} = 0\,.$$

When one sets:

$$\lambda_1 = \frac{-C + \sqrt{C^2 + 4ac}}{2a}\,, \qquad \lambda_2 = \frac{-C - \sqrt{C^2 + 4ac}}{2a}\,,$$

its general integral will be represented by:

$$F(p, \mathfrak{p}) = \varphi(p - \lambda_1 \mathfrak{p}) + \psi(p - \lambda_2 \mathfrak{p}),$$

in which φ and ψ mean arbitrary functions, such that from (22) and (16), H will assume the form:

$$H = a\,p'^2 + c\,\mathfrak{p}'^2 + \varphi(p - \lambda_1 \mathfrak{p}) + \psi(p - \lambda_2 \mathfrak{p}).$$

We will then get the following theorem:

*The necessary and sufficient condition for the **Lagrange** equation* (2) *to not include* p'' *and* p', *and for the kinetic potential to not include time explicitly, as well as for the elimination of p from the **Lagrange** equations to produce a differential equation that possesses a second-order kinetic potential is that the original kinetic potential (except for a complete differential quotient with respect to t of an arbitrary function of the coordinates, as always) must have the form:*

$$H = a\,p'^2 + c\,\mathfrak{p}'^2 + \varphi(p - \lambda_1 \mathfrak{p}) + \psi(p - \lambda_2 \mathfrak{p}).$$

*Therefore, the **Lagrange** equations will read:*

$$\begin{aligned}0 &= 2a\,p'' - \varphi'(p - \lambda_1 \mathfrak{p}) - \psi'(p - \lambda_1 \mathfrak{p}),\\ 0 &= 2a\,\mathfrak{p}'' - \lambda_1 \varphi'(p - \lambda_1 \mathfrak{p}) - \lambda_2 \psi'(p - \lambda_2 \mathfrak{p}),\end{aligned}$$

in which a and c mean constants, $\lambda_1 \cdot \lambda_2 = - c / a$, *and* φ, *as well as* ψ, *mean arbitrary functions of their arguments.*

*However, if the **Lagrange** equation* (1) *includes* p'', *but not* p', *and* p' *is also missing from equation* (2), such that after eliminating p'', p can also be expressed in terms of t, $\mathfrak{p}$, $\mathfrak{p}'$, $\mathfrak{p}''$, here as well, then following conditions must be satisfied:

$$\frac{d}{dt}\frac{\partial^2 H}{\partial p'^2} = 0 \quad \text{and} \quad \frac{\partial^2 H}{\partial \mathfrak{p}\,\partial p'} - \frac{d}{dt}\frac{\partial^2 H}{\partial \mathfrak{p}'\,\partial p'} - \frac{\partial^2 H}{\partial \mathfrak{p}'\,\partial p} = 0,$$

and it will easily follow that:

*The necessary and sufficient condition for the two **Lagrange** equations to not include* p' *is that the kinetic potential must take the form:*

(23) $$H = a\,p'^2 + p'\,\mathfrak{p}'\,\omega_1(t,\mathfrak{p}) + p'\omega_2(t,\mathfrak{p}) + \mathfrak{p}'\int \frac{\partial\omega_2}{\partial\mathfrak{p}}\,dp - \mathfrak{p}'\,p\,\frac{\partial\omega_1}{\partial t} + \omega_3(t,\mathfrak{p},\mathfrak{p}') + \omega_4(t,p,\mathfrak{p})\ ,$$

in which a means a constant, and ω_1, ω_2, ω_3, ω_4 mean arbitrary functions of their arguments, such that the two **Lagrange** equations will read:

(24) $$-\ 2a\,p'' - \omega_1\,\mathfrak{p}'' - \mathfrak{p}'^2\,\frac{\partial\omega_1}{\partial\mathfrak{p}} - 2\,\mathfrak{p}'\,\frac{\partial\omega_1}{\partial t} - \frac{\partial\omega_2}{\partial t} + \frac{\partial\omega_4}{\partial p} = P\ ,$$

(25) $$-\ \omega_1\,\mathfrak{p}'' - \frac{\partial^2\omega_1}{\partial\mathfrak{p}'^2}\,\mathfrak{p}'' - \mathfrak{p}'\,\frac{\partial^2\omega_3}{\partial\mathfrak{p}'\,\partial\mathfrak{p}} - \frac{\partial^2\omega_3}{\partial\mathfrak{p}'\,\partial t} + \frac{\partial\omega_3}{\partial\mathfrak{p}} - \int \frac{\partial^2\omega_3}{\partial\mathfrak{p}'\,\partial t}\,dp + p\,\frac{\partial^2\omega_1}{\partial t^2} + \frac{\partial\omega_4}{\partial\mathfrak{p}} = \mathfrak{P}\ .$$

If t is not included in the kinetic potential explicitly, so ω_1, ω_2, ω_3, ω_4 are independent of t, then the two equations of motion will not include p at all when $\omega_4 = 0$, and eliminating p'' will produce a second-order differential equation in $\mathfrak{p}$, which is the case that was treated above in the theory of hidden motion. However, if t is included in H or ω_4 is non-zero then one can eliminate p'' from those two equations. The result of that elimination will then yield the expression:

$$p = \omega\,(t,\,\mathfrak{p},\,\mathfrak{p}',\,\mathfrak{p}'')\ .$$

When that is substituted in one of the two equations, that will in turn yield a fourth-order differential equation for which (H) nonetheless includes the term that is quadratic in $\mathfrak{p}'''$:

$$a\left(\frac{\partial\omega}{\partial\mathfrak{p}''}\right)^2\,\mathfrak{p}'''^{\,2}\ .$$

In that case, *the fourth-order differential equation will not possess a second-order potential*, in general (*).

(*) For example, let:

$$\omega_1 = \mathfrak{p}\ ,\quad \omega_2 = t\,\mathfrak{p}\ ,\quad \omega_3 = 0\ ,\quad \omega_4 = 0\ ,\quad a = 1\ .$$

The kinetic potential will then read:

$$H = p'^2 + \mathfrak{p}\,p'\,\mathfrak{p}' + t\,\mathfrak{p}\,p' + t\,\mathfrak{p}'\,p$$

or

$$H = p'^2 + \mathfrak{p}\,p'\,\mathfrak{p}' - \mathfrak{p}\,p\ ,$$

and for vanishing external forces, the associated **Lagrange** equations will read:

$$2\,p'' + \mathfrak{p}\,\mathfrak{p}'' + \mathfrak{p}'^2 + \mathfrak{p} = 0 \quad\text{and}\quad \mathfrak{p}\,p'' + p = 0\ ,$$

such that eliminating p will make the fourth-order differential equation:

We now raise the question of when the fourth-order differential equation that is obtained by eliminating p and p'' from the last two **Lagrange** equations (24) and (25) will possess a second-order kinetic potential, under the assumption that the external forces P_1 and P_2 are zero, and the functions included in H do not include t explicitly, so that kinetic potential will take the form:

$$H = a\,p'^2 + p'\,\mathfrak{p}'\,\omega_1(\mathfrak{p}) + p'\,\omega_2(p, \mathfrak{p}) + \mathfrak{p}' \int \frac{\partial \omega_2}{\partial \mathfrak{p}}\, dp + \omega_3(p, \mathfrak{p}') + \omega_4(p, \mathfrak{p})\,,$$

or when we drop the complete differential quotient with respect to t :

$$p'\,\omega_2(p, \mathfrak{p}) + \mathfrak{p}' \int \frac{\partial \omega_2}{\partial \mathfrak{p}} = \frac{d}{dt} \int \omega_2(p, \mathfrak{p})\, dp\,,$$

so that the kinetic potential will have the form:

$$H = a\,p'^2 + p'\,\mathfrak{p}'\,\omega_1(\mathfrak{p}) + \omega_3(p, \mathfrak{p}') + \omega_4(p, \mathfrak{p})\,. \tag{26}$$

The **Lagrange** equations will then read:

$$-2a\,p'' - \omega_1\,\mathfrak{p}'' - \mathfrak{p}'^2\,\frac{\partial \omega_1}{\partial \mathfrak{p}} + \frac{\partial \omega_4(p, \mathfrak{p})}{\partial p} = 0\,, \tag{27}$$

$$-\,\omega_1\,p''\,\frac{\partial^2 \omega_3(p, \mathfrak{p}')}{\partial \mathfrak{p}'^2}\,\mathfrak{p}'' - \mathfrak{p}'\,\frac{\partial^2 \omega_3(p, \mathfrak{p}')}{\partial \mathfrak{p}'\,\partial \mathfrak{p}} + \frac{\partial \omega_3(p, \mathfrak{p}')}{\partial \mathfrak{p}} + \frac{\partial \omega_4(p, \mathfrak{p})}{\partial \mathfrak{p}} = 0\,, \tag{28}$$

and eliminating p'' will next yield:

$$\left(\omega_1^2 - 2a\,\frac{\partial^2 \omega_3}{\partial \mathfrak{p}'^2}\right)\mathfrak{p}'' + \mathfrak{p}'^2\,\omega_1\,\frac{\partial \omega_1}{\partial \mathfrak{p}} - 2a\,\mathfrak{p}'\,\frac{\partial^2 \omega_3}{\partial \mathfrak{p}\,\partial \mathfrak{p}'} + 2a\,\frac{\partial \omega_4}{\partial \mathfrak{p}} - \omega_1\,\frac{\partial \omega_4}{\partial p} = 0\,. \tag{29}$$

$$\mathfrak{p}^2\,\mathfrak{p}^{IV} + 6\,\mathfrak{p}\,\mathfrak{p}'\,\mathfrak{p}''' + 4\,\mathfrak{p}\,\mathfrak{p}''^{\,2} + 7\,\mathfrak{p}'^{\,2}\,\mathfrak{p}'' + 3\,\mathfrak{p}\,\mathfrak{p}'' + 3\,\mathfrak{p}'^{\,2} + \mathfrak{p} = 0\,.$$

The fact that this does not possess a second-order kinetic potential emerges from the fact that when its left-hand side is denoted by Q, the equation that must necessarily be satisfied, according to Lemma 4:

$$\frac{\partial Q}{\partial \mathfrak{p}'''} - 2\,\frac{d}{dt}\,\frac{\partial Q}{\partial \mathfrak{p}''''} = 0$$

is not fulfilled. The kinetic potential H will then go to:

$$(H) = \frac{1}{2}(\mathfrak{p}^2\mathfrak{p}''' + 4\mathfrak{p}\mathfrak{p}'\mathfrak{p}'' + \mathfrak{p}'^3 + 2\mathfrak{p}\mathfrak{p}')^2 + \frac{1}{2}\mathfrak{p}\mathfrak{p}'(\mathfrak{p}^2\mathfrak{p}''' + 4\mathfrak{p}\mathfrak{p}'\mathfrak{p}'' + \mathfrak{p}'^3 + 2\mathfrak{p}\mathfrak{p}') - \frac{1}{2}\mathfrak{p}\,(\mathfrak{p}^2\mathfrak{p}'' + \mathfrak{p}\mathfrak{p}'^2 + \mathfrak{p}^2)\,.$$

If that equation implies that:

$$p = F\,(\mathfrak{p}, \mathfrak{p}', \mathfrak{p}'')\,, \tag{30}$$

so (H) goes to a function that is quadratic in $\mathfrak{p}'''$, then substituting the values (30) in (27) will put the result of the elimination into the form:

$$N = -\;2a\,\frac{d^2F}{dt^2} - \omega_1(\mathfrak{p})\,\mathfrak{p}'' - \mathfrak{p}'^2\,\frac{\partial\omega_1(\mathfrak{p})}{\partial\mathfrak{p}} + \left(\frac{\partial\omega_4}{\partial p}\right)_{p=F} = 0\,.$$

One must address the question of whether the expression N, which has order four in the derivative of $\mathfrak{p}$, possesses a second-order kinetic potential. It will then follow from the second of the two conditions (23) and (24) in **§ 3** that are necessary and sufficient for the existence of a second-order kinetic potential that because the auxiliary formulas (2) and (3) of **§ 3** say that:

$$\frac{\partial N}{\partial\mathfrak{p}'''} = -2a\left(\frac{d}{dt}\frac{\partial F}{\partial\mathfrak{p}''} + \frac{\partial F}{\partial\mathfrak{p}'}\right) \qquad \text{and} \qquad \frac{\partial N}{\partial\mathfrak{p}''''} = -2a\,\frac{\partial F}{\partial\mathfrak{p}''}\,,$$

the previous equation will be fulfilled if and only if:

$$\frac{\partial F}{\partial\mathfrak{p}'} = 0\,.$$

Thus, F must be independent of $\mathfrak{p}'$, or the left-hand side of (29) must not include $\mathfrak{p}'$. It will then follow immediately that one must have:

$$\frac{\partial^3\omega_3}{\partial\mathfrak{p}'^3} = 0 \qquad \text{or} \qquad \omega_3\,(\mathfrak{p}, \mathfrak{p}') = \Omega_1\,(\mathfrak{p})\,\mathfrak{p}'^2 + \Omega_2\,(\mathfrak{p})\,\mathfrak{p}' + \Omega_3\,(\mathfrak{p})\,,$$

and

$$\omega_1^2 - 4a\,\Omega_1 = c\,,$$

in which c means a constant. The kinetic potential will then go to:

$$H = a\,p'^2 + \omega_1\,p'\,\mathfrak{p}' + \frac{\omega_1^2 - c}{4a}\,\mathfrak{p}'^2 + \Omega_3 + \omega_4\,, \tag{31}$$

and the **Lagrange** equations will go to:

(32)
$$\begin{cases} -2a\,p'' - \omega_1\,\mathfrak{p}'' - \mathfrak{p}'^2\,\dfrac{\partial\omega_1}{\partial\mathfrak{p}} + \dfrac{\partial\omega_4}{\partial p} = 0, \\ -\omega_1\,p'' - \dfrac{\omega_1^2 - c}{2a}\,\mathfrak{p}'' - \dfrac{\omega_1}{2a}\dfrac{\partial\omega_1}{\partial\mathfrak{p}}\,\mathfrak{p}'^2 + \dfrac{\partial\Omega_3}{\partial\mathfrak{p}} + \dfrac{\partial\omega_4}{\partial p} = 0, \end{cases}$$

so upon eliminating p'':

(33)
$$c\,\mathfrak{p}'' + \frac{\partial\Omega_3}{\partial\mathfrak{p}} + 2a\frac{\partial\omega_4}{\partial\mathfrak{p}} - \omega_1\frac{\partial\omega_4}{\partial p} = 0 .$$

If one calculates:

$$p = f(\mathfrak{p}, \mathfrak{p}'')$$

and substitutes that value in the first of equations (32) then the result of an elimination will be:

(34)
$$Q = -2a\frac{d^2 f}{dt^2} - \omega_1\,\mathfrak{p}'' - \mathfrak{p}'^2\,\frac{\partial\omega_1}{\partial\mathfrak{p}} + \left(\frac{\partial\omega_4}{\partial p}\right)_{p=f} = 0 ,$$

and equation (24) of **§ 3** will be fulfilled by the left-hand side of that equation.

Now since the existence of a second-order potential must also satisfy equation (23) of **§ 3**, and in turn, from the auxiliary formulas (2) and (3) of **§ 2**:

$$\frac{\partial Q}{\partial\mathfrak{p}'} = -4a\frac{d}{dt}\frac{\partial f}{\partial\mathfrak{p}} - 2\,\mathfrak{p}'\,\omega_1' ,$$
$$\frac{\partial Q}{\partial\mathfrak{p}''} = -2a\frac{d^2}{dt^2}\frac{\partial f}{\partial\mathfrak{p}''} - 2a\frac{\partial f}{\partial\mathfrak{p}} - \omega_1 + \left(\frac{\partial^2\omega_4}{\partial p^2}\right)\frac{\partial f}{\partial\mathfrak{p}''} ,$$
$$\frac{\partial Q}{\partial\mathfrak{p}'''} = -4a\frac{d}{dt}\frac{\partial f}{\partial\mathfrak{p}''} , \qquad \frac{\partial Q}{\partial\mathfrak{p}''''} = -2a\frac{\partial f}{\partial\mathfrak{p}''} ,$$

the equation:

(35)
$$\frac{d}{dt}\left[2a\frac{\partial f}{\partial\mathfrak{p}} + \omega_1 + \left(\frac{\partial^2\omega_4}{\partial p^2}\right)\frac{\partial f}{\partial\mathfrak{p}''}\right] = 0$$

must then be fulfilled identically. However, if one remarks that due to (33), the equation:

(36)
$$c\,\mathfrak{p}'' + 2a\frac{\partial\Omega_3}{\partial\mathfrak{p}''} + 2a\left(\frac{\partial\omega_4}{\partial\mathfrak{p}}\right)_{p=f} - \omega_1\left(\frac{\partial\omega_4}{\partial p}\right)_{p=f} = 0$$

must be an identity then the differential quotients with respect to $\mathfrak{p}''$ and $\mathfrak{p}$ will yield the following relations:

$$c + \left\{ 2a \left(\frac{\partial^2 \omega_4}{\partial \mathfrak{p}\, \partial p} \right)_{p=f} - \omega_1 \left(\frac{\partial^2 \omega_4}{\partial p^2} \right)_{p=f} \right\} \frac{\partial f}{\partial \mathfrak{p}''} = 0 \,,$$

$$2a \frac{\partial^2 \Omega_3(\mathfrak{p})}{\partial \mathfrak{p}^2} + 2a \left(\frac{\partial^2 \omega_4}{\partial \mathfrak{p}^2} \right)_{p=f} - \omega_1 \left(\frac{\partial^2 \omega_4}{\partial p\, \partial \mathfrak{p}} \right)_{p=f} + \left\{ 2a \left(\frac{\partial^2 \omega_4}{\partial p\, \partial \mathfrak{p}} \right)_{p=f} - \omega_1 \left(\frac{\partial^2 \omega_4}{\partial p^2} \right)_{p=f} \right\} \frac{\partial f}{\partial \mathfrak{p}} - \frac{\partial \omega_1}{\partial \mathfrak{p}} \left(\frac{\partial \omega_4}{\partial p} \right)_{p=f} = 0 \,,$$

which are likewise identities. Thus, it is easy to see that substituting the values they imply for $\frac{\partial f}{\partial \mathfrak{p}}$ and $\frac{\partial f}{\partial \mathfrak{p}''}$ in the integrated equation (35) will give the necessary and sufficient condition for a second-order potential to give the result of elimination in the form of the equation:

$$(37) \qquad 4a^2 \frac{\partial^2 \Omega_3(\mathfrak{p})}{\partial \mathfrak{p}^2} + 4a^2 \frac{\partial^2 \omega_4}{\partial \mathfrak{p}^2} - 2a\, \omega_1 \frac{\partial^2 \omega_4}{\partial \mathfrak{p}\, \partial p} - 2a \frac{\partial \omega_1}{\partial \mathfrak{p}} \frac{\partial \omega_4}{\partial p} + c \frac{\partial^2 \omega_4}{\partial p^2} = (\omega_1 + C) \left(2a \frac{\partial^2 \omega_4}{\partial \mathfrak{p}\, \partial p} - \omega_1 \frac{\partial^2 \omega_4}{\partial p^2} \right),$$

which must be fulfilled identically in p and $\mathfrak{p}$, and in which C must mean an arbitrary constant. Therefore, differentiating with respect to p will give the partial differential equation for ω_4 and ω_1:

$$(38) \qquad 4a^2 \frac{\partial^3 \omega_4}{\partial \mathfrak{p}^2\, \partial p} - 2a\, \omega_1 \frac{\partial^3 \omega_4}{\partial \mathfrak{p}\, \partial p^2} - 2a \frac{\partial \omega_1}{\partial \mathfrak{p}} \frac{\partial^2 \omega_4}{\partial p^2} + c \frac{\partial^3 \omega_4}{\partial p^3} = (\omega_1 + C) \left(2a \frac{\partial^3 \omega_4}{\partial \mathfrak{p}\, \partial p^2} - \omega_1 \frac{\partial^3 \omega_4}{\partial p^3} \right) .$$

If one sets $\omega_1 = \mu$, a constant, and sets:

$$\frac{2\mu + C}{2a} = 2\alpha \,, \qquad \frac{c + \mu(\mu + C)}{4a^2} = \beta$$

then one will get the following partial differential equation for ω_1 :

$$\frac{\partial^3 \omega_4}{\partial \mathfrak{p}^2\, \partial p} - 2\alpha \frac{\partial^3 \omega_4}{\partial \mathfrak{p}\, \partial p^2} + \beta \frac{\partial^3 \omega_4}{\partial p^3} = 0 \,.$$

When one sets:

$$\lambda_1,\ \lambda_2 = \alpha \pm \sqrt{\alpha^2 - \beta} = \frac{(2\mu + C) \pm \sqrt{C^2 - 4c}}{4a} \,,$$

its general integral will be represented by:

$$\omega_4 = \Phi\,(p - \lambda_1\,\mathfrak{p}) + \Psi\,(p - \lambda_2\,\mathfrak{p}) + X\,(\mathfrak{p})\,,$$

when Φ, Ψ, X represent arbitrary functions of their arguments. Since it follows from (37) that:

$$\Omega_3\,(\mathfrak{p}) = -\,X\,(\mathfrak{p}) + C_1\,,$$

in which C_1 means an integration constant, the kinetic potential (31) will go to:

$$H = a\,p'^2 + \mu\,p'\,\mathfrak{p}' + \frac{\mu^2 - c}{4a}\,\mathfrak{p}'^2 + \Phi\,(p - \lambda_1\,\mathfrak{p}) + \Psi\,(p - \lambda_2\,\mathfrak{p})\,, \tag{39}$$

in which a, μ, c, C mean arbitrary constants. The two **Lagrange** equations will then assume the form:

$$-\,2a\,p'' - \mu\,\mathfrak{p}'' + \Phi'(p - \lambda_1\,\mathfrak{p}) + \Psi'(p - \lambda_2\,\mathfrak{p}) = 0\,,$$

$$-\,\mu\,\mathfrak{p}'' - \frac{\mu^2 - c}{2a}\,\mathfrak{p}'' - \lambda_1\,\Phi'(p - \lambda_1\,\mathfrak{p}) - \lambda_2\Psi'(p - \lambda_2\,\mathfrak{p}) = 0\,.$$

That will then imply that:

If p' *is not included in two* ***Lagrange*** *equations, for which the kinetic potential does not include time t explicitly and the external forces are zero, and the elimination of p from them produces a differential equation in* $\mathfrak{p}$ *that possesses a second-order kinetic potential, then the necessary and sufficient condition for that is that the original kinetic potential must take the form:*

$$H = a\,p'^2 + \omega_1(\mathfrak{p})\,p'\,\mathfrak{p}' + \frac{\omega_1^2 - c}{4a}\,\mathfrak{p}'^2 + \Omega_3\,(\mathfrak{p}) + \omega_1\,(p, \mathfrak{p})\,.$$

The condition is then expressed by the associated ***Lagrange*** *equations:*

$$-\,2a\,p'' - \omega_1\,\mathfrak{p}'' - \mathfrak{p}'^2\,\frac{\partial\omega_1}{\partial\mathfrak{p}} + \frac{\partial\omega_4}{\partial p} = 0\,,$$

$$-\,\omega_1\,p'' - \frac{\omega_1^2 - c}{2a}\,\mathfrak{p}'' - \frac{\omega_1}{2a}\,\frac{\partial\omega_1}{\partial\mathfrak{p}}\,\mathfrak{p}'^2 + \frac{\partial\Omega_3}{\partial\mathfrak{p}} + \frac{\partial\omega_4}{\partial\mathfrak{p}} = 0\,,$$

in which ω_1, ω_4, Ω *are coupled with each other by the differential equation* (37). *If* ω_1 *is a constant, so* $\mathfrak{p}'$ *will also be missing from the two equations, then it will be obvious that one now has expressions that are analogous to the ones in the previous case.*

The only possible case in which one coordinate – e.g., p – is missing from the two ***Lagrange*** *equations* can lead back to the investigations in the previous section, but it is also easy to deal with directly. Namely, if the first **Lagrange** equation is to be free of p then it will follow immediately that:

$$\frac{\partial H}{\partial p} = \varphi(t, p, \mathfrak{p})\, p' + \psi(t, p, \mathfrak{p}, \mathfrak{p}'), \tag{40}$$

in which:

$$\frac{\partial \psi}{\partial p} - \frac{\partial \varphi}{\partial t} - \mathfrak{p}'\frac{\partial \varphi}{\partial \mathfrak{p}} = 0 . \tag{41}$$

However, the independence of the second **Lagrange** equation of p will imply the further condition:

$$\frac{\partial \varphi}{\partial \mathfrak{p}} p' + \frac{\partial \psi}{\partial \mathfrak{p}} - \frac{\partial^2 \psi}{\partial \mathfrak{p}' \partial t} - p'\frac{\partial^2 \psi}{\partial \mathfrak{p}' \partial p} - \mathfrak{p}'\frac{\partial^2 \psi}{\partial \mathfrak{p} \partial \mathfrak{p}'} - \mathfrak{p}''\frac{\partial^2 \psi}{\partial \mathfrak{p}'^2} = 0 . \tag{42}$$

Since it must be an identity, that will give:

$$\frac{\partial^2 \psi}{\partial \mathfrak{p}'^2} = 0 ,$$

so

$$\psi = \mathfrak{p}'\ \omega(t, p, \mathfrak{p}) + \chi(t, p, \mathfrak{p}) .$$

However, it will follow from (41) and (42) that one must have:

$$\frac{\partial \omega}{\partial p} = \frac{\partial \varphi}{\partial \mathfrak{p}}, \qquad \frac{\partial \chi}{\partial p} = \frac{\partial \varphi}{\partial t}, \qquad \frac{\partial \chi}{\partial \mathfrak{p}} = \frac{\partial \omega}{\partial t} . \tag{43}$$

Therefore, from (40), the expression for the kinetic potential will be:

$$H = p'\int \varphi(t, p, \mathfrak{p})\, dp + \mathfrak{p}'\int \omega(t, p, \mathfrak{p})\, dp + \int \chi(t, p, \mathfrak{p})\, dp + F(t, \mathfrak{p}, p', \mathfrak{p}') .$$

Since (43) says that the expression:

$$p'\int \varphi\, dp + \mathfrak{p}'\int \omega\, dp + \int \chi\, dp$$

is a complete differential quotient with respect to t, that expression will go to:

$$H = F(t, \mathfrak{p}, p', \mathfrak{p}') .$$

Now, the two **Lagrange** equations that emerge from that:

$$\frac{d}{dt}\frac{\partial F}{\partial p'} = 0\,, \qquad \frac{\partial F}{\partial \mathfrak{p}} - \frac{d}{dt}\frac{\partial F}{\partial \mathfrak{p}'} = 0$$

are, in fact, independent of p, and the first of them implies that:

$$\frac{\partial F}{\partial p'} = h \qquad \text{or} \qquad p' = \Omega(t, \mathfrak{p}, \mathfrak{p}')\,,$$

in which h means the integration constant. Thus, substituting that value of p' in the second equation will produce the relation:

$$\left(\frac{\partial F}{\partial \mathfrak{p}}\right) - \frac{d}{dt}\left(\frac{\partial F}{\partial \mathfrak{p}'}\right) = 0\,,$$

with the well-known notation. Therefore, since:

$$\frac{\partial (F)}{\partial \mathfrak{p}} = \left(\frac{\partial F}{\partial \mathfrak{p}}\right) + \left(\frac{\partial F}{\partial p'}\right)\frac{\partial p'}{\partial \mathfrak{p}} = \left(\frac{\partial F}{\partial \mathfrak{p}}\right) + h\frac{\partial \Omega}{\partial \mathfrak{p}}\,,$$
$$\frac{\partial (F)}{\partial \mathfrak{p}'} = \left(\frac{\partial F}{\partial \mathfrak{p}'}\right) + \left(\frac{\partial F}{\partial p'}\right)\frac{\partial p'}{\partial \mathfrak{p}'} = \left(\frac{\partial F}{\partial \mathfrak{p}'}\right) + h\frac{\partial \Omega}{\partial \mathfrak{p}'}\,,$$

that equation will assume the form:

$$\frac{\partial}{\partial \mathfrak{p}}[(F) - h\Omega] - \frac{d}{dt}\frac{\partial}{\partial \mathfrak{p}'}[(F) - h\Omega] = 0\,.$$

Hence, the kinetic potential of the elimination equation:

$$\mathfrak{H} = (F) - h\,\Omega$$

will, in turn, have order one, which should have already been clear from the previous investigations.

We once more summarize the results that we obtained here in order for the elimination of one coordinate between two **Lagrange** equations with a first-order kinetic potential to imply that the resulting differential equation possesses a second-order kinetic potential:

1) *When the* ***Lagrange*** *equation for* p *is independent of* p''*, so the kinetic potential has the form:*

$$H = f(t, p, \mathfrak{p}, \mathfrak{p}')\, p' + f_1(t, p, \mathfrak{p}, \mathfrak{p}'),$$

the fourth-order differential equation in $\mathfrak{p}$ *that emerges by eliminating p and its derivatives will possess a second-order kinetic potential. For the case in which* p'' *is not included in the second* ***Lagrange*** *equation either, so one has:*

$$H = f(t, p, \mathfrak{p})\, p' + f_1(t, p, \mathfrak{p}, \mathfrak{p}'),$$

the resulting second-order differential equation in $\mathfrak{p}$ *will, in turn, possess a first-order kinetic potential*.

2) *When the* ***Lagrange*** *equation that belongs to* $\mathfrak{p}$ *does not include* p' *or* p''*, and the kinetic potential does not include time t explicitly, the necessary and sufficient condition for the result of the elimination of* $\mathfrak{p}$ *to possess a second-order kinetic potential is that H must have the form:*

$$H = a\, p'^2 + c\, \mathfrak{p}'^2 + \varphi(p - \lambda_1 \mathfrak{p}) + \psi(p - \lambda_2 \mathfrak{p}),$$

in which a and c are arbitrary constants, $\lambda_1 \cdot \lambda_2 = -c / a$*, and* φ*, as well as* ψ*, mean arbitrary functions of their arguments.*

3) *When* p' *is missing from both* ***Lagrange*** *equations, the kinetic potential does not include time t explicitly, and the external forces are zero, the necessary and sufficient condition for the existence of a second-order kinetic potential for the fourth-order differential equation in* $\mathfrak{p}$ *is that the form of the original kinetic potential must have been:*

$$H = a\, p'^2 + \omega_1(\mathfrak{p})\, p'\, \mathfrak{p}' + \frac{\omega_1^2 - c}{4a}\, \mathfrak{p}'^2 + \Omega_3(\mathfrak{p}) + \omega_4(p, \mathfrak{p}),$$

in which a and c are arbitrary constants, and ω_1*,* ω_4*,* Ω *are coupled to each other by the differential equation:*

$$4a^2 \frac{\partial^2 \Omega_3(\mathfrak{p})}{\partial \mathfrak{p}^2} + 4a^2 \frac{\partial^2 \omega_4}{\partial \mathfrak{p}^2} - 2a\, \omega_1 \frac{\partial^2 \omega_4}{\partial \mathfrak{p}\, \partial p} - 2a \frac{\partial \omega_1}{\partial \mathfrak{p}} \frac{\partial \omega_4}{\partial p} + c \frac{\partial^2 \omega_4}{\partial p^2} = (\omega_1 + C)\left(2a \frac{\partial^2 \omega_4}{\partial \mathfrak{p}\, \partial p} - \omega_1 \frac{\partial^2 \omega_4}{\partial p^2} \right),$$

in which C means an arbitrary constant.

4) *Finally, when p is not contained in either* ***Lagrange*** *equation, so one has:*

$$H = F(t, \mathfrak{p}, p', \mathfrak{p}') ,$$

the kinetic potential of the elimination equation will, in turn, have order one.

In all of the above, a complete differential quotient with respect to t of an arbitrary function of the coordinates p and $\mathfrak{p}$ *has been ignored.*

The fact that for two **Lagrange** equations of motion in the mechanics of ponderable masses, the elimination of one coordinate will not always lead to a **Lagrange** equation in one variable and a second-order kinetic potential already emerges in the simple case of a free point that moves in a plane and is subject to a force function. Its kinetic potential is:

$$H = -\tfrac{1}{2} m(x'^2 + y'^2) - U(x, y) ,$$

and its equations of motion will then be:

$$m\,x'' - \frac{\partial U}{\partial x} = 0 , \qquad m\,y'' - \frac{\partial U}{\partial y} = 0 .$$

As is clear from the third case that was treated in this section, eliminating x from the two equations of motion will yield a condition equation for the force function if the resulting fourth-order differential equation in y is to possess a second-order kinetic potential. With the introduction of complex variables, by which the kinetic potential for the two variables p and $\mathfrak{p}$ in the mechanics of ponderable masses can always be put into the form:

$$H = \omega(p, \mathfrak{p})\, p'\, \mathfrak{p}' + \Omega(p, \mathfrak{p}) ,$$

the **Lagrange** equations will then assume the form:

$$\omega\,\mathfrak{p}'' + \mathfrak{p}'^2 \frac{\partial \omega}{\partial \mathfrak{p}} + \frac{\partial \Omega}{\partial p} = 0 , \qquad \omega\, p'' + p'^2 \frac{\partial \omega}{\partial p} + \frac{\partial \Omega}{\partial \mathfrak{p}} = 0 ,$$

so the present situation will then belong to the case that was treated in this section in which the result of the elimination always possessed a second-order kinetic potential. We shall not go into the details of that here.

However, we can also exhibit the general conditions that a first-order kinetic potential H must be subject to in order for the elimination of the coordinate p from the two **Lagrange** equations:

$$\frac{\partial H}{\partial p} - \frac{d}{dt}\frac{\partial H}{\partial p'} = 0 , \tag{44}$$

$$\frac{\partial H}{\partial \mathfrak{p}} - \frac{d}{dt}\frac{\partial H}{\partial \mathfrak{p}'} = 0 \tag{45}$$

to lead to a fourth-order differential equation that possesses a second-order kinetic potential.

Namely, if one differentiates each of equations (44) and (45) twice with respect to t then, from (2) and (3) of **§ 2**, the six equations that thus arise can be put into the form:

$$\left\{\begin{array}{ll} 2\dfrac{\partial H}{\partial p} - \dfrac{\partial H'}{\partial p'} = 0, & 2\dfrac{\partial H}{\partial \mathfrak{p}} - \dfrac{\partial H'}{\partial \mathfrak{p}'} = 0, \\ 3\dfrac{\partial H'}{\partial p} - \dfrac{\partial H''}{\partial p'} = 0, & 3\dfrac{\partial H'}{\partial \mathfrak{p}} - \dfrac{\partial H''}{\partial \mathfrak{p}'} = 0, \\ 4\dfrac{\partial H''}{\partial p} - \dfrac{\partial H'''}{\partial p'} = 0, & 4\dfrac{\partial H''}{\partial \mathfrak{p}} - \dfrac{\partial H'''}{\partial \mathfrak{p}'} = 0. \end{array}\right. \tag{46}$$

If the result of eliminating the five quantities p, p', p'', p''', p^{IV} from the six equations is represented by $Q = 0$ then the necessary and sufficient condition for there to exist a second-order kinetic potential $\mathfrak{H}$, or that:

$$Q = \frac{\partial \mathfrak{H}}{\partial \mathfrak{p}} - \frac{d}{dt}\frac{\partial \mathfrak{H}}{\partial \mathfrak{p}'} + \frac{d^2}{dt^2}\frac{\partial \mathfrak{H}}{\partial \mathfrak{p}''} ,$$

is known to have the form:

$$2\frac{\partial Q}{\partial \mathfrak{p}'} - 2\frac{d}{dt}\frac{\partial Q}{\partial \mathfrak{p}''} + \frac{d^2}{dt^2}\frac{\partial Q}{\partial \mathfrak{p}'''} = 0 ,$$

$$\frac{\partial Q}{\partial \mathfrak{p}'''} - 2\frac{d}{dt}\frac{\partial Q}{\partial \mathfrak{p}''''} = 0 ,$$

or in turn, with the help of (2) and (3) in **§ 2**:

$$5\frac{\partial Q}{\partial \mathfrak{p}'} - 4\frac{\partial Q'}{\partial \mathfrak{p}''} + \frac{\partial Q''}{\partial \mathfrak{p}'''} = 0 ,$$

$$3\frac{\partial Q}{\partial \mathfrak{p}'''} - 2\frac{\partial Q'}{\partial \mathfrak{p}''''} = 0 .$$

The representation of those conditions for the kinetic potential H with the help of equations (46) will imply the necessary and sufficient conditions for the result of the elimination to be a **Lagrange** equation for a second-order kinetic potential.

We conclude these studies with a general remark that might be based upon a kinetic potential H of order ν in only two variables p and $\mathfrak{p}$, for brevity, but without altering the result. Under the assumption that the external forces are zero, the two **Lagrange** equations will read:

$$\frac{\partial H}{\partial p} - \frac{d}{dt}\frac{\partial H}{\partial p'} + \frac{d^2}{dt^2}\frac{\partial H}{\partial p''} - \cdots + (-1)^\nu \frac{d^\nu}{dt^\nu}\frac{\partial H}{\partial p^{(\nu)}} = 0 ,$$

$$\frac{\partial H}{\partial \mathfrak{p}} - \frac{d}{dt}\frac{\partial H}{\partial \mathfrak{p}'} + \frac{d^2}{dt^2}\frac{\partial H}{\partial \mathfrak{p}''} - \cdots + (-1)^\nu \frac{d^\nu}{dt^\nu}\frac{\partial H}{\partial \mathfrak{p}^{(\nu)}} = 0 .$$

In order to obtain the differential equation in the variable $\mathfrak{p}$ that results from eliminating p and its derivatives from them, one differentiates each equation 2ν times with respect to t and eliminates the $4\nu + 1$ quantities $p, p', p'', \ldots, p^{(4\nu)}$ from the $4\nu + 2$ equations that thus arise. The result of that elimination will take the form:

(47) $$F(t, \mathfrak{p}, \mathfrak{p}', \mathfrak{p}'', \ldots, \mathfrak{p}^{(4\nu)}) = 0 .$$

However, since the **Lagrange** equations make:

$$\delta \int_{t_0}^{t_1} H\, dt = 0 ,$$

from **Hamilton**'s principle, when one substitutes the values of $p, p', p'', \ldots, p^{(\nu)}$ as functions of $t, \mathfrak{p}, \mathfrak{p}', \mathfrak{p}'', \ldots, \mathfrak{p}^{(4\nu)}$ that are derived from the system of differentiated **Lagrange** equations in the expression H and denotes the function of $\mathfrak{p}$ and its derivatives up to order 4ν that one obtains by $\mathfrak{H}$, one will have:

$$\delta \int_{t_0}^{t_1} \mathfrak{H}\, dt = 0 .$$

Therefore, one will have *the differential equation* (47) *of order* 4ν, *although it will not possess a kinetic potential of order* 2ν, *in general, but an integral function of the differential equation of order* 8ν (*):

(*) For example, let:

$$H = -\frac{1}{2}(p'^2 + \mathfrak{p}'^2) - p^3 - p\mathfrak{p} ,$$

so the equations of motion read:

$$p'' = 3p^2 + \mathfrak{p} , \qquad \mathfrak{p}'' = p .$$

The elimination equation in the variable $\mathfrak{p}$ will then be:

(6) $$Q = \mathfrak{p}^{IV} - 3\mathfrak{p}''^2 - \mathfrak{p} = 0 ,$$

and since:

$$\frac{\partial \mathfrak{H}}{\partial \mathfrak{p}} - \frac{d}{dt}\frac{\partial \mathfrak{H}}{\partial \mathfrak{p}'} + \frac{d^2}{dt^2}\frac{\partial \mathfrak{H}}{\partial \mathfrak{p}''} - \cdots + \frac{d^{4\nu}}{dt^{4\nu}}\frac{\partial \mathfrak{H}}{\partial \mathfrak{p}^{(4\nu)}} = 0 .$$

$$\frac{\partial Q}{\partial \mathfrak{p}'} - \frac{d}{dt}\frac{\partial Q}{\partial \mathfrak{p}''} + \frac{d^2}{dt^2}\frac{\partial Q}{\partial \mathfrak{p}'''} - \frac{d^3}{dt^3}\frac{\partial Q}{\partial \mathfrak{p}''''}$$

is not identically zero, that will not have a second-order kinetic potential. If one substitutes $p = \mathfrak{p}''$, $p' = \mathfrak{p}'''$ in H according to the second Lagrange equation then that will give:

$$\mathfrak{H} = \frac{\partial \mathfrak{H}}{\partial \mathfrak{p}} - \frac{d}{dt}\frac{\partial \mathfrak{H}}{\partial \mathfrak{p}'} + \frac{d^2}{dt^2}\frac{\partial \mathfrak{H}}{\partial \mathfrak{p}''} - \frac{d^3}{dt^3}\frac{\partial \mathfrak{H}}{\partial \mathfrak{p}'''} = \mathfrak{p}^{(IV)} - 6\mathfrak{p}'''^{\,2} - 6\mathfrak{p}''\mathfrak{p}^{(IV)} - \mathfrak{p}'' = 0 ,$$

so the quantity Q will be an integral function.

§ 18. – The extended Newtonian potential and the generalization of the Laplace-Poisson differential equation.

Let W be an entire function of the derivatives with respect to t, namely, r', r'', …, $r^{(\nu)}$, in which:

$$r^2 = (x-a)^2 + (y-b)^2 + (z-c)^2 .$$

Assume that r itself might enter arbitrarily into that function, and that the function has even degree $2k$ relative to $r^{(\nu)}$. If one denotes:

$$\frac{\partial^2}{\partial x^{(\nu)2}} + \frac{\partial^2}{\partial y^{(\nu)2}} + \frac{\partial^2}{\partial z^{(\nu)2}} \qquad \text{by} \qquad \Delta_{\nu\nu}$$

then the relations:

$$\frac{\partial r^{(\nu)}}{\partial x^{(\nu)}} = \frac{\partial r}{\partial x}, \qquad \frac{\partial r^{(\nu)}}{\partial y^{(\nu)}} = \frac{\partial r}{\partial y}, \qquad \frac{\partial r^{(\nu)}}{\partial z^{(\nu)}} = \frac{\partial r}{\partial z},$$

will, since one has:

$$\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2 + \left(\frac{\partial r}{\partial z}\right)^2 = 1 ,$$

imply that:

$$\Delta_{\nu\nu} = \frac{\partial^{2W}}{\partial r^{(\nu)2}} ,$$

as is obvious. Thus, it has degree $2k - 2$ relative to $r^{(\nu)}$, such that the k-times iterated expression:

$$\Delta_{\nu\nu}^k W = V$$

will no longer include the quantity $r^{(\nu)}$ at all, and therefore the coefficient of $r^{(\nu)2k}$ in W will be multiplied by $(2k)!$.

By contrast, if W has odd degree $2k + 1$ relative to $r^{(\nu)}$ then:

$$\Delta_{\nu\nu}^k W = W_1$$

will still have degree one relative to $r^{(\nu)}$. One then denotes:

$$\frac{\partial^2}{\partial x^{(\nu)}\,\partial x^{(\nu-1)}} + \frac{\partial^2}{\partial y^{(\nu)}\,\partial y^{(\nu-1)}} + \frac{\partial^2}{\partial z^{(\nu)}\,\partial z^{(\nu-1)}} \qquad \text{by} \qquad \Delta_{\nu\nu-1}$$

and observes that when $\nu > 1$, the relation that follows from (2) and (3) in **§ 2**:

$$\frac{\partial r^{(\nu)}}{\partial x^{(\nu-1)}} = \nu \frac{\partial r'}{\partial x},$$

and the relation:

$$\frac{\partial r}{\partial x}\frac{\partial r}{\partial x'} + \frac{\partial r}{\partial y}\frac{\partial r}{\partial y'} + \frac{\partial r}{\partial z}\frac{\partial r}{\partial z'} = 0$$

will imply that the expression:

$$\Delta_{\nu\nu-1}\, W_1 = \frac{\partial^2 W_1}{\partial r^{(\nu)}\,\partial r^{(\nu-1)}}$$

is independent of $r^{(\nu)}$. That will then yield:

$$\Delta_{\nu\nu-1}\,\Delta_{\nu\nu}^k W = V,$$

in which V depends upon only $r,\ r',\ r'',\ \ldots,\ r^{(\nu)}$ and represents the partial differential quotients of the coefficients of $r^{(\nu)2k+1}$ with respect to $r^{(\nu-1)}$, multiplied by $(2k + 1)!$. For the case in which $v = 1$, it will follow from:

$$\Delta_{11}\, W = W_1$$

and the relation:

$$\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} + \frac{\partial^2 r}{\partial z^2} = \frac{2}{r}$$

that:

$$\Delta_{10}\, W_1 = \frac{\partial^2 W_1}{\partial r\,\partial r'} + \frac{2}{r}\frac{\partial W_1}{\partial r'},$$

as is easy to see. Thus, it will once more follow that:

$$\Delta_{10}\,\Delta_{11}^k W = V,$$

in which V depends upon only r, and if the coefficient of $r'^{\,2k+1}$ in W is $\varphi(r)$ then W will have the value:

$$(2k+1)!\left(\varphi'(r) + \frac{2}{r}\,\varphi(r)\right).$$

Since one can now apply the same argument to the functions V that include only $r,\ r',\ \ldots,\ r^{(\nu-1)}$, it will follow that since only the terms in W that include the highest power of $r^{(\nu)}$ will come under consideration in the formation of the equation for V above, the continuation of the process to the lower-order terms will again affect only those terms that include the highest power of $r^{(\nu-1)}$, etc. Moreover, it will then follow upon repeating that process when the only terms in *W* that come under consideration are denoted by:

$$r^{(\nu)\alpha_\nu} r^{(\nu-1)\alpha_{\nu-1}} \cdots r''^{\alpha_2} r'^{\alpha_1} \varphi(r),$$

and one ultimately ignores a constant that will be specified later, one will be led to either:

$$\varphi(r) \quad \text{or} \qquad \frac{\partial \varphi(r)}{\partial r} + \frac{2}{r}\varphi(r),$$

whose common form can be represented by:

$$\frac{\partial^{\varepsilon_1}\varphi(r)}{\partial r^{\varepsilon_1}} + \varepsilon_1 \frac{2}{r}\varphi(r),$$

when $\varepsilon_1 = 0$ or 1, resp. However, since one finally has that for any function *V* of *r* :

$$\Delta_{00} V = \frac{\partial^2 V}{\partial r^2} + \frac{2}{r}\frac{\partial V}{\partial r},$$

one will then have:

$$\Delta_{00}\left\{\frac{\partial^{\varepsilon_1}\varphi(r)}{\partial r^{\varepsilon_1}} + \varepsilon_1 \frac{2}{r}\varphi(r)\right\} = \frac{\partial^{2+\varepsilon_1}\varphi(r)}{\partial r^{2+\varepsilon_1}} + \varepsilon_1 \frac{2}{r}\frac{\partial^2\varphi(r)}{\partial r^2} + \frac{2}{r}\frac{\partial^{1+\varepsilon_1}\varphi(r)}{\partial r^{1+\varepsilon_1}}.$$

One will then get the following extension of the known transformation of a function *W* that depends upon only *r* :

$$\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} + \frac{\partial^2 W}{\partial z^2} = \frac{\partial^2 W}{\partial r^2} + \frac{2}{r}\frac{\partial W}{\partial r}.$$

Let W be an entire function in the quantities r, r', …, $r^{(\nu)}$, *into which r itself can enter arbitrarily, and one selects those terms from it:*

$$r^{(\nu)\alpha_\nu} r^{(\nu-1)\alpha_{\nu-1}} \cdots r''^{\alpha_2} r'^{\alpha_1} \varphi(r)$$

that have the property that $r^{(\nu)\alpha_\nu}$ *is the highest power of* $r^{(\nu)}$ *that occurs in W,* $r^{(\nu-1)\alpha_{\nu-1}}$ *is the highest power of* $r^{(\nu-1)}$ *that occurs in conjunction with* $r^{(\nu)\alpha_\nu}$, $r^{(\nu-2)\alpha_{\nu-2}}$ *is the highest power of* $r^{(\nu-2)}$ *that occurs in conjunction with* $r^{(\nu)\alpha_\nu} r^{(\nu-1)\alpha_{\nu-1}}$, *etc., and which shall be called the highest power of W. One then sets:*

$$\alpha_\nu = 2\kappa_\nu + \varepsilon_\nu,$$

$$\begin{aligned}
\alpha_{\nu-1} - \varepsilon_\nu &= 2\kappa_{\nu-1} + \varepsilon_{\nu-1} , \\
\alpha_{\nu-2} - \varepsilon_{\nu-1} &= 2\kappa_{\nu-2} + \varepsilon_{\nu-2} , \\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
\alpha_2 - \varepsilon_3 &= 2\kappa_2 + \varepsilon_2 , \\
\alpha_1 - \varepsilon_2 &= 2\kappa_1 + \varepsilon_1 ,
\end{aligned}$$

in which the quantities $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_\nu$ *mean the numbers* 0 *or* 1. *The transformation that is then analogous to the one above will then assume the form:*

$$\begin{aligned}
&\Delta_{00}\,\Delta_{10}^{\varepsilon_1}\,\Delta_{11}^{\kappa_1}\,\Delta_{21}^{\varepsilon_2}\,\Delta_{22}^{\kappa_2}\cdots\Delta_{\nu-1,\nu-2}^{\varepsilon_{\nu-1}}\,\Delta_{\nu-1,\nu-1}^{\kappa_{\nu-1}}\,\Delta_{\nu,\nu-1}^{\varepsilon_\nu}\,\Delta_{\nu\nu}^{\kappa_\nu}\,W \\
&= \alpha_\nu\,!\,\alpha_{\nu-1}!\cdots\alpha_2\,!\alpha_1\,!\left\{\frac{\partial^{2+\varepsilon_1}\varphi(r)}{\partial r^{2+\varepsilon_1}} + \varepsilon_1\,\frac{2}{r}\,\frac{\partial^2\varphi(r)}{\partial r^2} + \frac{2}{r}\,\frac{\partial^{1+\varepsilon_1}\varphi(r)}{\partial r^{1+\varepsilon_1}}\right\}.
\end{aligned}$$

Now, in order to obtain the differential equation for a force function (in the sense that was defined in **§ 4**) with a finite or infinite number of centers at a point that lies outside the masses in a form that is the same for all r and any number of them, the right-hand side of the equation above must vanish, and therefore $\varphi(r)$ will satisfy one of the following two differential equations:

$$\frac{\partial^2\varphi(r)}{\partial r^2} + \frac{2}{r}\,\frac{\partial\varphi(r)}{\partial r} = 0$$

or

$$\frac{\partial^3\varphi(r)}{\partial r^3} + \frac{4}{r}\,\frac{\partial^2\varphi(r)}{\partial r^2} = 0 .$$

It will then have the form:

$$\varphi(r) = \frac{c}{r} + c_1 \qquad \text{or} \qquad \varphi(r) = \frac{c}{r^2} + c_1\,r + c_2 ,$$

and we will then find that:

The extended ***Laplace*** *equation for all functions W that depend upon r and its derivatives with respect to t up to order* ν *that are entire functions of those derivatives and whose highest term has the coefficient:*

$$\frac{c}{r} + c_1 \qquad \text{or} \qquad \frac{c}{r^2} + c_1\,r + c_2$$

assumes the form:

(1) $$\Delta_{00}\,\Delta_{10}^{\varepsilon_1}\,\Delta_{11}^{\kappa_1}\,\Delta_{21}^{\varepsilon_2}\,\Delta_{22}^{\kappa_2}\cdots\Delta_{\nu-1,\nu-2}^{\varepsilon_{\nu-1}}\,\Delta_{\nu-1,\nu-1}^{\kappa_{\nu-1}}\,\Delta_{\nu,\nu-1}^{\varepsilon_\nu}\,\Delta_{\nu\nu}^{\kappa_\nu}\,W = 0 .$$

The assumption that W is an entire function of $r, r', \dots, r^{(\nu)}$, into which r itself should enter arbitrarily, excludes the possibility that W can become infinitely large for an arbitrary value of r and finite values of the derivatives.

Suppose that:

$$R(r, r', r'', \dots, r^{(2\nu)})$$

is a force that depends upon the distance and its derivatives with respect to time up to order 2ν *and satisfies the equations:*

$$[1-(-1)^{\rho}]\frac{\partial R}{\partial r^{(\rho)}} - (\rho+1)_1 \frac{d}{dt}\frac{\partial R}{\partial r^{(\rho+1)}} + (\rho+2)_2 \frac{d^2}{dt^2}\frac{\partial R}{\partial r^{(\rho+2)}} - \dots + (-1)^{2\nu-\rho}(2\nu)_{2\nu-2}\frac{d^{2\nu-\rho}}{dt^{2\nu-\rho}}\frac{\partial R}{\partial r^{(2\nu)}} = 0$$

identically for $\rho = 1, 3, 5, \dots, 2\nu - 1$. *We shall call a force function W that satisfies the equation:*

$$R(r, r', r'', \dots, r^{(2\nu)}) = \frac{\partial W}{\partial r} - \frac{d}{dt}\frac{\partial W}{\partial r'} + \dots + (-1)^{\nu}\frac{d^{\nu}}{dt^{\nu}}\frac{\partial W}{\partial r^{(\nu)}}$$

a ***potential*** *when the highest-order term (in the sense that was defined above) in that function that depends upon r and its* ν *derivatives is expressed in the form:*

$$r^{(\nu)\alpha_\nu} r^{(\nu-1)\alpha_{\nu-1}} \cdots r''^{\alpha_2} r'^{\alpha_1}\left(\frac{c}{r} + c_1\right)$$

or

$$r^{(\nu)\alpha_\nu} r^{(\nu-1)\alpha_{\nu-1}} \cdots r''^{\alpha_2} r'^{\alpha_1}\left(\frac{c}{r^2} + c_1\, r + c_2\right),$$

according to whether the quantities are determined by the equations:

$$\alpha_\nu = 2\kappa_\nu + \varepsilon_\nu, \quad \alpha_{\nu-1} - \varepsilon_\nu = 2\kappa_{\nu-1} + \varepsilon_{\nu-1}, \quad \dots, \quad \alpha_2 - \varepsilon_3 = 2\kappa_2 + \varepsilon_2, \quad \alpha_1 - \varepsilon_2 = 2\kappa_1 + \varepsilon_1,$$

in which the quantities ε *mean the numbers* 0 *or* 1 *with:*

$$\varepsilon_1 = \alpha_1 - \alpha_2 + \alpha_3 - \alpha_4 + \dots + (-1)^{\nu-1}\alpha_\nu \qquad (\text{mod } 2),$$

have the value 0 *or* 1, *resp. The extended* ***Laplace*** *equation for the generalized* ***Newton****ian potential will then read:*

$$\Delta_{00}\,\Delta_{10}^{\varepsilon_1}\,\Delta_{11}^{\kappa_1}\,\Delta_{21}^{\varepsilon_2}\,\Delta_{22}^{\kappa_2}\cdots\Delta_{\nu-1,\nu-2}^{\varepsilon_{\nu-1}}\,\Delta_{\nu-1,\nu-1}^{\kappa_{\nu-1}}\,\Delta_{\nu,\nu-1}^{\varepsilon_\nu}\,\Delta_{\nu\nu}^{\kappa_\nu}\,W = 0 .$$

For the force function of **Weber**'s law:

$$W = \frac{m\,m_1}{r}\left(1 + \frac{r'^2}{k^2}\right),$$

the highest-order term is:

$$\frac{m\,m_1}{k^2}\frac{r'^2}{r}\ .$$

Since $\nu = 1$, $\alpha_1 = 2$, so $\kappa_1 = 1$ and $\varepsilon_1 = 0$, W will have the desired form of an extended **Newton**ian potential and will satisfy the partial differential equation (*):

(2) $$\Delta_{00}\,\Delta_{11}\,W = 0$$

or

$$\frac{\partial^4 W}{\partial x^2\,\partial x'^2} + \frac{\partial^4 W}{\partial x^2\,\partial y'^2} + \frac{\partial^4 W}{\partial x^2\,\partial z'^2}$$
$$+ \frac{\partial^4 W}{\partial y^2\,\partial x'^2} + \frac{\partial^4 W}{\partial y^2\,\partial y'^2} + \frac{\partial^4 W}{\partial y^2\,\partial z'^2}$$
$$+ \frac{\partial^4 W}{\partial z^2\,\partial x'^2} + \frac{\partial^4 W}{\partial z^2\,\partial y'^2} + \frac{\partial^4 W}{\partial z^2\,\partial z'^2} = 0\ .$$

(*) Let a force function have the form:

$$W = \frac{m\,m_1}{r}\left(1 + \frac{r'^{\lambda}}{k^2}\right),$$

for which $\nu = 1$, $\alpha_1 = \lambda$, so ε_1 will be zero or unity according to whether λ is even or odd, resp., which will then be a potential in the sense that was defined above only for even λ. Since:

$$\frac{\partial^2 r'}{\partial x^2} + \frac{\partial^2 r'}{\partial y^2} + \frac{\partial^2 r'}{\partial z^2} = -\ 2\,\frac{r'}{r^2}\,,$$

when one sets:

$$x'^2 + y'^2 + z'^2 = \nu^2\,,$$

so

$$\left(\frac{\partial r'}{\partial x}\right)^2 + \left(\frac{\partial r'}{\partial y}\right)^2 + \left(\frac{\partial r'}{\partial z}\right)^2 = \frac{\nu^2 - r'^2}{r^2}\,,$$

the relation will follow:

$$\Delta_{00}\,\Delta_{00}\,W = \lambda\,(\lambda-1)\,(\lambda-2)\,(\lambda-3)\,\frac{r'^{\lambda-4}\nu^4}{r^5} - 2\lambda\,(\lambda-1)\,(\lambda-2)\,(\lambda+1)\,\frac{r'^{\lambda-2}\nu^2}{r^5} - \lambda\,(\lambda+1)\,(\lambda-2)\,(\lambda+3)\,\frac{r'^{\lambda}}{r^5}\,,$$

such that for $\lambda = 0$ and $\lambda = 2$, one will have:

$$\Delta_{00}\,\Delta_{00}\,W = 0$$

for the **Newtonian** and **Weber** potentials.

In general, for any force function that depends upon only r and the first derivatives with respect to time (so the force is a function of r, r', r''), the existence of an extended **Newton**ian potential will require that it must take the form:

$$W = \varphi_0(r) + \varphi_1(r)\,r' + \varphi_2(r)\,r'^2 + \cdots + \varphi_{2\kappa-1}(r)\,r'^{2\kappa-1} + \left(\frac{c}{r} + c_1\right) r'^{\,2\kappa}$$

or

$$W = \varphi_0(r) + \varphi_1(r)\,r' + \varphi_2(r)\,r'^2 + \cdots + \varphi_{2\kappa}(r)\,r'^{2\kappa} + \left(\frac{c}{r^2} + c_1\,r + c_2\right) r'^{\,2\kappa+1}, \tag{4}$$

in which $\varphi_0(r)$, $\varphi_1(r)$, …, $\varphi_{2\kappa}(r)$ mean arbitrary functions of r, and the corresponding **Laplace** equations will read:

$$\Delta_{00}\,\Delta_{11}^{\kappa}\,W = 0 \quad \text{and} \quad \Delta_{00}\,\Delta_{10}\,\Delta_{11}^{\kappa}\,W = 0\,. \tag{5}$$

In order to determine the extended **Poisson** differential equation for the potentials (3) and (4), as is known, it is sufficient to determine the potential of a homogeneous hollow sphere at a point that lies inside of it, and for that reason, we shall address the somewhat-more-general problem:

Calculate the potential of a spherical shell with concentric layers of homogeneous mass-elements that act according to the potentials (3) *or* (4) *at point that lies inside or outside of it.*

If one puts the origin of a rectangular coordinate system at the center of the spherical shell, whose inner and outer radii might be denoted by R_0 and R_1, resp., and lays the YZ-plane along the direction in which the point moves at the moment in question with a velocity whose magnitude and direction are given then if its components are denoted by x', y', z', one will have:

$$x' = 0\,, \qquad v^2 = y'^2 + z'^2.$$

If one further denotes the coordinates of the spherical shell by a, b, c, and lets r be the distance from the attracted point to a point on the ring then it will follow from:

$$r^2 = (x-a)^2 + (y-b)^2 + (z-c)^2\,,$$

in which one sets $x = 0$, $y = 0$, upon differentiating with respect to t and preserving the coordinates a, b, c, that:

$$r\,r' = (x-a)\,x' + (y-b)\,y' + (z-c)\,z'\,,$$

or at the attracted point, which is characterized by $x = 0$, $y = 0$, $x' = 0$:

$$r\,r' = -b\,y' - c\,z' + z\,z'\,.$$

If one introduces polar coordinates for the coordinates of the spherical shell then one will have:

$$a = \rho \sin \vartheta \sin \varphi, \qquad b = \rho \sin \vartheta \cos \varphi, \qquad c = \rho \cos \vartheta,$$

with the well-known notations, and the relation above will go to:

(6) $$r' = z z' \frac{1}{r} - z' \frac{\rho \cos \vartheta}{r} - y' \frac{\rho \sin \vartheta \cos \varphi}{r}.$$

Since the essentially-positive r is defined by:

(7) $$r = \sqrt{z^2 + \rho^2 - 2\rho z \cos \vartheta},$$

when one recalls (3), (4), (6), (7), the potential of the spherical shell, which is constant in concentric layers, so its density, which varies only with ρ, will be denoted by σ, will be given by the expression:

(8) $$W = \int_{R_0}^{R_1} \int_0^{\pi} \int_0^{2\pi} \sigma \rho^2 \sin \vartheta \sum_{i=1}^{\lambda} \left\{ \frac{\psi_i \left(\sqrt{z^2 + \rho^2 - 2\rho z \cos \vartheta} \right)}{(z^2 + \rho^2 - 2\rho z \cos \vartheta)^{1/2}} (z z' - z \rho \cos \vartheta - y' \rho \sin \vartheta \cos \vartheta)^i \right\} d\varphi \, d\vartheta \, d\rho,$$

when we take the mass of the attracting point equal to unity. The potential W can be put into the form that is common to (3) and (4):

$$W = m \left\{ \psi_0(r) + \psi_1(r) r' + \cdots + \psi_\lambda(r) r'^{\lambda} \right\},$$

in which m is the mass of the attracting point, and according to whether:

$$\lambda = 2\kappa \qquad \text{or} \qquad \lambda = 2\kappa + 1,$$

one sets:

$$\psi_{2\kappa}(r) = \frac{c}{r} + c_1 \qquad \text{or} \qquad \psi_{2\kappa+1}(r) = \frac{c}{r^2} + c_1 r + c_2,$$

respectively.

In order to simplify the integrations, we base our calculations upon **Weber**'s law, for which we have:

$$\lambda = 2, \quad \psi_0(r) = \frac{1}{r}, \quad \psi_1(r) = 0, \quad \psi_2(r) = \frac{1}{k^2 r}.$$

After performing the integration over φ, we will then have:

(9) $$W = 2\pi \int_{R_0}^{R_1}\int_0^{\pi} \frac{\sigma\,\rho^2 \sin\vartheta\,d\vartheta\,d\rho}{(z^2+\rho^2-2\rho\,z\cos\vartheta)^{1/2}} + \frac{\pi}{k^2}\int_{R_0}^{R_1}\int_0^{\pi} \frac{2\,(z\,z'-\rho\,z'\cos\vartheta)^2 + y'^2\rho^2\sin^2\vartheta}{(z^2+\rho^2-2\rho\,z\cos\vartheta)^{1/2}}\,\sigma\,\rho^2\sin\vartheta\,d\vartheta\,d\rho\,.$$

If we denote the integral Θ over the variable ϑ between the limits 0 and π by Θ_a when it is taken for a point lies outside the spherical shell and by Θ_i for one that lies inside its interior then it is easy to see that:

for $\Theta = \int_0^{\pi} \frac{\sin\vartheta\,d\vartheta}{(z^2+\rho^2-2\rho\,z\cos\vartheta)^{1/2}}$, $\quad \Theta_a = \frac{2}{z}$, $\quad \Theta_i = \frac{2}{\rho}$,

for $\Theta = \int_0^{\pi} \frac{\sin\vartheta\,d\vartheta}{(z^2+\rho^2-2\rho\,z\cos\vartheta)^{1/2}}$, $\quad \Theta_a = \frac{2}{z}\frac{1}{z^2-\rho^2}$, $\quad \Theta_i = \frac{2}{z}\frac{1}{\rho^2-z^2}$,

for $\Theta = \int_0^{\pi} \frac{\sin\vartheta\cos\vartheta\,d\vartheta}{(z^2+\rho^2-2\rho\,z\cos\vartheta)^{1/2}}$, $\quad \Theta_a = \frac{2\rho}{z^2(z^2-\rho^2)}$, $\quad \Theta_i = \frac{2\,z}{z^2(z^2-\rho^2)}$,

for $\Theta = \int_0^{\pi} \frac{\sin\vartheta\cos^2\vartheta\,d\vartheta}{(z^2+\rho^2-2\rho\,z\cos\vartheta)^{1/2}}$, $\quad \Theta_a = \frac{2}{3z^3}\frac{z^2+2\rho^2}{z^2-\rho^2}$, $\quad \Theta_i = \frac{2}{3\rho^3}\frac{\rho^2+2z^2}{z^2-\rho^2}$.

If we also denote the corresponding values of the potential by W_a and W_i, resp., then the substitution of those values of the integral in the expression (9) will give:

$$W_a = M\left(\frac{1}{z}+\frac{z'^2}{k^2 z}\right) - \frac{4\pi}{3k^2}\frac{3z'^2-v^2}{z^3}\int_{R_0}^{R_1}\sigma\,\rho^4\,d\rho$$

and

$$W_i = 4\pi\int_{R_0}^{R_1}\sigma\,\rho\,d\rho + \frac{4\pi}{3k^2}v^2\int_{R_0}^{R_1}\sigma\,\rho\,d\rho\,,$$

after a simple calculation, when the total mass of the spherical shell is denoted by M.

Therefore, if a point that lies outside of a spherical shell that has concentric layers of constant density is at a distance l from the center of the sphere and possesses the velocity v, and l' *is the projection of v onto the direction of l then the value of the potential will be given by the expressions:*

(10) $$W_a = M\left(\frac{1}{z}+\frac{z'^2}{k^2 z}\right) - \frac{4\pi}{3k^2}\frac{3z'^2-v^2}{z^3}\int_{R_0}^{R_1}\sigma\,\rho^4\,d\rho$$

and

(11) $$W_i = 4\pi \int_{R_0}^{R_1} \sigma\,\rho\,d\rho + \frac{4\pi}{3k^2} v^2 \int_{R_0}^{R_1} \sigma\,\rho\,d\rho\,.$$

Thus (as should already be obvious from symmetry), the potential depends upon only the distance from the attracted point to the center, the magnitude of its velocity, and the direction of the latter with respect to the connecting line to the center.

The first term in the potential W_a is nothing but the value of the **Weber** potential when the total mass of the spherical ring is concentrated at the center, and that is also the total value of the potential when $v^2 = 3l'^2$, or when the angle that the velocity makes with the connecting line to the center is 54° 44′.

It is further obvious that for a point that lies in the interior space, the potential will be independent of the position of the point and the direction of the velocity. Therefore, it will have the form:

$$W_i = a + b\,v^2\,.$$

If the point lies in the concentrated spherical shell itself then the potential might be denoted by W_m . When one again denotes the distance from the point to the center of the sphere by l and composes the potential from the W_a of the spherical shell that belongs to l and R_0 and the W_i of the spherical shell that belongs to R_1 and l, then its value will be given by:

(12) $$W_m = 4\pi\left(\frac{1}{l} + \frac{l'^2}{k^2 l}\right)\int_{R_0}^{l} \sigma\,\rho^2\,d\rho - \frac{4\pi}{3k^2}\frac{3l'^2 - v^2}{l^3}\int_{R_0}^{l} \sigma\,\rho^4\,d\rho + 4\pi\int_{l}^{R_1} \sigma\,\rho\,d\rho + \frac{4\pi}{3k^2} v^2 \int_{l}^{R_1} \sigma\,\rho\,d\rho\,.$$

Once more, as for the **Newton**ian potential, the validity of the theorem will prove that the potential:

$$\int dm\left(\frac{1}{r} + \frac{r'^2}{k^2\,r}\right)$$

is finite and continuous for all of infinite space and for finite values of r', even for the case in which the point enters into the mass itself. However, that will follow immediately from the fact that when one lays the origin of the coordinates at the attracted point x, y, z and introduces polar coordinates by the relations:

$$a - x = r\sin\vartheta\sin\varphi\,, \qquad b - y = r\sin\vartheta\cos\varphi\,, \qquad c - z = r\cos\vartheta\,,$$

the potential will assume the form:

$$\iiint \sigma\, r^2 \sin\vartheta \left(\frac{1}{r} + \frac{r'^2}{k^2\, r}\right) dr\, d\vartheta\, d\varphi \,,$$

which is clear that from the fact that it must be finite even when $r = 0$. Its continuity in regard to r and r' follows in a known, precisely-analogous way.

The fact that the potentials W_a and W_i satisfy the extended **Laplace** differential equation is obvious, since:

$$\frac{\partial W_a}{\partial x'} = \frac{2M}{k^2}\frac{l'\,x}{l^2} - \frac{4\pi}{3k^2}\frac{6\,l'\,x - 2x'\,l}{l^4}\int_{R_0}^{R_1} \sigma\,\rho^4\, d\rho \,,$$

$$\frac{\partial^2 W_a}{\partial x'^2} = \frac{2M}{k^2}\frac{x^2}{l^3} - \frac{4\pi}{3k^2}\frac{6\,x^2 - l^2}{l^5}\int_{R_0}^{R_1} \sigma\,\rho^4\, d\rho \,,$$

so

$$\Delta_{11}\, W_a = \frac{2M}{k^2\, l} \,,$$

and

$$\frac{\partial W_i}{\partial x'} = \frac{8\pi}{3k^2}\, x' \int_{R_0}^{R_1} \sigma\,\rho\, d\rho \,, \qquad \frac{\partial^2 W_i}{\partial x'^2} = \frac{8\pi}{3k^2}\int_{R_0}^{R_1} \sigma\,\rho\, d\rho \,,$$

so

$$\Delta_{11}\, W_i = \frac{8\pi}{3k^2}\int_{R_0}^{R_1} \sigma\,\rho\, d\rho \,,$$

and therefore, one will have:

$$\Delta_{00}\, \Delta_{11}\, W_a = \Delta_{00}\, \Delta_{11}\, W_i = 0 \,.$$

We shall now examine the potential of a solid homogeneous ball at a point inside of it. From (12), it will assume the form:

$$W_m = 2\pi\sigma\left(R^2 - \frac{l^2}{3}\right) + \frac{8\pi\sigma}{15k^2}\, l^2\, l'^2 + \frac{2\pi\sigma}{3k^2}\, R^2\, v^2 - \frac{2\pi\sigma}{5k^2}\, l^2\, v^2 \,,$$

in which R means the radius of the ball, σ is the constant density, and l is the distance from the attracted point to the center. The equations:

$$\frac{\partial W_m}{\partial x'} = \frac{16\pi\sigma}{15k^2}\, l\, l'\, x + \frac{8\pi\sigma}{15k^2}\, l^2\, l'^2 + \frac{2\pi\sigma}{3k^2}\, R^2\, v^2 - \frac{2\pi\sigma}{5k^2}\, l^2\, v^2 \,,$$

$$\frac{\partial^2 W_m}{\partial x'^2} = \frac{16\pi\sigma}{15k^2}\, x^2 + \frac{4\pi\sigma}{3k^2}\, R^2 - \frac{4\pi\sigma}{5k^2}\, l$$

will then imply that:

$$V = \Delta_{11}\, W = -\,\frac{4\pi\,\sigma}{3k^2}\,l^2 + \frac{4\pi\,\sigma}{k^2}\,R^2\,,$$

and since one has:

$$\frac{\partial V}{\partial x} = -\,\frac{8\pi\,\sigma}{3k^2}\,x\,, \qquad \frac{\partial^2 V}{\partial x^2} = -\,\frac{8\pi\,\sigma}{3k^2}\,,$$

it will follow that:

$$\Delta_{00}\,\Delta_{11}\,W_i = -\,\frac{8\pi}{k^2}\,\sigma\,.$$

If one employs the result that was just found by cutting out an infinitely-small ball in the mass that surrounds the attracted point in the mass and is assumed to be homogeneous then *for the **Weber** potential, which is defined by the expression:*

$$W = \frac{m}{r}\left(1 - \frac{r'^2}{k^2}\right)\,,$$

*that will yield the extended **Laplace-Poisson** differential equation in the form of:*

$$\Delta_{00}\,\Delta_{11}\,W_i = -\,\frac{8\pi}{k^2}\,\sigma\,,$$

in which σ means the density of the attracting mass at the location where one finds the attracted point.

One can also derive the constant in the **Poisson** equations for potentials of arbitrary order directly from the extended **Laplace** equation. However, we have likewise found the potential of a spherical shell for forces that act according to **Weber**'s law in that way, and we will then use its value in order to treat a problem of motion.

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§ 19. – The motion of a point under the influence of a first-order force function.

If a point of mass m_1 is subject to a force function:

$$m\,m_1\,F(r, r')$$

that originates from a point of mass m, in which r means the distance between the two points, or a force that is given by the expression:

$$m\,m_1\left(\frac{\partial F}{\partial r} - \frac{d}{dt}\frac{\partial F}{\partial r'}\right),$$

then the kinetic potential will be defined by:

$$H = -\tfrac{1}{2}m_1(x'^2 + y'^2) - m\,m_1\,F(r, r'),$$

and the equations of motion will become:

$$\frac{\partial H}{\partial x} - \frac{d}{dt}\frac{\partial H}{\partial x'} = 0, \qquad \frac{\partial H}{\partial y} - \frac{d}{dt}\frac{\partial H}{\partial y'} = 0,$$

since the motion of the point m_1 will proceed in a plane through its initial position and initial velocity.

Now since the conditions that were presented in **§§ 7** and **10** for the principle of *vis viva* and the law of areas to be valid are fulfilled by the form of the force function that was assumed here, the first integrals of the equations of motion will read:

$$y\frac{\partial H}{\partial x'} - x\frac{\partial H}{\partial y'} = \alpha$$

and

$$H - x'\frac{\partial H}{\partial x'} - y'\frac{\partial H}{\partial y'} = h,$$

in which the area constant and the constant of *vis viva* are determined from the initial position and velocity, or upon introducing polar coordinates, when one further sets $m_1 = 1$:

$$r^2\frac{d\vartheta}{dt} = \alpha$$

and

$$\frac{1}{2}\left[\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\vartheta}{dt}\right)^2\right] - m\,F(r, r') + m\,r'\,\frac{\partial F(r, r')}{\partial r'} = h\,.$$

Upon eliminating $d\vartheta / dt$, it will follow that:

$$r'^2 = 2h - \frac{2m\,\alpha^2}{r^2}\,F\,(r, r') - 2m\,r'\,\frac{\partial F\,(r, r')}{\partial r'}\,,$$

and therefore, t can be represented as a function of r by a quadrature for all of these problems, as in **Kepler**'s problem. For the motion of a point that is attracted to a fixed center with the force function:

$$F\,(r, r') = \varphi_0\,(r) + \varphi_1\,(r)\,r' + \varphi_2\,(r)\,r'^2\,,$$

one will get:

$$t = \int_{r_0} \frac{r\sqrt{1 + 2m\,\varphi_2(r)}}{\sqrt{2h\,r^2 - \alpha^2 + 2m\,r^2\,\varphi_0(r)}}\,dr\,.$$

If that force function is the **Weber** potential, so:

$$F\,(r, r') = \frac{1}{r} + \frac{1}{k^2\,r}\,r'^2\,,$$

then time will follow by the elliptic integral:

$$t = \int_{r_0} \frac{r^2 + \dfrac{2m}{k^2}\,r}{\sqrt{\left(r^2 + \dfrac{2m}{k^2}\,r\right)(2h\,r^2 + 2m\,r - \alpha^2)}}\,dr\,.$$

If we now exhibit the kinetic potential that acts upon a moving point in the more general form:

$$H = f\,(r, r', v)\,,$$

in which:

$$r^2 = x^2 + y^2 + z^2\,, \qquad v^2 = x'^2 + y'^2 + z'^2\,,$$

then the three area theorems (9) will be true, according to (10) in **§ 10**:

$$x\,\frac{\partial H}{\partial y'} - y\,\frac{\partial H}{\partial x'} = c_1\,, \quad y\,\frac{\partial H}{\partial z'} - z\,\frac{\partial H}{\partial y'} = c_2\,, \quad z\,\frac{\partial H}{\partial x'} - x\,\frac{\partial H}{\partial z'} = c_3$$

or

$$\frac{1}{v}\frac{\partial H}{\partial v}(x\,y' - y\,x') = c_1\,, \qquad \frac{1}{v}\frac{\partial H}{\partial v}(y\,z' - z\,y') = c_2\,, \qquad \frac{1}{v}\frac{\partial H}{\partial v}(z\,x' - x\,z') = c_3\,,$$

from which, upon introducing polar coordinates:

$$x = r \sin\vartheta \cos\varphi\,, \qquad y = r \sin\vartheta \sin\varphi\,, \qquad z = r \cos\vartheta\,,$$

and when one sets, in addition:

$$\frac{1}{v}\frac{\partial H}{\partial v} = H_1(r, r', v)\ ,$$

to abbreviate, that will yield the three integral equations:

$$(1)\qquad \left\{\begin{aligned} &H_1(r, r', \sqrt{r'^2 + r^2\,\vartheta'^2 + r^2 \sin^2\vartheta\,\varphi'^2})\cdot r^2\varphi' \sin^2\vartheta = c_1\,,\\ &H_1(r, r', \sqrt{r'^2 + r^2\,\vartheta'^2 + r^2 \sin^2\vartheta\,\varphi'^2})\cdot r^2(\vartheta' \sin\varphi + \varphi' \sin\vartheta\cos\vartheta\cos\varphi) = c_2\,,\\ &H_1(r, r', \sqrt{r'^2 + r^2\,\vartheta'^2 + r^2 \sin^2\vartheta\,\varphi'^2})\cdot r^2(\vartheta' \cos\varphi - \varphi' \sin\vartheta\cos\vartheta\sin\varphi) = c_3\,. \end{aligned}\right.$$

Eliminating ϑ' and φ' from them will give the equation:

$$c_1 \cos\vartheta - c_2 \sin\vartheta \cos\varphi + c_3 \sin\vartheta \sin\varphi = 0\,,$$

or

$$c_1\,z - c_2\,x + c_3\,y = 0\,,$$

which says that the point will then move in the plane that is thus defined.

However, since the energy principle will yield the equation:

$$H - x'\frac{\partial H}{\partial x'} - y'\frac{\partial H}{\partial y'} - z'\frac{\partial H}{\partial z'} = h\,,$$

or since:

$$x'\frac{\partial H}{\partial x'} = \frac{\partial H}{\partial v}\frac{x'^2}{v} + \frac{\partial H}{\partial r'}\frac{x\,x'}{r}\,, \quad y'\frac{\partial H}{\partial y'} = \frac{\partial H}{\partial v}\frac{y'^2}{v} + \frac{\partial H}{\partial r'}\frac{y\,y'}{r}\,, \quad z'\frac{\partial H}{\partial z'} = \frac{\partial H}{\partial v}\frac{y'^2}{v} + \frac{\partial H}{\partial r'}\frac{z\,z'}{r}\,,$$

one will have:

$$H - v\frac{\partial H}{\partial v} - r'\frac{\partial H}{\partial r'} = h\,.$$

In polar coordinates, when one sets:

$$r'\frac{\partial H}{\partial r'} = H_2(r, r', v)\ ,$$

that will go to:

$$(2)\qquad H\left(r, r', \sqrt{r'^2 + r^2\,\vartheta'^2 + r^2\,\varphi'^2 \sin^2\vartheta}\right)$$
$$- (r'^2 + r^2\,\vartheta'^2 + r^2\,\varphi'^2 \sin^2\vartheta)\, H_1\left(r, r', \sqrt{r'^2 + r^2\,\vartheta'^2 + r^2\,\varphi'^2 \sin^2\vartheta}\right)$$
$$- r'^2 H_2\left(r, r', \sqrt{r'^2 + r^2\,\vartheta'^2 + r^2\,\varphi'^2 \sin^2\vartheta}\right) = h\,.$$

One can then get back to the problem by quadratures easily with the help of the four first-order integral equations. Namely, if one sets:

$$\frac{c_2}{c_1} = \kappa_1$$

then it will follow from the first two equations in (1) that:

$$(3)\qquad \sin\varphi \frac{d\vartheta}{d\varphi} + \sin\,\vartheta \cos\,\vartheta \cos\,\varphi = \kappa_1 \sin^2\,\vartheta\,,$$

or when κ_2 means an integration constant:

$$(4)\qquad \cot\,\vartheta = \kappa_1 \sin\,\vartheta + \kappa_2 \cos\,\varphi\,,$$

such that the first equation in (1) and (4) will give:

$$(5)\qquad \sin^2\vartheta\left(\frac{d\varphi}{dt}\right)^2 = \frac{c_1^2}{r^4 H_1^2\left(r, r', \sqrt{r'^2 + r^2\,(\vartheta'^2 + \varphi'^2 \sin^2\vartheta)}\right)} \cdot [1 + (\kappa_2 \sin\varphi + \kappa_1 \sin\varphi)^2]\,,$$

while (5) and (4) will give:

$$(6)\qquad \left(\frac{d\vartheta}{dt}\right)^2 = \frac{c_1^2}{r^4 H_1^2\left(r, r', \sqrt{r'^2 + r^2\,(\vartheta'^2 + \varphi'^2 \sin^2\vartheta)}\right)} \cdot (\kappa_1 \sin\varphi - \kappa_2 \sin\varphi)^2\,,$$

and (5) and (6) will give:

$$(7)\qquad \vartheta'^2 + \varphi'^2 \sin^2\vartheta = \frac{c_1^2\,(1 + \kappa_1^2 + \kappa_2^2)}{r^4 H_1^2\left(r, r', \sqrt{r'^2 + r^2\,(\vartheta'^2 + \varphi'^2 \sin^2\vartheta)}\right)}\,.$$

However, since the last equation gives $\vartheta'^2 + \varphi'^2 \sin^2\vartheta$ as a function of r and r', it will follow from the *vis viva* equation (2) that there is a relation between r and r':

$$r' = \omega(r, c_1^2(1+\kappa_1^2+\kappa_2^2), h) .$$

One can then represent t by a quadrature in r:

$$t + \kappa = \int \frac{dr}{\omega(r, c_1^2(1+\kappa_1^2+\kappa_2^2), h)}$$

or

$$r = \Omega(t+\kappa, c_1^2(1+\kappa_1^2+\kappa_2^2), h) .$$

If one further observes that r, r', $\vartheta'^2 + \varphi'^2 \sin^2\vartheta$ are known functions of t, moreover, and that from (4) one has:

$$(\kappa_1 \sin\varphi - \kappa_2 \sin\varphi)^2 = \frac{(1+\kappa_1^2+\kappa_2^2)\sin^2\vartheta - 1}{\sin^2\vartheta},$$

then equation (6) will assume the form:

$$\int \frac{\sin\vartheta\, d\vartheta}{\sqrt{(1+\kappa_1^2+\kappa_2^2)\sin^2\vartheta - 1}} = \int \chi(t+\kappa, c_1^2(1+\kappa_1^2+\kappa_2^2), h)\, dt + \lambda .$$

Thus, ϑ will also be determined by quadratures, which will then give φ immediately from equation (4). The expressions for r, ϑ, φ as functions of t will then include the six integration constants κ, κ_1, κ_2, h, c_1, λ, and:

The integration of all equations of motion that are based upon a first-order kinetic potential that depends upon the distance from the moving point to a fixed center, the derivative with respect to time of the distance, and the velocity of the moving point can always be reduced to simply a kinetic potential that is composed of quadratures.

We would now like to use that theorem to investigate the motion of a point that is attracted to a mass-element in a spherical shell with concentric homogeneous layers according to **Weber**'s law and discover what it will be outside of the ring or inside of the cavity.

Suppose that the spherical shell is bounded by two spheres of radii R_0 and R_1, resp., and that σ is the density of the spherical layer as a function of the distance ρ from the center, moreover. One sets:

$$N = 4\pi \int_{R_0}^{R_1} \sigma\, \rho^4\, d\rho ,$$

while M denotes the mass of the spherical shell. If r denotes the distance from a point that is found outside the shell to the center of the latter, and r' denotes the derivative of r with respect to time, while v denotes the velocity of the point then, according to (10) in **§ 18**, the potential that the spherical shell exerts upon the point of mass 1 will be:

$$W_a = M\left(\frac{1}{r} + \frac{r'^2}{k^2 r}\right) - \frac{N}{3k^2}\frac{3r'^2 - v}{r^3},$$

and the kinetic potential:

$$H = - T - W_a$$

will then assume the form:

$$H = -\tfrac{1}{2}v^2 - M\left(\frac{1}{r} + \frac{r'^2}{k^2 r}\right) - \frac{N}{3k^2}\frac{3r'^2 - v}{r^3},$$

in this case, which is included in the one that was treated above, in which one had:

$$H = f(r, r', v) \; .$$

If one now remarks that from the definitions that were given above, one will have:

$$H_1 = \frac{1}{v}\frac{\partial H}{\partial v} = -1 - \frac{2N}{3k^2}\frac{1}{r^3} \; ,$$

$$H_2 = \frac{1}{r'}\frac{\partial H}{\partial r'} = -\frac{2M}{k^2}\frac{1}{r} + \frac{2N}{3k^2}\frac{1}{r^3}$$

then equation (2), which represents the energy principle, will read:

$$\text{(8)} \qquad \tfrac{1}{2}v^2 - M\left(\frac{1}{r} - \frac{r'^2}{k^2 r}\right) + \frac{N}{3k^2}\frac{v^2 - 3r'^2}{r^3} = h \,,$$

while equation (7) will go to:

$$\text{(9)} \qquad v^2 = \frac{c_1^2(1 + \kappa_1^2 + \kappa_2^2)\, r^4}{\left(r^3 + \dfrac{2N}{3k^2}\right)^2} + r'^2$$

when one multiplies it by r^2 and adds r'^2 to both sides. If one now substitutes the value of v^2 from (9) in (8) then that will give:

$$t+\kappa=\int\frac{\sqrt{\left(r^3+\frac{2N}{3k^2}\right)\left(r^3+\frac{2M}{k^2}r^2-\frac{4N}{3k^2}\right)}}{r\sqrt{2\left(r^3+\frac{2N}{3k^2}\right)(M+hr)-c_1^2(1+\kappa_1^2+\kappa_2^2)r^2}}\,dr\,,$$

and (6) will give:

$$\int\frac{\sin\vartheta\,d\vartheta}{\sqrt{(1+\kappa_1^2+\kappa_2^2)\sin^2\vartheta-1}}$$

$$=\int\frac{c_1\sqrt{r^3+\frac{2M}{k^2}r^2-\frac{4N}{3k^2}}}{\sqrt{2\left(r^3+\frac{2N}{3k^2}\right)\left(2\left(r^3+\frac{2N}{3k^2}\right)(M+hr)-c_1^2(1+\kappa_1^2+\kappa_2^2)r^2\right)}}\,dr+\lambda\,.$$

All of the defining data have then been reduced to quadratures with that.

Finally, as far as the motion of a point that is found in the cavity is concerned, according to (11) of **§ 18**, when one sets:

$$4\pi\int_{R_0}^{R_1}\sigma\,\rho\,d\rho=A\,,$$

the potential of the spherical shell at an interior point will be:

$$W_i=A\left(1+\frac{v^2}{3k^2}\right),$$

so the equation of motion:

$$\frac{d^2x}{dt^2}=\frac{\partial W_i}{\partial x}-\frac{d}{dt}\frac{\partial W_i}{\partial x'}$$

and its two analogues will go to:

$$x''=\frac{2A}{3k^2}\,x''\,,\quad y''=\frac{2A}{3k^2}\,y''\,,\quad z''=\frac{2A}{3k^2}\,z''\,,$$

from which it will follow that $x''=0$, $y''=0$, $z''=0$. We then find that a point inside the hollow part of a spherical shell whose mass-element attracts the point according to **Weber**'s law will move along a straight line with constant velocity.

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§ 20. – Extending Poisson's discontinuity equation.

In connection with the extension of the **Laplace-Poisson** differential equation for higher-order potentials that was derived in **§ 18**, we shall ultimately examine the form of the extended **Poisson** discontinuity equation, along with some of its applications. It will suffice to apply those considerations to the **Weber** potential.

Let U denote the potential of a mass that fills up a space continually and attracts a point of mass 1 according to **Weber**'s potential:

$$W = \frac{m}{r}\left(1+\frac{r'^2}{k^2}\right) .$$

Let $d\tau$ be an element of that space, while σ is the density of the mass in it, and let r be its distance from the attracted point x, y, z. It will then follow that:

$$U = \iiint \frac{\sigma\,d\tau}{r}\left(1+\frac{r'^2}{k^2}\right) = \iiint \frac{\sigma}{r}\left(1+\frac{r'^2}{k^2}\right) da\,db\,dc$$

is finite and continuous for all points x, y, z of the space that is filled with mass. However, it will also follow, as the introduction of polar coordinates above shows, that the finitude and continuity of the potential will also persist inside of the mass for finite and continuous values of the derivative r'.

Now, if the attracted point lies outside of the attracting mass then it will follow upon differentiating with respect to the coordinates and their first derivatives that:

$$\Delta_{00}\,U - \frac{d}{dt}\Delta_{10}\,U = \iiint \sigma\left(\Delta_{00}\,W - \frac{d}{dt}\Delta_{10}\,W\right) da\,db\,dc\,,$$

in which the attracting mass is considered to be at rest. Now, since it will obviously follow from:

$$r^2 = (x-a)^2 + (y-b)^2 + (z-c)^2,$$
$$r\,r' = (x-a)\,x' + (y-b)\,y' + (z-c)\,z'$$

that:

$$\frac{\partial^2 W}{\partial x^2} = \frac{3(x-a)^2}{r^5} - \frac{1}{r^3} - \frac{3r'^2}{k^2 r^3} + \frac{15r'^2(x-a)^2}{k^2 r^5} - \frac{12r'(x-a)\,x'}{k^2 r^4} + \frac{2x'^2}{k^2 r^3}$$

and

$$\frac{\partial^2 W}{\partial x\,\partial x'} = -\frac{6r'(x-a)^2}{k^2 r^4} + \frac{2r'}{k^2 r^2} + \frac{2x'(x-a)}{k^2 r^3}\,,$$

along with corresponding expressions in y and z. When one sets:

$$x'^2 + y'^2 + z'^2 = v^2 ,$$

it will then follow that:

$$\Delta_{00}\, W = -\frac{6\,r'^2}{k^2\,r^3} + \frac{2\,v^2}{k^2\,r^3}, \qquad \Delta_{10}\, W = \frac{2\,r'}{k^2\,r^2},$$

and from that:

$$\Delta_{00}\, W - \frac{d}{dt}\Delta_{10}\, W = -\frac{2}{k^2\,r^2}\left(\frac{x-a}{r}\,x'' + \frac{y-b}{r}\,y'' + \frac{z-c}{r}\,z''\right).$$

We will then find that:

$$\Delta_{00}\, U - \frac{d}{dt}\Delta_{10}\, U = \frac{2}{k^2}(x''\,X + y''\,Y + z''\,Z) ,$$

in which:

$$X = -\iiint \frac{\sigma\,(x-a)}{r^3}\,da\,db\,dc ,$$

$$Y = -\iiint \frac{\sigma\,(y-b)}{r^3}\,da\,db\,dc ,$$

$$Z = -\iiint \frac{\sigma\,(z-c)}{r^3}\,da\,db\,dc$$

are the components of the force that the given mass-system exerts upon a point that is found outside of the mass according to **Newton***'s law.*

In order to see what value that expression will assume for a point that lies inside the mass-system, one forms the partial differential quotients of U with respect to the coordinate z. Due to the fact that:

$$\frac{\partial r}{\partial z} = -\frac{\partial r}{\partial c}, \qquad \frac{\partial r'}{\partial z} = -\frac{\partial r'}{\partial c},$$

it will go to:

$$\frac{\partial U}{\partial z} = -\iiint \sigma \frac{\partial}{\partial c}\left[\frac{1}{r}\left(1+\frac{r'^2}{k^2}\right)\right] da\,db\,dc$$

$$= -\iiint \frac{\partial}{\partial c}\left[\frac{\sigma}{r}\left(1+\frac{r'^2}{k^2}\right)\right] da\,db\,dc + \iiint \frac{\partial\sigma}{\partial c}\frac{1}{r}\left(1+\frac{r'^2}{k^2}\right) da\,db\,dc ,$$

or to:

$$\frac{\partial U}{\partial z} = -\iint \frac{\sigma}{r}\left(1+\frac{r'^2}{k^2}\right)\cos(n\,z)\,ds + \iiint \frac{\partial\sigma}{\partial c}\frac{1}{r}\left(1+\frac{r'^2}{k^2}\right) da\,db\,dc ,$$

by a known conversion. The ds in that is the surface element for the space that is filled with mass, and n means the inward-pointing normal to ds, such that the first integral can be regarded as a surface potential with the mass $\sigma\cos(n\,z)$, and the second one can be regarded as a spatial potential with a mass density of $\partial\sigma / \partial c$.

Now, in order to investigate the continuity of the expression $\partial U / \partial z$, it will be necessary to treat the continuity of a surface potential that acts according to **Weber**'s law:

$$V = \iint \frac{\delta}{r}\left(1 + \frac{r'^2}{k^2}\right) ds ,$$

in which the density δ is finite and should vary continuously on the surface, whose dimensions are finite, and which has a finite and continuous curvature.

It is obvious that the surface potential will still be finite and will suffer no jump for points that lie at a finite distance from the surface. Now, in order to see how it behaves when point is shifted infinitely close to the surface, we would like to use the method of proof that is ordinarily applied to the **Newton**ian surface potential and place the coordinate origin at the surface point that the attracted point approaches indefinitely and the z_1-axis along the normal to the surface, so the x_1 and y_1 axes are in the tangent plane. We now imagine cutting out an infinitely-small circle of radius R from the surface (the indicatrix, which can only be assumed to be a conic section as a result of the assumption that was made, will not introduce any consideration that deviates from the assumption of a circle), which is itself infinitely small, but infinitely large in comparison to the infinitely-small z_1 and can be considered to be independent of the latter. Therefore, when the surface potential of the circle that is covered with mass of constant density is denoted by V_1, while that of the rest of the surface is denoted by V_2, V_2 will also be finite and continuous when the point passes through the surface. Thus, we only need to examine the finitude and continuity of the potential V_1.

Now since:

$$V_1 = \delta \int_0^R \int_0^{2\pi} \frac{\rho \, d\rho \, d\varphi}{r_1}\left(1 + \frac{r_1'^2}{k^2}\right) ,$$

in which:

$$r_1^2 = (x_1 - \rho \cos \varphi)^2 + (y_1 - \rho \sin \varphi)^2 + z_1^2 ,$$
$$r_1 r_1' = (x_1 - \rho \cos \varphi)\, x_1' + (y_1 - \rho \sin \varphi)\, y_1' + z_1 z_1' ,$$

when the point whose velocity components are x_1', y_1', z_1' is found on the normal, so $x_1 = 0$, $y_1 = 0$, that will give:

$$V_1 = \delta \int_0^R \int_0^{2\pi} \frac{\rho \, d\rho \, d\varphi}{\sqrt{\rho^2 + z_1^2}}\left(1 + \frac{z_1^2 z_1'^2 - 2\rho\, z_1 z_1' (x_1' \cos\varphi + y_1' \sin\varphi) + \rho^2 (x_1' \cos\varphi + y_1' \sin\varphi)^2}{k^2 (\rho^2 + z_1^2)}\right) ,$$

or upon integrating over φ:

$$V_1 = \delta \int_0^R \frac{\rho \, d\rho}{\sqrt{\rho^2 + z_1^2}}\left(1 + \frac{z_1^2 z_1'^2 + \frac{1}{2}\rho^2 (x_1'^2 + y_1'^2)}{k^2 (\rho^2 + z_1^2)}\right) .$$

Therefore, upon performing the integration over ρ, one will have:

$$V_1 = 2\pi\,\delta\left[\sqrt{R^2+z_1^2}-\sqrt{z_1^2}\right]-\frac{2\pi\,\delta}{k^2}\,z_1^2\,z_1'^2\left[\frac{1}{\sqrt{R^2+z_1^2}}-\frac{1}{\sqrt{z_1^2}}\right]$$
$$+\frac{\pi\,\delta}{k^2}(x_1'^2+y_1'^2)\left[-\frac{R^2}{\sqrt{R^2+z_1^2}}+2\sqrt{R^2+z_1^2}-2\sqrt{z_1^2}\right],$$

in which the roots are taken to have the positive sign. If one now lets z_1 and R approach zero in such a way that one will also have $z_1 / R = 0$ then one can see from the latter expression that V_1 will converge to zero, and therefore the total surface potential V will again be finite and continuous when the point cuts through the surface along the normal. However, as a simple calculation will show, one will also get the following expression from the value of V_1 that was found above:

$$\frac{\partial V_1}{\partial z_1}-\frac{d}{dt}\frac{\partial V_1}{\partial z_1'} = 2\pi\,\delta\left[\frac{z_1}{\sqrt{R^2+z_1^2}}-\frac{z_1}{\sqrt{z_1^2}}\right]+\frac{4\pi\,\delta}{k^2}\,z_1\,z_1''\left[\frac{z_1}{\sqrt{R^2+z_1^2}}-\frac{z_1}{\sqrt{z_1^2}}\right]$$
$$+\frac{4\pi\,\delta}{k^2}\,z_1'^2\left[\frac{z_1}{\sqrt{R^2+z_1^2}}-\frac{z_1}{\sqrt{z_1^2}}\right]+\frac{2\pi\,\delta}{k^2}\,z_1'^2\left[-\frac{z_1^3}{(R^2+z_1^2)\sqrt{R^2+z_1^2}}+\frac{z_1^3}{z_1^2\sqrt{z_1^2}}\right]$$
$$+\frac{\pi\,\delta}{k^2}(x_1'^2+y_1'^2)\left[\frac{R^2\,z_1}{(R^2+z_1^2)\sqrt{R^2+z_1^2}}+\frac{2\,z_1}{\sqrt{R^2+z_1^2}}-\frac{2\,z_1}{\sqrt{z_1^2}}\right].$$

Since:

$$x_1'^2+y_1'^2+z_1'^2 = v^2,$$

for vanishing values of z, R, z_1 / R, that will give:

for $z_1 > 0$: $$\frac{\partial V_1}{\partial z_1}-\frac{d}{dt}\frac{\partial V_1}{\partial z_1'} = -\,2\pi\,\delta\left(1+\frac{v^2}{k^2}\right),$$

for $z_1 < 0$: $$\frac{\partial V_1}{\partial z_1}-\frac{d}{dt}\frac{\partial V_1}{\partial z_1'} = 2\pi\,\delta\left(1+\frac{v^2}{k^2}\right).$$

Thus, there will be a jump with a magnitude of $-4\pi\,\delta\left(1+\frac{v^2}{k^2}\right)$, and we will then get the *extended* ***Poisson*** *discontinuity theorem:*

$$\left(\frac{\partial V_1}{\partial n_1} - \frac{d}{dt}\frac{\partial V_1}{\partial n_1'}\right) + \left(\frac{\partial V_1}{\partial n_a} - \frac{d}{dt}\frac{\partial V_1}{\partial n_a'}\right) = -4\pi\,\delta\left(1 + \frac{v^2}{k^2}\right).$$

When the velocity of the point is directed along the normal, that will go to:

$$\left(\frac{\partial V_1}{\partial n_1} - \frac{d}{dt}\frac{\partial V_1}{\partial n_1'}\right) + \left(\frac{\partial V_1}{\partial n_a} - \frac{d}{dt}\frac{\partial V_1}{\partial n_a'}\right) = -4\pi\,\delta\left(1 + \frac{n'^2}{k^2}\right).$$

In order to determine the jump in discontinuity in the corresponding expressions for the x_1 and y_1 coordinates, we must return to the value that was defined above by the double integral:

$$V_1 = \delta \int_0^R \int_0^{2\pi} \frac{\rho\, d\rho\, d\varphi}{\sqrt{(x_1 - \rho\cos\varphi)^2 + (y_1 - \rho\sin\varphi)^2 + z_1^2}},$$
$$\times \left(1 + \frac{z_1^2\, z_1'^2 - 2\rho\, z_1\, z_1'\,(x_1'\cos\varphi + y_1'\sin\varphi) + \rho^2 (x_1'\cos\varphi + y_1'\sin\varphi)^2}{k^2(\rho^2 + z_1^2)}\right).$$

We then form:

$$\frac{\partial V_1}{\partial x_1}, \quad \frac{\partial V_1}{\partial x_1'}, \quad \frac{d}{dt}\frac{\partial V_1}{\partial x_1'}$$

and set $x_1 = 0$, $y_1 = 0$. As a simple calculation will, in turn, show, upon performing the integration over φ and then the one over r, we will get:

$$\frac{\partial V_1}{\partial x_1} - \frac{d}{dt}\frac{\partial V_1}{\partial x_1'} = -\frac{2\pi}{k^2}\, x_1'\, z_1' \left[\frac{z_1}{\sqrt{R^2 + z_1^2}} - \frac{z_1}{\sqrt{z_1^2}}\right]$$
$$+ \frac{6\pi}{k^2}\, x'\, z' \left[-\frac{1}{3}\frac{R^2\, z_1}{(R^2 + z_1^2)\sqrt{R^2 + z_1^2}} + \frac{2}{3}\frac{z_1}{\sqrt{R^2 + z_1^2}} - \frac{2}{3}\frac{z_1}{\sqrt{z_1^2}}\right]$$
$$- \frac{2\pi}{k^2}\, x_1'' \left[-\frac{R^2}{\sqrt{R^2 + z_1^2}} + 2\sqrt{R^2 + z_1^2} - 2\sqrt{z_1^2}\right].$$

In the infinitely-close neighborhood of the point on the surface, one will, in turn, have:

for $z_1 > 0$: $$\frac{\partial V_1}{\partial x_1} - \frac{d}{dt}\frac{\partial V_1}{\partial x_1'} = -\frac{2\pi}{k^2}\, x_1'\, z_1',$$

for $z_1 < 0$:
$$\frac{\partial V_1}{\partial x_1} - \frac{d}{dt}\frac{\partial V_1}{\partial x_1'} = \frac{2\pi}{k^2} x_1' z_1' .$$

Therefore, the expressions:
$$\frac{\partial V_1}{\partial x_1} - \frac{d}{dt}\frac{\partial V_1}{\partial x_1'} , \qquad \frac{\partial V_1}{\partial y_1} - \frac{d}{dt}\frac{\partial V_1}{\partial y_1'}$$

will make the jumps:
$$-\frac{4\pi\,\delta}{k^2} x_1' z_1' , \qquad -\frac{4\pi\,\delta}{k^2} y_1' z_1' ,$$

resp.

If we once more go back to the original coordinate system, whose coordinates have the following relationship to x_1, y_1, z_1 :

$$x_1 = x \cos(x\, x_1) + y \cos(y\, x_1) + z \cos(z\, x_1) ,$$
$$y_1 = x \cos(x\, y_1) + y \cos(y\, y_1) + z \cos(z\, y_1) ,$$
$$z_1 = x \cos(x\, z_1) + y \cos(y\, z_1) + z \cos(z\, z_1) ,$$

then it will follow from Lemma 2 of **§ 2** that:

$$\frac{\partial V}{\partial x} - \frac{d}{dt}\frac{\partial V}{\partial x'} = \left(\frac{\partial V}{\partial x_1} - \frac{d}{dt}\frac{\partial V}{\partial x_1'}\right)\cos(x_1\, x) + \left(\frac{\partial V}{\partial y_1} - \frac{d}{dt}\frac{\partial V}{\partial y_1'}\right)\cos(y_1\, x) + \left(\frac{\partial V}{\partial z_1} - \frac{d}{dt}\frac{\partial V}{\partial z_1'}\right)\cos(z_1\, x) .$$

The jumps in the expressions:

$$\frac{\partial V}{\partial x} - \frac{d}{dt}\frac{\partial V}{\partial x'} , \quad \frac{\partial V}{\partial y} - \frac{d}{dt}\frac{\partial V}{\partial y'} , \quad \frac{\partial V}{\partial z} - \frac{d}{dt}\frac{\partial V}{\partial z'}$$

upon crossing the surface will be given in the form:

$$-\frac{4\pi\,\delta}{k^2} n' x' - 4\pi\,\delta\left(1 + \frac{v^2 - n'^2}{k^2}\right)\cos(n\, x) ,$$
$$-\frac{4\pi\,\delta}{k^2} n' y' - 4\pi\,\delta\left(1 + \frac{v^2 - n'^2}{k^2}\right)\cos(n\, y) ,$$
$$-\frac{4\pi\,\delta}{k^2} n' z' - 4\pi\,\delta\left(1 + \frac{v^2 - n'^2}{k^2}\right)\cos(n\, z) .$$

If we return to our consideration of the spatial potential U that we defined above then that will show that U is continuous at the surface of the space itself, and the same thing will be true for $\partial U / \partial z$, since it is composed of a spatial potential and a surface potential, and V is continuous, as was proved above. It is obvious that upon switching z, c with x, a and y, b, we will then have that:

The expressions $\frac{\partial U}{\partial x}$, $\frac{\partial U}{\partial y}$, $\frac{\partial U}{\partial z}$ *are also continuous on the surface of the space.*

However, it will further follow from the continuity jumps that were just found that for a surface potential with the densities:

$$\sigma \cos(n\,x)\,, \quad \sigma \cos(n\,y)\,, \quad \sigma \cos(n\,z)\,,$$

which define one component of the expressions $\frac{\partial U}{\partial x}$, $\frac{\partial U}{\partial y}$, $\frac{\partial U}{\partial z}$, the expressions:

$$\frac{\partial^2 U}{\partial x^2} - \frac{d}{dt}\frac{\partial^2 U}{\partial x\,\partial x'}\,, \qquad \frac{\partial^2 U}{\partial y^2} - \frac{d}{dt}\frac{\partial^2 U}{\partial y\,\partial y'}\,, \qquad \frac{\partial^2 U}{\partial z^2} - \frac{d}{dt}\frac{\partial^2 U}{\partial z\,\partial z'}$$

will experience jump discontinuities of:

$$- \frac{4\pi\sigma}{k^2}\, n'\, x' \cos(n\,x) - 4\pi\sigma\left(1 + \frac{v^2 - n'^2}{k^2}\right)\cos^2(n\,x)\,,$$

$$- \frac{4\pi\sigma}{k^2}\, n'\, y' \cos(n\,y) - 4\pi\sigma\left(1 + \frac{v^2 - n'^2}{k^2}\right)\cos^2(n\,y)\,,$$

$$- \frac{4\pi\sigma}{k^2}\, n'\, z' \cos(n\,z) - 4\pi\sigma\left(1 + \frac{v^2 - n'^2}{k^2}\right)\cos^2(n\,z)\,,$$

resp., upon passing through the point on the surface. Therefore, *the expression:*

$$\Delta_{00}\, U - \frac{d}{dt}\; \Delta_{10}\, U$$

will experience a jump of:

$$- \frac{4\pi\sigma}{k^2}\, n'[x' \cos(n\,x) + y' \cos(n\,y) + z' \cos(n\,z)] - 4\pi\sigma\left(1 + \frac{v^2 - n'^2}{k^2}\right)$$

or

$$- 4\pi\sigma\left(1 + \frac{v^2}{k^2}\right).$$

If we combine that result with the one that was obtained above for the point that lies outside of the mass then we will find that we have:

$$\Delta_{00}\, U - \frac{d}{dt}\;\Delta_{10}\, U = \frac{2}{k^2}(x''X + y''Y + z''Z) - 4\pi\sigma\left(1 + \frac{v^2}{k^2}\right)$$

in the neighborhood of the surface. If the point is now found inside of the attracting mass and one lays a surface through the immediate neighborhood of that point then if the potential of the mass-system that includes that point is denoted by U_1, one will have:

$$\Delta_{00}\,(U - U_1) - \frac{d}{dt}\;\Delta_{10}\,(U - U_1) = \frac{2}{k^2}(x''X_2 + y''Y_2 + z''Z_2)\,,$$

when X_2, Y_2, Z_2 mean the force components, according to **Newton**’s law, of the mass-system in which the selected point does not lie. Since one has:

$$\Delta_{00}\, U_1 - \frac{d}{dt}\;\Delta_{10}\, U_1 = \frac{2}{k^2}(x''X_1 + y''Y_1 + z''Z_1) - 4\pi\sigma\left(1 + \frac{v^2}{k^2}\right),$$

one will then find from the above that:

In general, for any spatial potential that acts according to ***Weber****’s law:*

$$U = \iiint \frac{\sigma}{r}\left(1 + \frac{v^2}{k^2}\right) da\, db\, dc\,,$$

one will have the relation:

$$\Delta_{00}\, U - \frac{d}{dt}\;\Delta_{10}\, U = \frac{2}{k^2}(x''X + y''Y + z''Z) - 4\pi\sigma\left(1 + \frac{v^2}{k^2}\right),$$

in which σ means the mass density at the point at which the attracting point is found, v is its velocity, x'', y'', z'' are its accelerations, and X, Y, Z represent the force components of the total mass-system when it acts according to ***Newton****’s law.*

If one tests that relation for a homogeneous ball of density σ and radius whose element will attract a point in its interior at a distance of l from the center, and which possesses a velocity of v, according to **Weber**’s law then the potential that was developed in **§ 18** will be:

$$W_m = 2\pi\sigma\left(R^2 - \tfrac{1}{3}\,l^2\right) + \frac{8\pi\sigma}{15k^2}\,l^2\,l'^2 + \frac{2\pi\sigma}{3k^2}\,R^2\,v^2 - \frac{2\pi\sigma}{5k^2}\,l^2\,v^2\,.$$

The relations:

$$\Delta_{00}\, W_m = -4\pi\sigma\left(1+\frac{v^2}{k^2}\right), \qquad \Delta_{10}\, W_m = \frac{8\pi\sigma}{3k^2}\, l\, l' ,$$

will follow immediately from that, and thus, the relation:

$$\Delta_{00}\, W_m - \frac{d}{dt}\,\Delta_{10}\, W_m = -\frac{8\pi\sigma}{3k^2}(x\, x'' + y\, y'' + z\, z'') - 4\pi\sigma\left(1+\frac{v^2}{k^2}\right),$$

which agrees with the general relation that was found above when one observes that the components of the attraction of the ball can be represented by:

$$X = -\tfrac{4}{3}\,\pi\sigma\, x, \qquad Y = -\tfrac{4}{3}\,\pi\sigma\, y, \qquad Z = -\tfrac{4}{3}\,\pi\sigma\, z$$

when its elements act upon a point in its interior whose coordinates are x, y, z according to **Newton**'s law.

Deriving the **Poisson** discontinuity equation, as well as the other relations that were developed above for the general extended **Newton**ian potential of arbitrary order, requires no further analysis, and entirely similar considerations can be applied when the attracting mass is not assumed to be at rest.

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§ 21. – Review

The kinetic potential of a problem in the mechanics of ponderable masses has degree two relative to the derivatives of the mutually-independent coordinates, and linear terms enter into those quantities only when the constraint equations include time explicitly. In order to describe the motion that actually takes place in one part of that system separately from the others in terms of forces of a certain type and intensity, one must succeed in eliminating those coordinates and their first and second derivatives from the system of differential equations that should be dropped from the system. Moreover, one must investigate whether the differential equations of order two or higher that then appear in the coordinates under consideration will, in turn, possess a kinetic potential of order one or higher. However, such a process of elimination of variables between differential equations also requires the repeated differentiation of those differential equations that belong to the newly-formulated problem of motion, in general. As the investigations that were carried out above will show, the forces whose effect on a part of the system of points that is selected arbitrarily, but separate from the others, would produce the same motion that would be produced if the total system were subject to forces that belong to the mechanics of ponderable masses are generally higher-order forces, or the motion will be described by kinetic potentials of order higher than the one. The same thing will be true when one does not start from problems in the mechanics of ponderable masses, but from ones that are based upon general kinetic potentials of order one or arbitrarily higher, such that the question of the motion of the individual parts of a system when it is subjected to higher-order forces should be precisely the same as the question of when those parts that belong to a given system on which forces of any order act will lead to the study of higher-order kinetic potentials that will, in turn, give rise to extended principles of mechanics. However, all of those investigations are linked with the completion of a process of eliminating a number of independent coordinates by differentiating the equations of motion. That differentiation cannot be avoided as long as we do not assume certain properties of the given equations of motion or the kinetic potential that determines them. For that reason, the necessary and sufficient conditions that were developed above for the form of the first-order kinetic potential, as well as the way that one exhibits just those conditions for kinetic potentials of arbitrary order, will be determined only when the motion that actually takes place in a subsystem that constants of fewer points can seen to be produced by the effect of forces with the same or next-higher order with the help of algebraic elimination processes.

For the problems of motion in the mechanics of ponderable masses in which eliminating a number of coordinates cannot be achieved without differentiation processes, it is clear from the analysis in **§ 15** that for the case in which the left-hand sides of a number of **Lagrange** equations for the problem of motion are complete differential quotients with respect to time or the kinetic potential is independent of a number of coordinates (which would then coincide with the latter case), the first differential quotients of the hidden coordinates will be linear functions of the first differential quotients of those coordinates that the reduced problem should not include. Therefore, the new first-order potential, like the given one, which nonetheless possesses the current and potential energy separately, and in which only the first derivatives of the coordinates occur, will also once more possess derivatives of the coordinates that remain after eliminating the hidden coordinates only in the second degree, but one-dimensional terms will also enter, in general, and a separation of the current and potential energy will no longer be obvious. However, as a simple argument based upon the investigations that were carried out in **§§ 15, 16, 17** will show, it will then follow that this case of hidden motion will only lead to kinetic potentials that include the first derivatives of the remaining coordinates to a degree that is no higher than two, and that for the case in which extended force function for the coordinates of the subsystem exists in the sense that was given above, it will also be only quadratic in the derivatives of those quantities. Hence, when includes the coordinates of the points only in the form of mutual distances, it must be an entire function of degree two in the first derivatives of the distances and into whose coefficients the distances themselves can enter arbitrarily. That was shown above for, e.g., **Weber**'s law, which can replace **Newton**'s when yet a third point can be coupled with the system of two points in a certain way that will then alter the effect of **Newton**'s law by its inertia in such a way that for the case in which the third point remains hidden, the motion of the two system points seems to be produced by the **Weber** force function. By contrast, as would emerge from, e.g., the same expressions that were presented above, it can be easily shown that the motion of a point that moves in a resisting medium whose resistance is a function of the coordinates and velocity of the point cannot be produced by coupling it with other hidden ponderable mass and the action of forces that depend upon only the coordinates.

However, the case of hidden masses that was pointed out above is also not the only one in the mechanics of ponderable masses for which the motion of a subsystem can be, in turn, described by a first-order kinetic potential, but the current and potential energy obviously cannot be once more be seen as separate. For every problem in the mechanics of ponderable masses whose kinetic potential then consists of the *vis viva* and the force function, in the case of holonomic constraint equations, it is obvious from their form that the kinetic potential, when expressed in terms of the free or independent coordinates, will not have degree higher than two in the derivatives of the latter, but since the coefficients of the squares of those derivatives consist of sums of squares of the partial derivatives with respect to the free coordinates of all of the coordinate of the system when it is subject to no constraints, it will have a form that is endowed with the squares of the first derivatives of all independent coordinates with essentially-positive, non-vanishing coefficients that are independent of the coordinates themselves, in general. As a result of that, of the only possible cases of hidden motion that one treats with the general first-order kinetic potential, it would emerge from the forms of the kinetic potentials that one finds there that in the mechanics of ponderable masses, the only cases that one can consider to be ones of hidden motion are the ones for which either the kinetic potential is independent of a number of free coordinates or a series of **Lagrange** equations of motion is independent of a number of free coordinates and their first derivatives.

It seems to me that the foregoing considerations imply the necessity of introducing not merely the general first-order kinetic potential into mechanics, but also ones of higher order, as well as examining the extension and validity of the known principles from the mechanics of ponderable masses, and under that most general assumption, and presenting them as mathematical theorems of a more general nature. At the same time, with the introduction of the extended **Newton**ian potential and the **Laplace-Poisson** differential equation, it should be shown that a potential theory can also be developed in that extended theory of mechanics in a natural way, and that one can verify the applicability of those principles in some problems of motion.