"Zur Diracschen Theorie des Elektrons, I," Ann. Phys. (Leipzig) (5) **38** (1940), 421-435.

On Dirac's theory of the electron

I. Algebraic identities between the probability densities.

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In **Dirac**'s theory of the electron, the inner products $(\psi^{+}, \Gamma^{A}\psi)$ of the 16 **Clifford** numbers Γ^{A}:

$$(I, \psi^{1}, \psi^{2}, \psi^{3}, \psi^{4}, \psi^{5}, \psi^{[23]}, \psi^{[31]}, \psi^{[12]}, \psi^{[23]}, \psi^{[14]}, \psi^{[24]}, \psi^{[34]}, \psi^{[234]}, \psi^{[314]}, \psi^{[124]}, \psi^{[123]})$$

with a wave function ψ and its adjoint ψ^{+} are endowed with well-defined physical meanings. Relations exist between these inner products that arise, partly from the *Dirac equation* and partly from *the algebra of the chosen number system*. It is therefore sensible to carry out a systematic examination of the consequences of the algebra of **Clifford** number systems for **Dirac**'s theory, and to separate them from the consequences of the equations of motion, although the choice of the new number system is closely linked with the Dirac equation and is known to not be arbitrary. One will then recognize that many relations that one would be inclined to address as consequences of the relativistic character of the Dirac equation are already purely consequences of the choice of new number domain. These relations between the probability densities (charge, current, electric and magnetic moment, etc.) will be presented in Part I. They are *algebraic identities* that are always true, independently of the choice of the field in which the electron moves, and can be derived with no specialization of the Dirac matrices; in particular, their Hermiticity is not required.

There are even more field-independent relations between the probability densities, in addition to these algebraic identities. They exist upon the basis of the fact that the electromagnetic potentials by which the external field enters into the **Dirac** equation are *real*, while the wave equation is *complex*, moreover. These relations have the form of differential equations, and they will be derived in Part III under the name of *reality relations*. One will find six vectorial and four scalar relations of that type in III, § 2. The six vectorial relations can be traced back to four scalars, which will be shown in Part IV.

The conversion of the reality relations from the original form in which one first obtains them (III, § 1), and in which they contain *incomprehensible* quantities, into *vector notation* requires repeated utilization of the algebra of Dirac matrices. Therefore, in Part II, one will find the presentation of algebraic relations between quantities that are analogous to the probability densities of Part I, but differ from them by the appearance of a differential quotient ("forward" or "backward" differentiation, resp.). Part II then brings an essential extension of the wealth of formulas beyond that of the first part and contains mathematical tools that will be necessary if one is to carry out the elimination of *all* non-intuitive quantities from the reality relations in III.

Introduction and summary of I.

In Part I, the results of my previous paper [**1**] will be completed. According to **Pauli** [**2**], bilinear equations exist between the matrix elements of the Dirac matrices. In § 2, those equations will be brought into a form that is as simple as possible and is most convenient for the following applications [§ eq. (4) and (10)]. **Pauli**'s bilinear equations can be employed in order to obtain each factor from the 16 matrices of the **Dirac** matrix ring by multiplication, and the inner products define $16 \cdot 16 = 256$ bilinear relations between the 16 probability densities of **Dirac**'s theory. Many more relations appear among the 256 relations, although a considerable number of different relations can be traced back to nine primary ones from which one can derive all of the other ones. We refer to these relations as "algebraic identities," corresponding to their provenance, and thus systematically distinguish them from the "reality relations" that will be derived in Part III. § 3 includes a list of these primary and derived algebraic identities. Like **Pauli**'s equations, they are at least bilinear in their construction, at any rate. The most notable extension that I will offer beyond the aforementioned paper is the proof that there can be no more than nine primary identities (§ 4). In addition, the grounds for the presence of these identities will become clear here, and the reason for its simple nature will be sought in the laws of constructing inner products. Algebraic identities between the 16 inner products must then exist in any ring of linearly-independent matrices, but, according to **Pauli**, the special form of our identities [§ 3, eq. (15)-(16)] follows from the anti-commutativity of the Dirac matrices, and is not so simple in its nature. We pursue the effectiveness of the origin of the identities that was found in § 4 in a special example (§ 5). While neither Hermiticity nor any special form for the Dirac matrices is required for the derivations in all the remaining paragraphs, in § 5, we will employ the usual special representation of the α-matrices in order to derive the identities. § 6 brings some remarkable formulas that will find their application later on (Part II-IV), and § 7 contains the geometric representation (physical interpretation, resp.) of some identities.

§ 1. Notations.

The notations that will be used here are the same as in the previously-appearing publication [**1**]. However, some new notations will appear from now on, such that it would seem most reasonable to recall the ones that were used before in connection with the new ones.

A. We mean by (λ, μ, ν = 1, 2, 3, 4):

a) I, the four-rowed identity matrix.

b) $\gamma^\mu = (\gamma^1, \gamma^2, \gamma^3, \gamma^4)$, four anti-commutative basic matrices that satisfy the **Dirac** relations:

(1) $$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\delta_{\mu\nu} \cdot I.$$

c) $$\begin{aligned} \gamma^{[\mu\nu]} &= (i\gamma^2\gamma^3, i\gamma^3\gamma^1, i\gamma^1\gamma^2, i\gamma^1\gamma^4, i\gamma^2\gamma^4, i\gamma^3\gamma^4), \\ \gamma^{[\lambda\mu\nu]} &= (i\gamma^2\gamma^3\gamma^4, i\gamma^3\gamma^1\gamma^4, i\gamma^1\gamma^2\gamma^4, i\gamma^1\gamma^2\,\gamma^3), \\ \gamma^5 &= \gamma^1\gamma^2\gamma^3\gamma^4, \end{aligned}$$

which are all the linearly-independent product of the γ^μ, up to a factor, whose square is the identity matrix.

B. One obtains a second system of **Dirac** matrices α^μ that anti-commute:

(2) $$\alpha^\mu \alpha^\nu + \alpha^\nu \alpha^\mu = 2\delta_{\mu\nu} \cdot I$$

with the help of the substitution (k = 1, 2, 3):

(3) $$\alpha^k = -i\,\gamma^4\gamma^k, \qquad \alpha^4 = \gamma^4 \qquad (\gamma^k = i\,\alpha^4\alpha^k,\ \gamma^4 = \alpha^4).$$

In turn (μ, ν = 1, 2, 3, 4):

$$\begin{aligned} \alpha^{[\mu\nu]} &= (i\alpha^2\alpha^3, i\alpha^3\alpha^1, i\alpha^1\alpha^2, i\alpha^1\alpha^4, i\alpha^2\alpha^4, i\alpha^3\alpha^4), \\ \alpha^{[\lambda\mu\nu]} &= (i\alpha^2\alpha^3\alpha^4, i\alpha^3\alpha^1\alpha^4, i\alpha^1\alpha^2\alpha^4, i\alpha^1\alpha^2\,\alpha^3), \\ \alpha^5 &= (\alpha^1\alpha^2\alpha^3\alpha^4) \end{aligned}$$

denote all linear independent products of the α^μ, up to a factor, such that their squares are the identity matrix.

A matrix operator O defines a bilinear form:

$$(\Phi, O\psi) = \sum_{\rho,\sigma=1}^{4} \Phi_\rho O_{\rho\sigma} \psi_\sigma$$

by multiplying by the two systems of functions $\Phi = (\Phi_1, \Phi_2, \Phi_3, \Phi_4)$ and $\psi = (\psi_1, \psi_2, \psi_3, \psi_4)$. It is customary to say γ^μ or α^μ for the system of matrices that is employed in the Dirac equation according to whether the identity matrix appears in the mass term or before the time differentiation, resp. Certainly, bilinear forms that are constructed from them will then exhibit a well-defined behavior under Lorentz transformations. These bilinear forms, which take on the meaning of probability densities, are:

a) Two invariants:

$$\begin{aligned} \Omega &= (\psi^*, \alpha^4\psi) &&= -i\,(\psi^+, \psi) \quad (\textit{rest mass density}), \\ \hat{\Omega} &= -(\psi^*, \alpha^5\psi) &&= (\psi^+, \gamma^5\psi) \end{aligned}$$

b) A four-vector (*charge-current density*):

$$s_0 = (\psi^*, \psi) = -i\,(\psi^+, \gamma^4\psi),$$

$$s_k = -(\psi^*, \alpha^\kappa \psi) = (\psi^+, \gamma^\kappa \psi) \qquad (k = 1, 2, 3).$$

We combine the spatial components into the *current density vector* $\mathfrak{s} = (s_1, s_2, s_3)$.

e) A skew-symmetric tensor:

$$M_{ik} = (\psi^*, \alpha^{[ik4]}\psi) = -(\psi^+, \gamma^{[ik]}\psi),$$
$$M_{k0} = (\psi^*, \alpha^{[k4]}\psi) = -(\psi^+, \gamma^{[k4]}\psi).$$

In what follows, we will combine the components of this six-vector into two ordinary vectors, namely, the vector:

$$\mathfrak{M} = (M_{23}, M_{31}, M_{12}) \qquad (\textit{magnetic moment density})$$

and the vector that is dual to it:

$$\hat{\mathfrak{M}} = (M_{10}, M_{20}, M_{30}) \qquad (\textit{electric moment density}).$$

d) A spatial vector (a third-rank tensor that is skew-symmetric in all three indices λ, μ, ν) $l = k + 1$, $m = k + 2$ (mod 3):

$$\hat{s}_0 = -(\psi^*, \alpha^{[123]}\psi) = (\psi^+, \gamma^{[123]}\psi),$$
$$\hat{s}_k = (\psi^*, \alpha^{[lm]}\psi) = -i(\psi^+, \gamma^{[lm4]}\psi).$$

We combine its spatial components into the vector that is dual to the current vector:

$$\hat{\mathfrak{s}} = (\hat{s}_1, \hat{s}_2, \hat{s}_3) \qquad (\textit{spin density}).$$

ψ^* is the complex conjugate of ψ, while $\psi^+ = i\psi^*\gamma^4$ means the adjoint of ψ. The duality sign ^ means duality in the relativistic sense (cf., *loc. cit.* [**2**]). The given bilinear forms must still be multiplied by the dimensional factors $m, -e, -\frac{e\hbar}{2mc}, \frac{\hbar}{2}$ in order for them to correspond to the interpretations that they are given (cf., *loc. cit.* [**1**], § 1C). In what follows, they will be omitted for the sake of brevity.

C. In addition to the inner products that were mentioned up to now, in II-IV, it will be necessary to introduce inner products that include differential quotients. The appearance of a differential quotient with respect to x_μ in the first position of an inner product shall, for the sake of brevity, be called "backward differentiation" and will be denoted by an upper left index μ, while its appearance in the last position shall be called "forward differentiation," and will be denoted by an upper right index μ. We clarify this notation with the following examples:

$$^{\mu}\Omega = \left(\frac{\partial\psi^*}{\partial x_\mu}, \alpha^4\psi\right) = -\,i\left(\frac{\partial\psi^+}{\partial x_\mu}, \psi\right) = \sum_{\rho,\sigma=1}^{4}\frac{\partial\psi^*_\rho}{\partial x_\mu}\,\alpha^4_{\rho\sigma}\psi_\sigma \qquad \text{(backward diff.)}$$

$$\Omega^{\mu} = \left(\psi^*, \alpha^4\frac{\partial\psi}{\partial x_\mu}\right) = -\,i\left(\psi^+, \frac{\partial\psi}{\partial x_\mu}\right) = \sum_{\rho,\sigma=1}^{4}\psi^*_\rho\,\alpha^4_{\rho\sigma}\frac{\partial\psi_\rho}{\partial x_\mu} \qquad \text{(forward diff.)}$$

The vector notation:

$$^{\mu}\mathfrak{s},\ {}^{\mu}\hat{\mathfrak{s}},\ {}^{\mu}\mathfrak{M},\ {}^{\mu}\hat{\mathfrak{M}}\ \text{(backward diff.)}, \qquad \mathfrak{s}^{\mu},\ \hat{\mathfrak{s}}^{\mu},\ \mathfrak{M}^{\mu},\ \hat{\mathfrak{M}}^{\mu}\ \text{(forward diff.)}$$

might serve as a generic notation for any three components that are backward (forward, resp.) differentiated with respect to x_μ. The use of this notation in the context of other operations in the inner product will follow analogously from these examples. The sum of the backward and forward differentiation will correspond to the differential quotient of an inner product; e.g.:

$$^{\mu}M_{23} + M_{23}^{\mu} = \frac{\partial M_{23}}{\partial x_\mu},$$

while the difference – e.g., $({}^{\mu}M_{23} - M_{23}^{\mu})$ – will take on no special meaning. Such differences will appear quite often as in the course of this investigation (Part III) as "uninterpretable" quantities; the problem in Part II will be to prepare the mathematical tools for the elimination of those quantities.

§ 2. Pauli's bilinear equations in another form.

Without writing the **Dirac** matrices in a special representation, the bilinear equation:

$$\text{(4)} \qquad \boxed{\sum_{k=1}^{4}\gamma^k_{\rho\sigma}\gamma^k_{\bar\rho\bar\sigma} = \gamma^{[123]}_{\bar\rho\sigma}\gamma^{[123]}_{\rho\bar\sigma} - \gamma^{[123]}_{\rho\sigma}\gamma^{[123]}_{\bar\rho\bar\sigma} + \delta_{\bar\rho\sigma}\delta_{\rho\bar\sigma} - \gamma^5_{\bar\rho\sigma}\gamma^5_{\rho\bar\sigma} - \gamma^4_{\bar\rho\sigma}\gamma^4_{\rho\bar\sigma}.}$$

will follow from just the anti-commutativity of the four basic matrices γ^μ for the matrix elements of the matrix ring.

This equation encloses the two equations (20) and (21) of **Pauli**, *loc. cit.* [**2**]. Namely, if one replaces, e.g., ρ with $\bar{\bar\rho}$ and $\bar\rho$ with $\bar{\bar{\bar\rho}}$ in it [eq. (4) above], multiplies by $\gamma^{[123]}_{\rho\bar{\bar\sigma}}\gamma^{[123]}_{\bar\rho\bar{\bar{\bar\sigma}}}$, and sums over $\bar{\bar\rho}$, $\bar{\bar{\bar\rho}}$, then one will get:

$$\text{(5)} \qquad \gamma^{[23]}_{\rho\sigma}\gamma^{[23]}_{\bar\rho\bar\sigma} + \gamma^{[31]}_{\rho\sigma}\gamma^{[31]}_{\bar\rho\bar\sigma} + \gamma^{[12]}_{\rho\sigma}\gamma^{[12]}_{\bar\rho\bar\sigma} = \delta_{\bar\rho\sigma}\delta_{\rho\bar\sigma} - \delta_{\rho\sigma}\delta_{\bar\rho\bar\sigma} + \gamma^4_{\bar\rho\sigma}\gamma^4_{\rho\bar\sigma} + \gamma^5_{\bar\rho\sigma}\gamma^5_{\rho\bar\sigma} + \gamma^{[123]}_{\bar\rho\sigma}\gamma^{[123]}_{\rho\bar\sigma}.$$

If one multiplies by $\gamma^4_{\rho\bar{\bar{\sigma}}}\gamma^4_{\bar{\rho}\bar{\bar{\bar{\sigma}}}}$ and again sums over $\bar{\bar{\rho}}$, $\bar{\bar{\bar{\rho}}}$ then one will get:

$$(6) \qquad \gamma^{[14]}_{\rho\sigma}\gamma^{[14]}_{\bar{\rho}\bar{\sigma}} + \gamma^{[24]}_{\rho\sigma}\gamma^{[24]}_{\bar{\rho}\bar{\sigma}} + \gamma^{[34]}_{\rho\sigma}\gamma^{[34]}_{\bar{\rho}\bar{\sigma}} = \gamma^5_{\bar{\rho}\sigma}\gamma^5_{\rho\bar{\sigma}} - \gamma^5_{\rho\sigma}\gamma^5_{\bar{\rho}\bar{\sigma}} + \delta_{\bar{\rho}\sigma}\delta_{\rho\bar{\sigma}} - \gamma^4_{\bar{\rho}\sigma}\gamma^4_{\rho\bar{\sigma}} - \gamma^{[123]}_{\bar{\rho}\sigma}\gamma^{[123]}_{\rho\bar{\sigma}} .$$

Adding (5) and (6) will give [eq. (20) in **Pauli** [**2**] = eq. (α) in the paper [**1**]]:

$$(7) \qquad \sum_{[\mu\nu]} \gamma^{[\mu\nu]}_{\rho\sigma}\gamma^{[\mu\nu]}_{\bar{\rho}\bar{\sigma}} = 2(\delta_{\bar{\rho}\sigma}\delta_{\rho\bar{\sigma}} + \gamma^5_{\bar{\rho}\sigma}\gamma^5_{\rho\bar{\sigma}}) - \delta_{\rho\sigma}\delta_{\bar{\rho}\bar{\sigma}} - \gamma^5_{\rho\sigma}\gamma^5_{\bar{\rho}\bar{\sigma}} .$$

However, if one replaces ρ with $\bar{\bar{\rho}}$ and $\bar{\rho}$ with $\bar{\bar{\bar{\rho}}}$ in (4), multiplies it by $\gamma^5_{\rho\bar{\bar{\sigma}}}\gamma^5_{\bar{\rho}\bar{\bar{\bar{\sigma}}}}$, and sums over $\bar{\bar{\rho}}$, $\bar{\bar{\bar{\rho}}}$ then that will give:

$$(8) \qquad \gamma^{[234]}_{\rho\sigma}\gamma^{[234]}_{\bar{\rho}\bar{\sigma}} + \gamma^{[314]}_{\rho\sigma}\gamma^{[314]}_{\bar{\rho}\bar{\sigma}} + \gamma^{[124]}_{\rho\sigma}\gamma^{[124]}_{\bar{\rho}\bar{\sigma}} = \gamma^4_{\bar{\rho}\sigma}\gamma^4_{\rho\bar{\sigma}} - \gamma^4_{\rho\sigma}\gamma^4_{\bar{\rho}\bar{\sigma}} + \delta_{\bar{\rho}\sigma}\delta_{\rho\bar{\sigma}} - \gamma^5_{\bar{\rho}\sigma}\gamma^5_{\rho\bar{\sigma}} - \gamma^{[123]}_{\bar{\rho}\sigma}\gamma^{[123]}_{\rho\bar{\sigma}} .$$

By adding this equation to eq. (4), one will obtain [eq. (21) of **Pauli** [**2**] = the sum of eq. (β) and (γ) of the article [**1**]]:

$$(9) \qquad \sum_{\mu=1}^{4} \gamma^{\mu}_{\rho\sigma}\gamma^{\mu}_{\bar{\rho}\bar{\sigma}} + \sum_{[\lambda\mu\nu]} \gamma^{[\lambda\mu\nu]}_{\rho\sigma}\gamma^{[\lambda\mu\nu]}_{\bar{\rho}\bar{\sigma}} = 2(\delta_{\bar{\rho}\sigma}\delta_{\rho\bar{\sigma}} - \gamma^5_{\bar{\rho}\sigma}\gamma^5_{\rho\bar{\sigma}}) .$$

Conversely, (4) is also a consequence of eqs. (20) and (21) in **Pauli** [**2**]. Namely, (4) will arise when one replaces the index ρ with $\bar{\bar{\rho}}$ in (7), the index $\bar{\rho}$ with $\bar{\bar{\bar{\rho}}}$, multiplies by $\gamma^{[123]}_{\rho\bar{\bar{\rho}}}\gamma^{[123]}_{\bar{\rho}\bar{\bar{\bar{\rho}}}}$, sums over $\bar{\bar{\rho}}$, $\bar{\bar{\bar{\rho}}}$, and adds to (9).

If one then adds the second equation:

$$(10) \qquad \boxed{\begin{aligned} 2B_{\bar{\sigma}\sigma}B^{-1}_{\rho\bar{\rho}} = {} & (\delta_{\rho\sigma}\delta_{\bar{\rho}\bar{\sigma}} - \delta_{\bar{\rho}\sigma}\delta_{\rho\bar{\sigma}}) + (\gamma^5_{\rho\sigma}\gamma^5_{\bar{\rho}\bar{\sigma}} - \gamma^5_{\bar{\rho}\sigma}\gamma^5_{\rho\bar{\sigma}}) \\ & + (\gamma^4_{\rho\sigma}\gamma^4_{\bar{\rho}\bar{\sigma}} - \gamma^4_{\bar{\rho}\sigma}\gamma^4_{\rho\bar{\sigma}}) - (\gamma^{[123]}_{\rho\sigma}\gamma^{[123]}_{\bar{\rho}\bar{\sigma}} - \gamma^{[123]}_{\bar{\rho}\sigma}\gamma^{[123]}_{\rho\bar{\sigma}}) \end{aligned}}$$

to eq. (4) then one will have **Pauli**'s bilinear equations in the form that is most suitable for the following applications. The matrix B is the matrix [**2**] that makes the system of transposed matrices $\bar{\gamma}^{\mu}$ (i.e., the rows and columns have been switched) emerge from the system of γ^{μ} by similarity transformations:

$$(11) \qquad \bar{\gamma}^{\mu} = B\,\gamma^{\mu}\,B^{-1}.$$

The matrix B is skew-symmetric.

$$(12) \qquad \bar{B} = -B.$$

It follows from (11) and (12) that, along with B, the matrices $B\gamma^{\mu}$ are also skew-symmetric:

(13) $$\overline{(B\gamma^{\mu})} = -B\gamma^{\mu} \qquad (\mu = 1, 2, 3, 4, 5),$$

while the matrices $B\gamma^{[\mu\nu]}$ and $B\gamma^{[\lambda\mu\nu]}$ are ten symmetric matrices:

(14) $$\overline{(B\gamma^{[\mu\nu]})} = -B\gamma^{[\mu\nu]}, \quad \overline{(B\gamma^{[\lambda\mu\nu]})} = -B\gamma^{[\lambda\mu\nu]}.$$

One sees that (10) is equivalent to eq. (34) in **Pauli**'s paper when one takes the difference of eq. (4) and (10). The formulation (4) and (10) has the advantage of expressing the same thing in two equations that was previously expressed in three [viz., eqs. (α), (β), (γ), in *loc. cit.*].

§ 3. Algebraic identities between the probability densities.

If one multiplies the bilinear equation (4) with suitable matrices from the matrix ring and defines the inner products with the help of the functions ψ^+ and ψ then one will get a series of algebraic identities between the probability densities. They admit all of the ones that are derived from the nine primary identities whose derivation one will find in [**1**]. The Hermiticity of the γ^{μ} is not required in them.

Primary algebraic identities (*):

(15) $$(\mathfrak{s}, \mathfrak{s}) = s_0^2 - (\Omega^2 + \hat{\Omega}^2),$$

(16) $$(\hat{\mathfrak{s}}, \hat{\mathfrak{s}}) = \hat{s}_0^2 + (\Omega^2 + \hat{\Omega}^2),$$

(17) $$(\mathfrak{s}, \hat{\mathfrak{s}}) = s_0\, \hat{s}_0,$$

(18) $$\mathfrak{M} = \frac{\Omega\,(s_0\hat{\mathfrak{s}} - \hat{s}_0\mathfrak{s}) - \hat{\Omega}[\mathfrak{s}, \hat{\mathfrak{s}}]}{\Omega^2 + \hat{\Omega}^2},$$

(19) $$\hat{\mathfrak{M}} = \frac{\Omega[\mathfrak{s}, \hat{\mathfrak{s}}] + \hat{\Omega}\,(s_0\hat{\mathfrak{s}} - \hat{s}_0\mathfrak{s})}{\Omega^2 + \hat{\Omega}^2}.$$

The magnetic and electric moment can then be traced back to the remaining quantities s_0, $\hat{s}_0$, $\mathfrak{s}$, $\hat{\mathfrak{s}}$, Ω, $\hat{\Omega}$ completely [eqs. (18) and (19)], and they themselves are coupled to each other by three identities. From this viewpoint, $\mathfrak{M}$ and $\hat{\mathfrak{M}}$ can be derived from the remaining quantities, and one can calculate, e.g., the quantities s_0, $\hat{s}_0$, $\hat{\Omega}$ from the givens of $\mathfrak{s}$, $\hat{\mathfrak{s}}$, Ω, up to a sign that remains free, using (15)-(17). One will then get:

(*) **Correction:** In *loc. cit.* [**1**], pp. 98, formulas (II) and (IV) are lacking a duality sign. One will find the correct notation there on pp. 96.

$$s_0 = \pm\tfrac{1}{2}\{\,|\,(\mathfrak{s}+\hat{\mathfrak{s}})\,|\pm|\,(\mathfrak{s}-\hat{\mathfrak{s}})\,|\,\},\qquad \hat{s}_0 = \pm\tfrac{1}{2}\{\,|\,(\mathfrak{s}+\hat{\mathfrak{s}})\,|\mp|\,(\mathfrak{s}-\hat{\mathfrak{s}})\,|\},$$

$$\hat{\Omega} = \pm\left[\tfrac{1}{2}\left\{|\,\hat{\mathfrak{s}}\,|^2-|\,\mathfrak{s}\,|^2-2\Omega^2+\sqrt{(|\,\hat{\mathfrak{s}}\,|^2+|\,\mathfrak{s}\,|^2)-4(\mathfrak{s},\hat{\mathfrak{s}})^2}\right\}\right]^{1/2}.$$

Each of the four possibilities for choosing the sign in s_0 will yield a solution when one chooses the correspond one in $\hat{s}_0$. Therefore, the nine quantities $\mathfrak{M}$, $\hat{\mathfrak{M}}$, s_0, $\hat{s}_0$, $\hat{\Omega}$ can be calculated from the seven quantities $\mathfrak{s}$, $\hat{\mathfrak{s}}$, Ω, up to that sign uncertainty, with the help of our primary identities.

Derived algebraic identities.

Referring to eqs. (15)-(19) as the "primary" identities above is not the only possibility. One can, with equal justification, distinguish another system of equations besides the primary one from which the remaining ones are derived; e.g.:

$$(20)\qquad (\mathfrak{M},\mathfrak{M}) = s_0^2-\hat{s}_0^2-\hat{\Omega}^2,$$

$$(21)\qquad (\hat{\mathfrak{M}},\hat{\mathfrak{M}}) = s_0^2-\hat{s}_0^2-\Omega^2,$$

$$(22)\qquad (\mathfrak{M},\hat{\mathfrak{M}}) = \Omega\hat{\Omega},$$

$$(23)\qquad \mathfrak{s} = \frac{s_0[\mathfrak{M},\hat{\mathfrak{M}}]+\hat{s}_0(\Omega\,\mathfrak{M}+\Omega\,\hat{\mathfrak{M}})}{s_0^2-\hat{s}_0^2},$$

$$(24)\qquad \hat{\mathfrak{s}} = \frac{\hat{s}_0[\mathfrak{M},\hat{\mathfrak{M}}]+s_0(\Omega\,\mathfrak{M}+\Omega\,\hat{\mathfrak{M}})}{s_0^2-\hat{s}_0^2}.$$

In this system, the current and spin density lead back to the remaining quantities s_0, $\hat{s}_0$, Ω, $\hat{\Omega}$, $\mathfrak{M}$, $\hat{\mathfrak{M}}$, completely. However, eqs. (15) to (19) will be chosen to be the primary ones here, so eqs. (20)-(24) will then belong to the identities that can be derived from (15)-(19). We enumerate some further derived identities:

$$(25)\qquad (\mathfrak{M},\mathfrak{s}) = \hat{s}_0\,\Omega,\qquad\qquad (29)\qquad [\mathfrak{M},\mathfrak{s}] = \hat{\Omega}\,\hat{\mathfrak{s}}-s_0\,\hat{\mathfrak{M}},$$

$$(26)\qquad (\mathfrak{M},\hat{\mathfrak{s}}) = s_0\,\Omega,\qquad\qquad (30)\qquad [\mathfrak{M},\hat{\mathfrak{s}}] = \hat{\Omega}\,\mathfrak{s}-\hat{s}_0\,\hat{\mathfrak{M}},$$

$$(27)\qquad (\hat{\mathfrak{M}},\mathfrak{s}) = \hat{s}_0\,\hat{\Omega},\qquad\qquad (31)\qquad [\hat{\mathfrak{M}},\mathfrak{s}] = -\,\Omega\,\hat{\mathfrak{s}}+s_0\,\mathfrak{M},$$

$$(28)\qquad (\hat{\mathfrak{M}},\hat{\mathfrak{s}}) = s_0\,\hat{\Omega},\qquad\qquad (32)\qquad [\hat{\mathfrak{M}},\hat{\mathfrak{s}}] = -\,\Omega\,\mathfrak{s}+\hat{s}_0\,\mathfrak{M},$$

$$(33)\qquad [\mathfrak{M},\hat{\mathfrak{M}}] = s_0\,\mathfrak{s}-\hat{s}_0\,\hat{\mathfrak{s}},$$

$$(34)\qquad [\mathfrak{s},\hat{\mathfrak{s}}] = \Omega\,\hat{\mathfrak{M}}-\hat{\Omega}\,\mathfrak{M},$$

$$(35) \qquad s_0 \hat{\mathfrak{s}} - \hat{s}_0 \mathfrak{s} = \Omega \mathfrak{M} + \hat{\Omega} \hat{\mathfrak{M}} .$$

One further has:

$$(36) \qquad \begin{cases} s_0^2 - \hat{s}_0^2 - \Omega^2 - \hat{\Omega}^2 = \dfrac{1}{\Omega}(\hat{\mathfrak{M}}, [\mathfrak{s}, \hat{\mathfrak{s}}]) = -\dfrac{1}{\hat{\Omega}}(\mathfrak{M}, [\mathfrak{s}, \hat{\mathfrak{s}}]), \\ \qquad = \dfrac{1}{\hat{s}_0^2}(\hat{\mathfrak{s}}, [\mathfrak{M}, \hat{\mathfrak{M}}]) = \dfrac{1}{s_0^2}(\mathfrak{s}, [\mathfrak{M}, \hat{\mathfrak{M}}]), \\ \qquad = \dfrac{1}{s_0^2 - \hat{s}_0^2} \,|\, (s_0 \mathfrak{s} - \hat{s}_0 \hat{\mathfrak{s}}) \,|^2 = \dfrac{1}{\Omega^2 + \hat{\Omega}^2} \,|\, [\mathfrak{s}, \hat{\mathfrak{s}}) \,|^2 . \end{cases}$$

$$(37) \qquad |\, (s_0 \hat{\mathfrak{s}} - \hat{s}_0 \mathfrak{s}) \,|^2 = (\Omega^2 + \hat{\Omega}^2)(s_0^2 - \hat{s}_0^2) ,$$

$$(38) \qquad \Omega \mathfrak{M} - \hat{\Omega} \hat{\mathfrak{M}} = \frac{1}{\Omega^2 + \hat{\Omega}^2} \{ (\Omega^2 - \hat{\Omega}^2)(s_0 \hat{\mathfrak{s}} - \hat{s}_0 \mathfrak{s}) - 2 \Omega \hat{\Omega} [\mathfrak{s}, \hat{\mathfrak{s}}] \} ,$$

$$(39) \qquad \hat{\Omega} \mathfrak{M} + \Omega \hat{\mathfrak{M}} = \frac{1}{\Omega^2 + \hat{\Omega}^2} \{ 2 \Omega \hat{\Omega} (s_0 \hat{\mathfrak{s}} - \hat{s}_0 \mathfrak{s}) + (\Omega^2 - \hat{\Omega}^2)[\mathfrak{s}, \hat{\mathfrak{s}}] \} .$$

The vector form is inappropriate for a whole series of identities. We write them in components, in such a way that one will always find two more identities by cyclically permuting the indices 1, 2, 3 in the given ones:

$$(40) \qquad \hat{s}_1 \hat{s}_2 - s_1 s_2 = M_{10} M_{20} + M_{23} M_{31} ,$$

$$(41) \qquad s_1^2 - \hat{s}_1^2 = -\Omega^2 - M_{10}^2 + M_{31}^2 + M_{12}^2 = -\hat{\Omega}^2 - M_{10}^2 + M_{31}^2 + M_{12}^2 ,$$

$$(42) \qquad \begin{cases} s_0^2 - s_1^2 - \hat{s}_2^2 - \hat{s}_3^2 = \dfrac{1}{\Omega^2 + \hat{\Omega}^2} \{ (s_0 \hat{s}_1 - \hat{s}_0 s_1)^2 + (s_2 \hat{s}_3 - \hat{s}_2 s_3)^2 \} \\ \qquad = M_{23}^2 + M_{10}^2 = -\hat{s}_0^2 + \hat{s}_1^2 + s_2^2 + s_3^2 . \end{cases}$$

It is characteristic of this collection of formulas that it will go to itself under certain permutations of the quantities; e.g., the following one (viz., multiplication by γ^4):

Replace	Ω	$\hat{\Omega}$	s_0	$\hat{s}_0$	$\mathfrak{s}$	$\hat{\mathfrak{s}}$	$\mathfrak{M}$	$\hat{\mathfrak{M}}$
with	s_0	$\pm i \hat{s}_0$	Ω	$\mp i \hat{\Omega}$	$\mp i \hat{\mathfrak{M}}$	$\mathfrak{M}$	$\hat{\mathfrak{s}}$	$\pm i \mathfrak{s}$

Upper and lower signs always belong together in the bottom row.

§ 4. The maximum number of primary algebraic identities.

From § 3, there are nine primary identities in the components [eqs. (15)-(19)]. The proof that one therefore has all of them shall now be presented. In order to simplify the notation, in this paragraph we shall denote the 16 matrices of the matrix ring with the running number j = 1 to 16 in the sequence that is specified below:

$$\alpha^j =$$

$$(\alpha^1, \alpha^2, \alpha^3, \alpha^4, \alpha^5, I, \alpha^{[23]}, \alpha^{[31]}, \alpha^{[12]}, \alpha^{[14]}, \alpha^{[24]}, \alpha^{[34]}, \alpha^{[234]}, \alpha^{[314]}, \alpha^{[124]}, \alpha^{[123]})$$

and the 16 function doublets that arise by taking inner products with these matrices are specified by $z_{\mu\nu} = \psi_\mu^* \psi_\nu$ (μ, ν = 1, 2, 3, 4). With this notation, the 16 probability densities in § 1B *a*) to *d*) will take on the following appearance:

$$(43) \quad \left\{ \begin{array}{lll}
\sum_{\mu,\nu=1}^4 \alpha_{\mu\nu}^1 z_{\mu\nu} = -s_1, & \sum_{\mu,\nu=1}^4 \alpha_{\mu\nu}^7 z_{\mu\nu} = \hat{s}_1, & \sum_{\mu,\nu=1}^4 \alpha_{\mu\nu}^{12} z_{\mu\nu} = M_{30}, \\
\sum_{\mu,\nu=1}^4 \alpha_{\mu\nu}^2 z_{\mu\nu} = -s_2, & \sum_{\mu,\nu=1}^4 \alpha_{\mu\nu}^8 z_{\mu\nu} = \hat{s}_2, & \sum_{\mu,\nu=1}^4 \alpha_{\mu\nu}^{13} z_{\mu\nu} = M_{23}, \\
\sum_{\mu,\nu=1}^4 \alpha_{\mu\nu}^3 z_{\mu\nu} = -s_3, & \sum_{\mu,\nu=1}^4 \alpha_{\mu\nu}^9 z_{\mu\nu} = \hat{s}_3, & \sum_{\mu,\nu=1}^4 \alpha_{\mu\nu}^{14} z_{\mu\nu} = M_{31}, \\
\sum_{\mu,\nu=1}^4 \alpha_{\mu\nu}^4 z_{\mu\nu} = \Omega, & \sum_{\mu,\nu=1}^4 \alpha_{\mu\nu}^{10} z_{\mu\nu} = M_{10}, & \sum_{\mu,\nu=1}^4 \alpha_{\mu\nu}^{15} z_{\mu\nu} = M_{12}, \\
\sum_{\mu,\nu=1}^4 \alpha_{\mu\nu}^5 z_{\mu\nu} = -\hat{\Omega}, & \sum_{\mu,\nu=1}^4 \alpha_{\mu\nu}^{11} z_{\mu\nu} = M_{20}, & \sum_{\mu,\nu=1}^4 \alpha_{\mu\nu}^{16} z_{\mu\nu} = -\hat{s}_0. \\
\sum_{\mu,\nu=1}^4 \alpha_{\mu\nu}^6 z_{\mu\nu} = s_0, & &
\end{array} \right.$$

If one regards the $z_{\mu\nu}$ as unknowns then one will have 16 linear equations in 16 unknowns. This system has one and only one solution when the coefficient determinant:

$$(44) \qquad \Delta = \text{Det} \left| \alpha_{\mu\nu}^j \right| \qquad \begin{pmatrix} j = 1, 2, \cdots, 16 \\ \mu, \nu = 11, 12, \cdots, 44 \end{pmatrix}$$

is non-zero. That is, in fact, the case: Namely, one can, in turn, show, with the help of **Pauli**'s equation [*loc. cit.* [**2**], eq. (17)] that the square of this determinant is non-zero:

(45) $$\Delta^2 = \begin{vmatrix} \sum_{j=1}^{16} \alpha_{11}^j \alpha_{11}^j & \sum_{j=1}^{16} \alpha_{11}^j \alpha_{12}^j & \sum_{j=1}^{16} \alpha_{11}^j \alpha_{13}^j & \cdots & \sum_{j=1}^{16} \alpha_{11}^j \alpha_{44}^j \\ \sum_{j=1}^{16} \alpha_{12}^j \alpha_{11}^j & \sum_{j=1}^{16} \alpha_{12}^j \alpha_{12}^j & \sum_{j=1}^{16} \alpha_{12}^j \alpha_{13}^j & \cdots & \sum_{j=1}^{16} \alpha_{12}^j \alpha_{44}^j \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \sum_{j=1}^{16} \alpha_{44}^j \alpha_{11}^j & \sum_{j=1}^{16} \alpha_{44}^j \alpha_{12}^j & \sum_{j=1}^{16} \alpha_{44}^j \alpha_{13}^j & \cdots & \sum_{j=1}^{16} \alpha_{44}^j \alpha_{44}^j \end{vmatrix}$$

In fact, with this notation, one will have:

(46) $$\sum_{j=1}^{16} \alpha_{\mu\nu}^j \, \alpha_{\bar{\mu}\bar{\nu}}^j = \pm \delta_{\bar{\mu}\nu} \delta_{\mu\bar{\nu}} \qquad \text{for arbitrary } \mu, \nu, \bar{\mu}, \bar{\nu},$$

from which, it will emerge that a 4 will appear in each row of the determinant-squared at only the places with $\bar{\mu} = \nu$, $\bar{\nu} = \mu$, but only zeroes will appear everywhere else. One can bring Δ^2 into a form with 4's in the main diagonal, but 0 everywhere else by a suitable permutation of the rows, which can affect only the sign of Δ^2 (which is uninteresting to us here). The absolute value of the determinant-squared will then be:

(47) $$|\Delta^2| = 4^{16},$$

or

(48) $$|\Delta| = 2^{16} \neq 0.$$

With that, we have shown that none of the linear equations (43) are consequences of the remaining ones, and that the variables $z_{\mu\nu}$ can be calculated uniquely from the right-hand side. As long as the $z_{\mu\nu}$ are then free variables, no identities can then exist between the right-hand sides.

However, the $z_{\mu\nu}$ are not free variables. From the table:

(49) $$\begin{cases} z_{11} = \psi_1^* \psi_1, & z_{12} = \psi_1^* \psi_2, & z_{13} = \psi_1^* \psi_3, & z_{14} = \psi_1^* \psi_4, \\ z_{21} = \psi_2^* \psi_1, & z_{22} = \psi_2^* \psi_2 = \dfrac{z_{12} \cdot z_{21}}{z_{11}}, & z_{23} = \psi_2^* \psi_3 = \dfrac{z_{13} \cdot z_{21}}{z_{11}}, & z_{24} = \psi_2^* \psi_4 = \dfrac{z_{14} \cdot z_{21}}{z_{11}}, \\ z_{31} = \psi_3^* \psi_1, & z_{32} = \psi_3^* \psi_2 = \dfrac{z_{12} \cdot z_{31}}{z_{11}}, & z_{33} = \psi_3^* \psi_3 = \dfrac{z_{13} \cdot z_{31}}{z_{11}}, & z_{34} = \psi_3^* \psi_4 = \dfrac{z_{14} \cdot z_{31}}{z_{11}}, \\ z_{41} = \psi_4^* \psi_1, & z_{42} = \psi_4^* \psi_2 = \dfrac{z_{12} \cdot z_{41}}{z_{11}}, & z_{43} = \psi_4^* \psi_3 = \dfrac{z_{13} \cdot z_{41}}{z_{11}}, & z_{44} = \psi_4^* \psi_4 = \dfrac{z_{14} \cdot z_{41}}{z_{11}}, \end{cases}$$

it will become obvious that only the seven variables (e.g., z_{11}, z_{12}, z_{13}, z_{14}, z_{21}, z_{31}, z_{41}) in the upper and left-hand edges of this table are mutually independent, while the remaining nine will follow from them. Among those seven variables, one will find one real one (z_{11}) and three complex ones (z_{12}, z_{13}, z_{14}), whose real and imaginary parts correspond to the free choice of two variables. Naturally, seven other variables from this table can be chosen to be the independent ones, while nine other variables will be the dependent ones. However, it is important to note that seven variables must remain completely free to be chosen (on the basis of purely algebraic considerations; i.e., with no application of the Dirac equation!). Therefore, in the maximal case, nine primary identities can exist between the right-hand sides of (43) if no linear dependency of the eqs. (43) with each other exists ($\Delta \neq 0$) along with this bilinear dependency of the $z_{\mu\nu}$ with each other.

§ 5. The emergence of the primary identities in a special example.

From § 4, it must be possible to derive the primary identities from the relations (49) between the variables $z_{\mu\nu}$. While the arguments up to now have not required any specialization of the Dirac matrices, the following derivation shall be carried out with a special representation of the α-matrices:

$$\alpha_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \ \alpha_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \ \alpha_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

With that, one will have:

$$\begin{aligned}
&z_{11} + z_{22} - z_{33} - z_{44} = \Omega, && z_{14} + z_{23} + z_{32} + z_{41} = -\,s_1\,,\\
&z_{11} + z_{22} + z_{33} + z_{44} = s_0\,, && z_{14} - z_{23} + z_{32} - z_{41} = -\,s_1\,,\\
&z_{11} - z_{22} + z_{33} - z_{44} = -\,\hat{s}_3\,, && z_{14} + z_{23} - z_{32} - z_{41} = i\,M_{10}\,,\\
&z_{11} - z_{22} - z_{33} + z_{44} = -\,M_{12}\,, && z_{14} - z_{23} - z_{32} + z_{41} = -\,M_{20}\,,\\[1ex]
&z_{12} + z_{21} + z_{34} + z_{43} = -\hat{s}_1\,, && z_{13} - z_{24} + z_{31} - z_{42} = -\,s_3\,,\\
&z_{12} - z_{21} + z_{34} - z_{43} = -i\,\hat{s}_2\,, && z_{13} + z_{24} + z_{31} + z_{42} = \hat{s}_0\,,\\
&z_{12} + z_{21} - z_{34} - z_{43} = -\,M_{23}\,, && z_{13} - z_{24} - z_{31} + z_{42} = i\,M_{30}\,,\\
&z_{12} - z_{21} - z_{34} + z_{43} = -\,i\,M_{31}\,, && z_{13} + z_{24} - z_{31} - z_{42} = -\,i\hat{\Omega}\,.
\end{aligned}$$

One can solve these equations for the 16 variables $z_{\mu\nu}$. If one represents the $z_{\mu\nu} = \psi_\mu^* \psi_\nu$ as points in a square lattice then the image of each identity (primary or derived) will be a rectangle that has the four points μ, ν ; ρ, ν ; ρ, σ ; μ, σ for its corners, so one will indeed have:

$$z_{\mu\nu} = \frac{z_{\rho\nu} \cdot z_{\mu\sigma}}{z_{\mu\nu}} \qquad \text{or} \qquad z_{\mu\nu} \cdot z_{\rho\sigma} = \psi_\rho^* \psi_\mu^* \psi_\nu \psi_\sigma = z_{\rho\nu} \cdot z_{\mu\sigma}\,.$$

One can characterize each rectangle $R_{\mu\nu,\,\rho\sigma}$ in this square lattice by its upper left vertex (μ, ν) and its lower right vertex (ρ, σ). We would like to understand the meaning of the corresponding identity:

(50) $$R_{\mu\nu,\,\rho\sigma} \equiv z_{\mu\nu} \cdot z_{\rho\sigma} - z_{\rho\nu} \cdot z_{\mu\sigma} = 0$$

with this vertex notation. In the lattice point model, for each of the 16 lattice points, one can choose one of the nine other lattice points to be the opposite vertex of a rectangle, while the remaining seven lattice points that lie in the same row (column, resp.) will be unsuitable in that regard. One will then obtain 144 rectangles, four of which coincide, since every rectangle has four vertices. There are then 36 identities, from which one can look for the nine primary ones in them or define them by suitable combinations. In this way, one finds that:

$$s_0^2 - \sum_{i=1}^{3} s_i^2 - (\Omega^2 + \hat{\Omega}^2)$$
$$= 4\,\{R_{11,\,33} + R_{11,\,44} + R_{22,\,33} + R_{23,\,44} + R_{13,\,24} + R_{31,\,42}\} = 0,$$

$$\hat{s}_0^2 - \sum_{i=1}^{3} s_i^2 + (\Omega^2 + \hat{\Omega}^2)$$
$$= 4\,\{R_{11,\,33} + R_{12,\,43} + R_{21,\,34} + R_{22,\,44} + R_{33,\,44} + R_{11,\,22}\} = 0,$$

$$s_0\,\hat{s}_0 - \sum_{i=1}^{3} s_i\,\hat{s}_i$$
$$= 2\,\{R_{11,\,24} + R_{11,\,42} + R_{12,\,23} + R_{21,\,32} + R_{13,\,44} + R_{31,\,44} + R_{32,\,43} + R_{23,\,34}\} = 0.$$

These first three identities already agree with the primary ones (15), (16), (17), while the primary identities (18) and (19) will emerge from eliminating $\mathfrak{M}$ ($\hat{\mathfrak{M}}$, resp.) from the remaining six ones:

$$s_0\,\hat{s}_1 - \hat{s}_0\,s_1 - \Omega\,M_{23} - \hat{\Omega}\,M_{10}$$
$$= -\,2\,\{R_{11,\,34} + R_{11,\,43} + R_{21,\,33} + R_{12,\,33} + R_{12,\,44} + R_{21,\,44} + R_{22,\,43} + R_{22,\,34}\} = 0,$$

$$s_0\,\hat{s}_2 - \hat{s}_0\,s_2 - \Omega\,M_{31} - \hat{\Omega}\,M_{20}$$
$$= 2i\,\{R_{11,\,34} - R_{11,\,43} + R_{22,\,34} - R_{22,\,43} + R_{12,\,33} - R_{21,\,33} + R_{12,\,44} - R_{21,\,44}\} = 0,$$

$$s_0\,\hat{s}_3 - \hat{s}_0\,s_3 - \Omega\,M_{12} - \hat{\Omega}\,M_{30} = 4\,\{R_{22,\,44} - R_{11,\,33}\} = 0,$$

$$s_2\,\hat{s}_3 - \hat{s}_2\,s_3 - \Omega\,M_{10} + \hat{\Omega}\,M_{23}$$
$$= 2i\,\{R_{12,\,24} - R_{21,\,42} + R_{31,\,43} - R_{13,\,34} + R_{11,\,23} - R_{11,\,32} + R_{32,\,44} - R_{23,\,44}\} = 0,$$

$$s_3\,\hat{s}_1 - \hat{s}_3\,s_1 - \Omega\,M_{20} + \hat{\Omega}\,M_{31}$$
$$= 2\,\{R_{11,\,23} + R_{11,\,32} + R_{13,\,34} + R_{31,\,43} - R_{21,\,42} - R_{13,\,34} - R_{32,\,44} - R_{23,\,44}\} = 0,$$

$$s_1\,\hat{s}_2 - \hat{s}_1\,s_2 - \Omega\,M_{30} + \hat{\Omega}\,M_{12}$$
$$= 2i\,\{R_{11,\,24} - R_{11,\,42} + R_{13,\,44} - R_{31,\,44} + R_{21,\,32} - R_{12,\,23} + R_{32,\,43} - R_{23,\,34}\} = 0.$$

§ 6. Some formulas.

Due to the application that we have in mind, some formulas will find a place here, which, although they are easy to derive, still possess a surprising aspect.

For an entirely arbitrary vector $\mathfrak{k}$, one has the equation:

$$(51)\quad \begin{cases} (\hat{s}_0^2+\Omega^2+\hat{\Omega}^2)\,\mathfrak{s}\,(\mathfrak{s},\mathfrak{k})+(s_0^2-\Omega^2-\hat{\Omega}^2)\,\hat{\mathfrak{s}}\,(\hat{\mathfrak{s}},\mathfrak{k}) \\ \quad -s_0\,\hat{s}_0\{\hat{\mathfrak{s}}\,(\mathfrak{s},\mathfrak{k})+\mathfrak{s}\,(\hat{\mathfrak{s}},\mathfrak{k})\}+[\mathfrak{s},\hat{\mathfrak{s}}]([\mathfrak{s},\hat{\mathfrak{s}}],\mathfrak{k}) \\ \quad\quad +(\Omega^2+\hat{\Omega}^2)(\Omega^2+\hat{\Omega}^2-s_0^2+\hat{s}_0^2)\,\mathfrak{k}=0. \end{cases}$$

The sense of this vector equation will become apparent when one represents an arbitrary vector $\mathfrak{k}$ in a coordinate systems whose axes are $\mathfrak{s}$, $\hat{\mathfrak{s}}$, $[\mathfrak{s}, \hat{\mathfrak{s}}]$:

$$\mathfrak{k} = h_1\,\mathfrak{s} + h_2\,\hat{\mathfrak{s}} + h_3\,[\mathfrak{s}, \hat{\mathfrak{s}}].$$

The components will then become:

$$h_1 = \frac{(\hat{s}_0^2+\Omega^2+\hat{\Omega}^2)(\mathfrak{s},\mathfrak{k})-s_0\,\hat{s}_0(\hat{\mathfrak{s}},\mathfrak{k})}{(\Omega^2+\hat{\Omega}^2)(s_0^2-\hat{s}_0^2-\Omega^2-\hat{\Omega}^2)},$$

$$h_2 = \frac{(\hat{s}_0^2-\Omega^2-\hat{\Omega}^2)(\hat{\mathfrak{s}},\mathfrak{k})-s_0\,\hat{s}_0(\mathfrak{s},\mathfrak{k})}{(\Omega^2+\hat{\Omega}^2)(s_0^2-\hat{s}_0^2-\Omega^2-\hat{\Omega}^2)},$$

$$h_3 = \frac{([\mathfrak{s}\,\hat{\mathfrak{s}}],\mathfrak{k})}{(\Omega^2+\hat{\Omega}^2)(s_0^2-\hat{s}_0^2-\Omega^2-\hat{\Omega}^2)}.$$

Some further formulas are important in conversions, which either contain differential quotients or emerge by differentiating the identities in § 3. They are:

$$(52)\quad (\mathfrak{s}, (\hat{\mathfrak{s}}\text{ grad})\,\mathfrak{s}) = -(\hat{\mathfrak{s}}, \Omega\text{ grad}\,\Omega + \hat{\Omega}\text{ grad}\,\hat{\Omega}) + s_0\,(\hat{\mathfrak{s}}, \text{grad}\,s_0),$$

$$(53)\quad (\hat{\mathfrak{s}}, (\mathfrak{s}\text{ grad})\,\hat{\mathfrak{s}}) = (\mathfrak{s}, \Omega\text{ grad}\,\Omega + \hat{\Omega}\text{ grad}\,\hat{\Omega}) + \hat{s}_0\,(\mathfrak{s}, \text{grad}\,\hat{s}_0).$$

Proof: The application of the vector formula $(\mathfrak{A}, (\mathfrak{B}\text{ grad})\,\mathfrak{A}) = \left(\mathfrak{B}, \text{grad}\,\frac{\mathfrak{A}^2}{2}\right)$ to the left-hand side of the equations above, while recalling the identities (15) and (16), will give these formulas.

$$(54)\quad \left\{ \begin{aligned} &[\mathfrak{M}, \mathrm{rot}\,\mathfrak{M}] - \mathfrak{M}\,\mathrm{div}\,\mathfrak{M} + [\hat{\mathfrak{M}}, \mathrm{rot}\,\hat{\mathfrak{M}}] - \hat{\mathfrak{M}}\,\mathrm{div}\,\hat{\mathfrak{M}} + [\mathfrak{s}, \mathrm{rot}\,\mathfrak{s}] - \mathfrak{s}\,\mathrm{div}\,\mathfrak{s} \\ &\qquad - ([\hat{\mathfrak{s}}, \mathrm{rot}\,\hat{\mathfrak{s}}] - \hat{\mathfrak{s}}\,\mathrm{div}\,\hat{\mathfrak{s}}) + \tfrac{1}{2}\,\mathrm{grad}(\Omega^2 + \hat{\Omega}^2 - s_0^2 + \hat{s}_0^2) = 0. \end{aligned} \right.$$

Proof: It is simplest to do this in components. The 1 component of this equation can be written, when ordered after the differentiations:

$$\begin{aligned} &\frac{1}{2}\frac{\partial}{\partial x_1}(M_{31}^2 + M_{12}^2 - M_{23}^2 + M_{20}^2 + M_{30}^2 - M_{10}^2 \\ &\qquad + s_2^2 + s_3^2 - s_1^2 - \hat{s}_2^2 - \hat{s}_3^2 + \hat{s}_1^2 + \Omega^2 + \hat{\Omega}^2 - s_0^2 - \hat{s}_0^2) \\ &- \frac{\partial}{\partial x_2}(M_{23}M_{31} + M_{23}M_{31} + s_1\,s_2 - \hat{s}_1\,\hat{s}_2) \\ &- \frac{\partial}{\partial x_3}(M_{12}M_{23} + M_{30}M_{10} + s_3\,s_1 - \hat{s}_3\,\hat{s}_1) \equiv 0, \end{aligned}$$

since each individual term already vanishes: The first one vanishes with an application of eqs. (15), (16), (41), and the next two, from eq. (40).

In addition, some differential relations that will find diverse applications in III and IV will arise by taking the grad of the algebraic identities (15)-(17), (20)-(22), (25)-28), and taking the rot and div of formulas (29)-(35).

§ 7. Geometric representation and physical interpretation of some identities.

Instead of the one current density vector of **Schrödinger**'s theory, in **Dirac**'s theory of the electron, the four vectors of current density $\mathfrak{s}$, spin density $\hat{\mathfrak{s}}$, magnetic moment density $\mathfrak{M}$, and electric moment density $\hat{\mathfrak{M}}$ will appear at each point of space. From (18) and (19), the two planes that are spanned by $\mathfrak{s}$ and $\hat{\mathfrak{s}}$ ($\mathfrak{M}$ and $\hat{\mathfrak{M}}$, resp.) will be mutually perpendicular: The line of intersection of the two planes will have the direction of the vector:

$$s_0\hat{\mathfrak{s}} - \hat{s}_0\mathfrak{s} = \Omega\,\mathfrak{M} + \hat{\Omega}\hat{\mathfrak{M}}\,.$$

Nothing more can be said as long as one does not demand the Hermiticity of the **Dirac** matrices. However, if the **Dirac** matrices are Hermitian then the probability densities s_0, $\mathfrak{s}$, $\hat{\mathfrak{s}}$, $\mathfrak{M}$, $\hat{\mathfrak{M}}$, Ω, $\hat{\Omega}$ will be real. It will then follow from (36):

$$|\,[\mathfrak{s},\ \hat{\mathfrak{s}}\,]\,|^2 = (\Omega^2 + \hat{\Omega}^2)\,(s_0^2 - \hat{s}_0^2 - \hat{\Omega}^2 - \Omega^2)$$

that $s_0^2 - \hat{s}_0^2 - \hat{\Omega}^2 - \Omega^2 \geq 0$, the projection of $\hat{\mathfrak{M}} / \Omega$ onto $[\mathfrak{s}, \hat{\mathfrak{s}}\,]$ is positive, the projection of $\mathfrak{M} / \hat{\Omega}$ onto $[\mathfrak{s}, \hat{\mathfrak{s}}\,]$ is negative, and that the projections of $\mathfrak{s} / s_0$, $\hat{\mathfrak{s}} / \hat{s}_0$ onto $[\mathfrak{M}, \hat{\mathfrak{M}}\,]$ are both positive. However, that means that $\hat{\mathfrak{M}} / \Omega$ lies in the upper half of the plane that is spanned by $\mathfrak{s}$ and $\hat{\mathfrak{s}}$, and $\mathfrak{M} / \hat{\Omega}$ lies in the lower half, while $\mathfrak{s} / s_0$ and $\hat{\mathfrak{s}} / \hat{s}_0$ will be found on the same side of the plane that is spanned by $\mathfrak{M}$ and $\hat{\mathfrak{M}}$. The absolute values of all four projections will be the same. That very likely exhausts what one can say about the four vectors $\mathfrak{s}, \hat{\mathfrak{s}}$, $\mathfrak{M}, \hat{\mathfrak{M}}$ without choosing a special example as far as Hermiticity is concerned.

Eq. (29) in the form:

$$\hat{\mathfrak{M}} = \left[\frac{\mathfrak{s}}{s_0}, \mathfrak{M}\right] + \hat{\Omega}\frac{\hat{\mathfrak{s}}}{s_0}$$

is also especially interesting. One can introduce $\frac{\mathfrak{s}}{s_0} = \frac{\mathfrak{v}}{c}$ in this, in which $\mathfrak{v}$ is the velocity of the electron and c is that of light, and recognize that a connection between electric and magnetic moment will be prescribed in **Dirac**'s theory that is similar to the **Frenkel** connection:

$$\hat{\mathfrak{M}} = \left[\frac{\mathfrak{v}}{c}, \mathfrak{M}\right],$$

and comes solely from the *algebra* of the Dirac matrices. In general, a term $\hat{\Omega}\frac{\mathfrak{s}}{s_0}$ will be added that obviously specifies the contribution of spin to the electric moment. With the interpretation of $\frac{\mathfrak{s}}{s_0} = \frac{\mathfrak{v}}{c}$, $\mathfrak{v}$ generally also includes a rotatory part, in addition to the linear velocity, as would emerge in the **Gordon** decomposition of $\mathfrak{s}$ into a convection current and a polarization (magnetization, resp.) current. Cf., III, formula (12) for that.

An analogous argument is also valid for formula (31) in the form:

$$\mathfrak{M} = -\left[\frac{\mathfrak{s}}{s_0}, \hat{\mathfrak{M}}\right] + \Omega\frac{\hat{\mathfrak{s}}}{s_0} \qquad \rightarrow \qquad -\left[\frac{\mathfrak{v}}{c}, \hat{\mathfrak{M}}\right] + \Omega\frac{\hat{\mathfrak{s}}}{s_0},$$

whose first term represents the magnetic moment that is generated by an electric moment that moves with velocity $\mathfrak{v}$. The term $\Omega\frac{\hat{\mathfrak{s}}}{s_0}$ obviously represents the contribution of the spin to the magnetic moment that is produced.

What is intriguing in the foregoing is the fact that the relations in question are independent of the form of the equations of motion (i.e., the **Dirac** equation for the electron), since they already arise from the number system $\alpha^i \alpha^k + \alpha^k \alpha^i = 2\delta_{ik}$ that **Dirac** chose and the prescription for constructing the probability densities $(\psi^*, \alpha^j \psi)$.

References

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(Received 4 September 1940)