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On Dirac's theory of the electron

III. Consequences of the reality of the electromagnetic potentials.

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Introduction and summary

Now that the mathematical tools have been obtained in II for carrying out the plan that was described in the Introduction to II, all of the relations that follow from the reality of the electromagnetic potential for **Dirac**'s theory of the electron will be derived and put into a form that can be interpreted. We will find that *six vector relations* (§ 2, II-IV, XI-XIIIA, XI-XIIIB, XVII-XIXA, XVII-XIXB, XXIII to XXVB) and *four scalar relations* (§ 2, I, V, VIII, XA) arise from the reality of the potentials.

In § 1, after multiplying the Dirac equation by the 16 matrices of the **Dirac** matrix ring, forming inner products, and separating the real and imaginary parts, while considering the reality of the potentials, we will get 32 equations (1a)-(8b), when one counts components. In § 2, the four potentials $\mathfrak{A}$, V will be eliminated from those equations, whereby the 28 *potential-free* relations (§ 2, I-XXVIII) will remain. When one applies the mathematical tools of Parts I and II, they can be liberated of the uninterpretable quantities and will lead back to the aforementioned six vector relations and four scalar relations. In § 3, we will put two of the scalar relations and the six vector relations into forms that correspond to their peculiar symmetries.

Among the four scalar relations, one finds the continuity relations for charge density and mass density (§ 2, VII) and a similar relation for the temporal and spatial components of the spin density (§ 2, V). However, that still contains a supplementary term, which is why we shall occasionally refer to that relation as the "anti-continuity equation." Both relations have been derived before on the basis of a special choice of the Dirac matrices [**1**]. They are the two linear relations among all of the reality relations, and for that reason, they can be derived with no knowledge of **Pauli**'s bilinear equations. *All of the remaining ones are bilinear relations.* Among them, in turn, the two scalar relations (§ 2, I and XA) are the most important, since it can be shown that the six vector relations can be constructed from them (Part IV).

Whereas Hermiticity of the Dirac matrices was not required in I and II, from now on, that will be assumed; however, the generality of the representation of the matrices that will be obtained will then be lacking.

For the meaning of the symbols that occur here, confer I, § 1.

§ 1. Preliminary calculations

A. *The reality relations in their original form*

We split the terms in the Dirac equations that involve the potentials from the ones that do not and write:

(*) $$\underline{V\psi + \sum_{k=1}^{3} A_k \alpha^k \Psi = R\psi}\,,$$

in which the remainder operator R has the meaning:

$$R = -\frac{c}{e}\left\{ i\hbar\frac{\partial}{c\,\partial t} - i\hbar\sum_{k=1}^{3}\alpha^k\frac{\partial}{\partial x_k} - mc\,\alpha^4 \right\}.$$

We choose the following abbreviations for the inner products:

$$R_0 = (\psi^*, R\psi) = \sum_{\rho,\sigma=1}^{4} \psi_\rho^* R_{\rho\sigma}\psi_\sigma\,, \qquad R_j = (\psi^*, \alpha^j R\psi), \qquad j = 1, 2, \ldots, 5;$$

$$R_{k+1,k+2} = (\psi^*, \alpha^{k+1}\alpha^{k+2} R\psi), \quad R_{k4} = (\psi^*, \alpha^k\alpha^4 R\psi), \quad R_{123} = (\psi^*, \alpha^1\alpha^2\alpha^3 R\psi),$$

$$R_{k,k+1,4} = (\psi^*, \alpha^{k+1}\alpha^{k+2}\alpha^4 R\psi) \qquad (k = 1, 2, 3;\ k,\ k+1,\ k+2 \bmod 3).$$

If one left-multiplies eq. (*) successively by all matrices of the **Dirac** matrix ring and forms the inner products with the set of functions ψ^* then one will get:

1. Upon multiplying by the identity matrix I:

$$V s_0 - \sum_{k=1}^{3} A_k s_k = R_0\,.$$

2. Upon multiplying by the matrices α^k ($k = 1, 2, 3$):

$$-V s_k + A_k s_0 - i A_{k+1}\hat{s}_{k+2} + i A_{k+2}\hat{s}_{k+1} = R_k\,.$$

3. Upon multiplying by α^4:

$$V\Omega + i\sum_{k=1}^{3} A_k M_{k0} = R_4\,.$$

4. Upon multiplying by α^5:

$$V\hat{\Omega} - i\sum_{k=1}^{3} A_k M_{k+1,k+2} = R_5 \,.$$

5. Upon multiplying by $\alpha^{k+1}\,\alpha^{k+2}$ (k = 1, 2, 3; k, k + 1, k + 2 mod 3):

$$-\,i\,V\hat{s}_k + i\,A_k\,\hat{s}_0 + A_{k+1}\,s_{k+2} - A_{k+2}\,s_{k+1} = R_{k+1,k+2} \,.$$

6. Upon multiplication by $\alpha^k\,\alpha^4$ (k = 1, 2, 3):

$$-\,i\,V\,M_{k0} - A_k\,\Omega + i\,A_{k+1}\,M_{k,k+1} - i\,A_{k+2}\,M_{k+2,k} = R_{k4} \,.$$

7. Upon multiplication by $\alpha^{k+1}\,\alpha^{k+2}\,\alpha^4$ (k = 1, 2, 3):

$$-\,i\,V\,M_{k,+1k+2} - i\,A_{k+1}\,M_{k,+2,0} + i\,A_{k+2}\,M_{k,+1,0} + A_k\,\hat{\Omega} = R_{k+1,k+2,4} \,.$$

8. Upon multiplication by $\alpha^1\alpha^2\alpha^3$:

$$i\left\{V\,\hat{s}_0 - \sum_{k=1}^{3} A_k\hat{s}_k\right\} = R_{123} \,.$$

By adding (subtracting, resp.) the complex conjugates of these 16 equations, each of the two equations (a) and (b) will yield:

(1a) $$V\,s_0 - (\mathfrak{A},\,\mathfrak{s}) = \frac{1}{2}(R_0 + R_0^*) = P_0 \,,$$

(b) $$0 = -\frac{i}{2}(R_0 - R_0^*) = Q_0 \,,$$

(2a) $$\{-\,V\,\mathfrak{s} + \mathfrak{A}\,s_0\}_k = \frac{1}{2}(R_k + R_k^*) = P_k \,,$$

(b) $$[\mathfrak{A},\,\mathfrak{s}]_k = \frac{i}{2}(R_k - R_k^*) = Q_k \,,$$

(3a) $$V\,\Omega = \frac{1}{2}(R_4 + R_4^*) = P_4 \,,$$

(b) $$(\mathfrak{A},\,\hat{\mathfrak{M}}) = -\frac{i}{2}(R_4 - R_4^*) = Q_4 \,,$$

(4a) $$V\,\hat{\Omega} = -\frac{1}{2}(R_5 + R_5^*) = P_5 \,,$$

(b) $$(\mathfrak{A},\,\mathfrak{M}) = -\frac{i}{2}(R_5 - R_5^*) = Q_5 \,,$$

$$(5a)\qquad [\mathfrak{A},\ \mathfrak{s}]_k = \frac{1}{2}(R_{k+1,k+2} + R^*_{k+1,k+2}) = P_{k+1,k+2}\ ,$$

$$(\ b)\qquad \{V\ \hat{\mathfrak{s}} - \mathfrak{A}\,\hat{s}_0\}_k = \frac{i}{2}(R_{k+1,k+2} - R^*_{k+1,k+2}) = Q_{k+1,k+2}\ ,$$

$$(6a)\qquad A_k\,\Omega = -\frac{1}{2}(R_{k4} + R^*_{k4}) = P_{k4}\ ,$$

$$(\ b)\qquad \{V\ \hat{\mathfrak{M}} - [\mathfrak{A}\ \mathfrak{M}]\}_k = \frac{i}{2}(R_{k4} - R^*_{k4}) = Q_{k4}\ ,$$

$$(7a)\qquad A_k\ \hat{\Omega} = \frac{1}{2}(R_{k+1,k+2,4} + R^*_{k+1,k+2,4}) = P_{k+1,k+2,4}\ ,$$

$$(\ b)\qquad \{V\ \hat{\mathfrak{M}} - [\mathfrak{A}\ \mathfrak{M}]\}_k = \frac{i}{2}(R_{k+1,k+2,4} - R^*_{k+1,k+2,4}) = Q_{k+1,k+2,4}\ ,$$

$$(8a)\qquad 0 = \frac{1}{2}(R_{123} + R^*_{123}) = P_{123}\ ,$$

$$(\text{ b})\qquad V\ \hat{s}_0 - (\mathfrak{A},\hat{\mathfrak{s}}) = -\frac{1}{2}(R_{123} - R^*_{123}) = Q_{123}\ .$$

For the right-hand sides, one will get (the left superscript 0 means left-differentiation with respect to *t*, while the right superscript 0 means right-differentiation with respect to *t*):

$$(1a)\qquad P_0 = -\frac{\hbar c}{2e}\left\{-\frac{2mc}{\hbar}\Omega - \frac{i}{c}({}^0s_0 - s_0^0) - \sum_{k=1}^{3} i({}^k s_k - s_k^k)\right\},$$

$$(\text{ b})\qquad Q_0 = -\frac{\hbar c}{2e}\left\{\frac{\partial s_0}{c\,\partial t} + \operatorname{div}\mathfrak{s}\right\},$$

$$(2a)\qquad P_k = -\frac{\hbar c}{2e}\left\{-\operatorname{rot}_k\hat{\mathfrak{s}} + \frac{i}{c}({}^0s_k - s_k^0) + i\,({}^k s_0 - s_0^k)\right\},$$

$$(\text{ b})\qquad Q_k = -\frac{\hbar c}{2e}\left\{\left(-\frac{2mc}{\hbar}\hat{\mathfrak{M}} + \frac{\partial\mathfrak{s}}{c\,\partial t} + \operatorname{grad} s_0\right)_k + i({}^{k+1}\hat{s}_{k+2} - \hat{s}_{k+2}^{k+1}) - i\,({}^{k+2}\hat{s}_{k+1} - \hat{s}_{k+1}^{k+2})\right\},$$

$$(3a)\qquad P_4 = -\frac{\hbar c}{2e}\left\{-\frac{2mc}{\hbar}s_0 + \operatorname{div}\hat{\mathfrak{M}} - \frac{i}{c}({}^0\Omega - \Omega^0)\right\},$$

$$(\text{ b})\qquad Q_4 = -\frac{\hbar c}{2e}\left\{\frac{\partial\Omega}{c\,\partial t} + \sum_{k=1}^{3} i({}^k M_{k0} - M_{k0}^k)\right\},$$

$$(4a)\qquad P_5 = -\frac{\hbar c}{2e}\left\{-\operatorname{div}\mathfrak{M} - \frac{i}{c}({}^0\hat{\Omega} - \hat{\Omega}^0)\right\},$$

$$\text{(b)}\qquad Q_5 = -\frac{\hbar c}{2e}\left\{-\frac{2mc}{\hbar}\hat{s}_0 - \frac{\partial\hat{\Omega}}{c\,\partial t} + \sum_{k=1}^{3} i({}^k M_{k+1,k+2} - M_{k+1,k+2}^{k})\right\},$$

$$\text{(5a)}\qquad P_{k+1,k+2} = -\frac{\hbar c}{2e}\left\{\left(\frac{\partial\hat{\mathfrak{s}}}{c\,\partial t} + \operatorname{grad}\hat{s}_0\right)_k - i({}^{k+2}s_{k+1} - s_{k+1}^{k+2}) + i\,({}^{k+1}s_{k+2} - s_{k+2}^{k+1})\right\},$$

$$\text{(b)}\qquad Q_{k+1,k+2} = -\frac{\hbar c}{2e}\left\{\left(\frac{2mc}{\hbar}\mathfrak{M} + \operatorname{rot}\mathfrak{s}\right)_k - \frac{i}{c}({}^0\hat{s}_k - \hat{s}_k^0) - i\,({}^k\hat{s}_0 - \hat{s}_0^k)\right\},$$

$$\text{(6a)}\qquad P_{k4} = -\frac{\hbar c}{2e}\left\{\left(-\frac{2mc}{\hbar}\mathfrak{s} - \frac{\partial\hat{\mathfrak{M}}}{c\,\partial t} - \operatorname{rot}\mathfrak{M}\right)_k + i\,({}^k\Omega - \Omega^k)\right\},$$

$$\text{(b)}\qquad Q_{k4} = -\frac{\hbar c}{2e}\left\{-\operatorname{grad}_k\Omega - \frac{i}{c}({}^0M_{k0} - M_{k0}^0) - i({}^{k+1}M_{k,k+1} - M_{k,k+1}^{k+1}) + i({}^{k+2}M_{k+2,k} - M_{k+2,k}^{k+2})\right\},$$

$$\text{(7a)}\qquad P_{k+1,k+2,4} = -\frac{\hbar c}{2e}\left\{\left(\frac{\partial\mathfrak{M}}{c\,\partial t} - \operatorname{rot}\hat{\mathfrak{M}}\right)_k + i({}^k\hat{\Omega} - \hat{\Omega}^k)\right\},$$

$$\text{(b)}\qquad Q_{k+1,k+2,4} = -\frac{\hbar c}{2e}\left\{\left(-\frac{2mc}{\hbar}\hat{\mathfrak{s}} + \operatorname{grad}\hat{\Omega}\right)_k - \frac{i}{c}({}^0M_{k+1,k+2} - M_{k+1,k+2}^0)\right.$$
$$\left. + i({}^{k+1}M_{k+2,0} - M_{k+2,0}^{k+1}) - i({}^{k+2}M_{k+1,0} - M_{k+1,0}^{k+2})\right\},$$

$$\text{(8a)}\qquad P_{123} = -\frac{\hbar c}{2e}\left\{\frac{2mc}{\hbar}\hat{\Omega} - \frac{\partial\hat{s}_0}{c\,\partial t} - \operatorname{div}\hat{\mathfrak{s}}\right\},$$

$$\text{(b)}\qquad Q_{123} = -\frac{\hbar c}{2e}\left\{-\frac{i}{c}({}^0\hat{s}_0 - \hat{s}_0^0) - \sum_{k=1}^{3} i({}^k\hat{s}_k - \hat{s}_k^k)\right\}.$$

The right-hand sides of eqs. (1)-(8), except for (1b) and (8a), then contain the interpretable quantities, as well as the uninterpretable ones that were examined in II. One also finds these 32 relations in that form, but with a different notation in **W. Franz** [**2**]; in what follows, we shall apply the *algebraic* tools of I and II to them.

B. *Connection between the uninterpretable quantities and the potentials*

Two of the equations – namely, (1b) and (8a) – contain no potentials from the outset. In order to eliminate the potentials from the remaining ones, one can either:

A. Get $V = P_4 / \Omega$ from (3a) and $A_k = P_k / \Omega$ from (6a), or
B. Get $V = P_5 / \hat{\Omega}$ from (4a) and $A_k = P_{k+1,k+2,4} / \hat{\Omega}$ from (7a),

and substitute either solution into the remaining equations. It is preferable to substitute both solutions, since that will then yield a beautiful symmetry in the equations, in that half of them, after substituting solution A, will become identical to the other half, after substituting solution B.

That elimination will be carried out in the next paragraph, and the result will be that **all of the** *uninterpretable quantities can be removed from the 28 potential-free relations that arise by applying the algebraic identities of § 11, Part II.* One also sees from this that in **Dirac**'s theory, the electromagnetic potentials enter explicitly, e.g., into only the uninterpretable quantities:

$$\boxed{\begin{aligned} &\frac{i}{c}({}^0\Omega - \Omega^0) = \operatorname{div}\hat{\mathfrak{M}} - \frac{2mc}{\hbar}\mathfrak{s}_0 + \frac{2e}{\hbar c}\Omega V, && (9)\\ &\text{and} \\ &i({}^k\Omega - \Omega^k) = \left\{\frac{2mc}{\hbar}\mathfrak{s} + \frac{\partial\hat{\mathfrak{M}}}{c\,\partial t} + \operatorname{rot}\mathfrak{M}\right\} - \frac{2e}{\hbar c}\Omega A_k, && (10)\end{aligned}}$$

from (3a) and (6a), or into only the uninterpretable quantities:

$$\boxed{\begin{aligned} &\frac{i}{c}({}^0\hat{\Omega} - \hat{\Omega}^0) = -\operatorname{div}\mathfrak{M} + \frac{2e}{\hbar c}\hat{\Omega} V, && (11)\\ &i({}^k\hat{\Omega} - \hat{\Omega}^k) = \left\{-\frac{\partial\mathfrak{M}}{c\,\partial t} + \operatorname{rot}\hat{\mathfrak{M}}\right\} - \frac{2e}{\hbar c}\hat{\Omega} A_k, && (12)\end{aligned}}$$

from (4a) and (7a). (9) and (11) [(10) and (12), resp.] are connected by [II, eq. (93)] *algebraically*, and from [II, (87)-(92)], all of the remaining uninterpretable quantities are connected with (9) and (10).

By the way, eq. (10) corresponds to **Gordon**'s decomposition [**3**] $e\,\mathfrak{s}_k = s_k^e + s_k^p$ of the **Dirac** electrical current into a convection current:

$$s_k^e = -\frac{e\hbar}{2mc}\cdot i({}^k\Omega - \Omega^k) - \frac{e^2}{mc^2}\Omega A_k$$

and a polarization (magnetization, resp.) current:

$$s_k^p = \frac{e\hbar}{2mc}\left\{\frac{\partial\hat{\mathfrak{M}}}{c\,\partial t} + \operatorname{rot}\mathfrak{M}_k\right\}_k .$$

Our terminology "uninterpretable quantities" shall not call into question the many possible interpretations of those quantities, but shall mainly be a collective term for the quantities that cannot be represented as the differential quotient of a density.

One will obtain the remaining quantities that can be calculated from (9) and (10) using [II, eqs. (87-(93)] symmetrically from (9)-(12) when one constructs the quantities that were defined in [II, eq. (86)]:

$$(13)\qquad \frac{1}{c}U_0 = \Omega\,\mathrm{div}\,\hat{\mathfrak{M}} - \hat{\Omega}\,\mathrm{div}\,\mathfrak{M} - \frac{2mc}{\hbar}s_0\,\Omega + \frac{2e}{\hbar c}(\Omega^2 + \hat{\Omega}^2)\,V,$$

$$(14)\qquad U_k = \left\{\frac{2mc}{\hbar}\Omega\,\mathfrak{s} + \Omega\frac{\partial\hat{\mathfrak{M}}}{c\,\partial t} - \hat{\Omega}\frac{\partial\mathfrak{M}}{c\,\partial t} + \Omega\,\mathrm{rot}\,\mathfrak{M} + \hat{\Omega}\,\mathrm{rot}\,\hat{\mathfrak{M}}\right\}_k - \frac{2e}{\hbar c}(\Omega^2 + \hat{\Omega}^2)A_k\,,$$

and substitute these quantities in [II, eqs. (87)-(92)].

§ 2. Presentation of the reality relations in vector form

We eliminate the electromagnetic potentials from the 32 equations (1a)-(bb) and then obtain 28 potential-free reality relations. Combinations of uninterpretable quantities will then enter into them, each of which can be omitted completely with the help of the algebraic identities of Part I and II.

A. The electromagnetic potentials are contained most simply in eqs. (3a) and (6a) [(4a) and (7a), resp.], which is why we referred to those two cases as solutions A and B, resp. Four potential-free relations will arise by setting the two solutions equal to each other.

The scalar relation:

$$\text{I.}\qquad \boxed{\Omega\,\mathrm{div}\,\mathfrak{M} + \hat{\Omega}\,\mathrm{div}\,\hat{\mathfrak{M}} - \left(\mathfrak{s}, \frac{\partial\hat{\mathfrak{s}}}{c\,\partial t}\right) + s_0\left(\frac{\partial\hat{s}_0}{c\,\partial t} - \frac{2mc}{\hbar}\hat{\Omega}\right) = 0}$$

will follow by setting the V in (3a) equal to the V in (4a).

Proof. $\Omega P_5 - \hat{\Omega}P_4 = 0$. Substituting the values (3a) and (4a) for P_4 and P_5, resp., gives:

$$\Omega\,\mathrm{div}\,\mathfrak{M} + \hat{\Omega}\,\mathrm{div}\,\hat{\mathfrak{M}} - \hat{\Omega}\cdot\frac{2mc}{\hbar}s_0 = \hat{\Omega}\cdot({}^0\Omega - \Omega^0) - \Omega\cdot\frac{i}{c}({}^0\hat{\Omega} - \hat{\Omega}^0)\,,$$

$$= -s_0\frac{\partial\hat{s}_0}{c\,\partial t} + \left(\mathfrak{s}, \frac{\partial\hat{\mathfrak{s}}}{c\,\partial t}\right), \qquad \text{from [II, § 11, eq. (2).}$$

The vector relation:

$$\text{II.-IV.}\qquad \boxed{\begin{gathered}\Omega\left(\frac{\partial\mathfrak{M}}{c\,\partial t} - \mathrm{rot}\,\hat{\mathfrak{M}}\right) + \hat{\Omega}\left(\frac{\partial\hat{\mathfrak{M}}}{c\,\partial t} + \mathrm{rot}\,\mathfrak{M} + \frac{2mc}{\hbar}\mathfrak{s}\right)\\ = \hat{s}_0\,\mathrm{grad}\,s_0 - [\hat{\mathfrak{s}}, \mathrm{rot}\,\mathfrak{s}] - (\hat{\mathfrak{s}}, \mathrm{grad})\mathfrak{s}\end{gathered}}$$

follows by setting the A_k in (6a) equal to the A_k in (7a).

Proof. $\Omega\, P_{k+1,k+2,4} - \hat{\Omega}\, P_{k4} = 0$. Substituting the values (7a) and (6a) for $P_{k+1,k+2,4}$ and P_{k4}, resp., will yield:

$$\left\{\Omega\left(\frac{\partial\mathfrak{M}}{c\,\partial t} - \text{rot}\,\hat{\mathfrak{M}}\right) + \hat{\Omega}\left(\frac{\partial\hat{\mathfrak{M}}}{c\,\partial t} + \text{rot}\,\mathfrak{M} + \frac{2mc}{\hbar}\,\mathfrak{s}\right)\right\}_k$$
$$= \hat{\Omega}\cdot i\,({}^k\Omega - \Omega^k) - \Omega\cdot i\,({}^k\hat{\Omega} - \hat{\Omega}^k)$$
$$= \hat{s}_0\ \text{grad}_k\ s_0 - \left(\hat{\mathfrak{s}}, \frac{\partial\mathfrak{s}}{\partial x_k}\right), \quad \text{from [II, § 11, eq. (2)]}.$$

II.-IV. will follow when the vector formula [II, eq. (82)] is applied to $\left(\hat{\mathfrak{s}}, \dfrac{\partial\mathfrak{s}}{\partial x_k}\right)$.

B. The symmetry of the system of equations (1a)-(8b) will become clear when one simultaneously substitutes the two solutions A and B in the remaining equations. Some equations will yield the same relation by substituting the two solutions, so they will be *insensitive* to the exchange of the two solutions A and B. One can arrange the remaining ones into *pairs* for which the substitution of the A-solution in one equation of the pair in question will yield the same relations that arises by the substitution of the B-solution in the other equation, and conversely.

a) The insensitive equations are (1a) and (8b). If one substitutes solutions A and B in eq. (1a) then, under the assumption that $\Omega \neq 0$, $\hat{\Omega} \neq 0$, one will obtain the "anti-continuity equation":

V. (VI.)
$$\boxed{\frac{\partial\hat{s}_0}{c\,\partial t} + \text{div}\,\hat{\mathfrak{s}} - \frac{2mc}{\hbar}\,\hat{\Omega} = 0.}$$

in both cases.

Proof:

With A, it follows that:

$$P_4\, s_0 - \sum_{k=1}^{3} P_{k4}\, s_k = \Omega\, P_0\,.$$

Substituting P_4, P_{k4}, P_0 gives:

$$s_0\ \text{div}\,\hat{\mathfrak{M}} + \left(\mathfrak{s}, \frac{\partial\hat{\mathfrak{M}}}{c\,\partial t}\right) + (\mathfrak{s}, \text{rot}\,\mathfrak{M}) - \frac{2mc}{\hbar}\,\hat{\Omega}^2$$
$$= \frac{i}{c}({}^0\Omega - \Omega^0)\, s_0 - \frac{i}{c}({}^0 s_0 - s_0^0)\,\Omega$$

With B, it follows that:

$$P_5\, s_0 - \sum_{k=1}^{3} P_{k+1,k+2,4}\, s_k = \hat{\Omega}\, P_0\,.$$

Substituting P_5, $P_{k+1,k\_2,4}$, P_0 gives:

$$\frac{2mc}{\hbar}\,\Omega\hat{\Omega} - s_0\ \text{div}\,\mathfrak{M} - \left(\mathfrak{s}, \frac{\partial\mathfrak{M}}{c\,\partial t}\right)$$
$$+ (\mathfrak{s}, \text{rot}\,\hat{\mathfrak{M}})$$
$$= s_0\cdot\frac{i}{c}({}^0\hat{\Omega} - \hat{\Omega}^0) - \hat{\Omega}\cdot\frac{i}{c}({}^0 s_0 - s_0^0)$$

$$+\sum_{k=1}^{3}\{i\,({}^k\Omega-\Omega^k)\,s_k-i\,({}^ks_k-s_k^k)\,\Omega\},$$

which will become:

$$=\left(\mathfrak{s},\frac{\partial\hat{\mathfrak{M}}}{c\,\partial t}\right)-\hat{\Omega}\frac{\partial\hat{s}_0}{c\,\partial t}+s_0\,\mathrm{div}\,\hat{\mathfrak{M}}+\hat{\Omega}\,\mathrm{div}\,\hat{\mathfrak{s}}-\sum_{k=1}^{3}\left[\mathfrak{s},\frac{\partial\mathfrak{M}}{\partial x_k}\right]_k$$

with the use of the identities [II, §11, eq. (7) and (8)]. An application of the vector formula:

$$(15)\qquad -\sum_{k=1}^{3}\left[\mathfrak{A},\frac{\partial\mathfrak{B}}{\partial x_k}\right]_k=(\mathfrak{A},\,\mathrm{rot}\,\mathfrak{B})$$

to the last term will leave:

$$\hat{\Omega}\left\{\frac{\partial\hat{s}_0}{c\,\partial t}+\mathrm{div}\,\hat{\mathfrak{s}}-\frac{2mc}{\hbar}\hat{\Omega}\right\}=0.$$

$$+\sum_{k=1}^{3}\{s_k\cdot i({}^k\hat{\Omega}-\hat{\Omega}^k)-\hat{\Omega}\cdot i\,({}^ks_k-s_k^k)\},$$

which will become:

$$=\left(-\mathfrak{s},\frac{\partial\mathfrak{M}}{c\,\partial t}\right)+\Omega\frac{\partial\hat{s}_0}{c\,\partial t}-s_0\,\mathrm{div}\,\mathfrak{M}+\Omega\,\mathrm{div}\,\hat{\mathfrak{s}}-\sum_{k=1}^{3}\left[\mathfrak{s},\frac{\partial\hat{\mathfrak{M}}}{\partial x_k}\right]_k$$

with the use of the identities [II, §11, eq. (5) and (6)]. Analogous to A, with (15), one will have:

$$-\sum_{k=1}^{3}\left[\mathfrak{s},\frac{\partial\hat{\mathfrak{M}}}{\partial x_k}\right]_k=(\mathfrak{s},\,\mathrm{rot}\,\hat{\mathfrak{M}}),$$

from which, all that will remain will be:

$$\Omega\left\{\frac{\partial\hat{s}_0}{c\,\partial t}+\mathrm{div}\,\hat{\mathfrak{s}}-\frac{2mc}{\hbar}\hat{\Omega}\right\}=0.$$

Eq. (1a) then yields the same relation that is already present from the outset in (8a) free of potentials. The VI that we have put into brackets shall count as eq. (8a), with our enumeration the 28 relations.

If one substitutes the solutions A and B in eq. (8b) then, under the assumption that $\Omega\neq 0$, $\hat{\Omega}\neq 0$, one will obtain the "continuity equation":

VII (VIII, resp.) $$\boxed{\frac{\partial s_0}{c\,\partial t}+\mathrm{div}\,\mathfrak{s}=0.}$$

Proof:

With A, it follows that:

$$P_4\,\hat{s}_0-\sum_{k=1}^{3}P_{k4}\,\hat{s}_k=\Omega\,Q_{123}\,.$$

With consideration given to [I, (17)], substitution of P_4, P_{k4}, Q_{123} will yield:

With B, it follows that:

$$P_5\,\hat{s}_0-\sum_{k=1}^{3}P_{k+1,k+2,4}\,\hat{s}_k=\hat{\Omega}\,Q_{123}\,.$$

Substitution of P_5, $P_{k+1,k+2,4}$, Q_{123} will yield:

$$\hat{s}_0 \operatorname{div} \hat{\mathfrak{M}} + \left(\hat{\mathfrak{s}}, \frac{\partial \hat{\mathfrak{M}}}{c\,\partial t}\right) + (\hat{\mathfrak{s}}, \operatorname{rot} \mathfrak{M})$$
$$= \frac{i}{c}({}^0\Omega - \Omega^0)\,\hat{s}_0 - \frac{i}{c}({}^0\hat{s}_0 - \hat{s}_0^0)\,\Omega$$
$$+ \sum_{k=1}^{3}\left\{ i\,({}^k\Omega - \Omega^k)\,\hat{s}_k - i\,({}^k\hat{s}_k - \hat{s}_k^k)\,\Omega \right\},$$

and with the use of equations [II, § 11, eqs. (14) and (15)], this will become:

$$= \left(\hat{\mathfrak{s}}, \frac{\partial \hat{\mathfrak{M}}}{c\,\partial t}\right) - \hat{\Omega}\left(\frac{\partial s_0}{c\,\partial t} + \operatorname{div} \hat{\mathfrak{s}}\right) + \hat{s}_0 \operatorname{div} \hat{\mathfrak{M}}$$
$$- \sum_{k=1}^{3}\left[\hat{\mathfrak{s}}, \frac{\partial \mathfrak{M}}{\partial x_k}\right]_k .$$

If one applies the vector formula (15) to the last term then what will remain is:

$$\hat{\Omega}\left(\frac{\partial s_0}{c\,\partial t} + \operatorname{div} \mathfrak{s}\right) = 0.$$

$$-\hat{s}_0 \operatorname{div} \mathfrak{M} - \left(\hat{\mathfrak{s}}, \frac{\partial \mathfrak{M}}{c\,\partial t}\right) + (\hat{\mathfrak{s}}, \operatorname{rot} \hat{\mathfrak{M}})$$
$$= \hat{s}_0 \cdot \frac{i}{c}({}^0\hat{\Omega} - \hat{\Omega}^0) - \hat{\Omega} \cdot \frac{i}{c}({}^0\hat{s}_0 - \hat{s}_0^0)$$
$$+ \sum_{k=1}^{3}\left\{ \hat{s}_k \cdot i\,({}^k\hat{\Omega} - \hat{\Omega}^k) - \hat{\Omega} \cdot i\,({}^k\hat{s}_k - \hat{s}_k^k) \right\},$$

and with the use of equations [II, § 11, eqs. (12) and (13)], this will become:

$$= -\left(\hat{\mathfrak{s}}, \frac{\partial \mathfrak{M}}{c\,\partial t}\right) + \Omega\left(\frac{\partial s_0}{c\,\partial t} + \operatorname{div} \mathfrak{s}\right) - \hat{s}_0 \operatorname{div} \mathfrak{M}$$
$$- \sum_{k=1}^{3}\left[\hat{\mathfrak{s}}, \frac{\partial \hat{\mathfrak{M}}}{\partial x_k}\right]_k .$$

If one applies the vector formula (15) to the last term then what will remain is:

$$\Omega\left(\frac{\partial s_0}{c\,\partial t} + \operatorname{div} \mathfrak{s}\right) = 0.$$

Eq. (8b) then yields the same relation that already exists in (1b) from the outset, free from potentials. The VIII that is placed in brackets shall count as eq. (1b) with our enumeration of the 28 relations.

b) The first pair of equations are that linked with each other by the solutions A and B is (3b), (4b). Introducing solution A into eq. (3b) or solution B into eq. (4b) will lead to the same combination of an anti-continuity equation and a continuity equation. Introducing the mutually-permuted solutions will yield a new scalar reality relation:

IXA (XB)

$$\boxed{s_0\left(\frac{\partial s_0}{c\,\partial t} + \operatorname{div} \mathfrak{s}\right) - \hat{s}_0\left(\frac{\partial \hat{s}_0}{c\,\partial t} + \operatorname{div} \hat{\mathfrak{s}} - \frac{2mc}{\hbar}\hat{\Omega}\right) = 0.}$$

IXB (XA)

$$\boxed{\begin{aligned} &\left(\hat{\mathfrak{M}}, \frac{\partial \mathfrak{M}}{c\,\partial t} - \operatorname{rot} \hat{\mathfrak{M}}\right) - \left(\mathfrak{M}, \frac{\partial \hat{\mathfrak{M}}}{c\,\partial t} + \operatorname{rot} \mathfrak{M}\right) \\ &\qquad = \hat{\Omega}\frac{\partial \Omega}{c\,\partial t} - \Omega\frac{\partial \hat{\Omega}}{c\,\partial t} + (\mathfrak{s}, \operatorname{rot} \mathfrak{s}) - (\hat{\mathfrak{s}}, \operatorname{rot} \hat{\mathfrak{s}}).\end{aligned}}$$

Proof:

When solution A is substituted in eq. (3b), that will yield:

$$\sum_{k=1}^{3} P_{k4} M_{k0} = \Omega\, Q_4 \,.$$

With the values for P_{k4} and Q_4, that will give:

$$\frac{2mc}{\hbar}\hat{s}_0\,\hat{\Omega} + \frac{1}{2}\frac{\partial}{c\,\partial t}(\Omega^2 + \hat{\mathfrak{M}}^2) + (\hat{\mathfrak{M}}, \operatorname{rot}\hat{\mathfrak{M}})$$

$$= \left\{\sum_{k=1}^{3} M_{k0}\cdot i\,({}^k\Omega - \Omega^k) - \Omega\cdot i\,({}^k M_{k0} - M_{k0}^k)\right\},$$

from which the equation:

$$\frac{2mc}{\hbar}\hat{s}_0\,\hat{\Omega} + s_0\frac{\partial s_0}{c\,\partial t} - \hat{s}_0\frac{\partial \hat{s}_0}{c\,\partial t} + (\hat{\mathfrak{M}}, \operatorname{rot}\mathfrak{M})$$

$$= -\,s_0 \operatorname{div}\mathfrak{s} + \hat{s}_0\operatorname{div}\hat{\mathfrak{s}} - \sum_{k=1}^{3}\left[\hat{\mathfrak{M}}, \frac{\partial\mathfrak{M}}{\partial x_k}\right]_k$$

will arise with the help of [II, § 11, eq. (20)] and the identity [I, (21)]. An application of the vector formula (15) to the last term in this equation will lead to IXA.

When solution B is substituted in eq. (4b), that will yield:

$$\sum_{k=1}^{3} P_{k+1,k+2,4} M_{k+1,k+2} = \hat{\Omega}\, Q_5 \,.$$

With the values for $P_{k+1,k+2,4}$ and Q_5, that will give:

$$\left(\mathfrak{M}, \frac{\partial\mathfrak{M}}{c\,\partial t}\right) + \hat{\Omega}\frac{\partial\hat{\Omega}}{c\,\partial t} - (\mathfrak{M}, \operatorname{rot}\hat{\mathfrak{M}}) + \frac{2mc}{\hbar}\hat{\Omega}\,\hat{s}_0$$

$$= \left\{\sum_{k=1}^{3}\hat{\Omega}\cdot i\,({}^k M_{k+1,k+2} - M_{k+1,k+2}^k) - M_{k+1,k+2}\cdot i\,({}^k\hat{\Omega} - \hat{\Omega}^k)\right\},$$

which will become:

$$= -\,(\mathfrak{M}, \operatorname{rot}\hat{\mathfrak{M}}) - s_0 \operatorname{div}\mathfrak{s} + \hat{s}_0\operatorname{div}\hat{\mathfrak{s}}$$

with the help of [II, § 11, eq. (21). An application of the identity [I, (20)] will then lead to equation XB, which is identical to IXA.

3. When solution B is introduced into eq. (3b), that will yield:

$$\sum_{k=1}^{3} P_{k+1,k+2,4} M_{k0} = \hat{\Omega}\, Q_4 \,.$$

With the values for $P_{k+1,k+2,4}$ and Q_4, that will give:

$$\left(\frac{\partial\mathfrak{M}}{c\,\partial t}, \hat{\mathfrak{M}}\right) - \hat{\Omega}\,\frac{\partial\Omega}{c\,\partial t} - (\hat{\mathfrak{M}}, \operatorname{rot}\hat{\mathfrak{M}})$$

4. When solution A is introduced into eq. (4b), that will yield:

$$\sum_{k=1}^{3} P_{k4} M_{k+1,k+2} = \Omega\, Q_5 \,.$$

With the values for P_{k4} and Q_5, and consideration given to the identity [I, (15)], that will give:

$$\Omega\,\frac{\partial\hat{\Omega}}{c\,\partial t} - \left(\mathfrak{M}, \frac{\partial\hat{\mathfrak{M}}}{c\,\partial t}\right) - (\mathfrak{M}, \operatorname{rot}\mathfrak{M})$$

$$= \left\{\sum_{k=1}^{3} \hat{\Omega}\cdot i\,({}^k M_{k0} - M_{k0}^k) - M_{k0}\cdot i\,({}^k\hat{\Omega} - \hat{\Omega}^k)\right\},$$

and with the help of [II, § 11, eq. (44)], that will become:

$$= \tfrac{1}{2}\{(\mathfrak{M}, \text{rot}\,\mathfrak{M}) - (\hat{\mathfrak{M}}, \text{rot}\,\hat{\mathfrak{M}}) + (\mathfrak{s}, \text{rot}\,\mathfrak{s}) - (\hat{\mathfrak{s}}, \text{rot}\,\hat{\mathfrak{s}})\}\}.$$

The scalar relation IXB will emerge from this with consideration given to the identity [I, (22)]. It is identical with XA.

$$= \left\{\sum_{k=1}^{3} \Omega\cdot i\,({}^k M_{k+1,k+2} - M_{k+1,k+2}^k) - M_{k+1,k+2}\cdot i\,({}^k\Omega - \Omega^k)\right\},$$

and with the help of [II, § 11, eq. (43)] and the vector formula (15), that will become:

$$= \tfrac{1}{2}\{-(\mathfrak{M}, \text{rot}\,\mathfrak{M}) + (\hat{\mathfrak{M}}, \text{rot}\,\hat{\mathfrak{M}}) + (\mathfrak{s}, \text{rot}\,\mathfrak{s}) - (\hat{\mathfrak{s}}, \text{rot}\,\hat{\mathfrak{s}})\}\}.$$

XA will follow from this with the help of [I, (22)].

c) The second pair of equations that are coupled to each other by solutions A and B, is (2a), (5a), which will lead to two vector relations. Substitution of A in (2a) and B in (5a) will yield:

$$\left.\begin{array}{l}\text{XI - XIIIA}\\ \text{XIV - XVIB}\end{array}\right\}\quad \boxed{\begin{aligned}&\left[\mathfrak{s}, \frac{\partial\mathfrak{M}}{c\,\partial t} - \text{rot}\,\hat{\mathfrak{M}}\right] = -\hat{\Omega}\left(\frac{\partial\hat{\mathfrak{s}}}{c\,\partial t} + \text{grad}\,\hat{s}_0\right)\\ &-\hat{\mathfrak{M}}\,\text{div}\,\mathfrak{s} + (\hat{\mathfrak{M}}\,\text{grad})\mathfrak{s} - [\mathfrak{M}, \text{grad}\,s_0] + [\hat{\mathfrak{s}}, \text{grad}\,\Omega].\end{aligned}}$$

Substituting B in (2a) and A in (6b) will yield:

$$\left.\begin{array}{l}\text{XI - XIIIB}\\ \text{XIV - XVIA}\end{array}\right\}\quad \boxed{\begin{aligned}&\left[\mathfrak{s}, \frac{\partial\hat{\mathfrak{M}}}{c\,\partial t} + \text{rot}\,\mathfrak{M}\right] = \Omega\left(\frac{\partial\hat{\mathfrak{s}}}{c\,\partial t} + \text{grad}\,\hat{s}_0\right)\\ &+\mathfrak{M}\,\text{div}\,\mathfrak{s} - (\mathfrak{M}\,\text{grad})\mathfrak{s} - [\hat{\mathfrak{M}}, \text{grad}\,s_0] + [\hat{\mathfrak{s}}, \text{grad}\,\hat{\Omega}].\end{aligned}}$$

Proof:

1. Introducing solution A into eq. (2a) will give:

$$-P_4\,s_k + P_{k4}\,s_0 = \Omega\,P_k\,.$$

With the values for P_4, P_{k4}, P_k, that will yield:

$$\left\{\mathfrak{s}\,\text{div}\,\hat{\mathfrak{M}} + s_0\frac{\partial\hat{\mathfrak{M}}}{c\,\partial t} + s_0\,\text{rot}\,\mathfrak{M} - \Omega\,\text{rot}\,\hat{\mathfrak{s}}\right\}_k$$

2. Introducing solution B into eq. (5a) will give:

$$P_{k+2,k,4}\,s_{k+2} - P_{k,k+1,4}\,s_{k+1} = \hat{\Omega}\,P_{k+1,k+2}\,.$$

With the values for $P_{k+2,k,4}$, $P_{k,k+1,4}$, $P_{k+1,k+2}$, that will yield:

$$\left\{\left[\mathfrak{s}, \frac{\partial\hat{\mathfrak{M}}}{c\,\partial t} - \text{rot}\,\hat{\mathfrak{M}}\right] + \hat{\Omega}\left(\frac{\partial\hat{\mathfrak{s}}}{c\,\partial t} + \text{rot}\,\hat{s}_0\right)\right\}_k$$

$$= s_k \cdot \frac{i}{c}({}^0\Omega - \Omega^0) - \Omega \cdot \frac{i}{c}({}^0 s_k - s_k^0)$$
$$+ \Omega \cdot i\,({}^0 s_k - s_k^0) - s_0 \cdot i\,({}^k\Omega - \Omega^k),$$

and from [II, § 11, eq. (7) and (8)], that will:

$$= \left(\mathfrak{s}, \frac{\partial\hat{\mathfrak{M}}}{\partial x_k}\right) - \hat{\Omega}\frac{\partial\hat{s}_0}{\partial x_k}$$
$$- \left\{\hat{\Omega}\frac{\partial\mathfrak{s}}{c\,\partial t} - s_0\,\frac{\partial\hat{\mathfrak{M}}}{c\,\partial t} + \left[\mathfrak{s}, \frac{\partial\mathfrak{M}}{c\,\partial t}\right]\right\}_k .$$

An application of the vector formula [II, (82)] for the vectors $\mathfrak{s}$ and $\hat{\mathfrak{M}}$ lets one convert this equation into:

$$\left[\mathfrak{s}, \frac{\partial\mathfrak{M}}{c\,\partial t} - \mathrm{rot}\,\hat{\mathfrak{M}}\right]$$

$$= -\hat{\Omega}\left(\frac{\partial\hat{\mathfrak{s}}}{c\,\partial t} + \mathrm{grad}\,\hat{s}_0\right) - \hat{\mathfrak{s}}\,\mathrm{div}\,\hat{\mathfrak{M}}$$
$$- s_0\,\mathrm{rot}\,\mathfrak{M} + \Omega\,\mathrm{rot}\,\hat{\mathfrak{s}} + (\mathfrak{s}\,\mathrm{grad})\,\hat{\mathfrak{M}} .$$

If one draws upon the rot of the identity [I, (21)] in order to convert this then that will produce XI-XIIIA.

3. Introducing solution B into eq. (2a) will yield:

$$- P_5\, s_k + P_{k+1,k+2,4}\, s_5 = \hat{\Omega}\, P_k .$$

With the values of P_5, $P_{k+1,k+2,4}$, P_k, that will give:

$$\left\{\mathfrak{s}\,\mathrm{div}\,\mathfrak{M} + s_0\,\frac{\partial\mathfrak{M}}{c\,\partial t} - s_0\,\mathrm{rot}\,\hat{\mathfrak{M}} + \hat{\Omega}\,\mathrm{rot}\,\hat{\mathfrak{s}}\right\}_k$$
$$= \frac{i}{c}({}^0 s_k - s_k^0)\,\hat{\Omega} - \frac{i}{c}({}^0\hat{\Omega} - \hat{\Omega}^0)\, s_k$$

$$= i\,({}^{k+1}\hat{\Omega} - \hat{\Omega}^{k+1})\, s_{k+2} - \hat{\Omega}\cdot i\,({}^{k+1}s_{k+2} - s_{k+2}^{k+1})$$
$$- \left\{i\,({}^{k+2}\hat{\Omega} - \hat{\Omega}^{k+2})\, s_{k+1} - \hat{\Omega}\cdot i({}^{k+2}s_{k+1} - s_{k+1}^{k+2})\right\},$$

and with a double application of [II, § 11, eq. (6)], that will:

$$= \left[\hat{\mathfrak{M}}, \frac{\partial\mathfrak{s}}{\partial x_{k+2}}\right]_{k+1} - \left[\hat{\mathfrak{M}}, \frac{\partial\mathfrak{s}}{\partial x_{k+1}}\right]_{k+2}$$
$$- [\mathfrak{M}, \mathrm{grad}\, s_0]_k + [\hat{\mathfrak{s}}, \mathrm{grad}\,\Omega]_k .$$

Finally, an application of the vector formula:

$$(16)\quad \begin{cases} \left[\mathfrak{A}, \dfrac{\partial\mathfrak{B}}{\partial x_{k+1}}\right]_{k+2} - \left[\mathfrak{A}, \dfrac{\partial\mathfrak{B}}{\partial x_{k+2}}\right]_{k+1} \\ \qquad = A_k\,\mathrm{div}\,\mathfrak{B} - (\mathfrak{A}\,\mathrm{grad})\,\mathfrak{B}_k \end{cases}$$

to the vectors $\hat{\mathfrak{M}}$ and $\mathfrak{s}$ will lead to XIV-XVIB.

4. Introducing solution A into eq. (5a) will yield:

$$- P_{k+1,4}\, s_{k+2} - P_{k+2,4}\, s_{k+1} = \Omega\, P_{k+1,k+2} .$$

With the values of $P_{k+1,4}$, , $P_{k+2,4}$, $P_{k+1,k+2}$, that will give:

$$\left\{\left[\mathfrak{s}, \frac{\partial\hat{\mathfrak{M}}}{c\,\partial t} + \mathrm{rot}\,\mathfrak{M}\right] - \Omega\left(\frac{\partial\hat{\mathfrak{s}}}{c\,\partial t} + \mathrm{grad}\,\hat{s}_0\right)\right\}_k$$
$$= \Omega\cdot i\,({}^{k+1}s_{k+2} - s_{k+2}^{k+1}) - i({}^{k+1}\Omega - \Omega^{k+1})\, s_{k+2}$$

$$+\ i\,({}^k s_0 - s_0^k)\,\hat{\Omega} - i\,({}^k\hat{\Omega} - \hat{\Omega}^k)\,s_0\,,$$

and from [II, § 11, eqs. (5) and (6)], that:

$$= \left\{\left[\mathfrak{s}, \frac{\partial\hat{\mathfrak{M}}}{c\,\partial t}\right] + s_0\frac{\partial\mathfrak{M}}{c\,\partial t} - \Omega\frac{\partial\hat{\mathfrak{s}}}{c\,\partial t} - \Omega\,\mathrm{grad}\,\hat{s}_0\right\}_k + \left(\mathfrak{s}, \frac{\partial\mathfrak{M}}{\partial x_k}\right).$$

An application of the vector formula [II, (82)] to the vectors $\mathfrak{s}$ and $\mathfrak{M}$ will yield the following way of writing that:

$$\left[\mathfrak{s}, \frac{\partial\hat{\mathfrak{M}}}{c\,\partial t} + \mathrm{rot}\,\mathfrak{M}\right] - \Omega\left(\frac{\partial\hat{\mathfrak{s}}}{c\,\partial t} + \mathrm{grad}\,\hat{s}_0\right)$$
$$= \mathfrak{s}\,\mathrm{div}\,\mathfrak{M} - s_0\,\mathrm{rot}\,\hat{\mathfrak{M}} + \hat{\Omega}\,\mathrm{rot}\,\hat{\mathfrak{s}} - (\mathfrak{s}\,\mathrm{grad})\,\mathfrak{M},$$

from which, the relation XI-XIIIB will emerge with a conversion that uses the rot of [I, (29)].

$$-\left\{\Omega\cdot i\,({}^{k+2}s_{k+1} - s_{k+1}^{k+2}) - i({}^{k+2}\Omega - \Omega^{k+2})\,s_{k+1}\right\},$$

and with a double application of [II, § 11, eq. (8)], that will:

$$= \left[\mathfrak{M}, \frac{\partial\mathfrak{s}}{\partial x_{k+1}}\right]_{k+2} - \left[\mathfrak{M}, \frac{\partial\mathfrak{s}}{\partial x_{k+2}}\right]_{k+1}$$
$$-\ [\hat{\mathfrak{M}}, \mathrm{grad}\,s_0]_k - [\hat{\mathfrak{s}}, \mathrm{grad}\,\hat{\Omega}]_k\,.$$

An application of the vector formula (16) to the vectors $\mathfrak{M}$ and $\mathfrak{s}$ will let one rewrite this equation in the form XIV-XVIA.

d) The third pair of equations that are linked to each other by the solutions A and B is (2b), (5b), which likewise leads to two vector relations. Substituting A in eq. (5b) or B in eq. (2b) will yield:

$$\left.\begin{array}{l}\text{XVII - XIXA}\\ \text{XX - XXIIB}\end{array}\right\}\ \boxed{\begin{array}{c}\left[\hat{\mathfrak{s}}, \dfrac{\partial\mathfrak{M}}{c\,\partial t} - \mathrm{rot}\,\hat{\mathfrak{M}}\right] = -\,\hat{\Omega}\left(\dfrac{\partial\mathfrak{s}}{c\,\partial t} + \mathrm{grad}\ s_0\right) + \hat{\mathfrak{M}}\left(\dfrac{2mc}{\hbar}\hat{\Omega} - \mathrm{div}\ \hat{\mathfrak{s}}\right)\\ -[\mathfrak{M}, \mathrm{grad}\ \hat{s}_0] + [\hat{\mathfrak{s}}, \mathrm{grad}\,\Omega] + (\hat{\mathfrak{M}}\,\mathrm{grad})\hat{\mathfrak{s}}.\end{array}}$$

Substituting B in eq, (5b) or A in eq. (2b) will yield:

$$\left.\begin{array}{l}\text{XVII - XIXB}\\ \text{XX - XXIIA}\end{array}\right\}\ \boxed{\begin{array}{c}\left[\hat{\mathfrak{s}}, \dfrac{\partial\hat{\mathfrak{M}}}{c\,\partial t} + \mathrm{rot}\,\mathfrak{M}\right] = \Omega\left(\dfrac{\partial\mathfrak{s}}{c\,\partial t} + \mathrm{grad}\ s_0\right) + \mathfrak{M}\left(\dfrac{2mc}{\hbar}\hat{\Omega} - \mathrm{div}\ \hat{\mathfrak{s}}\right)\\ -[\hat{\mathfrak{M}}, \mathrm{grad}\ \hat{s}_0] + [\mathfrak{s}, \mathrm{grad}\,\hat{\Omega}] - (\mathfrak{M}\,\mathrm{grad})\hat{\mathfrak{s}}.\end{array}}$$

Proof:

1. Introducing solution A into eq. (5b) will yield:

$$P_4\,\hat{s}_k - P_{k4}\,\hat{s}_0 = \Omega\, Q_{k+1,k+2}\,,$$

and with the values for P_4, P_{k4}, $Q_{k+1,k+2}$, that will give:

$$\left\{\hat{\mathfrak{s}}\,\mathrm{div}\,\hat{\mathfrak{M}} + \hat{s}_0\left(\frac{\partial\hat{\mathfrak{M}}}{c\,\partial t} + \mathrm{rot}\,\hat{\mathfrak{M}}\right) - \Omega\,\mathrm{rot}\,\mathfrak{s} + \frac{2mc}{\hbar}(\Omega\,\mathfrak{M} - s_0\hat{\mathfrak{s}} + \hat{s}_0\mathfrak{s})\right\}_k$$

$$= \hat{s}_k\cdot\frac{i}{c}({}^0\Omega - \Omega^0) - \Omega\cdot\frac{i}{c}({}^0\hat{s}_k - \hat{s}_k^0) + \hat{s}_0\cdot\frac{i}{c}({}^k\Omega - \Omega^k) - \Omega\cdot i({}^k\hat{s}_0 - \hat{s}_0^k)\,.$$

[I, (35)] and [II, § 11, eq. (14) and (15)] make the following rewriting possible:

$$\left[\hat{\mathfrak{s}},\frac{\partial\mathfrak{M}}{c\,\partial t} - \mathrm{rot}\,\hat{\mathfrak{M}}\right] + \hat{\Omega}\left(\frac{\partial\mathfrak{s}}{c\,\partial t} + \mathrm{grad}\,\hat{s}_0\right) - \frac{2mc}{\hbar}\,\hat{\Omega}\,\hat{\mathfrak{M}}$$

$$= (\hat{\mathfrak{s}}\,\mathrm{grad})\,\hat{\mathfrak{M}} - \hat{\mathfrak{s}}\,\mathrm{div}\,\hat{\mathfrak{M}} - \hat{s}_0\mathrm{rot}\,\mathfrak{M} + \Omega\,\mathrm{rot}\,\mathfrak{s}\,,$$

and with the rot of [I, (32)], the right-hand side will become:

$$= (\hat{\mathfrak{M}}\,\mathrm{grad})\hat{\mathfrak{s}} - \hat{\mathfrak{M}}\,\mathrm{div}\,\hat{\mathfrak{s}} + [\mathfrak{s},\mathrm{grad}\,\Omega] - [\hat{\mathfrak{M}},\mathrm{grad}\,\hat{s}_0]\,,$$

which is (XVII-XIXA).

2. Introducing solution B into eq. (2b) will yield:

$$P_{k+2,k,4}\,\hat{s}_{k+2} - P_{k,k+1,4}\,\hat{s}_{k+1} = \hat{\Omega}\,Q_k\,,$$

and with the values for $P_{k+2,k,4}$, $P_{k,k+1,4}$, Q_k, that will give:

$$\left\{\left[\hat{\mathfrak{s}},\frac{\partial\hat{\mathfrak{M}}}{c\,\partial t} + \mathrm{rot}\,\hat{\mathfrak{M}}\right] + \frac{2mc}{\hbar}\hat{\Omega}\,\hat{\mathfrak{M}} - \hat{\Omega}\frac{\partial\mathfrak{s}}{c\,\partial t} - \hat{\Omega}\,\mathrm{grad}\,s_0\right\}_k$$

$$= \Omega\cdot i({}^{k+1}\hat{s}_{k+2} - \hat{s}_{k+2}^{k+1}) - \hat{s}_{k+2}\cdot i({}^{k+1}\hat{\Omega} - \hat{\Omega}^{k+1}) - \left\{\hat{\Omega}\cdot i({}^{k+2}\hat{s}_{k+1} - \hat{s}_{k+1}^{k+2}) - \hat{s}_{k+1}\cdot i({}^{k+2}\hat{\Omega} - \hat{\Omega}^{k+2})\right\}.$$

After applying [II, § 11, eq. (13)] twice, that will become:

$$= s_{k+2}\frac{\partial\Omega}{\partial x_{k+1}} - s_{k+1}\frac{\partial\Omega}{\partial x_{k+2}} + \left[\hat{\mathfrak{M}},\frac{\partial\hat{\mathfrak{s}}}{\partial x_{k+1}}\right]_{k+2} - \left[\hat{\mathfrak{M}},\frac{\partial\hat{\mathfrak{s}}}{\partial x_{k+2}}\right]_{k+1} + [\mathfrak{M},\mathrm{grad}\,\hat{s}_0]_k\,,$$

$$= \left\{-[\mathfrak{s},\mathrm{grad}\,\Omega] + \hat{\mathfrak{M}}\,\mathrm{div}\,\hat{\mathfrak{s}} - (\hat{\mathfrak{M}}\,\mathrm{grad})\hat{\mathfrak{s}} - [\mathfrak{M},\mathrm{grad}\,\hat{s}_0]\right\}_k.$$

3. Introducing solution B into eq. (5b) will give:

$$P_5\,\hat{s}_k - P_{k+1,k+2,4}\hat{s}_0 = \hat{\Omega}\,Q_{k+1,k+2},$$

and with the values for P_5, $P_{k+1,k+2,4}$, $Q_{k+1,k+2}$,

4. Introducing solution A into eq. (2b) will give:

$$P_{k+1}\,\hat{s}_{k+2} - P_{k+2,4}\hat{s}_{k+1} = \Omega\,Q_k,$$

and with the values for $P_{k+1,4}$, $P_{k+2,4}$, Q_k, that will give:

$$\left\{-\hat{\mathfrak{s}}\ \mathrm{div}\,\mathfrak{M} - \hat{s}_0 \frac{\partial\mathfrak{M}}{c\,\partial t} + \hat{s}_0\ \mathrm{rot}\,\hat{\mathfrak{M}} + \frac{2mc}{\hbar}\hat{\Omega}\,\mathfrak{M} - \hat{\Omega}\ \mathrm{rot}\ \mathfrak{s}\right\}_k$$

$$= \hat{s}_k\cdot\frac{i}{c}({}^0\hat\Omega - \hat\Omega^0) - \hat\Omega\cdot\frac{i}{c}({}^0\hat s_k - \hat s_k^0) + \hat s_0\cdot\frac{i}{c}({}^k\hat\Omega-\hat\Omega^k) - \hat\Omega\cdot\frac{i}{c}({}^k\hat s_0 - \hat s_0^k),$$

and, from [II, § 11, eq. (12) and (13)], that:

$$= \left\{\Omega\frac{\partial\mathfrak{s}}{c\,\partial t} - \left[\hat{\mathfrak{s}}, \frac{\partial\hat{\mathfrak{M}}}{c\,\partial t}\right] - \hat s_0\frac{\partial\mathfrak{M}}{c\,\partial t} + \Omega\ \mathrm{grad}\ s_0\right\}_k - \left(\hat{\mathfrak{s}}, \frac{\partial\mathfrak{M}}{\partial x_k}\right).$$

An application of the vector formula [II, (82)] to the vectors $\hat{\mathfrak{s}}$ and $\mathfrak{M}$ makes it possible to write that as follows:

$$\left[\hat{\mathfrak{s}},\frac{\partial\hat{\mathfrak{M}}}{c\,\partial t}+\mathrm{rot}\,\mathfrak{M}\right] - \Omega\left(\frac{\partial\mathfrak{s}}{c\,\partial t}+\mathrm{grad}\ s_0\right)+\frac{2mc}{\hbar}\hat\Omega\,\mathfrak{M}$$
$$= \hat{\mathfrak{s}}\,\mathrm{div}\,\mathfrak{M} - \hat s_0\,\mathrm{rot}\,\hat{\mathfrak{M}} + \Omega\,\mathrm{rot}\,\mathfrak{s} - (\hat{\mathfrak{s}}\ \mathrm{grad})\,\mathfrak{M},$$

whose right-hand side can be converted by an application of rot [I, (30)] in such a way that XVII-XIXB will arise.

that will give:

$$\left\{\frac{2mc}{\hbar}(\Omega\hat{\mathfrak{M}} - [\mathfrak{s},\hat{\mathfrak{s}}]) + \left[\hat{\mathfrak{s}},\frac{\partial\hat{\mathfrak{M}}}{c\,\partial t}+\mathrm{rot}\,\hat{\mathfrak{M}}\right] - \Omega\left(\frac{\partial\mathfrak{s}}{c\,\partial t}+\mathrm{grad}\ s_0\right)\right\}_k$$

$$= \Omega\cdot i({}^{k+1}\hat s_{k+2} - \hat s_{k+2}^{k+1}) - \hat s_{k+2}\cdot i({}^{k+1}\Omega - \Omega^{k+1}) - \left\{\Omega\cdot i({}^{k+2}\hat s_{k+1} - \hat s_{k+1}^{k+2}) - \hat s_{k+1}\cdot i({}^{k+2}\Omega-\Omega^{k+2})\right\},$$

and with a double application of [II, § 11, eq. (15)], it will emerge that this:

$$= \left\{[\mathfrak{s}, \mathrm{grad}\,\hat\Omega] - [\hat{\mathfrak{M}}, \mathrm{grad}\,\hat s_0]\right\}_k + \left[\mathfrak{M}, \frac{\partial\hat{\mathfrak{s}}}{\partial x_{k+1}}\right]_{k+2} - \left[\mathfrak{M},\frac{\partial\hat{\mathfrak{s}}}{\partial x_{k+2}}\right]_{k+1}.$$

An application of the identity [I, (34)] and the vector formula (16) to the vectors $\mathfrak{M}$ and $\hat{\mathfrak{s}}$ in the last two terms of the right-hand side of which will lead to XX-XXIIA.

e) The fourth pair of equations that are coupled to each other by solutions A and B is (6b), (7b), which will lead to a linear combination of the continuity equation and the anti-continuity equation, as well as a further vector relation. Substituting A in eq. (6b) and B in eq. (7b) will give:

$$\left.\begin{array}{l}\text{XXIII - XXVA}\\ \text{XXVI - XXVIIIB}\end{array}\right\} \qquad \boxed{\mathfrak{s}\left(\frac{\partial s_0}{c\,\partial t}+\mathrm{div}\ \mathfrak{s}\right) - \hat{\mathfrak{s}}\left(\frac{\partial\hat s_0}{c\,\partial t}+\mathrm{div}\ \hat{\mathfrak{s}} - \frac{2mc}{\hbar}\hat\Omega\right) = 0.}$$

Substituting B in eq, (6b) and A in eq. (7b) will give:

$$\left.\begin{array}{l}\text{XXIII - XXVB}\\ \text{XXVI - XXVIIIA}\end{array}\right\}\quad \boxed{\begin{array}{l}\left[\mathfrak{M},\dfrac{\partial\mathfrak{M}}{c\,\partial t}-\operatorname{rot}\hat{\mathfrak{M}}\right]+\left[\hat{\mathfrak{M}},\dfrac{\partial\hat{\mathfrak{M}}}{c\,\partial t}+\operatorname{rot}\mathfrak{M}\right]\\ -\left[\mathfrak{s},\dfrac{\partial\mathfrak{s}}{c\,\partial t}+\operatorname{grad}s_0\right]+\left[\hat{\mathfrak{s}},\dfrac{\partial\hat{\mathfrak{s}}}{c\,\partial t}+\operatorname{grad}\hat{s}_0\right]\\ =\Omega\operatorname{grad}\hat{\Omega}-\hat{\Omega}\operatorname{grad}\Omega+\hat{\mathfrak{M}}\operatorname{div}\mathfrak{M}\\ \quad -\mathfrak{M}\operatorname{div}\hat{\mathfrak{M}}+s_0\operatorname{rot}\mathfrak{s}-\hat{s}_0\operatorname{rot}\hat{\mathfrak{s}}.\end{array}}$$

Proof:

1. Introducing solution A into eq. (6b) yields:

$$P_4 M_{k0}-P_{k+1,4}M_{k,k+1}+P_{k+2,4}M_{k+2,k}=\Omega Q_{k4},$$

and with the values for P_4, $P_{k+1,4}$, $P_{k+2,4}$ that will yield:

$$\left\{-\frac{2mc}{\hbar}(s_0\hat{\mathfrak{M}}+[\mathfrak{M},\mathfrak{s}])-\left[\mathfrak{M},\frac{\partial\hat{\mathfrak{M}}}{c\,\partial t}+\operatorname{rot}\mathfrak{M}\right]+\hat{\mathfrak{M}}\operatorname{div}\hat{\mathfrak{M}}+\Omega\operatorname{div}\Omega\right\}_k$$

$$\begin{aligned}&=M_{k0}\cdot\frac{i}{c}({}^0\Omega-\Omega^0)-\Omega\cdot\frac{i}{c}\,({}^0M_{k0}-M^{\,0}_{k0})\\ &\quad -M_{k,k+1}\cdot i\,({}^{k+1}\Omega-\Omega^{k+1})\\ &\qquad -\Omega\cdot i\,({}^{k+1}M_{k,k+1}-M^{\,k+1}_{k,k+1})\\ &\quad -\{M_{k+2,k}\cdot i\,({}^{k+2}\Omega-\Omega^{k+2})\\ &\qquad -\Omega\cdot i\,({}^{k+2}M_{k+2,k}-M^{\,k+2}_{k,k+2})\}.\end{aligned}$$

When the uninterpretable quantities are replaced with the help of [II, § 11, eq. (20) and (43)], this will become:

$$=\left\{-\left[\mathfrak{M},\frac{\partial\hat{\mathfrak{M}}}{c\,\partial t}\right]+\mathfrak{s}\frac{\partial s_0}{c\,\partial t}-\hat{\mathfrak{s}}\frac{\partial\hat{s}_0}{c\,\partial t}\right\}_k$$

2. Introducing solution B into eq. (7b) yields:

$$P_5 M_{k+1,k+2}+P_{k+2,k,4}M_{k+2,0}-P_{k,k+1,4}M_{k+1,0}=\hat{\Omega}\;Q_{k+1,k+2,4},$$

and with the values for P_5, $P_{k+2,k,4}$, $P_{k,k+1,4}$, $Q_{k+1,k+2,4}$, that will yield:

$$\left\{\left[\hat{\mathfrak{M}},\frac{\partial\mathfrak{M}}{c\,\partial t}-\operatorname{rot}\hat{\mathfrak{M}}\right]+\mathfrak{M}\operatorname{div}\mathfrak{M}-\frac{2mc}{\hbar}\hat{\Omega}\,\hat{\mathfrak{s}}+\hat{\Omega}\operatorname{grad}\hat{\Omega}\right\}_k$$

$$\begin{aligned}&=\frac{i}{c}({}^0M_{k+1,k+2}-M^{\,0}_{k+1,k+2})\,\hat{\Omega}\\ &\quad -\frac{i}{c}\,({}^0\hat{\Omega}-\hat{\Omega}^0)\,M_{k+1,k+2}\\ &\quad +i\,({}^{k+2}M_{k+1,0}-M^{\,k+2}_{k+1,0})\,\hat{\Omega}\\ &\qquad -i\,({}^{k+2}\hat{\Omega}-\hat{\Omega}^{k+2})\,M_{k+1,0}\\ &\quad -\{i\,({}^{k+2}M_{k+2,0}-M^{\,k+1}_{k+2,0})\,\hat{\Omega}\\ &\qquad -i\,({}^{k+2}\hat{\Omega}-\hat{\Omega}^{k+2})\,M_{k+2,0}\}.\end{aligned}$$

When the uninterpretable quantities are replaced with the help of [II, § 11, eq. (31) and (44)], this will become:

$$=\left\{\left[\hat{\mathfrak{M}},\frac{\partial\mathfrak{M}}{c\,\partial t}\right]+\mathfrak{s}\frac{\partial s_0}{c\,\partial t}-\hat{\mathfrak{s}}\frac{\partial\hat{s}_0}{c\,\partial t}\right\}_k$$

$$+\frac{1}{2}\left\{\left[\mathfrak{M},\frac{\partial\mathfrak{M}}{\partial x_{k+2}}\right]-\left[\hat{\mathfrak{M}},\frac{\partial\hat{\mathfrak{M}}}{\partial x_{k+2}}\right]-\left[\mathfrak{s},\frac{\partial\mathfrak{s}}{\partial x_{k+2}}\right]+\left[\hat{\mathfrak{s}},\frac{\partial\hat{\mathfrak{s}}}{\partial x_{k+2}}\right]\right\}_{k+1}$$
$$+\frac{1}{2}\left\{\left[\mathfrak{M},\frac{\partial\mathfrak{M}}{\partial x_{k+1}}\right]-\left[\hat{\mathfrak{M}},\frac{\partial\hat{\mathfrak{M}}}{\partial x_{k+1}}\right]-\left[\mathfrak{s},\frac{\partial\mathfrak{s}}{\partial x_{k+1}}\right]+\left[\hat{\mathfrak{s}},\frac{\partial\hat{\mathfrak{s}}}{\partial x_{k+1}}\right]\right\}_{k+2}.$$

$$+\frac{1}{2}\left\{\left[\mathfrak{M},\frac{\partial\mathfrak{M}}{\partial x_{k+1}}\right]-\left[\hat{\mathfrak{M}},\frac{\partial\hat{\mathfrak{M}}}{\partial x_{k+1}}\right]+\left[\mathfrak{s},\frac{\partial\mathfrak{s}}{\partial x_{k+1}}\right]+\left[\hat{\mathfrak{s}},\frac{\partial\hat{\mathfrak{s}}}{\partial x_{k+1}}\right]\right\}_{k+2}$$
$$-\frac{1}{2}\left\{\left[\mathfrak{M},\frac{\partial\mathfrak{M}}{\partial x_{k+2}}\right]-\left[\hat{\mathfrak{M}},\frac{\partial\hat{\mathfrak{M}}}{\partial x_{k+2}}\right]+\left[\mathfrak{s},\frac{\partial\mathfrak{s}}{\partial x_{k+2}}\right]+\left[\hat{\mathfrak{s}},\frac{\partial\hat{\mathfrak{s}}}{\partial x_{k+2}}\right]\right\}_{k+1}.$$

With the use of the vector formula (16), the identity [I, (29)}, and an application of grad to the identities [I, (15), (16), (20), (21)], one will get:

$$-\mathfrak{s}\left(\frac{\partial s_0}{c\,\partial t}+\operatorname{div}\mathfrak{s}\right)+\hat{\mathfrak{s}}\left(\frac{\partial\hat{s}_0}{c\,\partial t}+\operatorname{div}\hat{\mathfrak{s}}-\frac{2mc}{\hbar}\hat{\Omega}\right)$$
$$=\tfrac{1}{2}\{[\mathfrak{M},\operatorname{rot}\mathfrak{M}]-\mathfrak{M}\operatorname{div}\mathfrak{M}+[\hat{\mathfrak{M}},\operatorname{rot}\hat{\mathfrak{M}}]-\hat{\mathfrak{M}}\operatorname{div}\hat{\mathfrak{M}}+[\mathfrak{s},\operatorname{rot}\mathfrak{s}]-\mathfrak{s}\operatorname{div}\mathfrak{s}$$
$$-([\hat{\mathfrak{s}},\operatorname{rot}\hat{\mathfrak{s}}]-\hat{\mathfrak{s}}\operatorname{div}\hat{\mathfrak{s}})+\tfrac{1}{2}\operatorname{grad}(\Omega^2+\hat{\Omega}^2+s_0^2-\hat{s}_0^2)\},$$

in both cases. The right-hand side vanishes, from {I, (54)], such that one will be dealing with XXIII-XXVA (XXCI-XXVIIIB, resp.).

3. Introducing solution B into eq. (6b) will yield:

$$P_5\,M_{k0}-P_{k+2,k,4}\,M_{k,k+1}+P_{k,k+1,4}\,M_{k+2,k}=\hat{\Omega}\,Q_{k4}\,,$$

and with the values for P_5, $P_{k+2,k,4}$, $P_{k,k+1,4}$, Q_{k4} that will give:

$$\left\{\left[\mathfrak{M},\frac{\partial\mathfrak{M}}{c\,\partial t}-\operatorname{rot}\hat{\mathfrak{M}}\right]-\hat{\mathfrak{M}}\operatorname{div}\mathfrak{M}+\hat{\Omega}\operatorname{grad}\Omega\right\}_k$$

$$=\frac{i}{c}({}^0\hat{\Omega}-\hat{\Omega}^0)\,M_{k0}-\frac{i}{c}({}^0M_{k0}-M_{k0}^0)\,\hat{\Omega}+i({}^{k+1}\hat{\Omega}-\hat{\Omega}^{k+1})\,M_{k,k+1}-i({}^{k+1}M_{k,k+1}-M_{k,k+1}^{k+1})\,\hat{\Omega}$$

4. Introducing solution A into eq. (7b) will yield:

$$P_4\,M_{k+1,k+2}+P_{k+1,4}\,M_{k+1,0}-P_{k+2,4}\,M_{k+1,0}=\Omega\,Q_{k+1,k+2,4}\,,$$

and with the values for P_4, $P_{k+1,4}$, $P_{k+2,4}$, $Q_{k+1,k+2,4}$ that will give:

$$\left\{\frac{2mc}{\hbar}([\hat{\mathfrak{M}},\mathfrak{s}]+\Omega\,\hat{\mathfrak{s}}-s_0\mathfrak{M})+\mathfrak{M}\operatorname{div}\hat{\mathfrak{M}}+\left[\hat{\mathfrak{M}},\frac{\partial\hat{\mathfrak{M}}}{c\,\partial t}+\operatorname{rot}\mathfrak{M}\right]-\Omega\operatorname{grad}\hat{\Omega}\right\}_k$$

$$=\frac{i}{c}({}^0\Omega-\Omega^0)\,M_{k+1,k+2}-\frac{i}{c}({}^0M_{k+1,k+2}-M_{k+1,k+2}^0)\,\Omega$$

$$-\{i({}^{k+2}\hat{\Omega}-\hat{\Omega}^{k+2})M_{k+2,k} \\ -i({}^{k+2}M_{k+2,k}-M_{k+2,k}^{k+2})\hat{\Omega}\}.$$

With the help of [II, § 11, eq. (21) and (44)], the uninterpretable quantities on the right-hand side of this can be eliminated, and after a double application of the vector formula (16), one will get:

$$\left[\mathfrak{M},\frac{\partial\mathfrak{M}}{c\,\partial t}-\text{rot}\,\hat{\mathfrak{M}}\right]-\hat{\mathfrak{M}}\,\text{div}\,\mathfrak{M}+\hat{\Omega}\,\text{grad}\,\Omega$$

$$=\frac{1}{2}\left\{\left[\mathfrak{M},\frac{\partial\mathfrak{M}}{c\,\partial t}\right]-\left[\hat{\mathfrak{M}},\frac{\partial\hat{\mathfrak{M}}}{c\,\partial t}\right]\right.$$
$$+\left[\mathfrak{s},\frac{\partial\mathfrak{s}}{c\,\partial t}\right]-\left[\hat{\mathfrak{s}},\frac{\partial\hat{\mathfrak{s}}}{c\,\partial t}\right]$$
$$-\mathfrak{M}\,\text{div}\,\hat{\mathfrak{M}}-\hat{\mathfrak{M}}\,\text{div}\,\mathfrak{M}$$
$$+s_0\,\text{rot}\,\mathfrak{s}-\hat{s}_0\,\text{rot}\,\hat{\mathfrak{s}}$$
$$+[\mathfrak{s},\text{grad}\,s_0]-[\hat{\mathfrak{s}},\text{grad}\,\hat{s}_0]$$
$$\left.+(\mathfrak{M}\,\text{grad})\,\hat{\mathfrak{M}}+(\hat{\mathfrak{M}}\,\text{grad})\,\mathfrak{M}\right\}.$$

$$+i({}^{k+2}\hat{\Omega}-\hat{\Omega}^{k+2})M_{k+1,0} \\ -i({}^{k+2}M_{k+1,0}-M_{k+1,0}^{k+2})\Omega$$
$$-\{i({}^{k+1}\Omega-\Omega^{k+1})M_{k+2,0} \\ -i({}^{k+1}M_{k+2,k}-M_{k+2,k}^{k+1})\hat{\Omega}\}.$$

With the help of [II, § 11, eq. (20) and (43)], one can eliminate the uninterpretable quantities on the right-hand side of this, and if one observes the identity [I, (31)], one will get:

$$\left[\hat{\mathfrak{M}},\frac{\partial\hat{\mathfrak{M}}}{c\,\partial t}+\text{rot}\,\mathfrak{M}\right]+\mathfrak{M}\,\text{div}\,\hat{\mathfrak{M}}-\Omega\,\text{grad}\,\hat{\Omega}$$

$$=-\frac{1}{2}\left\{\left[\mathfrak{M},\frac{\partial\mathfrak{M}}{c\,\partial t}\right]-\left[\hat{\mathfrak{M}},\frac{\partial\hat{\mathfrak{M}}}{c\,\partial t}\right]\right.$$
$$-\left[\mathfrak{s},\frac{\partial\mathfrak{s}}{c\,\partial t}\right]+\left[\hat{\mathfrak{s}},\frac{\partial\hat{\mathfrak{s}}}{c\,\partial t}\right]$$
$$-\mathfrak{M}\,\text{div}\,\hat{\mathfrak{M}}-\hat{\mathfrak{M}}\,\text{div}\,\mathfrak{M}$$
$$-s_0\,\text{rot}\,\mathfrak{s}+\hat{s}_0\,\text{rot}\,\hat{\mathfrak{s}}$$
$$-[\mathfrak{s},\text{grad}\,s_0]+[\hat{\mathfrak{s}},\text{grad}\,\hat{s}_0]$$
$$\left.+(\mathfrak{M}\,\text{grad})\,\hat{\mathfrak{M}}+(\hat{\mathfrak{M}}\,\text{grad})\,\mathfrak{M}\right\}.$$

In both cases, the replacement of the last two terms by means of the equation that arises from taking the grad of the identity [I, (22)] will lead one to drop the factor 1/2, and will give those equations the form of XXIII-XXVB (XXVI-XXVIIIA, resp.)

We then see that among the 28 relations that are obtained as a result of the multiple occurrences of the continuity equation and anti-continuity equation, only 22 distinct ones will be present – namely, four scalar relations and six vector relations.

§ 3. Symmetric construction of the reality relations

Some of the relations that were obtained in § 2 still contain pieces of the continuity equation VII and anti-continuity equation V. After eliminating those pieces, the relations will assume a symmetric characteristic form. Namely, each of them will consist of a sum of 16 expressions of the type:

$$F\frac{\partial G}{\partial x}-G\frac{\partial F}{\partial x},$$

in which F and G are two of the **Dirac** densities Ω, $\hat{\Omega}$, s_0, $\hat{s}_0$, $\mathfrak{s}$, $\hat{\mathfrak{s}}$, $\mathfrak{M}$, $\hat{\mathfrak{M}}$, and those expressions will appear in groups of four that are differentiated by x_1, x_2, x_3, ct. The fact that this notation for the remaining two scalar relations and six vector relations is natural will emerge from the arguments in Part IV.

We eliminate the mass term from relation I by applying V. With the use of relation VII, the divergence of the identity [I, (35)], and temporal derivative of the identity [I, (17)], the scalar relation I can be converted into the symmetric notation.

$$
(17)\qquad \left\{
\begin{aligned}
& s_0 \frac{\partial \hat{s}_0}{c\,\partial t} - \hat{s}_0 \frac{\partial s_0}{c\,\partial t} + \left(\mathfrak{s}, \frac{\partial \hat{\mathfrak{s}}}{c\,\partial t} \right) - \left(\hat{\mathfrak{s}}, \frac{\partial \mathfrak{s}}{c\,\partial t} \right) \\
& \qquad + s_0 \operatorname{div} \hat{\mathfrak{s}} - (\hat{\mathfrak{s}}, \operatorname{grad} s_0) - \{ \hat{s}_0 \operatorname{div} \mathfrak{s} - (\mathfrak{s}, \operatorname{grad} \hat{s}_0) \} \\
& \qquad + (\mathfrak{M}, \operatorname{grad} \Omega) - \Omega \operatorname{div} \mathfrak{M} + (\hat{\mathfrak{M}}, \operatorname{grad} \Omega) - \hat{\Omega} \operatorname{div} \hat{\mathfrak{M}} = 0.
\end{aligned}
\right.
$$

The fourth scalar relation (IXB, XA) is already of that type:

$$
(18)\qquad \left\{
\begin{aligned}
& \hat{\Omega} \frac{\partial \Omega}{c\,\partial t} - \Omega \frac{\partial \hat{\Omega}}{c\,\partial t} + \left(\mathfrak{M}, \frac{\partial \hat{\mathfrak{M}}}{c\,\partial t} \right) - \left(\hat{\mathfrak{M}}, \frac{\partial \mathfrak{M}}{c\,\partial t} \right) \\
& \qquad + (\mathfrak{s}, \operatorname{rot} \mathfrak{s}) - (\hat{\mathfrak{s}}, \operatorname{rot} \hat{\mathfrak{s}}) + (\mathfrak{M}, \operatorname{rot} \mathfrak{M}) + (\hat{\mathfrak{M}}, \operatorname{rot} \hat{\mathfrak{M}}) = 0.
\end{aligned}
\right.
$$

We convert the six vector relations in an analogous way with an application of the relations V and VII, the rotations of the identities [I, (29), (30), (31), (32), (34)] and the gradients of the identities [I, (17), (25), (26), (27), (28)]. The vector relations II-IV then imply that:

$$
\left\{
\begin{aligned}
& \mathbf{a} \equiv \{ \Omega \operatorname{rot} \hat{\mathfrak{M}} - [\operatorname{grad} \Omega, \hat{\mathfrak{M}}] \} - \{ \hat{\Omega} \operatorname{rot} \mathfrak{M} - [\operatorname{grad} \hat{\Omega}, \mathfrak{M}] \} \\
& \qquad + \{ \hat{s}_0 \operatorname{grad} s_0 - s_0 \operatorname{grad} \hat{s}_0 \} + \{ [\mathfrak{s}, \operatorname{rot} \hat{\mathfrak{s}}] - [\hat{\mathfrak{s}}, \operatorname{rot} \mathfrak{s}] \} + \{ \hat{\mathfrak{s}} \operatorname{div} \mathfrak{s} - \mathfrak{s} \operatorname{div} \hat{\mathfrak{s}} \} \\
& \qquad + \left\{ \mathfrak{M} \frac{\partial \Omega}{c\,\partial t} - \Omega \frac{\partial \mathfrak{M}}{c\,\partial t} \right\} + \left\{ \hat{\mathfrak{M}} \frac{\partial \hat{\Omega}}{c\,\partial t} - \hat{\Omega} \frac{\partial \hat{\mathfrak{M}}}{c\,\partial t} \right\} \\
& \qquad + \left\{ \hat{\mathfrak{s}} \frac{\partial s_0}{c\,\partial t} - s_0 \frac{\partial \hat{\mathfrak{s}}}{c\,\partial t} \right\} + \left\{ \hat{s}_0 \frac{\partial \hat{\Omega}}{c\,\partial t} - \mathfrak{s} \frac{\partial \hat{s}_0}{c\,\partial t} \right\} = 0.
\end{aligned}
\right.
$$

The vector relations XXIII-XXVB imply that:

$$\left\{\begin{aligned} \mathbf{b} \equiv & \{[\hat{\mathfrak{M}}, \operatorname{rot} \mathfrak{M}] - [\mathfrak{M}, \operatorname{rot} \hat{\mathfrak{M}}]\} - \{\mathfrak{M} \operatorname{div} \hat{\mathfrak{M}} - \hat{\mathfrak{M}} \operatorname{div} \mathfrak{M}\} \\ & - \{s_0 \operatorname{rot} \mathfrak{s} - [\operatorname{grad} s_0, \mathfrak{s}\} + \{\hat{s}_0 \operatorname{rot} \hat{\mathfrak{s}}] - [\operatorname{grad} \hat{s}_0, \hat{\mathfrak{s}}]\} + \{\hat{\Omega} \operatorname{grad} \Omega - \Omega \operatorname{div} \hat{\Omega}\} \\ & + \left[\mathfrak{M}, \frac{\partial \mathfrak{M}}{c\,\partial t}\right] + \left[\hat{\mathfrak{M}}, \frac{\partial \hat{\mathfrak{M}}}{c\,\partial t}\right] - \left[\mathfrak{s}, \frac{\partial \mathfrak{s}}{c\,\partial t}\right] + \left[\hat{\mathfrak{s}}, \frac{\partial \hat{\mathfrak{s}}}{c\,\partial t}\right] = 0. \end{aligned}\right.$$

(19)

The vector relations XI-XIIIA imply that:

$$\left\{\begin{aligned} \mathbf{c} \equiv & \{\Omega \operatorname{rot} \hat{\mathfrak{s}} - [\operatorname{grad} \Omega, \hat{\mathfrak{s}}]\} - \{\hat{\mathfrak{M}} \operatorname{div} \mathfrak{s} - \mathfrak{s} \operatorname{div} \hat{\mathfrak{M}}\} \\ & + \{\hat{s}_0 \operatorname{grad} \hat{\Omega} - \hat{\Omega} \operatorname{grad} \hat{s}_0\} + \{[\mathfrak{s}, \operatorname{rot} \hat{\mathfrak{M}}] - [\hat{\mathfrak{M}}, \operatorname{rot} \mathfrak{s}]\} \\ & + \{s_0 \operatorname{rot} \hat{\mathfrak{M}} - [\operatorname{grad} s_0, \mathfrak{M}]\} - \left\{\left[\mathfrak{M}, \frac{\partial \mathfrak{s}}{c\,\partial t}\right] - \left[\frac{\partial \mathfrak{M}}{c\,\partial t}, \mathfrak{s}\right]\right\} \\ & + \left\{\hat{\mathfrak{s}} \frac{\partial \hat{\Omega}}{c\,\partial t} - \hat{\Omega} \frac{\partial \hat{\mathfrak{s}}}{c\,\partial t}\right\} + \left\{s_0 \frac{\partial \hat{\mathfrak{M}}}{c\,\partial t} - \hat{\mathfrak{M}} \frac{\partial s_0}{c\,\partial t}\right\} = 0. \end{aligned}\right.$$

The vector relations XVII-XIXB imply that:

$$\left\{\begin{aligned} \mathbf{d} \equiv & \{\hat{\Omega} \operatorname{rot} \mathfrak{s} - [\operatorname{grad} \hat{\Omega}, \mathfrak{s}]\} - \{\mathfrak{M} \operatorname{div} \hat{\mathfrak{s}} - \hat{\mathfrak{s}} \operatorname{div} \mathfrak{M}\} \\ & + \{s_0 \operatorname{grad} \Omega - \Omega \operatorname{grad} s_0\} + \{[\hat{\mathfrak{s}}, \operatorname{rot} \mathfrak{M}] - [\mathfrak{M}, \operatorname{rot} \hat{\mathfrak{s}}]\} \\ & + \{\hat{s}_0 \operatorname{rot} \hat{\mathfrak{M}} - [\operatorname{grad} \hat{s}_0, \hat{\mathfrak{M}}]\} - \left\{\left[\hat{\mathfrak{M}}, \frac{\partial \hat{\mathfrak{s}}}{c\,\partial t}\right] - \left[\frac{\partial \hat{\mathfrak{M}}}{c\,\partial t}, \hat{\mathfrak{s}}\right]\right\} \\ & + \left\{\mathfrak{s} \frac{\partial \Omega}{c\,\partial t} - \Omega \frac{\partial \mathfrak{s}}{c\,\partial t}\right\} + \left\{\hat{s}_0 \frac{\partial \mathfrak{M}}{c\,\partial t} - \mathfrak{M} \frac{\partial \hat{s}_0}{c\,\partial t}\right\} = 0. \end{aligned}\right.$$

The vector relations XI-XIIIB imply that:

$$\left\{\begin{aligned} \mathbf{e} \equiv & \{\hat{\Omega} \operatorname{rot} \hat{\mathfrak{s}} - [\operatorname{grad} \hat{\Omega}, \hat{\mathfrak{s}}]\} - \{\mathfrak{M} \operatorname{div} \mathfrak{s} - \mathfrak{s} \operatorname{div} \mathfrak{M}\} \\ & - \{\hat{s}_0 \operatorname{grad} \Omega - \Omega \operatorname{grad} \hat{s}_0\} - \{[\mathfrak{s}, \operatorname{rot} \mathfrak{M}] - [\mathfrak{M}, \operatorname{rot} \mathfrak{s}]\} \\ & - \{s_0 \operatorname{rot} \hat{\mathfrak{M}} - [\operatorname{grad} s_0, \hat{\mathfrak{M}}]\} - \left\{\left[\hat{\mathfrak{M}}, \frac{\partial \mathfrak{s}}{c\,\partial t}\right] - \left[\frac{\partial \hat{\mathfrak{M}}}{c\,\partial t}, \mathfrak{s}\right]\right\} \\ & - \left\{\hat{\mathfrak{s}} \frac{\partial \Omega}{c\,\partial t} - \Omega \frac{\partial \hat{\mathfrak{s}}}{c\,\partial t}\right\} - \left\{s_0 \frac{\partial \mathfrak{M}}{c\,\partial t} - \mathfrak{M} \frac{\partial s_0}{c\,\partial t}\right\} = 0. \end{aligned}\right.$$

(19)

The vector relations XI-XIIIB imply that:

$$
\left\{
\begin{aligned}
\mathbf{f} \equiv & \{\Omega \text{ rot } \mathfrak{s} - [\text{grad } \Omega, \mathfrak{s}]\} + \{\hat{\mathfrak{M}} \text{ div } \hat{\mathfrak{s}} - \hat{\mathfrak{s}} \text{ div } \hat{\mathfrak{M}}\} \\
& + \{s_0 \text{ grad } \hat{\Omega} - \hat{\Omega} \text{ grad } s_0\} + \{[\hat{\mathfrak{s}}, \text{rot } \hat{\mathfrak{M}}] - [\hat{\mathfrak{M}}, \text{rot } \hat{\mathfrak{s}}]\} \\
& - \{\hat{s}_0 \text{ rot } \mathfrak{M} - [\mathfrak{M}, \text{grad } \hat{s}_0]\} - \left\{\left[\mathfrak{M}, \frac{\partial \hat{\mathfrak{s}}}{c\,\partial t}\right] - \left[\frac{\partial \mathfrak{M}}{c\,\partial t}, \hat{\mathfrak{s}}\right]\right\} \\
& + \left\{\mathfrak{s}\frac{\partial \hat{\Omega}}{c\,\partial t} - \hat{\Omega}\frac{\partial \mathfrak{s}}{c\,\partial t}\right\} - \left\{\hat{s}_0 \frac{\partial \hat{\mathfrak{M}}}{c\,\partial t} - \hat{\mathfrak{M}}\frac{\partial \hat{s}_0}{c\,\partial t}\right\} = 0.
\end{aligned}
\right.
$$

In contrast to the two relations V and VII, which are linear expressions, all of the remaining relations (17)-(19f) have a bilinear form. That is based upon the fact that in their original form, they contained uninterpretable quantities whose elimination must be purchased by the transition to the bilinear forms.

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