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On Dirac's theory of the electron

IV. Relations between the reality relations.

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Introduction and summary

In Part III, four scalar and six vector relations were derived from the reality of the electromagnetic potentials for **Dirac**'s theory of the electron. We shall now show that the six vectorial relations can be constructed from the four scalar ones. After the splitting of the two simpler scalar relations [Part III, eqs. (V) and (VII)] that was carried out already in III, § 3, and which was contained in the remaining relations in an easily-recognizable form, that problem reduced to *proving that the six vector relations* [Part III, eqs. (19a)-(f)] *can be constructed from the other two scalar relations with a bilinear structure* [III, eqs. (17) and (18)].

One would also expect the reducibility of *all* reality relations to *four of them* would be exist in the present treatment with no specialization of the Dirac matrices, so in the case of a special representation (cf., e.g., *loc. cit.* [**1**]), the four **Dirac** differential equations will by doubled by going to the complex conjugates, such that four potential-free relations must remain after eliminating the four potentials.

Nevertheless, the six vector relations that our general treatment will yield are also not without interest. Their appearance is connected with the elimination of the uninterpretable quantities, and their reducibility to the two scalar relations is not trivial, such that knowing that will be of value, despite their decomposability. In order to decompose them, one must make extensive use of the *second* **Pauli** bilinear equation [I, (10)] between the matrix elements of the Dirac matrices. One will then find (§ 1) a complex scalar τ [Definition eq. (9)], by which one can reduce the six vector relations with the help of three complex vectors $\mathfrak{T}'$, $\mathfrak{T}''$, $\mathfrak{T}'''$ [defined in eqs. (2), (2′)]. In that formulation, the six vector relations will read [III, (19a-f)]:

a)	$\mathfrak{T}'\tau + \mathfrak{T}'^*\tau^* = 0,$	c)	$\mathfrak{T}''\tau + \mathfrak{T}''^*\tau^* = 0,$	e)	$\mathfrak{T}'''\tau + \mathfrak{T}'''^*\tau^* = 0,$
b)	$\mathfrak{T}'\tau - \mathfrak{T}'^*\tau^* = 0,$	d)	$\mathfrak{T}''\tau - \mathfrak{T}''^*\tau^* = 0,$	f)	$\mathfrak{T}'''\tau - \mathfrak{T}'''^*\tau^* = 0.$

They can be satisfied then by $\tau = 0$ ($\tau^* = 0$, resp.), or also by:

$$\tau + \tau^* = 0,$$
$$i\,(\tau - \tau^*) = 0.$$

The agreement of the last two equations with the two bilinear scalar relations [III, (17) and (18)] can be proved (§ 2), and will thus lead to the construction of the six vector equations from the two scalar ones.

One can combine the two scalar relations into complex form:

$$-\tau^* = \left(\psi, T\frac{\partial\psi}{c\,\partial t}\right) - \sum_{k=1}^{3}\left(\psi, T\alpha^k\frac{\partial\psi}{\partial x_k}\right) = 0,$$

which is equivalent to [III, eqs. (17) and (18)] or [IV, eqs. (16) and (17)]. Like the simpler scalar relations ($l = k+1$, $m = k+2$ mod 3):

$$\frac{\partial s_0}{c\,\partial t} + \operatorname{div}\mathfrak{s} = \frac{\partial}{c\,\partial t}(\psi^*, \psi) - \sum_{k=1}^{3}\frac{\partial}{\partial x_k}(\psi^*, \alpha^k\psi) = 0,$$

$$\frac{\partial \hat{s}_0}{c\,\partial t} + \operatorname{div}\hat{\mathfrak{s}} - \frac{2mc}{\hbar}\hat{\Omega} = -\frac{\partial}{c\,\partial t}(\psi^*, \alpha^{[123]}\psi) + \sum_{k=1}^{3}\frac{\partial}{\partial x_k}(\psi^*, \alpha^{[lm]}\psi) + \frac{2mc}{\hbar}(\psi^*, \alpha^5\psi) = 0,$$

they will be linear expressions, but will contain no interpretable quantities.

For the meaning of all symbols that occur, cf., I, § 1.

§ 1. Application of the second Pauli bilinear equation to the decomposition of the six vector relations

1. In Part I, we mentioned a matrix B that took the system of γ^μ to the transposed ones $\overline{\gamma^\mu}$ (exchange of rows of columns) by a similarity transformation. Analogously, we define a matrix T for the system of α^μ that does the same thing to that system:

$$\overline{\alpha^\mu} = T\,\alpha^\mu\,T^{-1}. \tag{1}$$

From [I, (10)], its matrix elements will fulfill the second **Pauli** bilinear equation:

$$\boxed{\begin{aligned} 2T_{\nu\sigma}T^{-1}_{\rho\mu} &= (\alpha^4_{\rho\sigma}\alpha^4_{\mu\nu} - \alpha^4_{\mu\sigma}\alpha^4_{\rho\nu}) + (\alpha^5_{\rho\sigma}\alpha^5_{\mu\nu} - \alpha^5_{\mu\sigma}\alpha^5_{\rho\nu}) \\ &+ (\delta_{\rho\sigma}\delta_{\mu\nu} - \delta_{\mu\sigma}\delta_{\rho\nu}) - (\alpha^{[123]}_{\rho\sigma}\alpha^{[123]}_{\mu\nu} - \alpha^{[123]}_{\mu\sigma}\alpha^{[123]}_{\rho\nu}). \end{aligned}} \tag{***}$$

The six matrices T, $T\alpha^\mu$ (μ = 1, 2, 3, 4, 5) are antisymmetric, so the inner products $(\psi, T\psi)$ and $(\psi, T\alpha^\mu\psi)$ (μ = 1, 2, 3, 4, 5) will vanish. By contrast, the ten matrices $T\alpha^{[\mu\nu]}$, $T\alpha^{[\lambda\mu\nu]}$ are symmetric, and we call the corresponding inner products:

$$\left\{\begin{array}{c} T_{ik} = (\psi, T\alpha^{[ik]}\psi), \quad T_{k0} = (\psi, T\alpha^{[k4]}\psi), \quad T_k = (\psi, T\alpha^{[lm4]}\psi), \quad T_0 = (\psi, T\alpha^{[123]}\psi), \\ (i,k = 1,2,3),\ (k = 1,2,3),\ (k,l,m = 1,2,3 \text{ cyclically mod } 3). \end{array}\right. \tag{2}$$

The inner products differ from the ones that were constructed previously by the fact that the factor on the left also reads ψ, instead of ψ^*, as before. We symbolically combine the nine quantities into three complex vectors:

$$(2') \qquad \mathfrak{T}' = (T_{23}, T_{31}, T_{12}), \qquad \mathfrak{T}'' = (T_{10}, T_{20}, T_{30}), \qquad \mathfrak{T}''' = (T_1, T_2, T_3).$$

2. With the help of those quantities, the six vector relations [III, eqs. (19**a** to **f**)] can be reduced to the two scalar ones [III, (17) and (18)]. Namely ([…]* means the complex conjugate of the *directly* relevant square bracket in this):

$$\boxed{\begin{aligned} &(3) \qquad \frac{\mathbf{a}}{2} \equiv \left[\mathfrak{T}' \left\{ -\left(\frac{\partial \psi^*}{c\,\partial t}, T^{-1}\psi^* \right) + \sum_{k=1}^{3} \left(\frac{\partial \psi^*}{\partial x_k}, \alpha^k T^{-1}\psi^* \right) \right\} \right] + [\cdots]^* = 0, \\ &(4) \qquad \frac{\mathbf{b}}{2} \equiv \left[-i\mathfrak{T}'' \left\{ -\left(\frac{\partial \psi^*}{c\,\partial t}, T^{-1}\psi^* \right) + \sum_{k=1}^{3} \left(\frac{\partial \psi^*}{\partial x_k}, \alpha^k T^{-1}\psi^* \right) \right\} \right] + [\cdots]^* = 0. \end{aligned}}$$

A. **Proof of (3).** We select the first component a_1 of **a** and arrange it into parts according to the derivatives $\partial / \partial x_1$, $\partial / \partial x_2$, $\partial / \partial x_3$, $\partial / c\,\partial t$. We must then prove:

$$(\alpha_a) \qquad \left\{ \begin{aligned} &\hat{s}_0 \frac{\partial s_0}{\partial x_1} - s_0 \frac{\partial \hat{s}_0}{\partial x_1} + \hat{s}_1 \frac{\partial s_1}{\partial x_1} - \hat{s}_1 \frac{\partial s_1}{\partial x_1} + \hat{s}_2 \frac{\partial s_2}{\partial x_1} - \hat{s}_2 \frac{\partial s_2}{\partial x_1} + \hat{s}_3 \frac{\partial s_3}{\partial x_1} - \hat{s}_3 \frac{\partial s_3}{\partial x_1} \\ &\qquad = 2\left\{ \left[T_{23} \left(\frac{\partial \psi^*}{\partial x_1}, \alpha^1 T^{-1}\psi^* \right) \right] + [\cdots]^* \right\}, \end{aligned} \right.$$

$$(\beta_a) \qquad \left\{ \begin{aligned} &\Omega \frac{\partial M_{30}}{\partial x_2} - M_{30} \frac{\partial \Omega}{\partial x_2} + M_{12} \frac{\partial \hat{\Omega}}{\partial x_2} - \Omega \frac{\partial M_{12}}{\partial x_2} + \hat{s}_1 \frac{\partial s_2}{\partial x_2} - s_2 \frac{\partial \hat{s}_1}{\partial x_2} + \hat{s}_2 \frac{\partial s_1}{\partial x_2} - s_1 \frac{\partial \hat{s}_2}{\partial x_2} \\ &\qquad = 2\left\{ \left[T_{23} \left(\frac{\partial \psi^*}{\partial x_2}, \alpha^2 T^{-1}\psi^* \right) \right] + [\cdots]^* \right\}, \end{aligned} \right.$$

$$(\gamma_a) \qquad \left\{ \begin{aligned} &M_{20} \frac{\partial \Omega}{\partial x_3} - \Omega \frac{\partial M_{20}}{\partial x_3} + \hat{\Omega} \frac{\partial M_{31}}{\partial x_3} - M_{31} \frac{\partial \Omega}{\partial x_3} + \hat{s}_3 \frac{\partial s_1}{\partial x_3} - s_1 \frac{\partial \hat{s}_3}{\partial x_3} + \hat{s}_1 \frac{\partial s_3}{\partial x_2} - s_3 \frac{\partial \hat{s}_1}{\partial x_3} \\ &\qquad = 2\left\{ \left[T_{23} \left(\frac{\partial \psi^*}{\partial x_3}, \alpha^3 T^{-1}\psi^* \right) \right] + [\cdots]^* \right\}, \end{aligned} \right.$$

$$(\delta_{\mathrm{a}})\quad \left\{ \begin{aligned} & M_{23}\frac{\partial\Omega}{c\,\partial t}-\Omega\frac{\partial M_{23}}{c\,\partial t}+M_{10}\frac{\partial\hat{\Omega}}{c\,\partial t}-\hat{\Omega}\frac{\partial M_{10}}{c\,\partial t}+\hat{s}_1\frac{\partial s_0}{c\,\partial t}-s_0\frac{\partial\hat{s}_1}{c\,\partial t}+s_0\frac{\partial\hat{s}_1}{c\,\partial t}-s_3\frac{\partial\hat{s}_0}{c\,\partial t} \\ & \qquad =2\left\{\left[T_{23}\left(\frac{\partial\psi^*}{c\,\partial t},T^{-1}\psi^*\right)\right]+[\cdots]^*\right\}. \end{aligned} \right.$$

Proof of (α_{a}): Multiplying $2T_{\nu\bar{\sigma}}\,T^{-1}_{\bar{\rho}\mu}$ by $\alpha^{[23]}_{\bar{\sigma}\sigma}\,\alpha^{1}_{\rho\bar{\rho}}$ and summing over the indices $\bar{\rho}$, $\bar{\sigma}$ yields:

$$\begin{aligned} 2(T\alpha^{[23]})_{\nu\sigma}\,(\alpha^1T^{-1})_{\rho\mu} &= i\left(\alpha^5_{\rho\sigma}\alpha^4_{\mu\nu}-\alpha^4_{\rho\sigma}\alpha^5_{\mu\nu}+\alpha^{[14]}_{\rho\sigma}\alpha^{[234]}_{\mu\nu}-\alpha^{[234]}_{\rho\sigma}\alpha^{[14]}_{\mu\nu}\right) \\ &\quad +\alpha^{[123]}_{\rho\sigma}-\delta_{\rho\sigma}\alpha^{[123]}_{\mu\nu}+\alpha^{[23]}_{\rho\nu}\alpha^{1}_{\mu\sigma}-\alpha^{1}_{\rho\nu}\alpha^{[23]}_{\mu\sigma}. \end{aligned}$$

Multiplying by $\psi_\nu\,\psi_\sigma\,\dfrac{\partial\psi^*_\rho}{\partial x_1}\psi^*_\mu$ and summing over the indices μ, ν, ρ, σ leads to:

$$\begin{aligned} 2T_{23}\left(\frac{\partial\psi^*}{\partial x_1},\alpha^1T^{-1}\psi^*\right) &= i\,(-{}^1\hat{\Omega}\Omega+{}^1\Omega\hat{\Omega}+{}^1M_{10}\,M_{23}+{}^1M_{23}\,M_{10}) \\ &\quad -{}^1\hat{s}_0\,s_0+{}^1s_0\,\hat{s}_0+{}^1s_1\,\hat{s}_1-{}^1\hat{s}_1\,s_1, \end{aligned}$$

so, from [II, eqs. (2) and (50′)], that:

$$= {}^1s_0\,\hat{s}_0-{}^1\hat{s}_0\,s_0+{}^1s_1\,\hat{s}_1-{}^1\hat{s}_1\,s_1+{}^1\hat{s}_2\,s_2-{}^1s_2\,\hat{s}_2+{}^1\hat{s}_3\,s_3-{}^1s_3\,\hat{s}_3 \qquad \text{(notation 1)},$$

and from [II, eqs. (24′) and (41)], that:

$$= 2i\,(-{}^1\hat{\Omega}\Omega+{}^1M_{10}\,M_{23}+{}^1M_{12}\,M_{30}+{}^1M_{31}\,M_{20}) \qquad \text{(notation 2)}.$$

One will get (α_{a}) when one adds the first notation to its complex conjugate.

Proof of (β_{a}): Multiplying $2T_{\nu\bar{\sigma}}\,T^{-1}_{\bar{\rho}\mu}$ by $\alpha^{[23]}_{\bar{\sigma}\sigma}\,\alpha^{2}_{\rho\bar{\rho}}$ and summing over the indices $\bar{\rho}$, $\bar{\sigma}$ yields:

$$\begin{aligned} 2(T\alpha^{[23]})_{\nu\sigma}\,(\alpha^2T^{-1})_{\rho\mu} &= \alpha^{[34]}_{\rho\sigma}\alpha^4_{\mu\nu}-\alpha^{[124]}_{\rho\sigma}\alpha^5_{\mu\nu}-\alpha^{[31]}_{\rho\nu}\alpha^1_{\mu\sigma}-\alpha^2_{\rho\nu}\alpha^{[23]}_{\mu\sigma} \\ &\quad +i\left(\alpha^{[24]}_{\rho\nu}\alpha^{[234]}_{\mu\sigma}+\alpha^3_{\rho\sigma}\delta_{\mu\nu}+\alpha^{[12]}_{\rho\sigma}\alpha^{[123]}_{\mu\nu}-\alpha^{[314]}_{\rho\nu}\alpha^{[14]}_{\mu\sigma}\right). \end{aligned}$$

Multiplying by $\psi_\nu\,\psi_\sigma\,\dfrac{\partial\psi^*_\rho}{\partial x_2}\psi^*_\mu$ and summing over the indices μ, ν, ρ, σ leads to:

$$2T_{23}\left(\frac{\partial\psi^*}{\partial x_2},\alpha^2T^{-1}\psi^*\right)= {}^2M_{30}\,\Omega-{}^2M_{30}\,\hat{\Omega}\;+\;{}^2\hat{s}_2s_1+{}^2s_2\hat{s}_1$$

$$+\, i\, \{{}^2M_{20}\, M_{23} - {}^2M_{31}\, M_{10} - {}^2 s_3 s_0 + {}^2\, \hat{s}_3\, \hat{s}_0\},$$

so, from [II, eq. (20′) and (42′)], that:

$$= 2\, ({}^2M_{30}\, \Omega - {}^2M_{12}\, \hat{\Omega} + {}^2 s_2 \hat{s}_1 - {}^2\, \hat{s}_2 s_1) \qquad \text{(notation 1)}$$
$$= 2i\, ({}^2M_{20}\, M_{23} - {}^2M_{31}\, M_{10} - {}^2 s_3 s_0 + {}^2\, \hat{s}_3\, \hat{s}_0) \qquad \text{(notation 2).}$$

One will get (β_a) when one adds the first notation to its complex conjugate and employs [I, (34)] to symmetrize it.

Proof of (γ_a): Multiplying $2T_{\nu\bar{\sigma}}\, T^{-1}_{\bar{\rho}\mu}$ by $\alpha^{[23]}_{\bar{\sigma}\sigma}\, \alpha^{3}_{\rho\bar{\rho}}$ and summing over the indices $\bar{\rho}$, $\bar{\sigma}$ yields:

$$\begin{aligned} 2(T\alpha^{[23]})_{\nu\sigma}\, (\alpha^3 T^{-1})_{\rho\mu} &= -\, \alpha^{[24]}_{\rho\sigma}\alpha^{4}_{\mu\nu} - \alpha^{[314]}_{\rho\sigma}\alpha^{5}_{\mu\nu} - \alpha^{3}_{\rho\nu}\alpha^{[23]}_{\mu\sigma} + \alpha^{[12]}_{\rho\nu}\alpha^{1}_{\mu\sigma} \\ &\quad + i\left(\alpha^{[34]}_{\rho\nu}\alpha^{[234]}_{\mu\sigma} - \alpha^{[124]}_{\rho\nu}\alpha^{[14]}_{\mu\sigma} - \alpha^{2}_{\rho\sigma}\delta_{\mu\nu} + \alpha^{[31]}_{\rho\sigma}\alpha^{[123]}_{\mu\nu}\right). \end{aligned}$$

Multiplying by $\psi_\nu\, \psi_\sigma \dfrac{\partial \psi^*_\rho}{\partial x_3} \psi^*_\mu$ and summing over the indices μ, ν, ρ, σ leads to:

$$\begin{aligned} 2T_{23}\left(\frac{\partial \psi^*}{\partial x_3}, \alpha^3 T^{-1}\psi^*\right) &= -\, {}^3M_{20}\, \Omega + {}^3M_{31}\, \hat{\Omega} + {}^3 s_3\, \hat{s}_1 - {}^3\, \hat{s}_3\, s_1 \\ &\quad + i\, \{{}^3M_{30}\, M_{23} - {}^3M_{12}\, M_{10} + {}^3 s_2 s_0 - {}^2\, \hat{s}_2\, \hat{s}_0\}, \end{aligned}$$

so, from [II, eqs. (20′) and (41′)], that:

$$= \Omega\, {}^3M_{20} - {}^3M_{20}\, \Omega + {}^3M_{31}\, \hat{\Omega} - \hat{\Omega}\, M_{31} + {}^3 s_2\, \hat{s}_1 - {}^3\, \hat{s}_1\, s_3 + {}^3\, s_1\, \hat{s}_3 - {}^3\, \hat{s}_3\, s_1) \qquad \text{(notation 1),}$$

and from [II, eqs. (3′) and (39′)], that:

$$= 2i\, \{{}^3M_{30}\, M_{23} - {}^3M_{12}\, M_{10} + {}^3 s_2\, s_0 - {}^3\, \hat{s}_3\, \hat{s}_0) \qquad \text{(notation 2).}$$

One will get (γ_a) when one adds the first notation to its complex conjugate.

Proof of (δ_a): Multiplying $2T_{\nu\bar{\sigma}}\, T^{-1}_{\bar{\rho}\mu}$ by $\alpha^{[23]}_{\bar{\sigma}\sigma}$ and summing over the index $\bar{\sigma}$ yields:

$$\begin{aligned} 2(T\alpha^{[23]})_{\nu\sigma}\, T^{-1}{}_{\rho\mu} &= -\, \alpha^{[23]}_{\rho\sigma}\, \delta_{\mu\nu} - \delta_{\rho\nu}\, \alpha^{[23]}_{\mu\sigma} + \alpha^{[234]}_{\rho\sigma}\alpha^{4}_{\mu\nu} - \alpha^{4}_{\rho\nu}\alpha^{[234]}_{\mu\sigma} \\ &\quad + \alpha^{5}_{\rho\nu}\alpha^{[14]}_{\mu\sigma} - \alpha^{[14]}_{\rho\sigma}\alpha^{5}_{\mu\nu} + \alpha^{[123]}_{\rho\nu}\alpha^{1}_{\mu\sigma} - \alpha^{1}_{\rho\sigma}\alpha^{[123]}_{\mu\nu}. \end{aligned}$$

Multiplying by $\psi_\nu\, \psi_\sigma \dfrac{\partial \psi^*_\rho}{c\,\partial t} \psi^*_\mu$ and summing over the indices μ, ν, ρ, σ leads to:

$$2T_{23}\left(\frac{\partial\psi^*}{c\,\partial t},T^{-1}\psi^*\right)=\frac{1}{c}\,(\,{}^0\hat{s}_1\,s_0-{}^0\,\hat{s}_0\,\hat{s}_1+{}^0M_{23}\,\Omega-\Omega\,{}^0M_{23}$$
$$+\,{}^0M_{10}\,\hat{\Omega}\,-\,\hat{\Omega}\,{}^0M_{10}-{}^0s_1\,\hat{s}_0+{}^0\,\hat{s}_0\,s_1),\qquad\text{(notation 1)},$$

so, from [II, eqs. (45′) and (46)], that:

$$=\frac{2i}{c}\,\{\,{}^0s_3\,s_2-{}^0\,\hat{s}_3\,\hat{s}_2\,+\,{}^0M_{31}\,M_{12}+{}^0M_{20}\,M_{30})\qquad\text{(notation 2)}.$$

One will get (δ_a) then one adds the first notation to its complex conjugate. With that, the validity of the given construction of the component a_1 of the vector relation **a** is proved. Cyclic permutation will extend the proof to the remaining two components a_1, a_2.

B. **Proof of** (4): The first component of the vector relation **b** can be constructed with the help of the second notation in (α_a) to (δ_a). We arrange b_1 according to the derivatives $\partial / \partial x_1$, $\partial / \partial x_2$, $\partial / \partial x_3$, $\partial / c\,\partial t$ and must then prove that:

$$(\alpha_b)\quad\left\{\begin{aligned}&M_{23}\frac{\partial M_{10}}{\partial x_1}-M_{10}\frac{\partial M_{23}}{\partial x_1}+M_{20}\frac{\partial M_{31}}{\partial x_1}-M_{31}\frac{\partial M_{20}}{\partial x_1}\\&\quad M_{20}\frac{\partial M_{12}}{\partial x_1}-M_{12}\frac{\partial M_{30}}{\partial x_1}+\hat{\Omega}\frac{\partial\Omega}{\partial x_1}-\Omega\frac{\partial\hat{\Omega}}{\partial x_1}\\&\quad=2\left\{\left[-iT_{23}\left(\frac{\partial\psi^*}{\partial x_1},\alpha^1T^{-1}\psi^*\right)+[\cdots]^*\right]\right\},\end{aligned}\right.$$

$$(\beta_b)\quad\left\{\begin{aligned}&s_3\frac{\partial s_0}{\partial x_2}-s_0\frac{\partial s_3}{\partial x_2}+\hat{s}_0\frac{\partial\hat{s}_3}{\partial x_2}-\hat{s}_3\frac{\partial\hat{s}_0}{\partial x_2}\\&\quad+M_{31}\frac{\partial M_{10}}{\partial x_2}-M_{10}\frac{\partial M_{31}}{\partial x_2}+M_{23}\frac{\partial M_{20}}{\partial x_2}-M_{20}\frac{\partial M_{23}}{\partial x_2}\\&\quad=2\left\{\left[-iT_{23}\left(\frac{\partial\psi^*}{\partial x_2},\alpha^2T^{-1}\psi^*\right)+[\cdots]^*\right]\right\},\end{aligned}\right.$$

$$(\gamma_b)\quad \begin{cases} s_0 \dfrac{\partial s_2}{\partial x_3} - s_2 \dfrac{\partial s_0}{\partial x_3} + \hat{s}_2 \dfrac{\partial \hat{s}_0}{\partial x_3} - \hat{s}_0 \dfrac{\partial \hat{s}_2}{\partial x_3} \\ \quad + M_{12} \dfrac{\partial M_{10}}{\partial x_3} - M_{10} \dfrac{\partial M_{12}}{\partial x_3} + M_{23} \dfrac{\partial M_{30}}{\partial x_3} - M_{20} \dfrac{\partial M_{23}}{\partial x_3} \\ \quad = 2\left\{\left[-iT_{23}\left(\dfrac{\partial \psi^*}{\partial x_3}, \alpha^3 T^{-1}\psi^* \right) + [\cdots]^* \right]\right\}, \end{cases}$$

$$(\delta_b)\quad \begin{cases} M_{31} \dfrac{\partial M_{12}}{c\,\partial t} - M_{12} \dfrac{\partial M_{31}}{c\,\partial t} + M_{23} \dfrac{\partial M_{30}}{c\,\partial t} - M_{20} \dfrac{\partial M_{23}}{c\,\partial t} \\ \quad - s_2 \dfrac{\partial s_3}{c\,\partial t} + s_3 \dfrac{\partial s_2}{c\,\partial t} + \hat{s}_2 \dfrac{\partial \hat{s}_3}{c\,\partial t} - \hat{s}_3 \dfrac{\partial \hat{s}_2}{c\,\partial t} \\ \quad = 2\left\{\left[iT_{23}\left(\dfrac{\partial \psi^*}{c\,\partial t}, T^{-1}\psi^* \right) + [\cdots]^* \right]\right\}. \end{cases}$$

We multiply the expression in A that is referred to as the second notation by i, add it to its complex conjugate, and apply the symmetrization of the identities [I, (22), (23), (40)] in order to construct the first component of the vector relation **b** from the second notation of that expression.

3. Furthermore, the following conversion of the vector relations [I, eqs. (19**c** and **d**)] is true:

$$\boxed{\begin{aligned} &(5)\quad \frac{\mathbf{c}}{2} \equiv \left[\mathfrak{T}'' \left\{ -\left(\frac{\partial \psi^*}{c\,\partial t}, T^{-1}\psi^* \right) + \sum_{k=1}^{3} \left(\frac{\partial \psi^*}{\partial x_k}, \alpha^k T^{-1}\psi^* \right) \right\} \right] + [\cdots]^* = 0, \\ &(6)\quad \frac{\mathbf{d}}{2} \equiv \left[i\mathfrak{T}'' \left\{ -\left(\frac{\partial \psi^*}{c\,\partial t}, T^{-1}\psi^* \right) + \sum_{k=1}^{3} \left(\frac{\partial \psi^*}{\partial x_k}, \alpha^k T^{-1}\psi^* \right) \right\} \right] + [\cdots]^* = 0. \end{aligned}}$$

A. Proof of (5): We arrange the first component c_1 of **c** in parts that involve the derivatives with respect to x_1, x_2, x_3, ct, and then have to prove that:

$$(\alpha_c)\quad \begin{cases} M_{10} \dfrac{\partial s_1}{\partial x_1} - s_1 \dfrac{\partial M_{10}}{\partial x_1} + \hat{s}_0 \dfrac{\partial \hat{\Omega}}{\partial x_1} - \hat{\Omega} \dfrac{\partial \hat{s}_0}{\partial x_1} + s_2 \dfrac{\partial M_{20}}{\partial x_1} - M_{20} \dfrac{\partial s_2}{\partial x_1} + s_3 \dfrac{\partial M_{30}}{\partial x_1} - M_{30} \dfrac{\partial s_3}{\partial x_1} \\ \qquad = 2\left\{\left[T_{10}\left(\dfrac{\partial \psi^*}{\partial x_1}, \alpha^1 T^{-1}\psi^* \right) + [\cdots]^* \right]\right\}; \end{cases}$$

$$(\beta_c)\quad \begin{cases} \Omega\dfrac{\partial\hat{s}_3}{\partial x_2}-\hat{s}_3\dfrac{\partial\Omega}{\partial x_2}+M_{10}\dfrac{\partial s_2}{\partial x_2}-s_2\dfrac{\partial M_{10}}{\partial x_2}+M_{20}\dfrac{\partial s_1}{\partial x_2}-s_1\dfrac{\partial M_{20}}{\partial x_2}+M_{12}\dfrac{\partial s_0}{\partial x_2}-s_0\dfrac{\partial M_{12}}{\partial x_2} \\ \qquad\qquad =2\left\{\left[T_{10}\left(\dfrac{\partial\psi^*}{\partial x_2},\alpha^2T^{-1}\psi^*\right)+[\cdots]^*\right]\right\}; \end{cases}$$

$$(\gamma_c)\quad \begin{cases} \hat{s}_2\dfrac{\partial\Omega}{\partial x_3}-\Omega\dfrac{\partial\hat{s}_2}{\partial x_3}+M_{10}\dfrac{\partial s_3}{\partial x_3}-s_3\dfrac{\partial M_{10}}{\partial x_2}+M_{30}\dfrac{\partial s_1}{\partial x_3}-s_1\dfrac{\partial M_{30}}{\partial x_3}+s_0\dfrac{\partial M_{31}}{\partial x_3}-M_{31}\dfrac{\partial s_0}{\partial x_3} \\ \qquad\qquad =2\left\{\left[T_{10}\left(\dfrac{\partial\psi^*}{\partial x_3},\alpha^3T^{-1}\psi^*\right)+[\cdots]^*\right]\right\}; \end{cases}$$

$$(\delta_c)\quad \begin{cases} s_3\dfrac{\partial M_{31}}{c\,\partial t}-M_{31}\dfrac{\partial s_3}{c\,\partial t}+M_{12}\dfrac{\partial s_2}{c\,\partial t}-s_2\dfrac{\partial M_{12}}{c\,\partial t}+\hat{s}_1\dfrac{\partial\hat{\Omega}}{c\,\partial t}-\hat{\Omega}\dfrac{\partial\hat{s}_2}{c\,\partial t}+M_{10}\dfrac{\partial s_0}{c\,\partial t}-s_0\dfrac{\partial M_{10}}{c\,\partial t} \\ \qquad\qquad =-2\left\{\left[T_{10}\left(\dfrac{\partial\psi^*}{c\,\partial t},T^{-1}\psi^*\right)+[\cdots]^*\right]\right\}. \end{cases}$$

Proof of (α_c): Multiply $2T_{\nu\bar{\sigma}}T^{-1}_{\bar{\rho}\mu}$ by $\alpha^{[14]}_{\bar{\sigma}\sigma}\alpha^1_{\rho\bar{\rho}}$ and sum over the indices $\bar{\rho}$, $\bar{\sigma}$. That will give:

$$\begin{aligned} 2(T\alpha^{[14]})_{\nu\sigma}(\alpha^1T^{-1})_{\rho\mu} &= \alpha^5_{\rho\sigma}\alpha^{[123]}_{\mu\nu}-\alpha^{[123]}_{\rho\sigma}\alpha^5_{\mu\nu}+\alpha^{[14]}_{\rho\nu}\alpha^1_{\mu\sigma}-\alpha^1_{\rho\nu}\alpha^{[14]}_{\mu\sigma} \\ &+ i\left(\alpha^4_{\rho\sigma}\delta_{\mu\nu}-\delta_{\rho\sigma}\alpha^4_{\mu\nu}+\alpha^{[23]}_{\rho\nu}\alpha^{[234]}_{\mu\sigma}-\alpha^{[234]}_{\rho\nu}\alpha^{[23]}_{\mu\sigma}\right). \end{aligned}$$

Taking the inner product with $\psi_\nu\,\psi_\sigma\dfrac{\partial\psi^*_\rho}{\partial x_1}\psi^*_\mu$ will give:

$$\begin{aligned} 2T_{10}\left(\frac{\partial\psi^*}{\partial x_1},\alpha^1T^{-1}\psi^*\right) &= {}^1\hat{\Omega}\,\hat{s}_0-{}^1\hat{s}_0\,\hat{\Omega}-{}^1M_{10}\,s_1+{}^1s_1\,M_{10} \\ &+ i\,({}^1\Omega\,s_0-{}^1s_0\,\Omega+{}^1\hat{s}_0\,M_{23}-{}^1M_{23}\,\hat{s}_1), \end{aligned}$$

so, from [II, eqs. (7′) and (29′)], that:

$= {}^1\hat{\Omega}\,\hat{s}_0-{}^1\hat{s}_0\,\hat{\Omega}+{}^1s_1\,M_{10}-{}^1M_{10}\,s_1+{}^1M_{20}\,s_2-{}^1s_2\,M_{20}+{}^1M_{30}\,s_3-{}^1s_3\,M_{30}$ (notation 1),
$= i\{{}^1\Omega\,s_0-s_0\,{}^1\Omega+{}^1\hat{s}_1M_{23}-{}^1M_{23}\hat{s}_1+{}^1M_{31}\,\hat{s}_2-{}^1\hat{s}_2\,M_{31}+{}^1M_{12}\,\hat{s}_3-{}^1\hat{s}_3\,M_{12}\}$ (notation 2).

(α_c) will arise from the first notation when one adds it to its complex conjugate.

Proof of (β_c): We multiply $2T_{\nu\bar{\sigma}}T^{-1}_{\bar{\rho}\mu}$ by $\alpha^{[14]}_{\bar{\sigma}\sigma}\alpha^2_{\rho\bar{\rho}}$ and sum over the indices $\bar{\rho}$, $\bar{\sigma}$. That will give:

$$2(T\alpha^{[14]})_{\nu\sigma}(\alpha^2T^{-1})_{\rho\mu} = \alpha^{[12]}_{\rho\sigma}\alpha^4_{\mu\nu}+\alpha^{[24]}_{\rho\nu}\alpha^1_{\mu\sigma}-\alpha^{[124]}_{\rho\sigma}\,\delta_{\mu\nu}-\alpha^2_{\rho\nu}\alpha^{[14]}_{\mu\sigma}$$

$$+\ i\left(\alpha^{[31]}_{\rho\sigma}\alpha^{[234]}_{\mu\sigma}-\alpha^{[34]}_{\rho\sigma}\alpha^{[123]}_{\mu\nu}-\alpha^{3}_{\rho\sigma}\alpha^{5}_{\mu\nu}-\alpha^{[314]}_{\rho\nu}\alpha^{[23]}_{\mu\sigma}\right).$$

Taking the inner product with $\psi_\nu\ \psi_\sigma \dfrac{\partial\psi^*_\rho}{\partial x_2}\psi^*_\mu$ will give:

$$2T_{10}\left(\frac{\partial\psi^*}{\partial x_2},\alpha^2T^{-1}\psi^*\right) = {}^2\hat{s}_3\,\Omega\ -{}^2M_{20}\,s_1-{}^2M_{12}\,s_0+{}^2s_2\,M_{10}$$
$$+\,i\,({}^2\hat{s}_2\,M_{23}+{}^2M_{30}\,\hat{s}_0\ -{}^2s_3\ \hat{\Omega}-{}^2M_{31}\,\hat{s}_1),$$

so, from [II, eqs. (17′) and (36′)], that:

$$= 2\,({}^2\hat{s}_3\,\Omega\ -{}^2M_{20}\,s_1-{}^2M_{12}\,s_0+{}^2s_2\,M_{10}) \qquad \text{(notation 1)},$$
$$= 2i\{\,{}^2\hat{s}_2\,M_{23}+{}^2M_{30}\,\hat{s}_0\ -{}^2s_3\,\hat{\Omega}\ -{}^2M_{31}\,\hat{s}_1\ \} \qquad \text{(notation 2)}.$$

(β_c) will arise from the first notation when one adds it to its complex conjugate and considers the identity [I, (31)] for symmetrization.

Proof of (χ_c): We multiply $2T_{\nu\bar\sigma}\,T^{-1}_{\bar\rho\mu}$ by $\alpha^{[14]}_{\bar\sigma\sigma}\,\alpha^{3}_{\rho\bar\rho}$ and sum over the indices $\bar\rho$, $\bar\sigma$. That will give:

$$2(T\alpha^{[14]})_{\nu\sigma}\,(\alpha^3T^{-1})_{\rho\mu} = \alpha^{[34]}_{\rho\nu}\alpha^{1}_{\mu\sigma}-\alpha^{[31]}_{\rho\sigma}\alpha^{4}_{\mu\nu}+\alpha^{[314]}_{\rho\sigma}\,\delta_{\mu\nu}-\alpha^{3}_{\rho\nu}\alpha^{[14]}_{\mu\sigma}$$
$$+\ i\left(\alpha^{2}_{\rho\sigma}\alpha^{5}_{\mu\sigma}-\alpha^{[124]}_{\rho\nu}\alpha^{[23]}_{\mu\sigma}+\alpha^{[24]}_{\rho\sigma}\alpha^{[123]}_{\mu\nu}+\alpha^{[12]}_{\rho\nu}\alpha^{[234]}_{\mu\sigma}\right).$$

Taking the inner product with $\psi_\nu\ \psi_\sigma \dfrac{\partial\psi^*_\rho}{\partial x_3}\psi^*_\mu$ will give:

$$2T_{10}\left(\frac{\partial\psi^*}{\partial x_3},\alpha^3T^{-1}\psi^*\right) = -\,{}^3M_{30}\,s_1-{}^3\hat{s}_2\,\Omega+{}^3M_{31}\,s_0+{}^3s_3\,M_{10}$$
$$+\,i\,\{{}^3s_2\ \hat{\Omega}-{}^3M_{12}\,\hat{s}_1-\ {}^3M_{20}\,\hat{s}_0+{}^3\hat{s}_2\,M_{23}\},$$

so, from [II, eqs. (17′) and (35′)], that:

$$= 2\ \{-\,{}^3M_{30}\,s_1-{}^3\hat{s}_2\,\Omega\ +{}^3M_{31}\,s_0+{}^3s_3\,M_{10}\} \qquad \text{(notation 1)},$$
$$= 2i\{{}^3s_2\ \hat{\Omega}-{}^3M_{12}\ \hat{s}_1-{}^3M_{20}\,\hat{s}_0\ +{}^3\hat{s}_3\,M_{23}\} \qquad \text{(notation 2)}.$$

(χ_c) will arise from the first notation when one adds it to its complex conjugate and considers the identity [I, (31)] for symmetrization.

Proof of (δ_c): We multiply $2T_{\nu\bar{\sigma}}\,T^{-1}_{\bar{\rho}\mu}$ by $\alpha^{[14]}_{\bar{\sigma}\sigma}$ and sum over the index $\bar{\sigma}$. That will give:

$$\begin{aligned} 2(T\alpha^{[14]})_{\nu\sigma}\,T^{-1}_{\rho\mu} &= \alpha^{5}_{\rho\nu}\alpha^{[23]}_{\mu\sigma} - \alpha^{[23]}_{\rho\sigma}\alpha^{5}_{\mu\nu} + \alpha^{[14]}_{\rho\sigma}\,\delta_{\mu\nu} - \delta_{\rho\nu}\,\alpha^{[14]}_{\mu\sigma} \\ &+ i\left(\alpha^{4}_{\rho\nu}\alpha^{1}_{\mu\sigma} - \alpha^{1}_{\rho\sigma}\alpha^{4}_{\mu\nu} + \alpha^{[124]}_{\rho\nu}\alpha^{[234]}_{\mu\sigma} - \alpha^{[234]}_{\rho\sigma}\alpha^{[123]}_{\mu\nu}\right). \end{aligned}$$

Taking the inner product with $\psi_\nu\,\psi_\sigma\dfrac{\partial\psi^*_\rho}{c\,\partial t}\psi^*_\mu$ will give:

$$\begin{aligned} 2T_{10}\left(\frac{\partial\psi^*}{c\,\partial t}, T^{-1}\psi^*\right) &= \frac{1}{c}\{\,{}^0\hat{s}_1\,\hat{\Omega} - {}^0\hat{\Omega}\,\hat{s}_1 + {}^0M_{10}\,s_0 - {}^0s_0\,M_{10} \\ &+ i\,({}^0s_1\,\Omega - {}^0\Omega\,s_1 + {}^0M_{23}\,\hat{s}_0 - {}^0\hat{s}_0\,M_{23})\}, \end{aligned}$$

so, from [II, eqs. (8′) and (24′)], that:

$$= \frac{1}{c}\{\,{}^0\hat{s}_1\,\hat{\Omega} - {}^0\hat{\Omega}\,\hat{s}_1 + {}^0M_{10}\,s_0 - {}^0s_0\,M_{10} + {}^0M_{12}\,s_2 - {}^0s_2\,M_{12} + {}^0s_3\,M_{31} - {}^0M_{31}\,s_3\}\text{(notation 1)},$$

and from [II, eq. (13′)]:

$$= \frac{i}{c}\,({}^0\hat{s}_2\,M_{32} - {}^0M_{30}\,\hat{s}_2 + {}^0M_{20}\,\hat{s}_3 - {}^0\hat{s}_3\,M_{20} + {}^0s_1\,\Omega - {}^0\Omega\,s_1 + {}^0M_{23}\,\hat{s}_0 - {}^0\hat{s}_0\,M_{23})\ \text{(notation 2)}.$$

(δ_c) will arise from the first notation when one adds it to its complex conjugate.

With that, the proof for the entire expression (5) for the first component c_1 of **c** is complete. The proof can be extended to the two remaining components c_2, c_3 by cyclic permutation.

B. Proof of (6): The second notation in 3A allows one to construct the first component of **d**. We arrange it in derivatives with respect to x_1, x_2, x_3, ct, and have to show:

$$(\alpha_d)\quad \left\{\begin{aligned} &\hat{s}_1\frac{\partial M_{23}}{\partial x_1} - M_{23}\frac{\partial\hat{s}_1}{\partial x_1} + \Omega\frac{\partial s_0}{\partial x_1} - s_0\frac{\partial\Omega}{\partial x_1} + M_{31}\frac{\partial\hat{s}_2}{\partial x_1} - \hat{s}_2\frac{\partial M_{31}}{\partial x_1} + M_{12}\frac{\partial\hat{s}_3}{\partial x_1} - \hat{s}_3\frac{\partial M_{12}}{\partial x_1} \\ &\qquad = 2\left\{\left[iT_{10}\left(\frac{\partial\psi^*}{\partial x_1}, \alpha^1 T^{-1}\psi^*\right) + [\cdots]^*\right]\right\}; \end{aligned}\right.$$

$$(\beta_{\mathrm{d}})\quad \left\{\begin{aligned} &\hat{\Omega}\frac{\partial \hat{s}_3}{\partial x_2} - s_3\frac{\partial \hat{\Omega}}{\partial x_2} + \hat{s}_2\frac{\partial M_{23}}{\partial x_2} - M_{23}\frac{\partial \hat{s}_2}{\partial x_2} + \hat{s}_1\frac{\partial M_{31}}{\partial x_2} - M_{31}\frac{\partial \hat{s}_1}{\partial x_2} + M_{30}\frac{\partial \hat{s}_0}{\partial x_2} - \hat{s}_0\frac{\partial M_{30}}{\partial x_2} \\ &\qquad = 2\left\{\left[iT_{10}\left(\frac{\partial \psi^*}{\partial x_2}, \alpha^2 T^{-1}\psi^*\right) + [\cdots]^*\right]\right\}; \end{aligned}\right.$$

$$(\gamma_{\mathrm{d}})\quad \left\{\begin{aligned} &s_2\frac{\partial \hat{\Omega}}{\partial x_3} - \hat{\Omega}\frac{\partial s_2}{\partial x_3} + \hat{s}_3\frac{\partial M_{23}}{\partial x_3} - M_{23}\frac{\partial \hat{s}_3}{\partial x_3} + \hat{s}_1\frac{\partial M_{12}}{\partial x_3} - M_{12}\frac{\partial \hat{s}_1}{\partial x_3} + \hat{s}_0\frac{\partial M_{20}}{\partial x_3} - M_{20}\frac{\partial \hat{s}_0}{\partial x_3} \\ &\qquad = 2\left\{\left[iT_{10}\left(\frac{\partial \psi^*}{\partial x_3}, \alpha^3 T^{-1}\psi^*\right) + [\cdots]^*\right]\right\}; \end{aligned}\right.$$

$$(\delta_{\mathrm{d}})\quad \left\{\begin{aligned} &M_{30}\frac{\partial \hat{s}_2}{c\,\partial t} - \hat{s}_2\frac{\partial M_{30}}{c\,\partial t} + \hat{s}_3\frac{\partial M_{20}}{c\,\partial t} - M_{20}\frac{\partial \hat{s}_3}{c\,\partial t} + \Omega\frac{\partial s_1}{c\,\partial t} - s_1\frac{\partial \Omega}{c\,\partial t} + \hat{s}_0\frac{\partial M_{23}}{c\,\partial t} - M_{23}\frac{\partial \hat{s}_0}{c\,\partial t} \\ &\qquad = -2\left\{\left[iT_{10}\left(\frac{\partial \psi^*}{c\,\partial t}, T^{-1}\psi^*\right) + [\cdots]^*\right]\right\}. \end{aligned}\right.$$

If we multiply the second notation by i, symmetrize that expression with the help of the identity [I, (30)], and add it to its complex conjugate then we will get (α_{d}) to (δ_{d}). With that, the proof is complete for the first component d_1 of **d**. Cyclic permutation will extend it to the remaining two components d_1, d_2.

4. Furthermore, the following conversion of the vector relations [III, eqs. (19**e** and **f**)] is valid:

$$\boxed{\begin{aligned} &(7)\quad \frac{\mathbf{e}}{2} \equiv \left[-\mathfrak{T}'''\left\{-\left(\frac{\partial \psi^*}{c\,\partial t}, T^{-1}\psi^*\right) + \sum_{k=1}^{3}\left(\frac{\partial \psi^*}{\partial x_k}, \alpha^k T^{-1}\psi^*\right)\right\}\right] + [\cdots]^* = 0, \\ &(8)\quad \frac{\mathbf{f}}{2} \equiv \left[i\mathfrak{T}'''\left\{-\left(\frac{\partial \psi^*}{c\,\partial t}, T^{-1}\psi^*\right) + \sum_{k=1}^{3}\left(\frac{\partial \psi^*}{\partial x_k}, \alpha^k T^{-1}\psi^*\right)\right\}\right] + [\cdots]^* = 0. \end{aligned}}$$

A. Proof of (7): We arrange the first component c_1 of **c** in parts that have the derivatives with respect to x_1, x_2, x_3, ct, and then have to prove that:

$$(\alpha_{\mathrm{e}})\quad \left\{\begin{aligned} &s_1\frac{\partial M_{23}}{\partial x_1} - M_{23}\frac{\partial s_1}{\partial x_1} + \Omega\frac{\partial \hat{s}_0}{\partial x_1} - \hat{s}_0\frac{\partial \Omega}{\partial x_1} + M_{31}\frac{\partial s_2}{\partial x_1} - s_2\frac{\partial M_{31}}{\partial x_1} + M_{12}\frac{\partial s_3}{\partial x_1} - s_3\frac{\partial M_{12}}{\partial x_1} \\ &\qquad = 2\left\{\left[-T_1\left(\frac{\partial \psi^*}{\partial x_1}, \alpha^1 T^{-1}\psi^*\right) + [\cdots]^*\right]\right\}; \end{aligned}\right.$$

$$(\beta_{\mathrm{e}})\quad \left\{\begin{aligned} &\hat{\Omega}\frac{\partial \hat{s}_3}{\partial x_2}-\hat{s}_3\frac{\partial \Omega}{\partial x_2}+s_1\frac{\partial M_{31}}{\partial x_2}-M_{31}\frac{\partial s_1}{\partial x_2}+s_2\frac{\partial M_{23}}{\partial x_2}-M_{23}\frac{\partial s_2}{\partial x_2}+M_{30}\frac{\partial s_0}{\partial x_2}-s_0\frac{\partial M_{30}}{\partial x_2}\\ &\qquad =2\left\{\left[-T_1\left(\frac{\partial \psi^*}{\partial x_2},\alpha^2T^{-1}\psi^*\right)+[\cdots]^*\right]\right\};\end{aligned}\right.$$

$$(\gamma_{\mathrm{e}})\quad \left\{\begin{aligned} &\hat{s}_2\frac{\partial \hat{\Omega}}{\partial x_3}-\hat{\Omega}\frac{\partial \hat{s}_2}{\partial x_3}+s_3\frac{\partial M_{23}}{\partial x_3}-M_{23}\frac{\partial s_3}{\partial x_3}+s_0\frac{\partial M_{20}}{\partial x_3}-M_{20}\frac{\partial s_0}{\partial x_3}+s_1\frac{\partial M_{12}}{\partial x_3}-M_{12}\frac{\partial s_1}{\partial x_3}\\ &\qquad =2\left\{\left[-T_1\left(\frac{\partial \psi^*}{\partial x_3},\alpha^3T^{-1}\psi^*\right)+[\cdots]^*\right]\right\};\end{aligned}\right.$$

$$(\delta_{\mathrm{e}})\quad \left\{\begin{aligned} &s_3\frac{\partial M_{20}}{c\,\partial t}-M_{20}\frac{\partial s_3}{c\,\partial t}+M_{30}\frac{\partial s_2}{c\,\partial t}-s_2\frac{\partial M_{30}}{c\,\partial t}+\Omega\frac{\partial \hat{s}_1}{c\,\partial t}-\hat{s}_1\frac{\partial \Omega}{c\,\partial t}+s_0\frac{\partial M_{23}}{c\,\partial t}-M_{23}\frac{\partial s_0}{c\,\partial t}\\ &\qquad =-2\left\{\left[-T_1\left(\frac{\partial \psi^*}{c\,\partial t},T^{-1}\psi^*\right)+[\cdots]^*\right]\right\}.\end{aligned}\right.$$

Proof of (α_{e}): Multiply $2T_{\nu\bar{\sigma}}\,T^{-1}_{\bar{\rho}\mu}$ by $\alpha^{[234]}_{\bar{\sigma}\sigma}\,\alpha^{1}_{\rho\bar{\rho}}$, sum over the indices $\bar{\rho}$, $\bar{\sigma}$, and get:

$$\begin{aligned}2(T\alpha^{[234]})_{\nu\sigma}\,(\alpha^1T^{-1})_{\rho\mu} \;=\;& \alpha^{[123]}_{\rho\sigma}\alpha^{4}_{\mu\nu}-\alpha^{4}_{\rho\sigma}\alpha^{[123]}_{\mu\nu}+\alpha^{[234]}_{\rho\nu}\alpha^{1}_{\mu\sigma}-\alpha^{1}_{\rho\nu}\alpha^{[234]}_{\mu\sigma}\\ &+\;i\left\{\alpha^{[14]}_{\rho\nu}\alpha^{[23]}_{\mu\sigma}-\alpha^{[23]}_{\rho\nu}\alpha^{[14]}_{\mu\sigma}+\alpha^{5}_{\rho\sigma}\delta_{\mu\nu}-\delta_{\rho\sigma}\alpha^{5}_{\mu\nu}\right\}.\end{aligned}$$

Taking the inner product with $\psi_\nu\,\psi_\sigma\dfrac{\partial \psi^*_\rho}{\partial x_1}\psi^*_\mu$ will give:

$$\begin{aligned}2T_1\left(\frac{\partial \psi^*}{\partial x_1},\alpha^1T^{-1}\psi^*\right) \;=\;& {}^1\Omega\,\hat{s}_0-{}^1\hat{s}_0\,\Omega\;+\;{}^1s_1\,M_{23}-{}^1M_{23}\,s_1\\ &+\,i\,(\{\,{}^1M_{10}\,\hat{s}_1-{}^1\hat{s}_1\,M_{10}-\hat{\Omega}\,s_0+{}^1s_0\,\hat{\Omega}\},\end{aligned}$$

so, from [II, eqs. (5′) and (27′)], that:

$= {}^1\Omega\,\hat{s}_0-{}^1\hat{s}_0\,\Omega\;+\;{}^1s_1\,M_{23}-{}^1M_{23}\,s_1+{}^1M_{31}\,s_2-{}^1s_2\,M_{31}+{}^1M_{12}\,s_3-{}^1s_3\,M_{12}$ (notation 1),

and from [II, eqs. (15′) and (30′)], that:

$= i\{{}^1s_0\;\hat{\Omega}-{}^1\hat{\Omega}\,s_0+{}^1s_1\,M_{23}-{}^1M_{23}\,s_1+{}^1M_{31}\,s_2-{}^1s_2\,M_{31}+{}^1M_{12}\,s_3-{}^1s_3\,M_{12}\}$ (notation 2).

We add the first notation to its complex conjugate and get (–1)-times (α_{e}).

Proof of (β_e): We multiply $2T_{\nu\bar{\sigma}}\,T^{-1}_{\bar{\rho}\mu}$ by $\alpha^{[234]}_{\bar{\sigma}\sigma}\,\alpha^{2}_{\rho\bar{\rho}}$, sum over the indices $\bar{\rho}$, $\bar{\sigma}$, and get:

$$2(T\alpha^{[234]})_{\nu\sigma}\,(\alpha^2 T^{-1})_{\rho\mu} = \alpha^{[12]}_{\rho\sigma}\alpha^{5}_{\mu\nu} + \alpha^{[314]}_{\rho\nu}\alpha^{1}_{\mu\sigma} + \alpha^{[34]}_{\rho\sigma}\,\delta_{\mu\nu} - \alpha^{2}_{\rho\nu}\alpha^{[234]}_{\mu\sigma}$$
$$+\; i\left\{\alpha^{3}_{\rho\sigma}\alpha^{4}_{\mu\nu} + \alpha^{[24]}_{\rho\nu}\alpha^{[23]}_{\mu\sigma} - \alpha^{[124]}_{\rho\sigma}\alpha^{[123]}_{\mu\nu} - \alpha^{[31]}_{\rho\nu}\alpha^{[14]}_{\mu\sigma}\right\}.$$

Taking the inner product with $\psi_\nu\,\psi_\sigma\dfrac{\partial\psi^*_\rho}{\partial x_2}\psi^*_\mu$ will give:

$$2T_1\left(\frac{\partial\psi^*}{\partial x_2}, \alpha^2 T^{-1}\psi^*\right) = -\,{}^2\hat{s}_3\,\hat{\Omega} - {}^2M_{31}\,s_1 + {}^2M_{30}\,s_0 - {}^2s_2\,M_{23}$$
$$+\; i\,\{-\,{}^2s_3\,\Omega + {}^2M_{20}\,\hat{s}_1 + {}^2M_{12}\,\hat{s}_0 - {}^2\hat{s}_2\,M_{10}\},$$

so, from [II, eqs. (13′) and (31)], that:

$$= 2\,\{-\,{}^2\hat{s}_3\,\hat{\Omega} - {}^2M_{31}\,s_1 + {}^2M_{30}\,s_0 + {}^2s_2\,M_{23}\} \quad \text{(notation 1)},$$
$$= 2i\{-\,{}^2s_3\,\Omega + {}^2M_{20}\,\hat{s}_1 + {}^2M_{12}\,\hat{s}_0 - {}^2\hat{s}_2\,M_{10}\} \quad \text{(notation 2)}.$$

Adding the first notation to its complex conjugate, and considering the identity [I, (29)] for symmetrization, one will get (–1)-times (β_e).

Proof of (γ_e): We multiply $2T_{\nu\bar{\sigma}}\,T^{-1}_{\bar{\rho}\mu}$ by $\alpha^{[234]}_{\bar{\sigma}\sigma}\,\alpha^{3}_{\rho\bar{\rho}}$, sum over the indices $\bar{\rho}$, $\bar{\sigma}$, and get:

$$2(T\alpha^{[234]})_{\nu\sigma}\,(\alpha^3 T^{-1})_{\rho\mu} = \alpha^{[124]}_{\rho\nu}\alpha^{1}_{\mu\sigma} - \alpha^{[31]}_{\rho\sigma}\alpha^{5}_{\mu\nu} + \alpha^{[24]}_{\rho\sigma}\,\delta_{\mu\nu} - \alpha^{3}_{\rho\nu}\alpha^{[234]}_{\mu\sigma}$$
$$+\; i\left\{\alpha^{[34]}_{\rho\nu}\alpha^{[23]}_{\mu\sigma} - \alpha^{2}_{\rho\sigma}\alpha^{4}_{\mu\nu} + \alpha^{[314]}_{\rho\sigma}\alpha^{[123]}_{\mu\nu} - \alpha^{[12]}_{\rho\nu}\alpha^{[14]}_{\mu\sigma}\right\}.$$

Taking the inner product with $\psi_\nu\,\psi_\sigma\dfrac{\partial\psi^*_\rho}{\partial x_3}\psi^*_\mu$ will give:

$$2T_1\left(\frac{\partial\psi^*}{\partial x_3}, \alpha^3 T^{-1}\psi^*\right) = -\,{}^3M_{12}\,s_1 + {}^3\hat{s}_2\,\hat{\Omega} - {}^3M_{20}\,s_0 + {}^3s_3\,M_{23}$$
$$+\; i\,\{{}^3M_{30}\,\hat{s}_1 + {}^3s_2\,\Omega - {}^3M_{31}\,\hat{s}_0 - {}^3\hat{s}_3\,M_{10}\},$$

so, from [II, eqs. (13′) and (32)], that:

$$= 2\{-\,{}^3M_{12}\,s_1 + {}^3\hat{s}_2\,\hat{\Omega} - {}^3M_{20}\,s_0 + {}^3s_3\,M_{23}\} \quad \text{(notation 1)},$$
$$= 2i\{{}^3M_{30}\,\hat{s}_1 + {}^3s_2\,\Omega - {}^3M_{31}\,\hat{s}_0 - {}^3\hat{s}_3\,M_{10}\} \quad \text{(notation 2)}.$$

After adding first notation to its complex conjugate and considers the identity [I, (29)] for symmetrization, one will get (–1)-times (γ_e).

Proof of (δ_e): We multiply $2T_{\nu\bar{\sigma}}\,T^{-1}_{\bar{\rho}\mu}$ by $\alpha^{[234]}_{\bar{\sigma}\sigma}$ and sum over the index $\bar{\sigma}$, and get:

$$\begin{aligned} 2(T\alpha^{[234]})_{\nu\sigma}\,T^{-1}_{\rho\mu} \;&=\; \alpha^{[23]}_{\rho\sigma}\alpha^{5}_{\mu\nu} - \alpha^{4}_{\rho\nu}\alpha^{[23]}_{\mu\sigma} + \alpha^{[234]}_{\rho\sigma}\,\delta_{\mu\nu} - \delta_{\rho\nu}\,\alpha^{[234]}_{\mu\sigma} \\ &\quad + \; i\left(-\alpha^{1}_{\rho\sigma}\alpha^{5}_{\mu\nu} + \alpha^{5}_{\rho\nu}\alpha^{1}_{\mu\sigma} - \alpha^{[14]}_{\rho\sigma}\alpha^{[123]}_{\mu\nu} + \alpha^{[123]}_{\rho\nu}\alpha^{[14]}_{\mu\sigma}\right). \end{aligned}$$

Taking the inner product with $\psi_\nu\,\psi_\sigma\dfrac{\partial\psi^*_\rho}{c\,\partial t}\psi^*_\mu$ will give:

$$\begin{aligned} 2T_1\left(\frac{\partial\psi^*}{c\,\partial t}, T^{-1}\psi^*\right) = \frac{1}{c}\,\{\,{}^0\hat{s}_1\,\Omega - {}^0\Omega\,\hat{s}_1 + {}^0M_{23}\,s_0 - {}^0s_0\,M_{23} \\ + i\,({}^0\hat{\Omega}\,s_1 - {}^0s_1\,\hat{\Omega} + {}^0\hat{s}_0\,M_{10} - {}^0M_{10}\,\hat{s}_0)\},\end{aligned}$$

so, from [II, eqs. (8′) and (18′)], that:

$$= \frac{1}{c}\,\{\,{}^0\hat{s}_1\,\Omega - {}^0\Omega\,\hat{s}_1 + {}^0M_{23}\,s_0 - {}^0s_0\,M_{23} + {}^0s_2\,M_{30} - {}^0M_{30}\,s_2 + {}^0M_{20}\,s_3 - {}^0s_3\,M_{20}\}\text{(notation 1)},$$

and from [II, eq. (15′) and (17′)]:

$$= -\,\frac{i}{c}\,(\,{}^0s_1\,\hat{\Omega} - {}^0\hat{\Omega}\,s_1 + {}^0M_{10}\,\hat{s}_0 - {}^0\hat{s}_0\,M_{10} + {}^0M_{12}\,\hat{s}_2 - {}^0\hat{s}_2\,M_{12} + {}^0\hat{s}_3\,M_{31} - {}^0M_{31}\,\hat{s}_3)$$

(notation 2).

After adding the first notation to its complex conjugate, we will get (–1)-times (δ_e).

With that, the proof for the entire expression (7) for the first component e_1 of **e** is complete. The proof can be extended to the two remaining components e_2, e_3 by cyclic permutation.

A. Proof of (8): The second notation in 4A allows us to construct the first component f_1 of **f**. We arrange it in parts that have the derivatives with respect to x_1, x_2, x_3, ct, and then have to prove that:

$$(\alpha_f)\quad \left\{\begin{aligned} &\hat{s}_2\frac{\partial M_{20}}{\partial x_1} - M_{20}\frac{\partial\hat{s}_2}{\partial x_1} + M_{10}\frac{\partial\hat{s}_1}{\partial x_1} - \hat{s}_1\frac{\partial M_{10}}{\partial x_1} + s_0\frac{\partial\hat{\Omega}}{\partial x_1} - \hat{\Omega}\frac{\partial s_0}{\partial x_1} + \hat{s}_3\frac{\partial M_{20}}{\partial x_1} - M_{30}\frac{\partial\hat{s}_2}{\partial x_1} \\ &\qquad = 2\left\{\left[iT_1\left(\frac{\partial\psi^*}{\partial x_1}, \alpha^1 T^{-1}\psi^*\right) + [\cdots]^*\right]\right\}; \end{aligned}\right.$$

$$(\beta_{\mathrm{f}})\quad \left\{\begin{aligned} &\Omega\frac{\partial s_3}{\partial x_2}-s_3\frac{\partial\Omega}{\partial x_2}+M_{10}\frac{\partial\hat{s}_2}{\partial x_2}-\hat{s}_2\frac{\partial M_{10}}{\partial x_2}+M_{20}\frac{\partial\hat{s}_1}{\partial x_2}-\hat{s}_1\frac{\partial M_{20}}{\partial x_2}+M_{12}\frac{\partial\hat{s}_0}{\partial x_2}-\hat{s}_0\frac{\partial M_{12}}{\partial x_2}\\ &\qquad=2\left\{\left[iT_1\left(\frac{\partial\psi^*}{\partial x_2},\alpha^2T^{-1}\psi^*\right)+[\cdots]^*\right]\right\};\end{aligned}\right.$$

$$(\gamma_{\mathrm{f}})\quad \left\{\begin{aligned} &s_2\frac{\partial\Omega}{\partial x_3}-\Omega\frac{\partial s_2}{\partial x_3}+M_{10}\frac{\partial\hat{s}_3}{\partial x_3}-\hat{s}_3\frac{\partial M_{10}}{\partial x_3}+M_{30}\frac{\partial\hat{s}_1}{\partial x_3}-\hat{s}_1\frac{\partial M_{30}}{\partial x_3}+\hat{s}_0\frac{\partial M_{31}}{\partial x_3}-M_{31}\frac{\partial\hat{s}_0}{\partial x_3}\\ &\qquad=2\left\{\left[iT_1\left(\frac{\partial\psi^*}{\partial x_3},\alpha^3T^{-1}\psi^*\right)+[\cdots]^*\right]\right\};\end{aligned}\right.$$

$$(\delta_{\mathrm{f}})\quad \left\{\begin{aligned} &M_{12}\frac{\partial\hat{s}_2}{c\,\partial t}-\hat{s}_2\frac{\partial M_{12}}{c\,\partial t}+\hat{s}_3\frac{\partial M_{31}}{c\,\partial t}-M_{31}\frac{\partial\hat{s}_3}{c\,\partial t}+s_1\frac{\partial\hat{\Omega}}{c\,\partial t}-\hat{\Omega}\frac{\partial s_1}{c\,\partial t}+M_{10}\frac{\partial\hat{s}_0}{c\,\partial t}-\hat{s}_0\frac{\partial M_{10}}{c\,\partial t}\\ &\qquad=-2\left\{\left[iT_1\left(\frac{\partial\psi^*}{c\,\partial t},T^{-1}\psi^*\right)+[\cdots]^*\right]\right\}.\end{aligned}\right.$$

We multiply the second notation by i and add it to its complex conjugate and consider the expression in the identity [I, (32)] for symmetrization. In that way, we will get (α_{f}) to (δ_{f}), whose sum will yield the first component f_1 of **f**. We will get the remaining components f_2, f_3, in turn, by cyclic permutation.

5. Looking back at paragraphs **2-4**, we see from the form of eqs. (3)-(8) that our six vector equations contain only two scalar equations as their nucleus under the assumption that not all three complex vectors $\mathfrak{T}'$, $\mathfrak{T}''$, $\mathfrak{T}'''$ vanish simultaneously. In the general case, they will be non-zero, and our six vector equations will then be fulfilled in such a way that the real part, as well as the imaginary of the scalar:

$$(9)\qquad \boxed{\tau\equiv-\left(\frac{\partial\psi^*}{c\,\partial t},T^{-1}\psi^*\right)+\sum_{k=1}^{3}\left(\frac{\partial\psi^*}{\partial x_k},\alpha^kT^{-1}\psi^*\right)}$$

will vanish:

$$(10)\qquad \tau+\tau^*=0,$$

$$(11)\qquad i(\tau-\tau^*)=0.$$

As a glance back at [III, eqs. (19**a**) to (**f**)] will show, it is not simple to see the decomposability of the vector relation in that form. With the help of the second **Pauli** bilinear equation, we have now been led back to eqs. (10) and (11), and in the next paragraph, we will prove the agreement of those equations with the two scalar reality relations [III, eqs. (17) and (18)].

§ 2. Conversion of the complex scalar τ

In order to express eqs. (10) and (11) in the running quantities Ω, $\hat{\Omega}$, s_0, $\hat{s}_0$, $\hat{\mathfrak{s}}$, $\mathfrak{M}$, $\hat{\mathfrak{M}}$, and their derivatives, each of the six vector equations (3)-(8) can be selected, and the corresponding vector $\mathfrak{T}'$, $\mathfrak{T}''$, $\mathfrak{T}'''$ can be separated from the scalar τ by using the algebraic tools of Part II. We choose eq. (3) and carry out certain conversions on the expressions (α_a)-(δ_a) that will allow the complex vector $\mathfrak{T}'$ to emerge from those expressions.

In order to do that, we introduce the solutions [II, eqs. (61) and (65)] for ${}^k s_0$, ${}^k\mathfrak{s}$, ${}^k\hat{s}_0$, ${}^0\hat{\mathfrak{s}}$ into those expressions:

$$(\alpha_a)\qquad 2T_{23}\left(\frac{\partial\psi^*}{\partial x_1}, \alpha^1 T^{-1}\psi^*\right) = {}^1s_0\,\hat{s}_0 - {}^1\hat{s}_0\,s_0 + {}^1s_1\,\hat{s}_1 - {}^1\hat{s}_1\,s_1 + {}^1s_2\,\hat{s}_2 - {}^1\hat{s}_2\,s_2 + {}^1s_3\,\hat{s}_3 - {}^1\hat{s}_3\,s_3$$

$$= -\,i\,{}^1B\,\{\hat{s}_0{}^2 + \hat{s}_1{}^2 - \hat{s}_2{}^2 - \hat{s}_3{}^2 - s_0{}^2 - s_1{}^2 + s_2{}^2 + s_3{}^2\} + {}^1\xi\,(\hat{s}_0 + \hat{s}_1 q_1 - \hat{s}_2 q_2 - \hat{s}_3 q_3) - {}^1\eta\,(s_0 + s_1 q_1 - s_2 q_2 - s_3 q_3).$$

With consideration given to the identities [I, (15) and (16)] and eqs. [II, (63a) and (b)], that will become:

$$= 2\,\{-\,i\,{}^1B\,(\hat{s}_1{}^2 - s_1{}^2 - \Omega^2 - \hat{\Omega}^2) + {}^1\xi\,\hat{s}_1 - {}^1\eta\,s_1\},$$

from which, with [II, eq. (71)], it will emerge that:

$$= 2\,q_1\,\{i\,{}^1B\,(\hat{s}_0{}^2 - \hat{s}_1{}^2 - \Omega^2 - \hat{\Omega}^2)\,p_1 + {}^1\xi\,\hat{s}_1 - {}^1\eta\,s_1\}.$$

When one eliminates ${}^2M_{30}$ and ${}^2M_{12}$ from:

$$(\beta_a)\qquad 2T_{23}\left(\frac{\partial\psi^*}{\partial x_2}, \alpha^2 T^{-1}\psi^*\right) = 2({}^2M_{30}\,\Omega - {}^2M_{12}\,\hat{\Omega} + {}^2s_2\,\hat{s}_1 - {}^2\hat{s}_2\,s_1)$$

by using [II, eq. (3′)], that will become:

$$= 2\{i\,({}^2s_0\,s_3 - {}^2s_3\,s_0) + {}^2s_2\,\hat{s}_1 - {}^2\hat{s}_2\,s_2\},$$

and with consideration given to the identities [II, eqs. (63a), (72), (75)], that will become:

$$= 2\,q_1\,\{i\,{}^2B\,(s_0{}^2 - \hat{s}_0{}^2 - \Omega^2 - \hat{\Omega}^2)\,p_2 + {}^2\xi\,\hat{s}_2 - {}^2\eta\,s_2\}.$$

When one eliminates ${}^2M_{20}$ and ${}^2M_{31}$ from:

$$(\gamma_a)\qquad 2T_{23}\left(\frac{\partial\psi^*}{\partial x_3},\alpha^3T^{-1}\psi^*\right) = {}^3\Omega\, M_{20} - {}^3M_{20}\,\hat{\Omega} + {}^3M_{31}\,\hat{\Omega} - {}^3\hat{\Omega}\, M_{31}$$
$$+ {}^3s_3\,\hat{s}_1 - {}^3\hat{s}_1 s_3 + {}^3s_1\,\hat{s}_3 - {}^3\hat{s}_3\, s_1$$

by using [II, eq. (3) and (11′)], that will become:

$$= i\,({}^3s_2\, s_0 - {}^3s_0\, s_2 + {}^3\hat{s}_0\,\hat{s}_2 - {}^3\hat{s}_2\,\hat{s}_0) + {}^3s_2\,\hat{s}_1 - {}^3\hat{s}_1\, s_3 + {}^3s_1\,\hat{s}_3 - {}^3\hat{s}_3\, s_1$$
$$= 2\,{}^3B\{(s_0\,\hat{s}_2 - \hat{s}_0\, s_2) + i\,(s_3\, s_1 - \hat{s}_3\,\hat{s}_1)\} + {}^3\xi\,\{i\,(q_2\, s_0 - s_2) + (q_3\,\hat{s}_1 + q_1\,\hat{s}_3)\}$$
$$- {}^3\eta\,\{i\,(q_2\,\hat{s}_0 - \hat{s}_2) + (q_3\, s_3 + q_3\, s_1)\},$$

and with consideration given to the identities [II, eqs. (63c and d), (72), (75)], that will become:

$$= 2\,q_1\,\{\,(\,s_0{}^2 - \hat{s}_0{}^2 - \Omega^2 - \hat{\Omega}^2\,)\; i\,{}^3B\,p_3 + {}^3\xi\,\hat{s}_3 - {}^3\eta\, s_3\}.$$

When one eliminates ${}^0M_{23}$ and ${}^0M_{10}$ from:

$$(\delta_a)\qquad 2T_{23}\left(\frac{\partial\psi^*}{c\,\partial t},T^{-1}\psi^*\right) = {}^0\hat{s}_1 s_0 - {}^0s_0\,\hat{s}_1 - {}^0s_1\,\hat{s}_0 + {}^0\hat{s}_0\, s_1$$
$${}^0M_{23}\,\Omega - {}^0\Omega\, M_{23} + {}^0M_{10}\,\hat{\Omega} - {}^0\hat{\Omega}\, M_{10}$$

by using [II, eq. (43) and (44′)], that will become:

$$= {}^0\hat{s}_1\, s_0 - {}^0s_0\,\hat{s}_1 + {}^0\hat{s}_0\, s_1 - {}^0s_1\,\hat{s}_0 + i\,({}^0s_3\, s_2 - {}^0s_2\, s_3 + {}^0\hat{s}_2\,\hat{s}_3 - {}^0\hat{s}_3\,\hat{s}_2)$$
$$= 2\,{}^0B\{(s_2\,\hat{s}_3 - \hat{s}_2\, s_3) - i\,(s_0\, s_1 - \hat{s}_0\,\hat{s}_1)\} + {}^0\xi\,\{i\,(s_2\, q_3 - s_3\, q_2) - (q_1\,\hat{s}_0 + \hat{s}_1)\}$$
$$+ {}^0\eta\,\{i\,(q_2\,\hat{s}_3 - q_3\,\hat{s}_2) + (q_1\, s_0 + s_1)\},$$

and with consideration given to the identities [II, eqs. (63e and d), (64)], that will become:

$$= 2\,q_1\,\{-\,i\,{}^0B\,(\,s_0{}^2 - \hat{s}_0{}^2 - \Omega^2 - \hat{\Omega}^2\,) - \hat{s}_0\,{}^0\xi + s_0\,{}^3\eta\}.$$

Upon comparing the four cases, one will see that q_1 enters into all of the expressions multiplicatively:

$$(12)\quad \begin{cases} 2T_{23}\left\{-\left(\dfrac{\partial\psi^*}{c\,\partial t},T^{-1}\psi^*\right)+\displaystyle\sum_{k=1}^{3}\left(\dfrac{\partial\psi^*}{\partial x_k},\alpha^k T^{-1}\psi^*\right)\right\} \\ = 2q_1\displaystyle\sum_{\mu=0}^{3}\{i(s_0{}^2-\hat{s}_0{}^2-\Omega^2-\hat{\Omega}^2)\,{}^{\mu}Bp_\mu+\hat{s}_\mu\cdot{}^{\mu}\xi-s_\mu\cdot{}^{\mu}\eta\}. \end{cases}$$

(For the sake of symmetry, the term $\mu = 0$ is carried along with the other ones, so the upper index $\mu = 0$ will then means differentiation with respect to ct, and $p_0 = 1$). The sum in (12) will be preserved by cyclically permuting the indices 1, 2, 3, while T_{23} will run through the components of the vector $\mathfrak{T}'$, and q_1 will run through those of the vector $\mathfrak{q}$. Those two vectors are connected by:

$$(13)\qquad \boxed{\mathfrak{T}' = -qT_0,}$$

and one will get:

$$(14)\qquad \boxed{\tau = -\frac{1}{T_0}\sum_{\mu=0}^{3}\{i(s_0{}^2-\hat{s}_0{}^2-\Omega^2-\hat{\Omega}^2)\,{}^{\mu}B\,p_\mu+\hat{s}_\mu\cdot{}^{\mu}\xi-s_\mu\cdot{}^{\mu}\eta}$$

for the scalar τ that was mentioned in (9). In that, $T_0 = (\psi, T\,\alpha^{[123]}\,\psi)$ is the last of the quantities that were defined in (2).

In order to prove eq. (13), one makes use of the second **Pauli** bilinear equation § 1, eq. (***). One will obtain:

$$\text{a)}\qquad T_0\,T_0^* = (\psi, T\,\alpha^{[123]}\,\psi)(\psi^*, \alpha^{[123]}\,T^{-1}\psi^*) = s_0^2-\hat{s}_0^2-\Omega^2-\hat{\Omega}^2$$

from that equation when one forms 2 $(\psi, T\,T^{-1}\psi^*)^2$ with it and observes the symmetry behavior in the resulting summands. Moreover, one will find the first component from:

$$\text{b)}\qquad \mathfrak{T}'T_0^* = -\{s_0\,\mathfrak{s}-\hat{s}_0\,\hat{\mathfrak{s}}+i\,[\mathfrak{s},\hat{\mathfrak{s}}]\}$$

when one forms $(\psi, T\,\alpha^{[23]}\,\psi)(\psi^*, \alpha^{[123]}\,T^{-1}\psi^*)$ with it. Combining both equations, in conjunction with [II, eq. (64)], will yield (13).

By the way, one will find the connections between the remaining two vectors $\mathfrak{T}''$ and $\mathfrak{T}'''$ that were defined in (2) and T_0 in a similar way; they read:

$$\mathfrak{T}'' = \frac{[\hat{\mathfrak{s}},\mathfrak{M}]-i[\mathfrak{s},\hat{\mathfrak{M}}]}{s_0{}^2-\hat{s}_0{}^2-\Omega^2-\hat{\Omega}^2}\,T_0, \qquad \mathfrak{T}''' = -\frac{[\hat{\mathfrak{s}},\hat{\mathfrak{M}}]+i[\mathfrak{s},\mathfrak{M}]}{s_0{}^2-\hat{s}_0{}^2-\Omega^2-\hat{\Omega}^2}\,T_0.$$

It follows from τ (with $q_0 = 1$) that:

(15) $$\tau^* = -\frac{1}{T_0^*}\sum_{\mu=0}^{3}\{-i(s_0{}^2 - \hat{s}_0{}^2 - \Omega^2 - \hat{\Omega}^2)B^\mu \cdot q_\mu + \hat{s}_\mu \cdot \xi^\mu - s_\mu \cdot \eta^\mu\}.$$

For the further conversion of (14) [(15), resp.], we recall the meanings of the quantities ${}^\mu B$, ${}^\mu\xi$, ${}^\mu\eta$, p_μ [B^μ, ξ^μ, η^μ, q_μ, resp.] in Part II and define:

$${}^\mu B + B^\mu = \frac{1}{\Omega^2 + \hat{\Omega}^2}\left(\hat{\Omega}\frac{\partial\Omega}{\partial x_\mu} - \Omega\frac{\partial\hat{\Omega}}{\partial x_\mu}\right),$$

according to [II, eq. (62)], define:

$$i\,({}^\mu B - B^\mu) = \frac{1}{\Omega^2 + \hat{\Omega}^2}\left\{-s_0\frac{\partial\hat{s}_0}{\partial x_\mu} + \left(\mathfrak{s}, \frac{\partial\hat{\mathfrak{s}}}{\partial x_\mu}\right)\right\}, \qquad \mu = 0, 1, 2, 3,$$

according to [II, eqs. (62) and (68)], and define:

$${}^k B\, p_k + B^k\, q_k = \frac{1}{s_0{}^2 - \hat{s}_0{}^2 - \Omega^2 - \hat{\Omega}^2}\left\{(s_0\,\mathfrak{s} - \hat{s}_0\,\hat{\mathfrak{s}})_k\,({}^k B + B^k) - [\mathfrak{s}, \hat{\mathfrak{s}}]_k \cdot i\,({}^k B - B^k)\right\}$$

and

$$i\,({}^k B\, p_k - B^k\, q_k) = \frac{1}{s_0{}^2 - \hat{s}_0{}^2 - \Omega^2 - \hat{\Omega}^2}\left\{(s_0\,\mathfrak{s} - \hat{s}_0\,\hat{\mathfrak{s}})_k \cdot i\,({}^k B - B^k) + [\mathfrak{s}, \hat{\mathfrak{s}}]_k \cdot ({}^k B + B^k)\right\},$$

$$k = 1, 2, 3,$$

according to [II, eqs. (62), (64), (68), (69)].

We convert the real and imaginary part of τ with the help of those expressions. We would not like to satisfy ourselves with a mere confirmation that those quantities are identical with the scalar reality relations [III, eqs. (17) and (18)], but, at the same time, we would like to take the opportunity to write them in a form [IV, eq. (16), {(17) and (18), resp.}] in which they no longer contain the electric and magnetic moments. It will then contain only quantities that also enter into the continuity equation and the anti-continuity equation.

Due to the vanishing of τ and τ^*, one can drop the non-vanishing factors $-1 / T_0$ [$-1 / T_0^*$, resp.] in τ [τ^*, resp.]. One will then get:

a) *The real part* is:

$$\tau + \tau^* = (s_0{}^2 - \hat{s}_0{}^2 - \Omega^2 - \hat{\Omega}^2) \cdot i\,({}^0 B - B^0) + \hat{s}_0\,({}^0\xi + \xi^0) - s_0\,({}^0\eta + \eta^0)$$
$$+ \sum_{k=1}^{3}\{(s_0{}^2 - \hat{s}_0{}^2 - \Omega^2 - \hat{\Omega}^2) \cdot i\,({}^k B\, p_k - B^k q_k) + \hat{s}_k\,({}^k\xi + \xi^k) - s_k\,({}^k\eta + \eta^k)\} = 0,$$

and with consideration given to [II, eqs. (83) and (85)] and some calculations, it will go to:

$$(16)\qquad \boxed{\begin{aligned}&(\Omega^2+\hat{\Omega}^2)\left\{-\hat{s}_0\frac{\partial s_0}{c\,\partial t}+\left(\mathfrak{s},\frac{\partial\hat{\mathfrak{s}}}{c\,\partial t}\right)+(\mathfrak{s},\operatorname{grad}\hat{s}_0)-(\hat{\mathfrak{s}},\operatorname{grad}s_0)\right\}\\&+(s_0\,\hat{\mathfrak{s}}-\hat{s}_0\,\mathfrak{s},\Omega\operatorname{grad}\Omega+\hat{\Omega}\operatorname{grad}\hat{\Omega})+([\mathfrak{s},\hat{\mathfrak{s}}],\Omega\operatorname{grad}\Omega-\hat{\Omega}\operatorname{grad}\hat{\Omega})=0.\end{aligned}}$$

The fact that this form for the scalar relation agrees with that of [III, eq. (17)] will become clear when one considers the fact that the last two terms are:

$$\begin{aligned}&(s_0\,\hat{\mathfrak{s}}-\hat{s}_0\,\mathfrak{s},\Omega\operatorname{grad}\Omega+\hat{\Omega}\operatorname{grad}\hat{\Omega})+([\mathfrak{s},\hat{\mathfrak{s}}],\Omega\operatorname{grad}\Omega-\hat{\Omega}\operatorname{grad}\hat{\Omega})\\&\quad=(\Omega^2+\hat{\Omega}^2)\,\{(\mathfrak{M},\operatorname{grad}\Omega)+(\hat{\mathfrak{M}},\operatorname{grad}\hat{\Omega})\}.\end{aligned}$$

After dropping the non-zero factor $\Omega^2+\hat{\Omega}^2$, one will then obtain:

$$-\hat{s}_0\frac{\partial s_0}{c\,\partial t}+\left(\mathfrak{s},\frac{\partial\hat{\mathfrak{s}}}{c\,\partial t}\right)+(\mathfrak{s},\operatorname{grad}\hat{s}_0)-(\hat{\mathfrak{s}},\operatorname{grad}s_0)+(\mathfrak{M},\operatorname{grad}\Omega)+(\hat{\mathfrak{M}},\operatorname{grad}\hat{\Omega})=0,$$

which is an equation that will go to the relation [III, eq. (17)] when one applies the identities [I, (17) and (35)].

b) *The imaginary part* is*:*

$$\begin{aligned}&-i\,(\tau-\tau^*)=(s_0{}^2-\hat{s}_0{}^2-\Omega^2-\hat{\Omega}^2)\,({}^0B+B^0)-\hat{s}_0\cdot i\,({}^0\xi-\xi^0)+s_0\cdot i\,({}^0\eta-\eta^0)\\&+\sum_{k=1}^{3}\{(s_0{}^2-\hat{s}_0{}^2-\Omega^2-\hat{\Omega}^2)({}^kB\,p_k+B^kq_k)-\hat{s}_k\cdot i\,({}^k\xi-\xi^k)+s_k\cdot i\,({}^k\eta-\eta^k)\}=0,\end{aligned}$$

and with consideration given to [II, eqs. (81) and (94)] and some calculations, it will go to:

$$(17)\qquad \boxed{\begin{aligned}&(s_0{}^2-\hat{s}_0{}^2-\Omega^2-\hat{\Omega}^2)\left(\hat{\Omega}\frac{\partial\Omega}{c\,\partial t}-\Omega\frac{\partial\hat{\Omega}}{c\,\partial t}\right)\{(,)+(\mathfrak{s},)-(\,)\}\\&+(s_0\,\hat{\mathfrak{s}}-\hat{s}_0\,\mathfrak{s},\hat{\Omega}\operatorname{grad}\Omega-\Omega\operatorname{grad}\hat{\Omega})\\&\left([\mathfrak{s},\hat{\mathfrak{s}}],s_0\frac{\partial\hat{\mathfrak{s}}}{c\,\partial t}-\hat{s}_0\frac{\partial\mathfrak{s}}{c\,\partial t}+s_0\operatorname{grad}\hat{s}_0-[\mathfrak{s},\operatorname{rot}\mathfrak{s}]-(\hat{\mathfrak{s}}\operatorname{grad})\mathfrak{s}\right)=0.\end{aligned}}$$

That equation is equivalent to the scalar relation {III, eq. (18)].

Proof: It will follow from the identities [I, (18) and (19)], after differentiation by *ct* and consideration of some other identities [I, (22), (25), (26), (27), (28), (34), (35), (36)], that the time-differentiated part of the scalar relation [III, eq. (18)] will be:

$$1.\quad \left(\hat{\mathfrak{M}}, \frac{\partial \mathfrak{M}}{c\,\partial t}\right) - \left(\mathfrak{M}, \frac{\partial \hat{\mathfrak{M}}}{c\,\partial t}\right) - \hat{\Omega}\frac{\partial \Omega}{c\,\partial t} + \Omega\frac{\partial \hat{\Omega}}{c\,\partial t}$$

$$= \frac{1}{\Omega^2 + \hat{\Omega}^2}\left\{2(s_0{}^2 - \hat{s}_0{}^2 - \Omega^2 - \hat{\Omega}^2)\left(\hat{\Omega}\frac{\partial \Omega}{c\,\partial t} - \Omega\frac{\partial \hat{\Omega}}{c\,\partial t}\right)\right.$$

$$\left. + \left([\mathfrak{s}, \hat{\mathfrak{s}}], \frac{\partial}{c\,\partial t}(s_0\hat{\mathfrak{s}} - \hat{s}_0\mathfrak{s})\right) - \left(s_0\hat{\mathfrak{s}} - \hat{s}_0\mathfrak{s}, \frac{\partial}{c\,\partial t}[\mathfrak{s}, \hat{\mathfrak{s}}]\right)\right\}.$$

Analogously, with consideration given to the identities [I, (34) and (35)], the spatially-differentiated part of that scalar relation that contains $\mathfrak{M}$ and $\hat{\mathfrak{M}}$ will follow from [I, (18) and (19)]:

$$2.\quad (\mathfrak{M}, \operatorname{rot} \mathfrak{M}) + (\hat{\mathfrak{M}}, \operatorname{rot} \hat{\mathfrak{M}}) = \frac{1}{\Omega^2 + \hat{\Omega}^2}\{-2\,(s_0\,\mathfrak{s} - \hat{s}_0\,\hat{\mathfrak{s}},\ \hat{\Omega}\operatorname{grad}\Omega - \Omega\operatorname{grad}\hat{\Omega})$$
$$+ ([\mathfrak{s}, \hat{\mathfrak{s}}], \operatorname{rot}[\mathfrak{s}, \hat{\mathfrak{s}}] - s_0 \operatorname{grad}\hat{s}_0 + \hat{s}_0 \operatorname{grad} s_0)$$
$$+ \hat{s}_0{}^2(\mathfrak{s}, \operatorname{rot}\mathfrak{s}) - s_0\,\hat{s}_0\{(\hat{\mathfrak{s}}, \operatorname{rot}\mathfrak{s}) + (\mathfrak{s}, \operatorname{rot}\hat{\mathfrak{s}}) + s_0{}^2(\hat{\mathfrak{s}}, \operatorname{rot}\hat{\mathfrak{s}})\}.$$

By substituting 1. and 2. in [III, eq. (18)] and after some minor calculations, what will arise is:

$$(18)\quad \left\{\begin{aligned}
&\left(\hat{\mathfrak{M}}, \frac{\partial \mathfrak{M}}{c\,\partial t} - \operatorname{rot}\hat{\mathfrak{M}}\right) - \left(\mathfrak{M}, \frac{\partial \hat{\mathfrak{M}}}{c\,\partial t} + \operatorname{rot}\mathfrak{M}\right)\\
&\qquad - \hat{\Omega}\frac{\partial \Omega}{c\,\partial t} + \Omega\frac{\partial \hat{\Omega}}{c\,\partial t} - (\mathfrak{s}, \operatorname{rot}\mathfrak{s}) + (\hat{\mathfrak{s}}, \operatorname{rot}\hat{\mathfrak{s}})\\
&= \frac{1}{\Omega^2 + \hat{\Omega}^2}\left\{2(s_0{}^2 - \hat{s}_0{}^2 - \Omega^2 - \hat{\Omega}^2)\left(\hat{\Omega}\frac{\partial \Omega}{c\,\partial t} - \Omega\frac{\partial \hat{\Omega}}{c\,\partial t}\right)\right.\\
&\qquad + 2\left([\mathfrak{s}, \hat{\mathfrak{s}}], s_0\frac{\partial \hat{\mathfrak{s}}}{c\,\partial t} - \hat{s}_0\frac{\partial \mathfrak{s}}{c\,\partial t}\right) + 2(s_0\,\mathfrak{s} - \hat{s}_0\,\hat{\mathfrak{s}}, \hat{\Omega}\operatorname{grad}\Omega - \Omega\operatorname{grad}\hat{\Omega})\\
&\qquad + ([\mathfrak{s}, \hat{\mathfrak{s}}], s_0 \operatorname{grad}\hat{s}_0 - \hat{s}_0 \operatorname{grad} s_0 - \operatorname{rot}[\mathfrak{s}, \hat{\mathfrak{s}}]) + (\Omega^2 + \hat{\Omega}^2 - s_0{}^2)(\hat{\mathfrak{s}}, \operatorname{rot}\hat{\mathfrak{s}})\\
&\qquad \left. + s_0\,\hat{s}_0((\hat{\mathfrak{s}}, \operatorname{rot}\mathfrak{s}) + (\mathfrak{s}, \operatorname{rot}\hat{\mathfrak{s}})) - (\Omega^2 + \hat{\Omega}^2 + \hat{s}_0{}^2)(\mathfrak{s}, \operatorname{rot}\mathfrak{s})\right\} = 0.
\end{aligned}\right.$$

The left-hand side of that equation is our scalar relation in the form [III, eq. (18)], while the right-hand side is a further representation that does not include the electric and magnetic moment. However, (17) will also go to that equation when one performs the conversions:

$$([\mathfrak{s}, \hat{\mathfrak{s}}], [\mathfrak{s}, \operatorname{rot}\hat{\mathfrak{s}}]) = (s_0{}^2 - \Omega^2 - \hat{\Omega}^2)(\hat{\mathfrak{s}}, \operatorname{rot}\hat{\mathfrak{s}}) - (\hat{\mathfrak{s}}, \operatorname{rot}\mathfrak{s}))$$

and

$$
\begin{aligned}
([\mathfrak{s}, \hat{\mathfrak{s}}], (\hat{\mathfrak{s}} \text{ grad})\, \mathfrak{s}) = \tfrac{1}{2}\{&([\mathfrak{s}, \hat{\mathfrak{s}}], \text{rot}\, [\mathfrak{s}, \hat{\mathfrak{s}}] + s_0 \text{ grad } \hat{s}_0 + \hat{s}_0 \text{ grad } s_0) \\
&+ (\Omega^2 + \hat{\Omega}^2 - s_0^2)\, (\hat{\mathfrak{s}}, \text{rot}\, \hat{\mathfrak{s}}) + s_0\, \hat{s}_0 ((\mathfrak{s}, \text{rot } \hat{\mathfrak{s}}) - (\hat{\mathfrak{s}}, \text{rot } \mathfrak{s})) \\
&+ (\Omega^2 + \hat{\Omega}^2 + \hat{s}_0^{\,2})(\mathfrak{s}, \text{rot } \mathfrak{s})
\end{aligned}
$$

on it and multiplies the entire equation by 2.

§ 3. Reducing the six vector relations to the scalar τ

Since $\tau = 0$ ($\tau^* = 0$, resp.), one can drop the factor $-T_0$ ($-T_0^*$, resp.) in the construction of the vector relations, and the expression (12), in conjunction with (14), will give the representations:

$$
\frac{\mathbf{a}}{2}: \mathfrak{q}\, \tau + \mathfrak{p}\, \tau^* = 0 \qquad \text{and} \qquad -\frac{\mathbf{b}}{2}: \quad i\,(\mathfrak{q}\, \tau - \mathfrak{p}\, \tau^*) = 0, \qquad \text{resp.}
$$

for the two vector relations (3) and (4), resp., or, with consideration given to the values for $\mathfrak{q}$ and $\mathfrak{p}$ [II, eqs. (64) and (69)], and dropping the non-vanishing factor $s_0^2 - \hat{s}_0^{\,2} - \Omega^2 - \hat{\Omega}^2$:

$$
\boxed{\begin{aligned}
&(s_0\, \mathfrak{s} - \hat{s}_0\, \hat{\mathfrak{s}})(\tau + \tau^*) + [\mathfrak{s}, \hat{\mathfrak{s}}] \cdot i(\tau - \tau^*) = 0, \\
&[\mathfrak{s}, \hat{\mathfrak{s}}](\tau + \tau^*) - (s_0\, \mathfrak{s} - \hat{s}_0\, \hat{\mathfrak{s}}) \cdot i(\tau - \tau^*) = 0,
\end{aligned}}
\qquad
\begin{aligned}
&(19a) \quad \text{(II - IV)}, \\
&(19b) \quad \text{(XXIII - XXV.B)}.
\end{aligned}
$$

One can get back to the remaining four vector equations (5)-(8) in the scalar τ along a path that is precisely analogous to the calculations that led to eq. (12) in § 2, and obtain:

$$
\boxed{\begin{aligned}
&-[\hat{\mathfrak{s}}, \mathfrak{M}](\tau + \tau^*) + [\mathfrak{s}, \hat{\mathfrak{M}}] \cdot i(\tau - \tau^*) = 0, \\
&-[\mathfrak{s}, \hat{\mathfrak{M}}](\tau + \tau^*) - [\hat{\mathfrak{s}}, \mathfrak{M}] \cdot i(\tau - \tau^*) = 0, \\
&-[\hat{\mathfrak{s}}, \hat{\mathfrak{M}}](\tau + \tau^*) - [\mathfrak{s}, \mathfrak{M}] \cdot i(\tau - \tau^*) = 0, \\
&-[\mathfrak{s}, \mathfrak{M}](\tau + \tau^*) - [\hat{\mathfrak{s}}, \hat{\mathfrak{M}}] \cdot i(\tau - \tau^*) = 0,
\end{aligned}}
\qquad
\begin{aligned}
&(19c) \quad \text{(XI - XIII.A)}, \\
&(19d) \quad \text{(XVII - XIX.B)}, \\
&(19e) \quad \text{(XI - XIII.B)}, \\
&(19f) \quad \text{(XVII - XIX.A)}.
\end{aligned}
$$

The Roman numerals on the right once more refer to the enumeration of the vector relations in III, § 2, which will lead back to the scalar τ here, after eliminating some parts of the continuity equation and anti-continuity equation [cf., III, eqs. (19**a**) to (**f**)]. A glance at those equations will show that the possibility of such a reduction there is not entirely obvious, but rather, **Dirac**'s theory will allow one the scalar relations $\tau + \tau^* = 0$, $i\,(\tau - \tau^*) = 0$ to be expressed in many ways in the six vector relations [III, eq. (19**a**) to (**f**)]. The form [IV, (19a)-(f)] that is obtained now also allows to see, more easily than before, that scalar multiplication of the vector relations by $\mathfrak{s}, \hat{\mathfrak{s}}$, $[\mathfrak{s}, \hat{\mathfrak{s}}]$ must always lead to a representation of one or the other scalar relation, except for a factor that is non-vanishing in the general case. When one scalar multiplies eq. (19a) by $\mathfrak{s}$ or $\hat{\mathfrak{s}}$, (b) by [$\mathfrak{s}$,

$\hat{\mathfrak{s}}$], (c) by $\mathfrak{s}$, (d) by $\hat{\mathfrak{s}}$, (e) by $\mathfrak{s}$, and (f) by $\hat{\mathfrak{s}}$, in succession, one will get s_0 or $\hat{s}_0$, $\Omega^2 + \hat{\Omega}^2$, $\hat{\Omega}$, Ω, $-\Omega$, $-\hat{\Omega}$, resp., times $(s_0^2 - \hat{s}_0^2 - \Omega^2 - \hat{\Omega}^2)$ times the one scalar relation $\tau + \tau^* = 0$, and when one scalar multiplies (a) by [$\mathfrak{s}$, $\hat{\mathfrak{s}}$], (b) by $\mathfrak{s}$ or $\hat{\mathfrak{s}}$, (c) by $\hat{\mathfrak{s}}$, (d) by $\mathfrak{s}$, (e) by $\hat{\mathfrak{s}}$, (f) by $\mathfrak{s}$, in turn, one will get $\Omega^2 + \hat{\Omega}^2$, $-s_0$ or $-\hat{s}_0$, $-\Omega$, $-\hat{\Omega}$, Ω, resp., times $(s_0^2 - \hat{s}_0^2 - \Omega^2 - \hat{\Omega}^2)$ times the other scalar relation $i\,(\tau - \tau^*) = 0$.

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References

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