"Beschleunigungsrelative Bewegungen und die konforme Gruppe der Minkowski'schen Welt," Sitz. Kais. Akad. Wiss. **125** (1916), 899-919.

Relatively-accelerated motions and the conformal group of Minkowski space

By

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(With 1 text figure)

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In an earlier paper ([1]), certain stationary acceleration states of a reference body were treated when it moved in such a way that a comoving observer did not need to be aware of them. Those motions, which are called, briefly, "relatively-accelerated" motions, possess the trajectories of a one-parameter orthogonal transformation group of Minkowski space S_4 for their worldlines. The comoving observer found a light velocity field that varied from place to place, and in general also from direction to direction, so it was tensorial. The case of a scalar light velocity field (so one that is merely a function of position) was present only for *hyperbolic motion*, which was investigated in a second paper ([2]). In the sense of the known Einstein equivalence hypothesis, which one can express as: "stationary acceleration states are equivalent to stationary force fields" (which is understood to mean *spatially* and *temporally* stationary), one can (heuristically) infer the existence of a homogeneous electrostatic or gravitational field for the processes that take place in the fields equivalent to hyperbolic motion, and in that fact one gets a glimpse of Byk's theory of matter and quanta.

Of the remaining types of motion, the orthogonal group represents, for example, the *uniform rotation* of a magnetostatic field. The application to the theory of matter is included in that as long as one regards the atomic forces as *central forces*. Up to now, an exception to that picture is found merely in a known hypothesis that **Ritz** assumed in order to explain the Zeeman effect.

([1]) **F. Kottler**, "Relativitätsprinzip und beschleunigte Bewegung," Ann. Phys. (Leipzig) **44** (1914), pp. 701 (cited as I).

([2]) **F. Kottler**, "Fallende Bezugssysteme vom Standpunkte des Relativitätsprinzips," Ann. Phys. (Leipzig) **45** (1914), pp. 481 (cited as II).

One can now ask whether hyperbolic motion is the only case that is equivalent to a static field of attraction. Are there not still other relatively-accelerated motions of that nature? **Ehrenfest** and **van Os** have *answered that question in the affirmative* for a certain class of motions.

In the present article, that question will be examined from the group-theoretic viewpoint, and that will show: The trajectories of a one-parameter orthogonal group are the *only* relatively-accelerated motions of Minkowski space, as long as one makes the assumption: *The Lorentz transformations are valid at infinity*; i.e., the yardsticks and clocks of the instantaneously resting accelerated observer and those of the continually-resting one coincide.

However, that assumption, which is also at the basis of the proper finite Lorentz transformation (for a translating observer, in that case), is not necessary. In fact, all that is necessary is that the law of propagation of light is the same for both observers; indeed, that is at the foundations of Minkowski space. However, it is known that the form of Maxwell's equations also remains the same for a much broader class of transformations than the (orthogonal) Lorentz ones, namely, for the conformal transformations, as **H. Bateman** first showed. A special case of the Bateman transformation was already available to **Lorentz** (Bucherer-Langevin electron).

That viewpoint is also natural, since the yardsticks and clocks of the moving observer are not given *a priori* but inferred from experiments. For example, the length is defined as the distance between two points of the reference body and must therefore not coincide with that of the continually-resting observer. Indeed, it must not appear to be a line for one when it is curved for the other. Only the law of propagation of light is established *a priori*. That must remain true of the metric of the moving observer in his immediate neighborhood.

If one allows the Bateman transformation then one will find that the possible relatively-accelerating motions include the trajectories of a *one-parameter group of conformal transformations* in which the orthogonal one is naturally included as a subgroup. *The curves that were found by Ehrenfest and van Os then prove to be special cases of the conformal group*.

Another natural question to ask is how those kinematically-defined motions are dynamically possible. In the first paper, the constant electromagnetic field was given as the dynamical cause of the orthogonal group, and according to **Einstein**, they are equivalent, in their own right. That question will not be investigated here for the conformal group.

Notations.

(Cf., I)

Upper indices: coordinate indices (contravariant)
Lower indices: covariant (hyperplane coordinates)

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§ 1. – Relatively-accelerated motions.

1. – In Newtonian mechanics ($c = \infty$). every motion can be regarded as relatively-accelerated; i.e., it will remain hidden from the comoving observer as long as the reference system is given by a Newtonian rigid body. This, the possibility of mutual relative motion of two parts of the body is excluded. In the well-known example of the train car, its rattling will be consistent with the intuition of the naïve observer that it proves the existence of motion, so that rattling will be excluded by the above postulate of a rigid body. By contrast, e.g., rotation of the Earth and the **Ptolemaic** viewpoint will be regarded as admissible relatively-accelerated motions ([1]).

2. – Now, Minkowskian mechanics (c finite) does not know of any rigid bodies; i.e., a system of mass-points whose mutual distances remains constant forever. In fact, that is implied *dynamically* by the finite propagation speed c of the force effects. The particles of a body on which a force begins to act experience relative displacements corresponding to whether the force wave affected them earlier or later. A *quasi-rigid* body is then realized only it is under the influence of *no* forces at all (uniform rectilinear translation) or *constant* forces when the stationary state has been established, so when the waves that are produced by variations of the force no longer perturb the body (worldlines of an orthogonal transformation, I, § **3**).

3. – Correspondingly, one would expect the relativity of acceleration in Minkowskian mechanics only for the worldlines of a motion of S_4 . The following *kinematical* considerations confirm that:

The assumption that is made about that: *The Lorentz transformation is valid infinitesimally;* i.e., the motion of an infinitely-small piece of the body – e.g., from the viewpoint of a comoving observer – can always be regarded as a *uniform translation* during an infinitely-small time, such that all processes in the immediate vicinity of the comoving observer are identical to the rest processes in a system in which the location considered is precisely at rest for an infinitely-small time. In particular, it follows from this that the yardsticks and clocks of the observer that is instantaneously at rest will provide the same information as those of an observer that is constantly at rest at the time and place where they coincide. Expressed differently: In the *immediate vicinity* of the comoving observer, the Minkowski arc-length element:

$$dS^2 = dx^2 + dy^2 + dz^2 - c^2\, dt^2$$

is always valid, in which x, y, z, t are referred to an ordinary Lorentz system.

4. – We support that hypothesis with this remark: *The law of propagation of light is independent of one's fundamental conception of the universe* (cf., II, § **5**, 1). One must then be able to put it into the form:

$$dx^2 + dy^2 + dz^2 - c^2\, dt^2 = 0$$

for every world-point as long as the observe restricts himself to his *immediate vicinity*. It still does not follow from this that the law possesses the Minkowski arc-length as a metric at infinitely-close points.

Rather, it can also possess an arc-length:

$$dS^2 = \lambda^2\,(dx^2 + dy^2 + dz^2 - c^2\, dt^2)\,.$$

([1]) The difference between those two cases (train car and Earth rotation) was discussed before by **H. Poincaré**, *La Science et l'Hypothesé*, chap. VII, pp. 138.

However, the λ in this is restricted by the condition (cf., *infra*, § **2**) that the map that we have here is conformal. Indeed, we have:

$$dS^2 = d\bar{x}^2 + d\bar{y}^2 + d\bar{z}^2 - c^2 d\bar{t}^2,$$

where $\bar{x}, \bar{y}, \bar{z}, \bar{t}$ are referred to a reference system of the accelerated observer, because that observer will also clearly be working with Euclidian geometry. It will then be possible that the metric of the observer that is *continually* at rest and that of the observer that is *instantaneously* at rest are not identical, but are *connected by a conformal transformation*, such that, e.g., the one perceives something to be a circle that exists as a straight line to the other. As is known, that is an extension of the Lorentz transformation that **H. Bateman** had already made ([1]), when he showed that under the assumption of invariance of charge, Maxwell's equations also keep the same form under that transformation, which he called a *spherical wave transformation,* that they has in a system at rest.

We next overlook the Bateman transformation and demand the validity of Maxwell's equations under just the proper Lorentz transformation. The metric of the observer that is instantaneously at rest and the metric of the one that is continually at rest will then be completely identical.

5. – One now imagines (cf., II, § **6**) that the accelerated observer has constructed a type of polar coordinate system whose radii are light rays that emanate from the mass-points of his reference body. When will the observer that can choose his viewpoint on the body *completely arbitrarily* come to believe that the body is at rest? For that to be true, obviously the polar coordinates of the different mass-points must remain constant. Those coordinates are the distance and the polar angles. The preservation of the latter demands that all points that lie along a light ray must remain along it. *The light path must then be preserved.* (*A*) The preservation of the former demands that *the length of the light ray must be preserved* (*B*). In the terminology of I, those postulates read: Constancy of the effective position, and in II, the constancy of apparent coordinates.

6. – If one now represents the motion in the system that is continually at rest in the known Minkowski form as a spacetime curve:

$$x^{(h)} = x^{(h)}(a, b, c, u), \qquad h = 1, 2, 3, 4,$$

in which the u establishes the connectedness of a configuration (e.g., for true or apparent coordinates, as in II), no matter how one agrees upon the suggested conception of the accelerated observer:

$$\frac{dx^{(h)}}{du} = \frac{dx^{(h)}}{du}(u)$$

will be the infinitesimal transformation that generates the spacetime curve.

([1]) **H. Bateman**, Proceedings of the London Mathematical Society (2) **8** (1910), pp. 224, *et seq.* A special case of the Bateman transformation already appeared in the work of **Lorentz** as the factor k (**Bucherer-Langevin!**).

Naturally, since the law of propagation of light is indeed absolute, the world-points that the accelerated observer sees along a light ray will also lie along a light ray in the system at rest – i.e., a minimal line in S_4 . The postulate (*A*) in **5** then demands that:

The transformation that defines the time evolution of the spacetime curve takes every minimal light to another one. It is therefore *conformal*.

The postulate (*B*) in **5** requires that:

This transformation must preserve certain lengths. It is then *orthogonal*, moreover.

It then follows that: In order for the observer on the reference body to appear to be at rest, it is necessary that the spacetime curves of the body consist of *trajectories of an one-parameter group of orthogonal transformations* of S_4 . They are once more the relatively-accelerated motions that were treated in I. (One recognizes the constancy of not only the effective positions, but also the "simultaneous" ones, so the true coordinates, such that above assertion of the observer, which was merely based upon his sense of sight, will also stand up to measurements.)

7. – In that way, the assumption that was made in **3** of the agreement between the measurements of the observer at rest and the accelerated one is justified at any moment. If we now drop that assumption and perform a Bateman transformation then it will still not follow from the preservation of lengths in accelerated systems that it is preserved in the system at rest, since both systems are conformally related. Therefore, it will follow only that the infinitesimal transformation is conformal. We will then obtain a trajectory of a *one-parameter conformal group* of S_4, and we will now have to establish whether the *preservation of lengths in the accelerated system can be realized by means of a suitable metric*. We shall address that problem once we have exhibited the equations of those motions (§ **3**).

8. – In connection with that, two papers of **Ehrenfest** ([1]) and **van Os** ([2]) should be mentioned. Starting from the assumption that the light rays in a stationary static field are curved, **Ehrenfest** demanded: ($\mathfrak{A}$), the preservation and ($\mathfrak{B}$), the reversibility of the light ray. Because of **Einstein**'s equivalence principle, the field should be replaceable with an accelerated state of a reference system, and therefore the accelerated motions that satisfy the postulates ($\mathfrak{A}$) and ($\mathfrak{B}$) will be examined. Now, the postulate ($\mathfrak{A}$) is precisely our postulate (*A*). The cases that **Ehrenfest** obtained must then be the trajectories of a one-parameter conformal group. It then follows from ($\mathfrak{B}$) that they must be sections of hyperspheres, so circles (hyperbolas in the real representation, respectively), through every two of which there passes one sphere (light hyperboloid, resp.). In what follows, with **Ehrenfest**, *loc. cit.*, we will restrict ourselves to such paths of a conformal group, although naturally **Ehrenfest**'s postulate ($\mathfrak{B}$) is not by any means required by the relativity of acceleration, but merely by the scalar, non-tensorial nature of the light velocity field.

([1]) **P. Ehrenfest**, "On Einstein's theory of the stationary gravitational field," Kon. Ned. Akad. Wet. **15** (1913), 1187-1191.

([2]) **C. H. van Os**, "On a system of curves occurring in Einstein's theory of gravitation," Kon. Ned. Akad. Wet. **16** (1913), pp. 40, *et seq.*.

§ 2. – The conformal group of S_4 .

1. – **Darboux** ([1]) showed that the conformal group of an S_n ($n > 2$) is composed of reflections, displacements, rotations, similarity transformations (stretchings), and transformations through reciprocal radii (inversions) ([2]).

For the sake of exhibiting the most general infinitesimal transformation, one must exclude the transfers (*Umlegungstransformations*) (negative functional determinant!). The reflections drop out, and in place of the inversion, one finds a double inversion; e.g., through two hyperspheres that are displaced from each other by infinitely little.

One gets the formulas:

$$\left.\begin{aligned}\frac{dx^{(1)}}{du} = \varepsilon^{(1)} \\ + * + \varepsilon_2^{(1)} x^{(2)} + \varepsilon_3^{(1)} x^{(3)} + \varepsilon_4^{(1)} x^{(4)} \\ + \gamma\, x^{(1)} \\ + \delta \sum x^2 - 2x^{(1)} \sum x\,\delta, \\ \frac{dx^{(1)}}{du} = \cdots\end{aligned}\right\} \quad (\varepsilon_k^{(h)} = -\varepsilon_h^{(k)}). \qquad (1)$$

The first two rows contain the translation (rotation, resp.) that is known from the orthogonal group (I, § **1**), the third row contains the stretching, and the fourth contains the double inversion (which is coupled with a translation). In regard to the applications to Minkowski S_4, one must assume that the coefficients that carry the index 4 are pure imaginary, while the others are real.

2. – As for the classification of (1), it is most convenient for one to start from Lie's theorem ([3]):

The general conformal group S_n is isomorphic to the general projective group of a manifold of degree two in S_{n+1} .

([1]) **G. Darboux**, *Systèmes orthogonaux*, Paris, 1910, pp. 166, *et seq*. **Darboux**'s proof rests upon **Lamé**'s method for curvilinear orthogonal coordinates. It can be given more simply by the methods of the *absolute differential calculus*, which have the advantage that they are true for arbitrary oblique coordinates. In order to do that one must perform a transformation:

$$\sum dx^2 = \frac{1}{h^2} \sum d\bar{x}^2$$

and exhibit **Christoffel**'s differential equations for the $\frac{\partial^2 \bar{x}}{\partial x\, \partial x}$ In order for them to be integrable, it is necessary and sufficient that the Riemann symbol of the quadratic differential form $\frac{1}{h^2} \sum d\bar{x}$ must vanish; that immediately implies that $h = a \sum \bar{x}^2$, etc.

([2]) In that way, one will get the most general *finite* Bateman transformations; cf., **H. R. Hassé**, Proc. Lond. Math. Soc. **12** (1913), pp. 181, *et seq.* In contrast to the rest system, the relevant reference system has an accelerated motion whose paths are conic sections that extend to infinity.

([3]) **S. Lie**, *Theorie der Transformationsgruppen*, v. III, Leipzig, 1893, pp. 281 and 357.

One arrives at the map of Minkowski S_4 to a four-fold extended hypersurface of degree two M_4^{II} in a five-dimensional space Σ_5 , namely, the so-called Klein hypersurface in Kleinian ([1]) Σ_5 , that is required for this.

The conformal group of S_4 takes hyperspheres to hyperspheres. The equation of a hypersphere is:

$$\lambda_5 \Sigma x^2 + 2 (\lambda_1 x^{(1)} + \lambda_2 x^{(2)} + \lambda_3 x^{(3)} + \lambda_4 x^{(4)}) + \lambda_6 = 0 .$$

It will reduce to a point (i.e., a null hypersphere) when:

$$\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2 - \lambda_5 \lambda_6 = 0.$$

One will then achieve a map of the conformal group to the projective group of an M_4^{II} when one maps the hyperspheres to the *hyperplanes* (Σ_4) in a Σ_5 whose six homogeneous coordinates are λ_1, λ_2, λ_3, λ_4, λ_5, λ_6 . The null hyperspheres are then, in particular, the tangential hyperplanes (Σ_4) to the desired M_4^{II}, whose equation reads:

$$\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2 - \lambda_5 \lambda_6 = 0 \tag{2}$$

in hyperplane coordinates. One must next introduce the homogeneous coordinates:

$$x^{(1)} = \frac{\xi^{(1)}}{\xi^{(5)}}, \quad x^{(2)} = \frac{\xi^{(2)}}{\xi^{(5)}}, \quad x^{(3)} = \frac{\xi^{(3)}}{\xi^{(5)}}, \quad x^{(4)} = \frac{\xi^{(4)}}{\xi^{(5)}}$$

for the points in Minkowski S_4 and then set:

$$\begin{gathered} \rho\,\kappa^{(1)} = \xi^{(1)}\,\xi^{(5)}, \qquad \rho\,\kappa^{(2)} = \xi^{(2)}\,\xi^{(5)}, \qquad \rho\,\kappa^{(3)} = \xi^{(3)}\,\xi^{(5)}, \qquad \rho\,\kappa^{(4)} = \xi^{(4)}\,\xi^{(5)}, \\ \rho\kappa^{(5)} = \tfrac{1}{2}(\xi^{(5)})^2, \quad \rho\kappa^{(6)} = \tfrac{1}{2}[(\xi^{(1)})^2 + (\xi^{(2)})^2 + (\xi^{(3)})^2 + (\xi^{(4)})^2], \end{gathered} \tag{3}$$

in which the $\kappa^{(1)}$, $\kappa^{(2)}$, $\kappa^{(3)}$, $\kappa^{(4)}$, $\kappa^{(5)}$, $\kappa^{(6)}$ are the homogeneous coordinates of Σ_5 , but they must fulfill the relation:

$$(\kappa^{(1)})^2 + (\kappa^{(2)})^2 + (\kappa^{(3)})^2 + (\kappa^{(4)})^2 - 4\,\kappa^{(5)}\,\kappa^{(6)} = 0 , \tag{4}$$

if they are to represent a point in Minkowski S_4 . S_4 is then, in fact, mapped to an M_4^{II}, whose equation is (2) in hyperplane coordinates and (4) in point coordinates, and the equation of a hypersphere actually assumes the form:

$$\lambda_1\,\kappa^{(1)} + \lambda_2\,\kappa^{(2)} + \lambda_3\,\kappa^{(3)} + \lambda_4\,\kappa^{(4)} + \lambda_5\,\kappa^{(5)} + \lambda_6\,\kappa^{(6)} = 0 ,$$

([1]) **F. Klein**, Math. Ann. **5** (1871), employed that map, whereby the ∞^4 lines in S_3 appear in place of the ∞^4 points of S_4 . The six **Klein**ian (i.e., generalized **Plücker**) coordinates of that space indeed fulfill a quadratic equation of the form (4) [(6) in the text, resp.].

which is the equation of a hyperplane (Σ_4) in Σ_5, although the relation (4) must be appended to it. The ∞^3 *points of a* ***Minkowski*** *hypersphere will then be mapped to the section of a hyperplane* (Σ_4) *in* ***Klein****ian* Σ_5 *with the* ***Klein*** *hypersurface* M_4^{II}; i.e., to the ∞^3 points of a triply-extended quadratic manifold M_4^{II}, etc.

3. – (3) then implies the following correspondences ([1]):

A point in Minkowski S_4 that is not at infinity	A point of the Klein M_4^{II} in Kleinian Σ_5
All ∞^3 points at infinity that do not lie on the absolute null-sphere [i.e., $\xi^{(5)} = 0$], but: $(\xi^{(1)})^2 + (\xi^{(2)})^2 + (\xi^{(3)})^2 + (\xi^{(4)})^2 \neq 0$	A point O in M_4^{II} with the coordinates (0, 0, 0, 0, 0, 1); every "direction" in S_4 corresponds to a tangent in M_4^{II}
A point on the absolute null-sphere: $\xi^{(5)} = 0$, $(\xi^{(1)})^2 + (\xi^{(2)})^2 + (\xi^{(3)})^2 + (\xi^{(4)})^2 = 0$	A line through O that belongs to M_4^{II} completely
Hypersphere	Section of a Σ_4 with M_4^{II} ; i.e., a M_3^{II}
Sphere	Section of a Σ_3 with M_4^{II}; i.e., a surface of degree two M_2^{II}
Circle	Section of a Σ_2 with M_4^{II} ; i.e., a conic section M_1^{II}
Minimal line	Line that belongs to M_4^{II} completely
Line	Conic section in M_4^{II} that goes through O and contacts the tangent to M_4^{II} at O, which represents the "direction" of the line in S_4 .

With this table, it is easy to show that the *conformal* group is mapped, without exception, to the group that transforms M_4^{II} into itself, namely, the *projective* group of M_4^{II} . Namely, as long as the points at infinity remain at rest – i.e., it is *affine* – that will be clear *a priori*. However, only the inversion in the conformal group is not affine. Nonetheless, one shows that the points at infinity transform the point O and the triply-extended hyper-cone Γ_3^{II} of lines in M_4^{II} precisely like the projective group of M_4^{II} .

4. – On the other hand, exhibiting the projective group of M_4^{II}, which then leaves the equation:

$$(\kappa^{(1)})^2 + (\kappa^{(2)})^2 + (\kappa^{(3)})^2 + (\kappa^{(4)})^2 - 4\,\kappa^{(5)}\,\kappa^{(6)} = 0 \qquad (4)$$

([1]) **C. M. Jessop**, *Treatise on the Line Complex*, Cambridge 1903, pp. 245, *et seq.*

invariant, is easy. In order to do that, we introduce the so-called hexa-spherical coordinates ([1]):

$$\begin{gathered}\mu^{(1)}=\kappa^{(1)}, \qquad \mu^{(2)}=\kappa^{(2)}, \qquad \mu^{(3)}=\kappa^{(3)}, \qquad \mu^{(4)}=\kappa^{(4)}, \\ \mu^{(5)}=\kappa^{(5)}-\kappa^{(6)}, \qquad i\,\mu^{(6)}=\kappa^{(5)}+\kappa^{(6)},\end{gathered} \tag{5}$$

with which, (4) will assume the form:

$$(\mu^{(1)})^2+(\mu^{(2)})^2+(\mu^{(3)})^2+(\mu^{(4)})^2+(\mu^{(5)})^2+(\mu^{(6)})^2=0\,. \tag{6}$$

In that form for M_4^{II}, the projective group can be written out directly. One gets the infinitesimal transformation:

$$\frac{d\mu^{(h)}}{du}=\sum_{k=1}^{6}\beta_k^{(h)}\,\mu^{(k)}, \qquad h=1,2,3,4,5,6, \tag{7}$$

with the known condition:

$$\beta_k^{(h)}=-\beta_h^{(k)}.$$

That gives the following infinitesimal transformation of the projective group of (4):

$$\begin{aligned}
\frac{d\kappa^{(1)}}{du}&= \;\;*\;\; +\beta_2^{(1)}\kappa^{(2)}+\beta_3^{(1)}\kappa^{(3)}+(\beta_5^{(1)}-i\,\beta_6^{(1)})\,\kappa^{(5)}+(-\beta_5^{(1)}-i\,\beta_6^{(1)})\,\kappa^{(6)}, \\
\frac{d\kappa^{(2)}}{du}&=\beta_1^{(2)}\,\kappa^{(1)}+\;\;*\;\;+\beta_3^{(2)}\,\kappa^{(3)}+\beta_4^{(2)}\,\kappa^{(4)}+(+\beta_5^{(2)}-i\,\beta_6^{(2)})\,\kappa^{(5)}+(-\beta_5^{(2)}-i\,\beta_6^{(2)})\,\kappa^{(6)}, \\
\frac{d\kappa^{(3)}}{du}&=\dots, \\
\frac{d\kappa^{(4)}}{du}&=\dots, \\
2\frac{d\kappa^{(5)}}{du}&=(-\beta_5^{(1)}-i\beta_6^{(1)})\,\kappa^{(1)}+(-\beta_5^{(2)}-i\beta_6^{(2)})\,\kappa^{(2)}+(-\beta_5^{(3)}-i\beta_6^{(3)})\,\kappa^{(3)}+(-\beta_5^{(4)}-i\,\beta_6^{(4)})\,\kappa^{(4)}+2i\,\beta_6^{(5)}\kappa^{(5)}+\;\;*, \\
2\frac{d\kappa^{(6)}}{du}&=(+\beta_5^{(1)}-i\beta_6^{(1)})\,\kappa^{(1)}+(+\beta_5^{(2)}-i\beta_6^{(2)})\,\kappa^{(2)}+(\beta_5^{(3)}-i\beta_6^{(3)})\,\kappa^{(3)}+(\beta_5^{(4)}-i\,\beta_6^{(4)})\,\kappa^{(4)}+\;\;*\;\;+2i\,\beta_6^{(5)}\kappa^{(6)}.
\end{aligned} \tag{8}$$

If one uses (3) to write:

$$\begin{gathered}\kappa^{(1)}=x^{(1)}\,(\xi^{(5)})^2, \qquad \kappa^{(2)}=x^{(2)}\,(\xi^{(5)})^2, \qquad \kappa^{(3)}=x^{(3)}\,(\xi^{(5)})^2, \qquad \kappa^{(4)}=x^{(4)}\,(\xi^{(5)})^2, \\ 2\kappa^{(5)}=(\xi^{(5)})^2, \qquad 2\kappa^{(6)}=(\xi^{(5)})^2\,\Sigma\,x^2\end{gathered}$$

([1]) **Jessop**, *loc. cit.*, pp. 255.

here then one will find that when $\xi^{(5)} \neq 0$, after calculating $d\xi^{(5)} / du$ from the fifth equation in (8), substituting that in the first four of equations (8), and dividing by $(\xi^{(5)})^2$, one get precisely the formula (1) with:

$$\left.\begin{aligned} \varepsilon^{(h)} &= \tfrac{1}{2}(\beta_5^{(h)} - i\beta_6^{(h)}), \\ \varepsilon_k^{(h)} &= \beta_k^{(h)}, \\ \delta^{(h)} &= \tfrac{1}{2}(-\beta_5^{(h)} - i\beta_6^{(h)}) \\ \gamma &= i\beta_6^{(5)}, \end{aligned}\right\} \quad h, k = 1, 2, 3, 4,$$

with which the complete correspondence between (1) and (7) [(8), respectively] is achieved ([1]).

§ 3. – The curves of Ehrenfest and van Os.

1. – As we stated before, we now have to look for those curves (1) that are circles, and indeed with the property that any two of them lie on a sphere. From the tables in §§ **2**, **3**, the map to the Klein Σ_5 yields those transformations of M_4^{II} to itself whose paths are planar, such that any two such planes lie in an S_3 (which corresponds to the sphere). As a result, two such planes intersect in a line g, and since the planes are transformed into themselves, the line g must be transformed into itself, so it is then common to all planes. *The paths of such a transformation of M_4^{II} into itself then lie in the ∞^3 planes through a line g in Σ_5 and are conic sections.* Those conic sections, which lie on M_4^{II}, then correspond to the desired curves (1). They are circles in S_4 with the following property: The aforementioned line g in Σ_5 generally intersects M_4^{II} in *two* (real or conjugate imaginary) points. In particular, they can also be a tangent to M_4^{II} (when the two points coincide) or lie on M_4^{II} completely (when they then correspond to a minimal line in S_4). The Ehrenfest-van Os result follows from that: *The desired circles in S_4 that can be linked pair-wise by a sphere generally all go through two points of S_4 .*

When one distinguishes the positions of the aforementioned lines g in Σ_5 with respect to the distinguished point O in M_4^{II}, one will get the special cases of **van Os**.

2. – We now choose a special case of (1), namely, the double inversion:

$$\frac{dx^{(h)}}{du} = \delta^{(h)} \sum x^2 - 2x^{(h)} \sum x\,\delta, \quad h = 1, 2, 3, 4. \tag{9}$$

Under the inversion:

$$\bar{x} = \frac{x}{\sum x^2}, \quad x = \frac{\bar{x}}{\sum \bar{x}^2},$$

that will emerge from the displacement:

([1]) The sixth equation in (7) [(8), respectively] is always a consequence of the first five. Indeed, the homogeneous coordinates κ determine a point of Σ_5 only by way of their ratios.

$$\frac{d\bar{x}^{(h)}}{du} = \delta^{(h)}, \quad h = 1, 2, 3, 4, \tag{9.a}$$

because:

$$\frac{dx^{(h)}}{du} = \frac{\dfrac{d\bar{x}^{(h)}}{du}}{\sum \bar{x}^2} - 2\frac{\sum \bar{x}\dfrac{d\bar{x}^{(h)}}{du}}{\left(\sum \bar{x}^2\right)}\bar{x}^{(h)}.$$

It is then clear from the outset that it belongs to the Ehrenfest-van Os curves, because the displacement has the ∞^3 parallel lines in S_4 as its trajectories, which map to Σ_5 as the ∞^3 conic sections in M_4^{II} by way of the point O and a tangent to M_4^{II} (which cannot be a generator, though), so it corresponds to the case "*g* is tangent." The inversion maps M_4^{II} as a projective transformation ([1]), so the curves that correspond to the curves (9) in Σ_5 are conic sections in M_4^{II} that lie in the ∞^3 planes through a tangent g to M_4^{II} ; i.e., *the curves* (9) *are circles in* S_4 *that have a pair of coincident points in common, so they have a common tangent.* ∞^1 *circles in each plane of* S_4 *go through that tangent that define a pencil of contact circles.* One likewise confirms that by integrating (9). (In order to do that, one chooses, e.g.:

$$\delta^{(1)} = \delta^{(2)} = \delta^{(3)} = 0, \qquad \delta^{(4)} \neq 0$$

and pure imaginary, since that is the direction of the common tangent, and that must be timelike for worldlines!)

3. – If one now imagines that the arc-length element for an observer that is found in uniform translation (9.a) reads:

$$dX^2 + dY^2 + dZ^2 - c^2\, dT^2$$

then one will find for an observer that is accelerated according to (9) that:

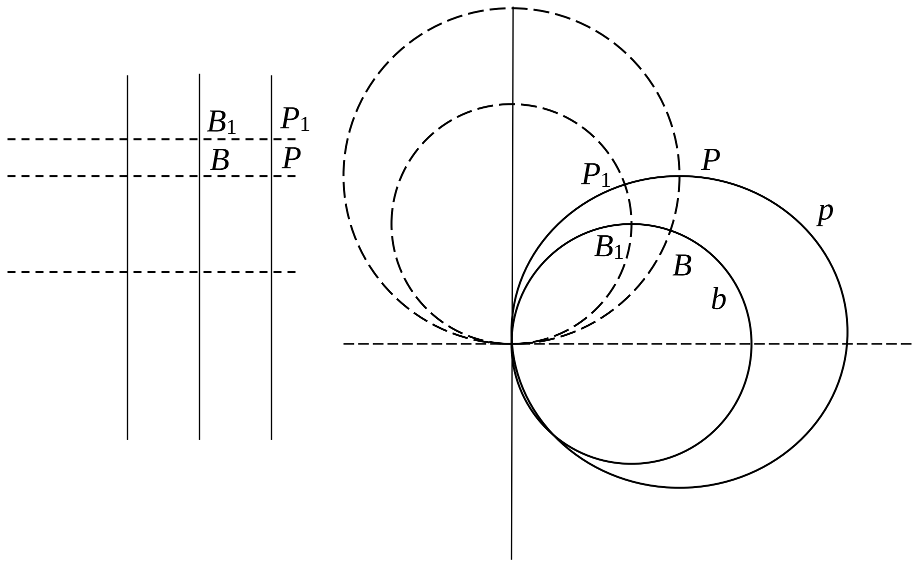
$$\frac{1}{\sum X'^2}(dX'^2 + dY'^2 + dZ'^2 - c^2 dT'^2).$$

Since that is supposed to be a Minkowskian element of arc-length, the X', Y', Z', T' are not Cartesian coordinates, but rather, they are curvilinear orthogonal coordinates in the **Darboux-Lamé** sense. The orthogonal system T' = const. is composed of hyperspheres here, and its ∞^3 orthogonal trajectories X' = const., Y' = const., Z' = const. are just the circles (9) ([2]).

([1]) Namely: The hypersphere of the inversion will be mapped as a hyperplane Σ_4 . Relative to M_4^{II}, it will then belong to a point P that is its pole, and which does not lie on M_4^{II}. One finds that the "inverse" of any point Q in M_4^{II} is the second point of intersection of the line PQ with M_4^{II}. Cf., **Jessop**, *loc. cit.*, pp. 252.

([2]) For a discussion of the orthogonal trajectories of a family of ∞^1 hyperspheres as a generalization of **Born**'s rigid body of the first kind (orthogonal trajectories of ∞^1 hyperplanes), see **H. Bateman**, Am. J. Math. **34** (1912), pp. 346, *et seq.*

The figure illustrates the two-dimensional analogue in the Euclidian plane, in which the *circle* is used as the basis for the metric as its *gauge curve* (*Eichkurve*) (II, § **1**). In that way the trajectory circles $X' =$ const., $Y' =$ const., $Z' =$ const. are *distinguished* lines that are shown as *dashed* in the figure in place of the circles that intersect the hyperspheres $T' =$ const. orthogonally.



Derivation of the trajectories of a double inversion (pencil of contact circles) by an inversion of the trajectories of a translation (pencil of parallel rays).

If one admits the conformal (Bateman) transformation:

$$dX^2 + dY^2 + dZ^2 - c^2\, dT^2 = \frac{1}{\sum X'^2}(dX'^2 + dY'^2 + dZ'^2 - c^2 dT'^2)$$

then the observer that participates in an Ehrenfest-van Os motion will have the arc-length element:

$$dX'^2 + dY'^2 + dZ'^2 - c^2\, dT'^2,$$

so his motion will appear to be a uniform translation or a state of rest to him. *The speed of light is constant.* (One gets a motion with variable light speed, e.g., by inverting the hyperbolic motion.) For example, an observer that moves along the "worldline" b will observe a normal "equidistance" $BP = BP$ in his metric for a second world-line from his reference body and can likewise indicate skew "equidistance."

§ 4. – Return to the trajectories of an orthogonal transformation group.

1. – Here, we would like to develop the arc-length element of an observer that corresponds to a general "motion" in S_4 . In order to do that, we recall the representation I, § **4**:

$$X = x + \Gamma^{(1)} c_1 + \Gamma^{(2)} c_2 + \Gamma^{(3)} c_3 + \Gamma^{(4)} c_4 .$$

In order to show that a "motion" is actually defined in that way, so, e.g., when one assumes that:

$$\frac{dx^{(h)}}{du} = \sum_{k=1}^{4} \varepsilon_k^{(h)} x^{(k)}, \quad h = 1, 2, 3, 4 \quad (\varepsilon_k^{(h)} = -\varepsilon_h^{(k)}),$$

so it will also follow that:

$$\frac{dX^{(h)}}{du} = \sum_{k=1}^{4} \varepsilon_k^{(h)} X^{(k)}, \quad h = 1, 2, 3, 4,$$

one recalls the representation of the direction cosines of the moving 4-frame that was given in Appendix 1 to I. For that, one has to calculate (*loc. cit.*, pp. 741) the differential quotients:

$$\frac{dx^{(h)}}{du} = \sum_k \varepsilon_k^{(h)} x^{(k)},$$

$$\frac{d^2 x^{(h)}}{du^2} = \sum \varepsilon_k^{(h)} \frac{dx^{(k)}}{du} = \sum \varepsilon_k^{(h)} \varepsilon_{k'}^{(k)} x^{(k')}, \quad \text{etc.},$$

and form the scalar products:

$$\sum \frac{dx}{du} \frac{dx}{du} = \sum_{k,k'} \varepsilon_k^{(h)} \varepsilon_{k'}^{(k)} x^{(k)} x^{(k')},$$

$$\sum \frac{dx}{du} \frac{d^2 x}{du^2} = \sum \varepsilon_k^{(h)} \varepsilon_{k'}^{(h)} \varepsilon_{k''}^{(k)} x^{(k')} x^{(k'')} = 0, \quad \text{etc.}$$

Due to the skew symmetry of the $\varepsilon_k^{(h)}$, all of the ones for which an odd number of ε appear will *vanish*. The ones with an even number of ε are *constant*. If will then follow from that and *loc. cit.*, pp. 741, that the direction cosines c_1, c_2, c_3, c_4 can be represented as linear functions of the $\frac{dx}{du}$, $\frac{d^2x}{du^2}$, $\frac{d^3x}{du^3}$, $\frac{d^4x}{du^4}$ thus:

$$\frac{dc_i^{(h)}}{du} = \sum_k \varepsilon_k^{(h)} c_i^{(k)}, \quad h, i = 1, 2, 3, 4,$$

with which, the proof is complete. Analogously, one will arrive at that proof when one assumes that:

$$\frac{dx^{(h)}}{du} = \varepsilon^{(h)} + \sum_k \varepsilon_k^{(h)} x^k, \quad h = 1, 2, 3, 4.$$

2. – One can then give the "true" representation (II, § **1**):

$$\begin{aligned} dX = {} & c_1 \, (1 - \xi' / R_1) \, d \, (i \, c \, \tau') \\ & c_2 \, (d\xi' - \eta' / R_2) \, d \, (i \, c \, \tau') \\ & c_3 \, (d\eta' + \xi' / R_2) \, d \, (i \, c \, \tau') - \zeta' / R_3 \, d \, (i \, c \, \tau') \end{aligned}$$

$$c_4\,(d\zeta' + \eta' / R_3)\, d\,(i\,c\tau')\,.$$

It then follows that:

$$\begin{aligned} dS^2 = {} & d\xi'^2 + d\eta'^2 + d\zeta'^2 \\ & - 2c\, d\tau'\, d\xi' \cdot \eta' i / R_2 \\ & + 2c\, d\tau'\, d\eta' \cdot (\xi' i / R_2 - \zeta' i / R_3) \\ & + 2c\, d\tau'\, d\zeta' \cdot \eta' i / R_3 \\ & - c^2 d\tau'^2 [(1 - \xi' / R_1)^2 + \eta'^2 / R_2^2 + \eta'^2 / R_3^2 + (\xi' / R_2 - \eta' / R_3)^2], \end{aligned}$$

in which one might recall that $1 / R_2$ and $1 / R_3$ are pure imaginary (I, Appendix 1, pp. 740).

The case of $1 / R_2 = 0$, $1 / R_2 = 0$ is the case of hyperbolic motion that was treated in II. It is the only one that yields a light velocity field that is based upon a scalar function of position. All others are based upon tensorial functions of position, which is why they seem to be unsuitable for the representation of potentials of central forces.

What is common to all of them is that the coefficients of dS^2 are free of τ'. *That characterizes the static field!* In that way, Einstein's equivalence hypothesis will become applicable as soon as one regards the latter as a heuristic tool (and not as the "reevaluation" of all mechanical "values") that should then be expressed as: *Stationary force fields behave like stationary states of acceleration*. In that, "stationary" is understood to mean *spatially* and *temporally* stationary. The hypothesis is true for fields that are temporally-stationary, but spatially-variable, only infinitesimally, where the fields are homogeneous; **Einstein** already had that possibility in mind; cf., **Ehrenfest**, *loc. cit.*

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