"Über Einsteins Äquivalenzhypothese und die Gravitation," Ann. Phys. (Leipzig) **50** (1916), 955-972.

On Einstein's equivalence hypothesis and gravitation

By Friedrich Kottler

Translated by D. H. Delphenich

In my previous articles ([1]), I have discussed the question of the relativity of acceleration or the possibility of an accelerated motion remaining hidden. Since that possibility is based upon **Einstein's** known equivalence hypothesis, his line of reasoning yielded an application to gravitation, but generally a very restricted one, since my aforementioned papers implied the admissibility of the relativity of acceleration for only one special accelerated motion if one wished to derive a gravitational field from it, namely, for falling motion.

Since then, **Einstein** abandoned the equivalence hypothesis. The reason for that was essentially based upon a special conception of his results that he formulated, which followed from giving the forces of the gravitational field an autonomous character, while *here* motion in a gravitational field shall be regarded as force-free, so the law of inertia will be altered, and gravitation will be interpreted as a pure inertial phenomenon. That conception of things seems to me to be a rigorous consequence of the equivalence hypothesis and is therefore objectionable only *simultaneously* with it.

Now, that is not to say that the equivalence hypothesis must remain true. Other theoretical or experimental objections against it can be raised. For the time being, they do not seem to exist to me, and therefore the attempt to once more take up and investigate the equivalence hypothesis is justified, even if its capacity to imply a theory of gravitation has shown to be quite limited.

The provisional results restrict to the case of the field of a mass point. On the surface of things, they differ from **Einstein**'s new theory only inessentially, besides having greater simplicity. The essential difference is fundamental, namely, the aforementioned kinematic, not dynamical, conception of gravitation.

Pursuing that investigation any further is not possible for the author at the moment, due to military service. Because of that, the character of this publication is also provisional.

([1]) **F. Kottler**, "Über die Raumzeitlinien der Minkowskischen Welt," Sitzber. Wiener Akad. **121** (1912), pp. 1659, *et seq.*

"Relativitätsprinzip und beschleunigte Bewegung," Ann. Phys. (Leipzig) **44** (1914), pp. 701 (cited as I)

"Fallende Bezugssysteme vom Standpunkte des Relativitätsprinzips," Ann. Phys. (Leipzig) **45** (1914), pp. 481 (cited as II)

"Beschleunigungsrelative Bewegungen und die konforme Gruppe der Minkowskischen Welt," Sitzber. Wiener Akad. (1916) (cited as III).

Contents

1. – In the previous-cited articles by the author, the following question was treated: How must an accelerated motion be formulated from the standpoint of the **Einstein-Minkowski** principle of relativity in order for a comoving observer to be able to believe that they are at rest?

The answered that was found was: When "rest" is interpreted to mean the constancy of the proper coordinates of all points of the accelerated reference body as viewed from the (arbitrary) viewpoint of the observers, on kinematical grounds, that can mean only those motions for which the reference body behaves like a (quasi-) rigid body, not only at each of its points, but in each position of those points. However, the associated world-lines are the trajectories of a one-parameter orthogonal group for **Minkowskian** S_4 . The associated mechanical trajectories are, up to second order, the ones with constant **Newtonian** acceleration, so the cases of falling, uniform rotation, and their combinations. The dynamical basis (if one is required) for the observer to not recognize accelerating forces that act upon him also follows, namely, that this force should be constant, so it should not be subjected to any variation that would once more lead to the orthogonal group of S_4 .

A generalization is conceivable only when one understands rest to merely mean the constancy of the line-of-sight angle of the accelerated reference body as seen from the comoving observer.

One then obtains the world-lines of a one-parameter conformal group of **Minkowskian** S_4 . That case was not subject to any dynamical considerations.

2. – That theory of "the relativity of acceleration" can be applied to **Einstein**'s equivalence hypothesis ([1]). Under the assumption that an observer on an accelerated reference body cannot prove that he is moving, so he will believe that his vicinity is (absolutely) at rest under the action of an accelerating field, **Einstein** proposed the hypothesis: *An actual force field and such an apparent acceleration field are physically (so also electromagnetically) completely equivalent.*

Einstein was mainly thinking of **Galileo**'s discovery, which has since been confirmed many times, that the acceleration of the homogeneous gravity field of the Earth is the same for all bodies. However, if the homogeneous field of gravity is equivalent to an acceleration field, from the above, then **Galilieo**'s law is not only self-explanatory, but also fundamental for a deeper understanding of gravitation, which has remained enigmatic up to now.

However, from what was said in § **1**, the assumption that is fundamental to **Einstein**'s equivalence hypothesis that the observer would not confirm their accelerated motion is *not sound* on the basis of the **Einstein-Minkowski** principle of relativity. Whereas every relativity of accelerated would be permissible on the basis of **Newton**'s principle of relativity because **Newtonian** mechanics has rigid bodies, that would no longer be permissible here for the quasi-rigid motions that were considered in § **1**. Thus, in particular, that would include free fall or its relativistic analogue, namely, **Born**'s hyperbolic motion, which is, in fact, identical to it up to second-order quantities.

Since **Einstein** restricted his consideration to free fall, which was taken to mean hyperbolic motion precisely ([2]), whereby he generally neglected second-order quantities, those considerations that are known to have led him to the curvature of light rays in a gravity field will also remain unimpeachable from the **Minkowskian** standpoint. The only thing that is not allowed is **Einstein**'s implicitly-postulated relativity of acceleration for other accelerated motions, which is an assertion that was repeated in his later work, without, of course, making any applications of it.

3. – Since the relativity of acceleration is valid exactly for at least hyperbolic motion, due to their fundamental significance, the exact results are also of interest to the restricted equivalence hypothesis for a theory of gravitation. They were given by the author in article II, and are briefly the following:

Let γ be the (**Minkowskian**) acceleration of the accelerated reference body along the positive Z-axis, let x', y', z', t' be the coordinates and time of the comoving observer, and let x, y, z, t be those of the observer at rest. One then has the following equations, which correspond to the Lorentz transformation here:

([1]) **A. Einstein**, Ann. Phys. (Leipzig) **35** (1911), pp. 898.
([2]) **A. Einstein**, Ann. Phys. (Leipzig) **38** (1912), pp. 355. In particular, cf. § 1, Einleitung.

$$\text{(1)}\qquad \left\{\begin{aligned} x &= x', \\ y &= y', \\ c^2/\gamma + z &= (c^2/\gamma + z')\cosh\frac{\gamma t'}{c}, \\ c\,t &= (c^2/\gamma + z')\sinh f\,\frac{\gamma t'}{c}. \end{aligned}\right.$$

If one neglects $(\gamma t')^2$ here in comparison to c^2 then one will again find **Einstein**'s formulas ([1]).

It further follows *exactly* that in place of the arc-length element:

$$ds^2 = dx^2 + dy^2 + dz^2 - c^2\,dt^2$$

of **Minkowski** space, one has:

$$\text{(2}a\text{)}\qquad ds'^2 = dx'^2 + dy'^2 + dz'^2 - c'^2\,dt'^2$$

here, where:

$$\text{(2}b\text{)}\qquad c' = c + \gamma / c \cdot z'$$

and

$$ds' = ds\,.$$

Hence, the "speed of light" c' in the comoving system is no longer invariant.

It ultimately follows that when:

$$d\sigma^2 = -\,ds^2 = -\,ds'^2 = c'^2\,dt'^2 - dx'^2 - dy'^2 - dz'^2$$

is the time-like arc-length along the world-line of the point, the equations of motion of force-free material point in the rest system whose rest energy is E_0 will be:

([1]) *Loc. cit.*, pp. 359. Equations (4).

$$
(3)\qquad
\begin{cases}
E_0 \dfrac{d^2 x'}{d\sigma^2} = 0, \\[2mm]
E_0 \dfrac{d^2 y'}{d\sigma^2} = 0, \\[2mm]
E_0 \dfrac{d^2 z'}{d\sigma^2} + E_0 \dfrac{\gamma}{c}\left(c + \dfrac{\gamma}{c} z'\right)\left(\dfrac{dt'}{d\sigma}\right)^2 = 0, \\[2mm]
E_0 \dfrac{d}{d\sigma}\left[\left(c + \dfrac{\gamma}{c} z'\right)^2 \dfrac{dt'}{d\sigma}\right] = 0,
\end{cases}
$$

from which, it follows naturally that the path in the primed system is not a line, but as a result of the *apparent* acceleration – γ, it will be the path of a falling or thrown body (up to second-order quantities).

Einstein also found that result before. However, **Einstein** gave *a different form* and *a different interpretation* to equations (3).

A different form ([1]): **Einstein** introduced the mass m_0 instead of the energy E_0 and the proper time τ' in place of the arc-length σ. Since the mass depends upon the speed of light c', it is not, however, an invariant or covariant, nor is the proper time τ', since one has $d\sigma = c'\,d\tau'$. However, if **Minkowskian** mechanics is to take on any physical meaning only in relation to constructions that are invariant (covariant, resp.) with respect to ds^2 then we must also demand that in the generalized mechanics that we aspire to here, as well. Mass no longer exists as a fundamental physical constant then since it depends upon the field. By contrast, when it is at rest in a space that is free from any field, the rest energy E_0 (i.e., the internal energy of the mass-point) will persist as a physical constant.

A different interpretation: **Einstein** posed the quantity:

$$
-\, E_0 \frac{\gamma}{c}\left(c + \frac{\gamma}{c} z'\right)\left(\frac{dt'}{d\sigma}\right)^2
$$

using the **Lagrange** process, and as a result of the fourth equation in (3) (viz., the so-called conservation of energy) ([2]), with A as an integration constant (dimension = gr cm^3 sec^{-3}), it can be written:

$$
-\, \frac{A^2}{E_0} \frac{1}{c'^3} \frac{\partial c'}{\partial z'} ,
$$

([1]) **Einstein**'s change of dimension by c will not be discussed here since it is inessential.

([2]) Strictly speaking, such a thing will not be spoken of here. It is not the energy $E' = m'c'^2$ that is conserved, but $A = m'c'^3$.

so it is equal to a "force" that the apparent field exerts on the mass-point. *Here*, that quantity will be regarded as an apparent force, in the sense of **d'Alembert**'s supplementary forces, such as the guiding acceleration, the centrifugal forces, the **Coriolis** forces, etc., of classical mechanics (i.e., a purely kinematical one), but it does not possess an autonomous dynamical existence. The left-hand sides of the equations of motion are *incompletely* lacking in those quantities. It then follows that (3) *are the equations of motion for the force-free mass-point in the moving system*.

4. – In order to apply the results that were set down in § **3** to the theory of the homogeneous field of gravity, one then agrees to accept **Einstein**'s equivalence hypothesis. According to (2*b*), since:

$$\Phi = \gamma z',$$

where Φ is the **Newtonian** potential of the field, one can write the curvature of light in the field of gravity as:

$$c' = c + \frac{\gamma}{c} z' = c + \frac{\Phi}{c}, \tag{4}$$

in which c is perhaps the value $(c')_{z'=0}$.

The difference from the interpretation that was given by **Einstein** that was mentioned in § **3** then has a kinematical conception of gravity, rather than a dynamical one, as a consequence, i.e., an alteration of **Galileo**'s law of inertia enters in place of the force field. The force-free point no longer moves uniformly and rectilinearly according to (3), but curvilinearly and non-uniformly. It seems quite paradoxical that this initially seems to be the consistent version of **Einstein**'s equivalence! In fact, if the source of the equality in the gravitational acceleration is kinematical for all masses then gravity itself must have a kinematical origin, i.e., it must be an inertial phenomenon! Experiments suggest that intuition: The lack of dispersion, propagation phenomena, etc., under gravitation, one might say the lack of energy in it (When has one seen any gravitational energy converted into any other forms of energy?), all of that only loses its mysterious character in that way.

However, the concepts of the potential, stresses, force, etc., of the field of gravity then *lose any physical meaning*, and that also comes to light in the *non-covariant* form of the expressions that **Einstein** postulated for them. At the same time, the difficulty that **Einstein** ([1]) raised in regard to his own equivalence hypothesis vanishes, since it arose from his dynamical conception of things.

5. – It is known that as a result of that dynamical conception, **Einstein** abandoned the equivalence hypothesis, and in particular, the consequence (4) and arrived at a theory of gravitation in which merely the change in the **Minkowski** ds^2 by the field of gravity was preserved, but was simultaneously raised to its most general form. The covariance that he postulated in that way relative to the new:

([1]) **A. Einstein**, Ann. Phys. (Leipzig) **38** (1912), pp. 452, *et seq.*

$$ds^2 = \sum_{i,k=1}^{4} g_{ik}\, dx_i\, dx_k \tag{5}$$

shall be connected with the relativity of acceleration. From a **Minkowskian** standpoint, that is initially possible only when ds^2 is Euclidian, i.e., it can be transformed into the form:

$$dx^2 + dy^2 + dz^2 - c^2\, dt^2$$

with constant *c*, i.e., when all of its Riemann symbols vanish. If that is fulfilled then we can infer only hyperbolic motion from § **1**, which defines **Einstein**'s relativity of acceleration. Therefore, there is no general relativity of acceleration, at least from a **Minkowskian** standpoint. It also seems that **Einstein**'s covariance of the physical quantities relative to (5) *has nothing to do with a relativity of acceleration, but with the older **Lorentzian** relativity of velocity*, insofar as a translation (that varies from position to position and time to time) must have no effect on the laws of physics ([1]).

If the relativity of acceleration ultimately seems to be ruined and lacking in physical significance then there is nonetheless an aspect of the situation that the complete breakdown of the equivalence hypothesis [i.e., more precisely, equations (1) to (3) for hyperbolic motion] has yet to point out. It is nothing but the attempt to extend the Lorentz transformation to accelerated systems and the fact that this was quite fruitful in the study of translatory moving systems in the way that it reveals, e.g., the peculiar connection between stresses, energy, impulse, and energy current (i.e., the general inertia of energy and the like), so one might expect that a relativity of acceleration that is supported by experiments would reveal something similar for accelerated systems and therefore for the gravitational field. The equivalence of inertial and gravitational mass does not generally emerge from the equivalence hypothesis alone. As a result of § **4**, it is the result of an alteration of the law of inertia in which the **Minkowskian** ds^2 is replaced with something like (5). The conception of gravitation as an inertial phenomenon is not connected with the equivalence hypothesis either. Rather, what is singles out is the fact that it has heuristic value and opens up an exact (if also less than humble) path to the theory of gravitation, while **Einstein**'s more recent theory must depend upon other, less obvious, hypotheses.

Since the equivalence hypothesis has not been contradicted by either theory or experiment up to now, while the evidence for it and its heuristic value is indisputable, it would not seem uninteresting to pursue it further in the direction that **Einstein**'s investigations have now embarked upon ([2]). The question is now: *How do the masses of the fields determine the coefficients of the* ds^2?

([1]) In order to see that, one makes the local transformation:

$$(ds^2)_0 = \sum (g_{ik})_0 (dx_i)_0 (dx_k)_0 = dx_0^2 + dy_0^2 + dz_0^2 - c^2 dt_0^2 .$$

([2]) **A. Einstein**, Ann. Phys. (Leipzig) **49** (1916), pp. 769.

6. – We shall first use that viewpoint to examine the isolated field that makes the equivalence hypothesis accessible to us, i.e., the static homogeneous field, which corresponds to the arc-length element:

(6a) $$ds^2 = dx^2 + dy^2 + dz^2 - c^2\, dt^2 ,$$

with

(6b) $$c = c_0 + \frac{\gamma}{c_0} z .$$

As is well-known from the theory of differential invariants, for:

$$ds^2 = \sum_{i,k}^{4} g_{ik}\, dx_i\, dx_k$$

one generally has the **Christoffel** three-index symbols:

$$\begin{bmatrix} i\,k \\ h \end{bmatrix} = \frac{1}{2}\left(\frac{\partial g_{ih}}{\partial x_k} + \frac{\partial g_{hk}}{\partial x_i} - \frac{\partial g_{ik}}{\partial x_h}\right),$$

as well as the **Riemann** four-index symbols:

$$(i\,k\,m\,n) = \frac{1}{2}\left(\frac{\partial^2 g_{in}}{\partial x_k\,\partial x_m} + \frac{\partial^2 g_{km}}{\partial x_i\,\partial x_n} - \frac{\partial^2 g_{im}}{\partial x_k\,\partial x_n} - \frac{\partial^2 g_{kn}}{\partial x_i\,\partial x_m}\right) + \sum_{r,s=1}^{4} g^{(rs)}\left(\begin{bmatrix} i\,n \\ r \end{bmatrix}\begin{bmatrix} k\,m \\ s \end{bmatrix} - \begin{bmatrix} i\,m \\ r \end{bmatrix}\begin{bmatrix} k\,n \\ s \end{bmatrix}\right) ,$$

in which $g^{(rs)}$ is the reciprocal form to g_{rs} , so it is equal to the subdeterminant that is adjoint to g_{rs} in the determinant:

$$g \mid g_{ik} \mid \qquad i, k = 1, 2, 3, 4,$$

divided by g.

Now, the **Riemann** four-index symbols, whose vanishing establishes the Euclidian nature of ds^2 , are fourth-order covariants ([1]), which are composed of the g_{ik} and their first (second, resp) differential quotients.

With **Einstein** ([2]), we construct a second-order covariant from that fourth-order one:

$$B_{ik} = \sum_{r,s=1}^{4} g^{(rs)} (i\,r\,s\,k) .$$

One has:

$$B_{ik} = B_{ki} ,$$

([1]) Cf., **F. Kottler**, "Raumzeitlinien der Minkowskian Welt," Sitzungsber. Wiener Akad. d. Wiss. (1912).
([2]) **A. Einstein**, *loc. cit.*, pp. 801.

due to the known properties of the Riemann symbols $(i\,r\,s\,k) = (k\,s\,r\,i)$ and $g^{(rs)} = g^{(sr)}$.

When that is applied to the field of the speed of light c, which might be a function of only z, so:

$$ds^2 = dx^2 + dy^2 + dz^2 - c^2(z)\,dt^2,$$

and one takes $x_1 = x$, $x_2 = y$, $x_3 = z$, $x_4 = i\,t$, that will imply that:

$$\left.\begin{aligned} &(1212) = (1313) = (2323) = 0, \\ &(1213) = (1223) = (1323) = 0, \end{aligned}\right\}$$

$$(1214) = (1224) = (1234) = (1314) = (1324) = (1334) = (2324) = (2334) = 0,$$

$$\left.\begin{aligned} &(1414) = (2424) = 0 \qquad (3434) = -\frac{1}{2}\frac{\partial^2(c^2)}{\partial z^2} + \frac{1}{4}\frac{1}{c^2}\left[\frac{\partial(c^2)}{\partial z}\right]^2 - c\,\frac{\partial^2 c^2}{\partial z^2}, \\ &(1424) = (1434) = (2434) = 0. \end{aligned}\right\}$$

The demand that ds^2 should be **Minkowskian**, hence Euclidian, then further implies that:

$$\frac{d^2c}{dz^2} = 0$$

or

$$c = c_0 + \frac{\gamma}{c_0}\,z,$$

which is form that the equivalence hypothesis is known to imply.

In that way, we know ([1]) that the masses are found to be at a great distance in the direction of negative z, and further that we must have:

$$z > -\frac{c_0^2}{\gamma},$$

which is a condition that excludes just the possibility of approaching the masses.

7. – We can then infer many worthwhile pointers for our problem from the equivalence hypothesis. First of all, the homogeneous field can be expressed in terms of a homogeneous light velocity field. Furthermore, upon approaching the masses, a certain barrier will be reached at which the speed of light vanishes. Finally, the speed of light satisfies certain covariant differential

([1]) **F. Kottler**, II, pp. 490.

equations, and in that way, the demand of the principle of relativity that only covariants should have any physical meaning will be satisfied.

Before we undertake the obvious generalization to the field of a mass-point, we shall bring up two issues:

First: What are the remaining acceleration-relative fields? We infer from III that their ds^2 has the form:

(7) $$ds^2 = dx^2 + dy^2 + dz^2 + 2g_{14}\,dx\,dit + 2g_{24}\,dy\,dit + 2g_{34}\,dz\,dit + g_{44}\,(dit)^2,$$

whereby the following conditions are fulfilled:

1. $g_{14}\ g_{24}\ g_{34}\ g_{44}$ is free of t (i.e., it is stationary),

$$\left.\begin{array}{ll} 2. & \dfrac{\partial g_{i4}}{\partial x_k} + \dfrac{\partial g_{k4}}{\partial x_i} = 0, \\[2ex] 3. & \dfrac{\partial g_{i4}}{\partial x_k} - \dfrac{\partial g_{k4}}{\partial x_i} = \text{const.}, \end{array}\right\}\quad i, k = 1, 2, 3,$$

4. $\sqrt{g}$ is linear in x, y, z,

as one easily derives. The latter conditions are connected with the orthogonal character of the basic infinitesimal transformation of S_4 . All of the Riemann symbols will vanish then. The arc-length element ds^2 is therefore Minkowskian or Euclidian.

1. and 4. are familiar to us from the case of the field that corresponds to hyperbolic motion in § **6**. By contrast, 2. and 3. do not come about naturally, because $g_{14}\ g_{24}\ g_{34}$ symbolizes the relative velocity compared to the observer, and no such thing exists in our falling reference system. That relative velocity, which originates in either a rotation or a ballistic motion or the like of the reference system according to I, also affects the field: The speed of light varies not only from place to place, but also with the direction at one and the same place: The field is *anisotropic*.

We have no reason to assume that the gravitational field is anisotropic, at least in the simplest cases.

We therefore exclude the cases $g_{14} \neq 0$, $g_{24} \neq 0$, $g_{34} \neq 0$.

Second: The (three-dimensional) space of the observer is Euclidian because it corresponds to the arc-length:

$$dx^2 + dy^2 + dz^2 .$$

That is consistent with all remaining hypotheses of physics and a change in that assumption, so an effect of the field on the nature of *space*, would not seem obvious, for the time being, since the field is expressed merely in terms of *temporal* variations, so it belongs to the realm of *mechanics*, not *geometry*. Consistent with the equivalence hypothesis, we then assume that:

$$g_{11} = g_{22} = g_{33} = 1, \qquad g_{12} = g_{13} = g_{23} = 0$$

in Cartesian coordinates and let only g_{44} depend upon the coordinates x, y, z . g_{11} will be free of t for a stationary field.

8. – We can now take the step that we had postponed, namely, generalizing the results of the equivalence hypothesis that are true for the homogeneous field to the radially-symmetric field of a mass-point. Following (7), we write:

$$ds^2 = dr^2 + r^2\, d\vartheta^2 + r^2 \sin^2 \vartheta\, d\varphi^2 - c^2(r)\, dt^2 \tag{7a}$$

in polar coordinates, or with:

$$x_1 = r, \qquad x_2 = \vartheta, \qquad x_3 = \varphi, \qquad x_4 = i\, t,$$

and

$$g_{11} = 1, \qquad g_{22} = r^2 = x_1^2, \qquad g_{33} = r^2 \sin^2 \vartheta = x_1^2 \sin^2 x_2\,, \qquad g_{33} = c^2\,(x_1)\,,$$

in the form:

$$ds^2 = g_{11}\, dx_1^2 + g_{22}\, dx_2^2 + g_{33}\, dx_3^2 + g_{44}\, dx_4^2\;.$$

Obviously, that ds^2 can no longer be Minkowskian, so will not be Euclidian, either. Certainly, not all of the Riemann symbols vanish. We shall construct those twenty Riemann symbols.

The first group, which are six in number, *do not* include the index 4. They vanish since they are all identical to the Riemann symbols of the three-dimensional "space" of the observer here, due to the assumed Euclidian nature of the latter.

The second group, which are eight in number, include the index 4 *once*. They all vanish, since $g_{14} = g_{24} = g_{34} = 0$, i.e., the field is isotropic.

The third group, which are six in number, include the index 4 *twice*. Three of them vanish, and what will remain are:

$$(1414) = -\frac{1}{2}\frac{\partial^2(c^2)}{\partial r^2} + \frac{1}{c^2}\cdot\frac{1}{4}\left[\frac{\partial(c^2)}{\partial r}\right]^2 = -c\,\frac{\partial^2 c}{\partial r^2}\,,$$

$$(2424) = -\frac{1}{4}\frac{\partial(r^2)}{\partial r}\cdot\frac{\partial(c^2)}{\partial r} = -c\,r\,\frac{\partial c}{\partial r}\,,$$

$$(3434) = -\frac{1}{4}\frac{\partial(r^2 \sin^2 \vartheta)}{\partial r}\cdot\frac{\partial(c^2)}{\partial r} = -c\,r\sin^2 \vartheta\,\frac{\partial c}{\partial r}\,.$$

If one constructs the B_{ik} then when one writes:

$$\frac{\partial c}{\partial r} = c', \qquad \frac{\partial^2 c}{\partial r^2} = c'',$$

it will follow that:

$$B_{11} = \frac{1}{g_{44}}(1441) = \frac{1}{c}c'',$$

$$B_{22} = \frac{1}{g_{44}}(2442) = \frac{r}{c}c',$$

$$B_{33} = \frac{1}{g_{44}}(3443) = \frac{r\sin^2\vartheta}{c}c',$$

$$B_{44} = \frac{1}{g_{11}}(4114) + \frac{1}{g_{22}}(4224) + \frac{1}{g_{44}}(4334) = c\left(c' + \frac{2}{r}c'\right) = c\,\Delta c,$$

$$B_{12} = B_{13} = \ldots = B_{34} = 0\,.$$

In those expressions, Δ is the three-dimensional **Laplace** operator for a function that depends upon merely r.

Finally, we form the invariant:

$$B = \sum_{i,k} g^{(ik)}\,B_{ik} = \frac{1}{g_{11}}B_{11} + \frac{1}{g_{22}}B_{22} + \frac{1}{g_{33}}B_{33} + \frac{1}{g_{44}}B_{44} = \frac{2}{r}\Delta c\,.$$

We have now acquired the necessary apparatus. From the Ansatz (2*b*):

$$c' = c + \frac{\gamma\,z'}{c},$$

which can be written:

$$c' = c + \frac{\Phi}{c},$$

Einstein had already concluded at the time that the speed of light was proportional to the **Newtonian** potential, except for a constant, only to abandon that simple assumption in favor of his later theory. On the basis of the equivalence hypothesis, that is also the most obvious generalization that is consistent with (2*b*). We then add the differential equation:

(8) $$B = \sum_{i,k} g^{(ik)} B_{ik} = \sum_{i,k,r,s} g^{(ik)} g^{(rs)} (i\,r\,s\,k) = \frac{2}{c}\Delta c = 0,$$

which has an invariant form. In fact, with the integration conditions:

1. c shall be a constant equal to c_∞ at infinity,

2. c has a pole at $r = 0$ (near $r = 0$, there is a "barrier" where $c = 0$),

it yields the desired form for c:

(7*b*) $$c = c_\infty \left(1 - \frac{\alpha}{r}\right),$$

which is valid for $r > \alpha$.

The fact that this form (7*b*) includes the form (2*b*) is easy to see. One sets r equal to something sufficiently large compared to α and writes:

$$r = r_0 + z,$$

where z is a quantity that is small compared to r_0. That gives:

$$c = c_\infty \left(1 - \frac{\alpha}{r}\right) = c_\infty \left(1 - \frac{\alpha}{r_0 + z}\right) = c_\infty \left(1 - \frac{\alpha}{r_0} + \frac{\alpha\, z}{r_0^2} + \cdots\right).$$

Furthermore, one has:

$$c_0 = c_\infty \left(1 - \frac{\alpha}{r_0}\right)$$

for $r = r_0$, so:

$$c + c_0 + \frac{\alpha\, c_\infty}{r_0^2} + \cdots$$

However, since one can clearly set:

(9) $$\Phi = -\frac{\alpha\, c_\infty^2}{r_0},$$

and due to the fact that:

$$\gamma = \frac{d\Phi}{dr} = \frac{\alpha\, c_\infty^2}{r_0^2},$$

one will have the formula:

$$c = c_0 + \frac{\gamma\, z}{c_\infty} + \cdots,$$

which can be written:

$$c = c_0 + \frac{\gamma z}{c_\infty},$$

approximately, which has the desired form.

One can deduce the order of magnitude for α for the field of an atom from (9). Indeed, one also has:

$$\Phi = -\frac{k\,m}{r},$$

in which $k = 6.7 \times 10^{-8}\ \mathrm{gr}^{-1}\ \mathrm{cm}^3\ \mathrm{sec}^{-2}$ is **Newton**’s gravitational constant and $m = 10^{-24}$ gr. With $c_\infty = 3 \times 10^{10}\ \mathrm{cm\ sec}^{-1}$, it will follow that ([1]):

$$\alpha = \frac{k\,m}{c_\infty^2} = 7.4 \times 10^{-53}\ \mathrm{cm}.$$

The “barrier” $r = \alpha$ is, in any event, an infinitely-small physical quantity ([2]).

In conclusion, we have highlighted the fact that the mass-point that corresponds to the kinematical-geometric foundation of our theory appears *only mathematically as a pole*, not *physically as a mass*, as in **Einstein**’s dynamical theory. On the basis of our theory that we derived from the equivalence hypothesis, *mass seems to be a discontinuity in space and time.*

Due to the barrier, it was reasonably for us to *not* go into the interior of the mass ([3]). The extensions of that theory must then assume the differential equation:

$$B = 0,$$

with the integration conditions:

1. c_∞ = const.,

2. each mass-point is a pole of c,

from which, the masses at the mass-points can be solved.

Naturally, the concepts of mass, density, etc., can play no role in this purely-geometric theory. Indeed, density is not an invariant or covariant, moreover, so from the standpoint of the principle of relativity, it has no absolute physical meaning. We cannot therefore expect something that would correspond to **Poisson**’s theory here.

([1]) Cf., **H. Reissner**, Ann. Phys. (Leipzig) **50** (1916), pp. 115.

([2]) Naturally, the aforementioned barrier $z > -\frac{c_0^2}{r} = -\left(\frac{r-\alpha}{\alpha}\right)^2$ merely applies to the *homogeneous* field.

([3]) The development that the author carried out at the conclusion of his article II, which led to **Byk**’s atomic model, breaks down with that.

It is not the goal of this discussion to carry out the suggested extension of the theory, but only to postulate its possibility, especially since it has not been established to be a unique consequence of the equivalence hypothesis. Rather, perhaps all of that might be discussed in a later paper.

9. – Finally, it is incumbent upon us to compare our results with those of **Einstein**'s theory in its most recent conception.

One knows that **Einstein** placed the completely general:

$$ds^2 = \sum_{i,k=1}^{4} g_{ik}\, dx_i\, dx_k$$

at its pinnacle. "Space," i.e.:

$$\sum_{i,k=1}^{3} g_{ik}\, dx_i\, dx_k ,$$

is no longer Euclidian for him, but seems to vary throughout the field. The field is then expressed in terms of ten particular potentials g_{11}, g_{12}, …, g_{44}. In matter-free space, they satisfy generalizations of the **Laplace** equation, namely, the ten differential equations:

$$B_{ik} = B_{ki} = \sum_{r,s} g^{(rs)}\,(i\,r\,s\,k) = 0, \qquad i, k = 1, 2, 3, 4 .$$

When **Einstein** multiplied that B_{ik} by $\partial g_{ik} / \partial x_m$ and summed over i, k :

$$\sum_{i,k} \frac{\partial g_{ik}}{\partial x_m} B_{ik} = 0, \qquad m = 1, 2, 3, 4,$$

it yielded four equations that he interpreted as the law of impulse on formal grounds, and from which he obtained certain quantities for the "stresses" of the "gravitational field" that were quadratic in the differential quotients $\partial g_{ik} / \partial x_m$, so they naturally could not be *covariant*. **Einstein** obtained the differential equations of the field for matter-filled spaces by introducing those field stresses and simply establishing on formal grounds that the stresses in the matter and the field *do not* superpose additively.

Finally, the case of the **Einstein-Schwarzschild** mass-point should be mentioned. One gets ([1]):

$$ds^2 = dr^2 + h^2 d\vartheta^2 + h^2 \sin^2\vartheta\, d\varphi^2 - c_\infty^2 \left(\frac{dh}{dr}\right)^2 dt^2 ,$$

in which:

$$r = \sqrt{h^2 - 2\alpha h} + \alpha \ln(h - \alpha + \sqrt{h^2 - 2\alpha h}),$$

([1]) **H. Reissner**, *loc. cit*, pp. 120.

and α is the constant that was already mentioned above (cf., § **8**):

$$\alpha = \frac{k\,m}{c_\infty^2}.$$

If one had $\alpha = 0$ then would have $h = r$, and:

$$c = c_\infty \frac{dh}{dr} = c_\infty \frac{\sqrt{h^2 - 2\alpha\,h}}{h}$$

would be constant. One would then have the gravitation-free field, i.e., the usual **Minkowskian** case. If one neglects α^2 in comparison to h^2 then one will have:

$$r = h - \alpha + \alpha \ln 2h\,,$$

$$c = c_\infty \left(1 - \frac{\alpha}{h}\right) + \cdots = c_\infty \left(1 - \frac{\alpha}{r}\right) + \cdots$$

Our results differ from those of **Einstein** only by quantities or order α^2 / h^2, at least relative to the speed of light. By contrast, a difference of order α / h is present in the geometric nature of space.

The numerical differences are then physically vanishing. By contrast, there exists a fundamental difference. To summarize once more, one has:

1. The equivalence hypothesis requires a change in the law of inertia that involves certain *apparent* field forces. **Einstein** interpreted them as *actual* field forces and preserved **Galileo**'s law of inertia for them but had to dispense with the covariant form (hence, **Minkowski**'s principle of physics) in the expression of those "forces."

2. For **Einstein**, it followed from the assumption that all $B_{ik} = 0$ (while here we need one that $\sum g^{(ik)} B_{ik} = 0$) that the equivalence hypothesis would no longer be true for him at all.

3. **Einstein** abandoned the Euclidian nature of space in the gravitational field.

4. For **Einstein**, the field would be anisotropic, in general.

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