"Fallende Bezugssysteme vom Standpunkt des Relativitätsprinzips," Ann. Phys. (Leipzig) (4) **45** (1914), 481-516.

Falling reference systems from the standpoint of the principle of relativity

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Translated by D. H. Delphenich

In an earlier paper ([1]), it was shown that a possible class of relatively-accelerated motions includes the ones that possess the paths of a "motion" in Minkowski space S_4 as their worldlines. In particular, the hyperbolic motions belong to it, which are the relativistic generalization of **Galilei**an free-fall.

In the present article, a falling reference system in a state of hyperbolic motion will be examined on the basis of the generalized Lorentz transformation that was presented in the previous one. In agreement with the known theory of **Einstein**, the curvature of the light rays (they are circles) will be verified, and two possible conceptions of observers that are imagined to have no knowledge of their accelerated motion will be discussed, namely, the "apparent" observer and the "true" one. For the former, the observer assumes that light rays are not curved, which is generally permissible since the deviation from the straight line is proportional to the acceleration divided by the square of the speed of light, so it is vanishingly small. He then sees falling processes exactly as we would on Earth, and in particular, the ballistic trajectory will appear to be the **Galile**an parabola. The latter observer will recognize the curvature of light rays and see that his world is governed by other laws of nature than the ones that are usually assumed. It will then be assumed that he must renounce **Fermat**'s principle of optics and **Galileo**'s law of inertia. The correct form for the ballistic trajectory is an almost parabolic ellipse.

One gets a third conception when one bases one's measurements on the assumption that light rays are straight lines. **Fermat**'s principle and **Galileo**'s law of inertia can then be justified, but the spatial geometry that one constructs in that way is no longer Euclidian, but hyperbolic.

In conclusion, the similarity with a theory of matter and quanta that was presented by **Byk** will be pointed out, and that will encompass an overview of a complete theory of matter, gravitation, quantum mechanics, and the cohesion of electricity.

([1]) **F. Kottler**, "Relativitätsprinzip und beschleunigte Bewegung," Ann. Phys. (Leipzig) **44** (1914), pp. 701. (Cited as I)

Notations

$\mathfrak{X}', \mathfrak{Y}', \mathfrak{Z}'$ true coordinates (normal arrangement), the "simultaneous" position in I.

X', Y', Z' apparent coordinates (minimal arrangement; i.e., by light rays), the "effective position in I.

Detailed table of contents

§ 1. – True and apparent coordinates in an accelerated reference system.

1. – Next recall the concept of the equidistance of a family of worldlines, which makes it possible to assign fixed “coordinates” to a worldline *p* for all times when an association is defined between the points of *p* and *b*, the worldline of the observer at just one time, which expresses just the state of rest relative to the observer on *b*.

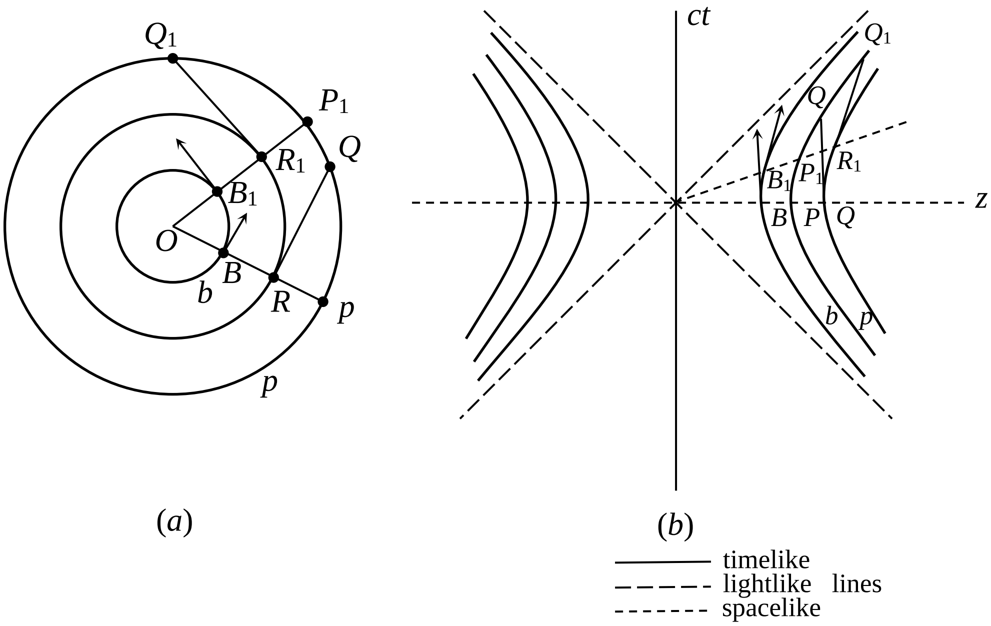

Figure 1. – Normal and oblique equidistance for spacelike rotation (*a*) and timelike rotation (*b*).

Fig. 1.a shows the analogy with the ordinary Euclidian plane, in which the *metric of the unit circle* around *O* is employed as the *gauge curve* ([1]): The trajectories of the rotation here are known to be concentric circles around the center of rotation *O*, and it is obvious that each figure will go to nothing but congruent figures under successive rotations. In particular, one then has $BP = B_1P_1$, and likewise $BQ = B_1Q_1$, which can be composed of two components $BR = B_1R_1$, which is parallel to the principal normal at *B* (B_1, resp.), and $RQ = R_1Q_1$, which is parallel to the tangent. We will appeal to $BP = B_1P_1 =$ const. as the “coordinate” of the circle *p* from the standpoint of an “observer” that moves along the circle *b* with the normal assignment and to $BR = B_1R_1 =$ const. (measured along the principal normal) and $RQ = R_1Q_1 =$ const. (measured along the tangent) as the “coordinates” with the assignment $Q \to B$.

Fig. 1.b shows the same thing for a Euclidian plane in which the *equilateral hyperbola:*

$$z^2 - (c\,t)^2 = 1 \qquad \text{(spacelike directions)}$$

([1]) The gauge curve (*Eichkurve*) (that terminology goes back to **Minkowski**’s geometry of numbers) gives the unit line segment in an arbitrary direction by way of the length of its radius that is parallel to that direction; cf., **L. Heffter** and **C. Koehler**, *Lehrbuch der analytische Geometrie*, Leipzig, 1905, pp. 367.

appears as *the gauge system*, or *its conjugate:*

$$z^2 - (c\,t)^2 = -\,1 \qquad\qquad \text{(timelike directions)},$$

respectively. The rotation has concentric equilateral hyperbolas around the center of rotation O as its trajectories here ([1]). An argument that is analogous to the one before implies the "coordinate":

$$\mathfrak{B}' = BP = B_1P_1 = \text{const.}$$

for the hyperbola p, as considered from the standpoint of an observer that moves along the hyperbola b, when the "normal assignment $P \rightarrow B$ is used, or the "coordinates":

$$\begin{aligned} Z' &= BR &&= B_1R_1 &&= \text{const.}, \\ c\,T' &= RQ &&= R_1Q_1 &&= \text{const.}, \end{aligned}$$

respectively, when the assignment $Q \rightarrow B$ is used.

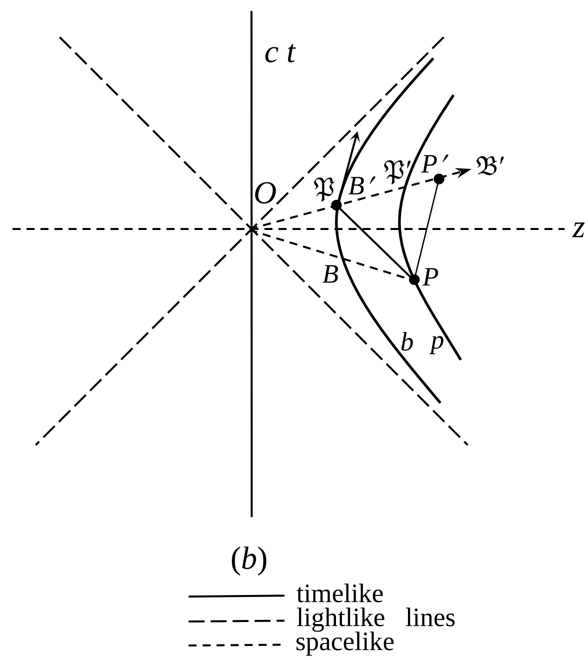

Figure 2. – True ($\mathfrak{P}'$) and apparent (P') coordinates of the worldline p on the reference body.

2. – Two assignments of the family of worldlines might be emphasized: The first one is the one for which the associated points are connected by *spacelike* lines that are *perpendicular* to all worldlines. (In Fig. 2, they are then the radii through O, so P belongs to B or $\mathfrak{P}'$ belongs to B'.)

([1]) The conjugate system of hyperbolas also belongs to them, but it yields no worldlines.

Since a cone of light rays goes through B (a pair of them, in the plane of Fig. 2), in order for that association to be one-to-one, it is necessary that one chooses half of that cone. Here, one obviously has to choose the *forward cone*, which includes the light rays that B *receives* from points that lie *before* it in time.

We call the first assignment the *true* one, while the second one is the *apparent* one. In order to justify that terminology, we remark that: The family of worldlines represents the motion of a body (Fig. 2: a simply-extended **Minkowski** spacetime curve). That body keeps a constant form in the proper system: e.g., the distance $BP = B'\mathfrak{P}'$ = const. (Fig. 2: tangent, principal normal = radius through O). Whenever an observer that moves along b performs that measurement with the help of a measuring tape, he will then find that the distance is *actually* equal to BP on a proper yardstick when he knows that direction from his viewpoint B to P ([1]). Thus, BP will be called the *true* distance, while $P \to B$ or $\mathfrak{P}' \to B'$ is the *true* assignment.

However, since the observer only sees with the help of light, he cannot *see* P at the *true* distance.

In fact, he will first see P when he himself arrives at B', so the latency time that the light needs to go from P to B' lies between the positions B, P, … of the reference body and those of B', $\mathfrak{P}'$, … However, since the accelerated motion of the observer has caused his proper system to rotate, the projection P' of P onto it will no longer yield the *true* distance BP, but the *apparent* one $B'P' > BP$. (Naturally, the *true* distance is $B'\mathfrak{P}' = BP$, where $\mathfrak{P}'$ is the position that belongs to B' under the *true* assignment, so it is the intersection point of the radius OB' with the worldline p.) ([2]) (Naturally, for a *uniform translation*, for which the family of worldlines consists of parallel lines, the proper system displaces parallel to itself, such that every light point *appears* to be at the *true* distance.)

3. – It might be remarked beforehand ([3]) that only the true ($\mathfrak{P}'$) coordinates are accessible to the measurement, which means that the apparent ones (P') are merely illusory. As far as time is concerned, P' is associated with the "apparent" latency time T', where $c\,T' = PP' = B'P'$, while the arc-length is BB' for the "true" latency time $c\,\mathfrak{T}'$ (both referred to proper time). Once more, only the latter is accessible to measurement ([3]), and therefore the former exists at most as something that is given indirectly by the "apparent" distance of the light point.

Since the curvature of the worldline b that is given by the reciprocal "radius" OB decreases with increasing distance (no line at all appears in place of the hyperbola at infinity), the difference between the apparent and true coordinate will then be even smaller. Expressed differently: *The smaller the acceleration, the less the apparent and true coordinate will differ.* We will see (§ **2**, subsection **5**) that the proper acceleration divided by the square of the speed of light is definitive of that difference.

([1]) However, cf., **§ 6**, subsection **2**.

([2]) In the case of the reference body with more than one dimension (indeed, Fig. 2 represents just such a thing), in addition to the change in distance of the apparent position with respect to the true position, one must also include the change in direction.

([3]) **§ 6**, subsection **1**.

§ 2. – Calculating the coordinates by means of the generalized Lorentz transformation.

1. – We recall the result (I, § **6**) that the proper coordinates imply the values $\Gamma^{(2)}$, $\Gamma^{(3)}$, $\Gamma^{(4)}$, and $\Gamma^{(1)}$ (the latter as an imaginary time coordinate) in the generalized Lorentz transformation:

$$X = x + \Gamma^{(1)} c_1 + \Gamma^{(2)} c_2 + \Gamma^{(3)} c_3 + \Gamma^{(4)} c_4 .$$

For that, we require the values of the direction cosines of the moving 4-frame. If we now take $x^{(1)} = x$, $x^{(2)} = y$, $x^{(3)} = z$, $x^{(4)} = i\,c\,t$, and the worldline b of the observer is given in the form (*loc. cit.*, Type III.b, column 2):

$$x = x_0 , \qquad y = y_0 , \qquad z = b \cosh u, \quad c\,t = b \sinh u$$

then Tab. 1, Type III.b, columns 13 to 16 (*loc,. cit.*) will give us the values:

Tangent:	$c_1^{(1)} = 0,$	$c_1^{(2)} = 0,$	$c_1^{(3)} = (1/i) \sinh u,$	$c_1^{(4)} = \cosh u$
Principal normal:	$c_2^{(1)} = 0,$	$c_2^{(2)} = 0,$	$c_2^{(3)} = -\cosh u,$	$c_2^{(4)} = -\sinh u$
Binormal:	$c_3^{(1)} = 1,$	$c_3^{(2)} = 0,$	$c_3^{(3)} = 0,$	$c_3^{(4)} = 0$
Trinormal:	$c_4^{(1)} = 0,$	$c_4^{(2)} = 1,$	$c_4^{(3)} = 0,$	$c_4^{(4)} = 0$

The terms "binormal" and "trinormal" are clearly arbitrary. We then have:

$$X = x_0 + \Gamma^{(3)}, \quad Y = y_0 + \Gamma^{(4)}, \quad Z = (b - \Gamma^{(2)}) \cosh u + \left(\frac{1}{i}\,\Gamma^{(1)}\right) \sinh u,$$

$$c\,T = (b - \Gamma^{(2)}) \sinh u + \left(\frac{1}{i}\,\Gamma^{(1)}\right) \cosh u .$$

We apply those formulas: First of all, to the determination of the *true* coordinates $\mathfrak{X}'$, $\mathfrak{Y}'$, $\mathfrak{Z}'$, and secondly, to the determination of the *apparent* coordinates X', Y', Z'.

2. – *True coordinates* $\mathfrak{X}'$, $\mathfrak{Y}'$, $\mathfrak{Z}'$: The corresponding assignment was (§ **1**, subsection **2**) the "normal" one. The position X, Y, Z, $c\,T$ then belongs to a well-defined value $\mathfrak{u}$ of the parameter of the worldline b of the observer, such that X, Y, Z, $c\,T$ will lie on the normal space to $\mathfrak{u}$. For that to be true, it is obviously necessary that:

$$\text{(1)} \qquad \sum_{h=1}^{4} [X^{(h)} - x^{(h)}(\mathfrak{u})] \cdot c_1^{(h)}(\mathfrak{u}) = i\,(Z \cdot \cosh \mathfrak{u} - cT \cdot \sinh \mathfrak{u}) = 0 .$$

If that condition for $\mathfrak{u}$ is fulfilled then one can write:

$$X = x + \Delta^{(2)} c_2 + \Delta^{(3)} c_3 + \Delta^{(4)} c_4 ,$$

so when one sets ([1]):

$$\mathfrak{Z}' = -\Delta^{(2)}, \qquad \mathfrak{X}' = \Delta^{(3)}, \qquad \mathfrak{Y}' = \Delta^{(4)}, \qquad \mathfrak{Z}' - c\,\tau = (\ldots)\,\Delta^{(1)} = 0 ,$$

one will have ([2]):

$$\text{(2)} \qquad \begin{cases} X = x_0 + \mathfrak{X}', \\ Y = y_0 + \mathfrak{Y}', \\ Z = (b + \mathfrak{Z}')\cosh \mathfrak{u}, \\ cT = (b + \mathfrak{Z}')\sinh \mathfrak{u}. \end{cases}$$

In order for these equations to be soluble for $\mathfrak{X}'$, $\mathfrak{Y}'$, $\mathfrak{Z}'$, it is necessary that the determinant must vanish:

$$\begin{vmatrix} 1 & 0 & 0 & X \\ 0 & 1 & 0 & Y \\ 0 & 0 & \cosh \mathfrak{u} & Z \\ 0 & 0 & \sinh \mathfrak{u} & cT \end{vmatrix} = 0,$$

which leads to the condition (1) for $\mathfrak{u}$, which is also sufficient.

3. – In (2), one must make the restriction that:

$$Z^2 - c^2 T^2 > 0, \qquad Z > 0,$$

since the spacetime curve of the reference body contains no world-point with $Z^2 - c^2 T^2 < 0$ (indeed, such a thing would have to move with superluminal speed) and even less so, world-points with $Z^2 - c^2 T^2 < 0$, but $Z \leq 0$. Otherwise, it would have to include the asymptotes through O on the grounds of continuity, which would mean the speed of light for its material points and would therefore be excluded. The observer *cannot see* such world-points *at all*. The ones for which $Z^2 - c^2 T^2 < 0$ do not lie at any proper time in his space. However, the world-points for which $Z^2 - c^2 T^2 > 0$, but $Z \leq 0$ will send their light to him only after an infinitely-long time, as one sees from e.g., Fig. 2. The light rays that come out of the other spacelike solid angle of the asymptotes, which includes the second branch of the hyperbola, obviously first penetrate the former solid angle at

([1]) For the sign of $\mathfrak{Z}'$, confer I, rem. 1 on pp. 741 in Appendix 4.

([2]) One will find the formula (4) that was given by **A. Einstein**, Ann. Phys. (Leipzig) **38** (1912), pp. 359, when one, with **Einstein**, restricts oneself to quantities of order two in $\mathfrak{u}$. The *falling motion* that he assumed is basically a *hyperbolic motion*; see his introductory remarks, *loc. cit.*, pp. 356: "let this acceleration be uniform in the **Born** sense."

infinity, since they all run parallel to those asymptotes that this solid angle includes and outside of it. Similar statements are also true for two *non*-coplanar hyperbolas that belong to *different* solid angles ([1]).

4. *Apparent coordinates X', Y', Z'.* – The corresponding assignment is the "minimal" one. The position $X, Y, Z, c\,T$ then belongs to a well-defined value of the parameter $u > \mathfrak{u}$, in such a way that $X, Y, Z, c\,T$ lies on the forward cone of the point $x(u), y(u), z(u), ct(u)$. In order for that to be true, one must obviously have:

$$\sum_{h=1}^{4}[X^{(h)} - x^{(h)}(u)]^2 = 0. \tag{3}$$

If that condition is fulfilled then one can write:

$$X = x + \Gamma^{(1)} c_1 + \Gamma^{(2)} c_2 + \Gamma^{(3)} c_3 + \Gamma^{(4)} c_4\,, \qquad \text{with} \qquad \sum_{h=1}^{4}(\Gamma^{(h)})^2 = 0,$$

so if one sets:

$$Z' = -\Gamma^{(2)}, \qquad X' = \Gamma^{(3)}, \qquad Y' = \Gamma^{(2)}, \qquad c\,T' = c\,\tau = -\sqrt{(X')^2 + (Y')^2 + (Z')^2}$$

then one will have:

$$\left\{\begin{aligned} X &= x_0 + X', \\ Y &= y_0 + Y', \\ Z &= (b + Z')\cosh u - \sqrt{(X')^2 + (Y')^2 + (Z')^2}\cdot\sinh u\,, \\ cT &= (b + Z')\sinh u - \sqrt{(X')^2 + (Y')^2 + (Z')^2}\cdot\cosh u\,. \end{aligned}\right. \tag{4}$$

In order for these quadratic equations in X', Y', Z' to be compatible, it is once more only necessary for the condition (3) on u to be satisfied.

5. – *Relationship between the values $\mathfrak{X}'$, $\mathfrak{Y}'$, $\mathfrak{Z}'$ and X', Y', Z' that belong to one and the same system of values for $X, Y, Z, c\,T$.* – Since solving (4) is not as simple as solving (2), we shall look for a relationship between the true and apparent coordinates that will allow us to merely perform the transformation (2), and from there to also arrive at the values of X', Y', Z'.

In order to do that, we have:

$$X = x_0 + \mathfrak{X}' \qquad = x_0 + X',$$
$$Y = y_0 + \mathfrak{Y}' \qquad = y_0 + Y',$$

([1]) We then call the plane $b + \mathfrak{Z}' = 0$ on the reference body the *limiting plane*; cf., **6**, subsection **4**. **M. Born**, Ann. Phys. (Leipzig) **30** (1909), pp. 1 deduced a maximal acceleration that is compatible with the radius of a corpuscle from the condition above.

$$Z = (b + \mathfrak{Z}')\cosh u = (b + Z')\cosh u - \sqrt{(X')^2 + (Y')^2 + (Z')^2}\,\sinh u,$$

$$cT = (b + \mathfrak{Z}')\sinh u = (b + Z')\sinh u - \sqrt{(X')^2 + (Y')^2 + (Z')^2}\,\cosh u,$$

so:

$$X' = \mathfrak{X}', \qquad Y' = \mathfrak{Y}',$$

and from the invariance of the two-dimensional distance from O ([1]) under our rotation, we will have:

$$Z^2 - c^2\,T^2 = (b + \mathfrak{Z}')^2 = (b + Z')^2 - [(X')^2 + (Y')^2 + (Z')^2]\,,$$

such that the desired relations read:

$$\text{(5)} \qquad \left\{ \begin{aligned} X' &= \mathfrak{X}', \\ Y' &= \mathfrak{Y}', \\ 2b\,Z' &= \mathfrak{X}'^2 + \mathfrak{Y}'^2 + \mathfrak{Z}'^2 + 2b\,\mathfrak{Z}'. \end{aligned} \right.$$

Mathematically speaking, that is a quadratic transformation of $\mathfrak{X}'$, $\mathfrak{Y}'$, $\mathfrak{Z}'$-space under which the origin (from the standpoint of the observer) remains invariant. We will learn about its remarkable properties in the following paragraphs. Here, it might only be pointed out that the final conclusions of the previous paragraphs remain true: The difference between the apparent and true coordinates is expressed by:

$$Z' - \mathfrak{Z}' = \frac{1}{2b}(\mathfrak{X}'^2 + \mathfrak{Y}'^2 + \mathfrak{Z}'^2) = \frac{g}{2c^2}(\mathfrak{X}'^2 + \mathfrak{Y}'^2 + \mathfrak{Z}'^2),$$

where g is the proper acceleration of the observer. It will get smaller as g gets smaller in comparison to c^2.

§ 3. – Light rays and force-free mass-points in an accelerated reference system.

1. – We fix a timelike line and represent its points as functions of time:

$$\text{(6)} \qquad X = X_0 + q_x\,T, \qquad Y = Y_0 + q_y\,T, \qquad Z = Z_0 + q_z\,T, \qquad c\,T = c\,T,$$

in which:

$$q^2 = q_x{}^2 + q_y{}^2 + q_z{}^2 \le c^2.$$

([1]) The point X, Y, 0, 0, resp.

If the equality sign is true then we are dealing with a light ray, and otherwise with the worldline of a force-free mass-point, since the **Galilean** law of inertia is also true in **Minkowksi**an mechanics.

We transform according to (2); that gives:

$$\begin{aligned} X &= X_0 + q_x\,T = x_0 + \mathfrak{X}', \\ Y &= Y_0 + q_y\,T = y_0 + \mathfrak{Y}', \\ Z &= Z_0 + q_z\,T = (b + \mathfrak{Z}')\cosh \mathfrak{u}, \\ c\,T &= \qquad\qquad\;\; = (b + \mathfrak{Z}')\sinh \mathfrak{u}, \end{aligned}$$

which allows us to obtain the value of $\mathfrak{u}$ from (1):

$$Z \sinh \mathfrak{u} - c\,T \cosh \mathfrak{u} = Z_0 \sinh \mathfrak{u} + (q_x \sinh \mathfrak{u} - c \cosh \mathfrak{u})\,T = 0\ .$$

That is then a point-wise association of lines with hyperbolas by the observer. In particular, $T = 0$ belongs with the value $\mathfrak{u} = 0$, which is self-explanatory, since at time $t = 0$, the normal space of the observer coincides with the space $c\,t = 0$.

We then get $\mathfrak{X}'$, $\mathfrak{Y}'$, $\mathfrak{Z}'$ as functions of T :

$$\text{(7.a)} \qquad \left\{ \begin{aligned} \mathfrak{X}' &= X_0 - x_0 + q_x T, \\ \mathfrak{Y}' &= Y_0 - y_0 + q_y T, \\ b + \mathfrak{Z}' &= \sqrt{(Z_0 + q_z T)^2 - c^2 T^2}, \end{aligned} \right.$$

and since $c\,\mathfrak{T}' = c\,\tau$, and from (I, Tab. 1, column 7):

$$c^2\,\tau^2 = b^2\,du^2,$$

we get the time $\mathfrak{T}'$ from (1) as:

$$c\,\mathfrak{T}' = b \arg\tanh \frac{c\,T}{Z_0 + q_z T}$$

or

$$\text{(7.b)} \qquad T = \frac{Z_0 \sinh \dfrac{c\,\mathfrak{T}'}{b}}{c \cdot \cosh \dfrac{c\,\mathfrak{T}'}{b} - q_z \cdot \sinh \dfrac{c\,\mathfrak{T}'}{b}}\,.$$

The parameter T appears in the representation (7.a), and from (6), it has *an invariant meaning*. Hence, the *arc-length element is also invariant* under the transformation (2), so:

$$d\sigma^2 = c^2\,dT^2 - dX^2 - dY^2 - dZ^2 = (c^2 - q^2)\,dT^2$$

will also be an invariant, such that one merely has to replace *T* with:

$$T = \frac{\sigma}{\sqrt{c^2 - q^2}}$$

in (7.a) in order to make the invariance emerge.

2. – We then represent the $\mathfrak{X}'$, $\mathfrak{Y}'$, $\mathfrak{Z}'$ as functions of the proper time. In order to do that, we set:

$$c = \sqrt{c^2 - q_z^2} \cdot \cosh \frac{c\,\mathfrak{T}_1'}{b}, \quad q_z = \sqrt{c^2 - q_z^2} \cdot \sinh \frac{c\,\mathfrak{T}_1'}{b},$$

such that:

(7.c)
$$T = \frac{Z_0 \sinh \frac{c\,\mathfrak{T}_1'}{b}}{\sqrt{c^2 - q_z^2} \cdot \cosh \frac{c\,(\mathfrak{T}' - \mathfrak{T}_1')}{b}} = \frac{Z_0\, c}{c^2 - q_z^2} \tanh \frac{c\,(\mathfrak{T}' - \mathfrak{T}_1')}{b} + \frac{Z_0\, q_z}{c^2 - q_z^2}$$

and

(8)
$$\begin{cases} \mathfrak{X}' = \mathfrak{X}_1' + (b + \mathfrak{Z}_1') \dfrac{q_x}{\sqrt{c^2 - q_z^2}} \tanh \dfrac{c\,(\mathfrak{T}' - \mathfrak{T}_1')}{b}, \\ \mathfrak{Y}' = \mathfrak{Y}_1' + (b + \mathfrak{Z}_1') \dfrac{q_y}{\sqrt{c^2 - q_z^2}} \tanh \dfrac{c\,(\mathfrak{T}' - \mathfrak{T}_1')}{b}, \\ \mathfrak{Z}' = \qquad (b + \mathfrak{Z}_1') \cdot \dfrac{1}{\cosh \frac{c\,(\mathfrak{T}' - \mathfrak{T}_1')}{b}}, \end{cases}$$

in which we have set:

$$\mathfrak{X}_1' = X_0 - x_0 + Z_0 \frac{q_x\, q_z}{c^2 - q_z^2}, \quad \mathfrak{Y}_1' = Y_0 - y_0 + Z_0 \frac{q_y\, q_z}{c^2 - q_z^2}, \quad b + \mathfrak{Z}_1' = \frac{Z_0\, c}{\sqrt{c^2 - q_z^2}} .$$

The meaning of the time-point $\mathfrak{T}' = \mathfrak{T}_1'$, so from (7.c):

$$T_1 = \frac{Z_0\, q_z}{c^2 - q_z^2}, \quad Z_1 = \frac{Z_0\, c^2}{c^2 - q_z^2},$$

is clarified as follows: The coordinate $\mathfrak{Z}'$ attains its maximum for $\mathfrak{T}' = \mathfrak{T}_1'$; i.e., since the direction $\mathfrak{Z}'$ represents the vertical to the reference body, the path reverses itself and *increases* in the direction of *falling*.

3. – In the *apparent* coordinates, (7.a), by means of (5), will give us:

$$(9)\qquad \left\{\begin{aligned} X' &= X_0' + q_x T, \\ Y' &= Y_0' + q_y T, \\ 2b\,Z' &= -(c^2 - q^2)T^2 + 2\,(X_0'\,q_x + Y_0'\,q_y + Z_0\,q_z)T + X_0'^2 + Y_0'^2 + Z_0^2 - b^2, \end{aligned}\right.$$

where we have $X'_0 = X_0 - x_0$, $Y'_0 = Y_0 - y_0$.

§ 4. – Discussion of the results of § 3.

1. *Light rays in the accelerated system:* $q^2 = c^2$. – If follows from (7.a) or (8) that the light rays are *circles* ([1]) on the reference body that all have their centers in the limiting plane:

$$b + \mathfrak{Z}' = 0$$

and are perpendicular to the limiting plane, so expressed more concisely, they intersect the limiting plane perpendicularly. In particular, the normal lines to that plane belong to them; they will be obtained when:

$$q_x = q_y = 0, \qquad q_z = c\ .$$

Therefore, the light rays that run parallel to the direction of falling are *rectilinear*, while all other are *curved* as a result of the falling motion of the accelerated system.

In the *apparent* coordinates, the light rays (9) are represented as *lines*. That was to be expected, because the apparent coordinates were defined by in such a way that the observer moved in the direction in which he *saw* them. It would then be self-explanatory that the light rays represented lines.

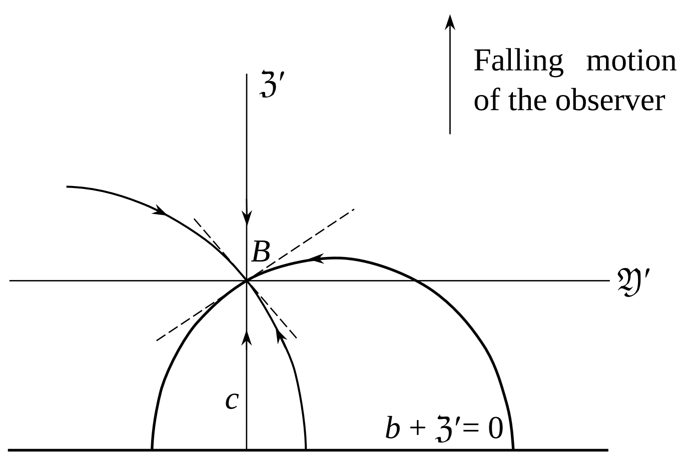

Figure 3. – True and apparent form of a light ray in a falling reference system.

([1]) Radius: $b + \mathfrak{Z}'_1$, center: $\mathfrak{X}'_1$, $\mathfrak{Y}'_1$, $b + \mathfrak{Z}' = 0$.

The true and apparent forms of the light rays that B receives are illustrated in Fig. 3. Naturally, the rays that are emitted by B are identical with the ones that arrive at B, since the speed of light is independent of the direction in our accelerated reference body (otherwise, it would be a *rotating* body). The figure is restricted to the $\mathfrak{Y}'\,\mathfrak{Z}'$-plane. The light rays in an arbitrary plane through the $\mathfrak{Z}'$-axis that goes through B obviously represent a *pencil of circles* whose real base-points are both B and the point that is symmetric to B relative to the base-plane $b + \mathfrak{Z}' = 0$, whose chord is the $\mathfrak{Z}'$-axis, which makes that element common to the pencil of ∞^1 circles in the ∞^1 planes through the $\mathfrak{Z}'$-axis.

2. – Moreover, we can also plot the apparent position P' at which the observer sees the point $\mathfrak{P}'$ of his reference body. We merely have to take the apparent light path instead of the true one, and since (5) says that we have:

$$X' = \mathfrak{X}', \qquad Y' = \mathfrak{Y}', \qquad 2b\,Z' = \mathfrak{X}'^2 + \mathfrak{Y}'^2 + \mathfrak{Z}'^2 + 2b\,\mathfrak{Z}',$$

we can then simply transfer the same $\mathfrak{X}'$, $\mathfrak{Y}'$ coordinates that the point $\mathfrak{P}'$ possesses to P' and thus find the apparent position P', as it is given in Fig. 4, which lies in the plane $\mathfrak{X}' = 0$.

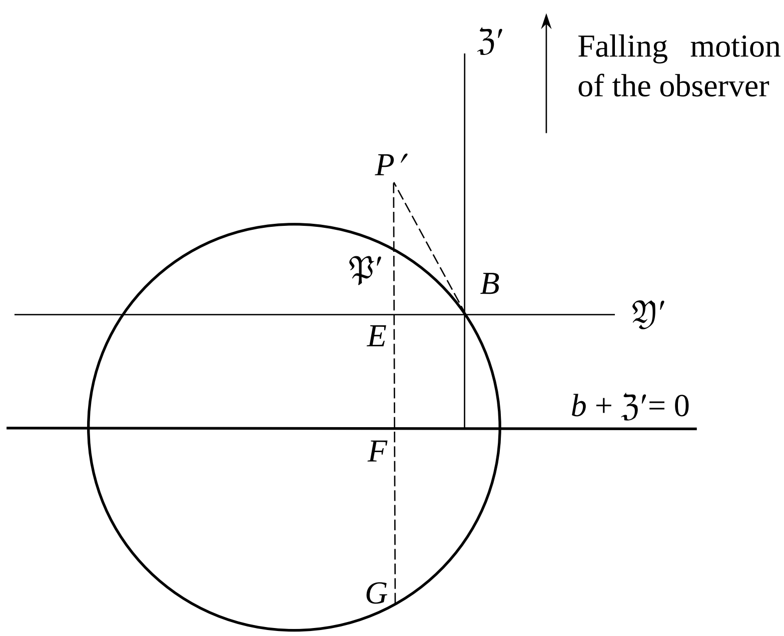

Figure 4. – Connection between the true ($\mathfrak{P}'$) and apparent (P') coordinates.

In fact, one finds that the potency (*Potenz*) of the point P' relative to the circles that represent the true light path from $\mathfrak{P}'$ to B is:

$$\overline{P'B}^2 = P'\mathfrak{P}' \cdot P'G.$$

However:

$$\overline{P'B}^2 = \overline{P'E}^2 + \overline{EB}^2 = Z'^2 + Y'^2 \quad \text{and} \quad P'\mathfrak{P}' = Z' - \mathfrak{Z}',$$

$$P'G = Z' + \mathfrak{Z}' + 2EF = Z' + \mathfrak{Z}' + 2b,$$

so:

$$Z'^2 + Y'^2 = (Z' - \mathfrak{Z}')(Z' + \mathfrak{Z}' + 2b),$$

from which the third relation in (5) will result.

3. *Force-free mass-points in an accelerated system:* $q^2 < c^2$. – Their paths are ellipses ([1]) in planes perpendicular to the limiting plane $b + \mathfrak{Z}' = 0$ with their major axes parallel to the vertical and their centers in the limiting plane. More briefly, they intersect the limiting plane perpendicularly. The ratio of the axes is:

$$\beta / \alpha = \sqrt{\frac{q_x^2 + q_y^2}{c^2 - q_z^2}}\,,$$

and the eccentricity is:

$$\varepsilon = \sqrt{\frac{c^2 - q^2}{c^2 - q_z^2}}\,.$$

One concludes: If $q_x = q_y = 0$ (so the horizontal velocity is zero) then the point will experience a pure "fall" in the vertical direction ($\beta = 0$) (viz., a rectilinear path). Otherwise, the path will always be an ellipse, except in the limiting case $q = c$ ($e = 0$), when it is a circle (viz., a light ray). The ellipse will then become the "ballistic trajectory" of the points, so since q is small compared to c, it will generally be approximately a *parabola* ($\varepsilon = 1$). Most likely, that will be true for pure horizontal velocity ($q_z = 0$), from the character of ellipses.

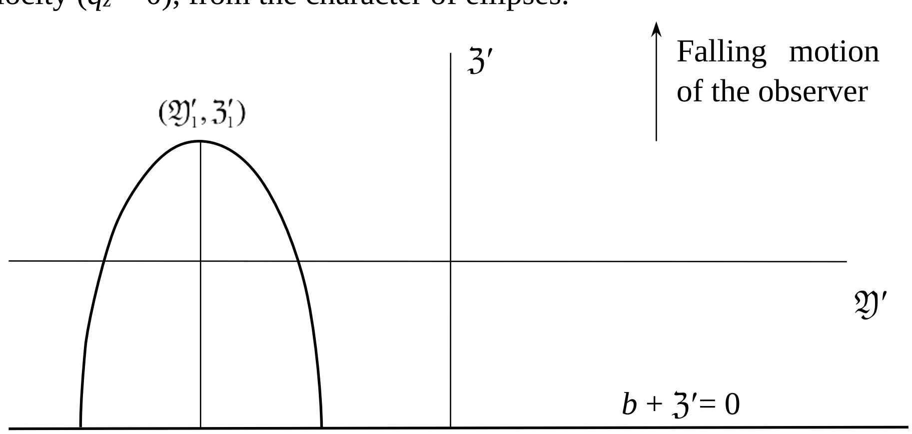

Figure 5. – The "ballistic trajectory" of a force-free point in an accelerated reference system (viz., an ellipse).

([1]) Semi-major axis: $\alpha = b + \mathfrak{Z}'_1$, semi-minor axis: $\beta = (\ldots)$, center: $\mathfrak{X}'_1$, $\mathfrak{Y}'_1$, $b + \mathfrak{Z}' = 0$.

Furthermore, one finds the confirmation of the statement of the previous paragraph that $\mathfrak{T}' = \mathfrak{T}'_1$ is the time-point of the point of reversal in Fig. 5.

4. – Formula (9) gives the apparent form of the falling (ballistic, resp.) trajectory. The former is again a vertical line, while the latter is a **Galilean** *parabola*. *The apparent forms of the ballistic and falling trajectories are then the same as in classical mechanics here*. That should not be surprising. Namely, one finds upon differentiating (9) with respect to T that:

$$\frac{d^2X'}{dT^2} = 0, \quad \frac{d^2Y'}{dT^2} = 0, \quad \frac{d^2Z'}{dT^2} = -\frac{c^2}{b^2}(1 - q^2/c^2),$$

and since the arc-length of the world-line is:

$$\sigma = cT\sqrt{1 - q^2/c^2},$$

one can also write that as:

$$\frac{d^2X'}{d\sigma^2} = 0, \quad \frac{d^2Y'}{d\sigma^2} = 0, \quad \frac{d^2Z'}{d\sigma^2} = -1/b. \tag{10}$$

That is nothing but an expression of the fact that the "apparent" acceleration is equal to minus the proper acceleration (cf., I, Appendix 4). That is clearly only true for the "apparent" coordinates. Indeed, they characterize an effective position that is referred to an ordinary Lorentz system that coincides with the moving 4-frame for a moment. However, in such a system, the observer is instantaneously at rest, which is why the **Minkowski** force reduces to the single Z'-component whose magnitude is c^2/b. The acceleration coincides with the proper acceleration in that system, and since the observer believes himself to actually be at rest, it will also become the "apparent" acceleration for all points that the observer *sees*, and naturally to *only* those points.

Since the apparent acceleration is constant then, equations (10) will give the **Galilean** parabola, and not, say, a catenary, which indeed enters in place of the **Galilean** parabola in the theory of relativity, from (I, § **2**, subsection 9). However, in that theory, that fact is based upon the difference between longitudinal and transverse mass, which does not actually come into play here, since we are treating only "apparent" accelerated motions that are force-free in "reality."

How the accelerations that can be represented in the "true" coordinates can be first discussed in the following paragraphs on the basis of the arc-length element.

§ 5. – Arc-length element and law of inertia.

1. – Being given the expression for the arc-length of the world means being given the law of light propagation, from which the speed of light can be calculated directly as a function of position, time, and direction. Conversely, knowing that law and certain boundary values of the coefficient will suffice to exhibit the expression for the arc-length. Furthermore, according to the theory of relativity, the propagation of force effects obeys the same laws.

Now, since that law of propagation is something that is given absolutely, it will follow that it will remain valid under any type of considerations that it even makes possible. For example, that is the most natural way of visualizing the Lorentz transformation. The apparent contractions, etc., that result under the transition to a moving system do not result *in order* to make the speed of light constant, but *because* it is. In fact, measurements (for example, simultaneity, etc.) can only be made with the help of light. On those grounds, all mathematical expressions whatsoever will not assume a certain invariant, covariant, or contravariant form *in order* for one to confirm no change in the law of propagation, but *because* it is measured only with the help of that law.

2. – If one assumes that the law of propagation varies from position to position and time to time ([1]) then the *form* of the expressions must be that of the theory of differential invariants ([2]). In fact, if one were to use a different way of looking at the world instead of the original spacetime system that would then come about under a spacetime transformation then one would once more need to find the law of propagation. However, that means, mathematically: The transformation preserves the arc-length, and by contrast, the desired form of the mathematical expressions cannot include any contradiction. In particular, certain invariants must be the same in both conceptions. Analogously to how one infers the vector property (transformation like coordinates or their products) in ordinary vector analysis, one now finds the spacetime vector property when one postulates that the components transform like the *differentials* of the coordinates or their products (contravariants), whereby one must also consult the reciprocal transformation (covariants). In fact, under a transformation of the x to $\bar{x}$:

$$d\bar{x}^i = \sum_{k=1}^{4} \frac{\partial \bar{x}^{(i)}}{\partial x^{(k)}} dx^{(k)}, \quad i = 1, 2, 3, 4$$

will be a linear transformation of the differentials for which the coefficients are not constant, in general.

3. – In order to develop the physics of a general arc-length element, one has a convenient tool. One can take (and this will suffice for our purposes) an arc-length:

$$dS^2 = \sum_{i,k=1}^{4} \bar{g}_{ik}(\bar{x}^{(1)}, \bar{x}^{(2)}, \bar{x}^{(3)}, \bar{x}^{(4)})\, d\bar{x}^{(i)}\, d\bar{x}^{(k)} = dx^2 + dy^2 + dz^2 - c^2 dt^2$$

that can be transformed into the Euclidian **Minkowski** form. One can then regard the $\bar{x}$ as arithmetic quantities, so one does not by any means need to assume anything but the **Minkowski**

([1]) Cf., **A. Einstein** and **M. Grossmann**, “Entwurf, etc.,” Leipzig, Teubner, 1913.

([2]) **G. Ricci** and **T. Levi-Civita**, Math. Ann. **54** (1901), pp. 125. – **J. E. Wright**, Cambridge tracts in Mathematics, etc., no. 9, Cambridge, 1908. – **Einstein-Grossmann**, *loc. cit.*, Part II. – **F. Kottler**, Wien. Ber. IIa, **121** (October 1912).

world initially. It is then clear that one merely has to rewrite **Minkowski** mechanics in the generalized coordinates $\bar{x}$.

For **Minkowski**, the law of inertia reads: force-free mass-points describe *straight* lines. Its conversion yields: force-free mass-points describe shortest ([1]) – i.e., *geodetic* – worldlines, since the geodetic property is indeed invariant under transformations that take the arc-length element to itself. If one then has the equations of motion for **Minkowski**:

(11) $$\frac{d^2 x^{(h)}}{d\sigma^2} = 0, \quad h = 1, 2, 3, 4$$

then they will now read ([2]):

(11.a) $$\frac{d}{d\sigma}\left(\sum_{k=1}^{4} \bar{g}_{hk} \frac{d\bar{x}^{(k)}}{d\sigma}\right) - \frac{1}{2} \sum_{k,l=1}^{4} \bar{g}_{hk} \frac{d\bar{x}^{(k)}}{d\sigma} \frac{d\bar{x}^{(l)}}{d\sigma} = 0, \quad h = 1, 2, 3, 4,$$

which will make:

$$d\sigma^2 = -dx^2 - dy^2 - dz^2 + c^2 dt^2 = -\sum_{k,l=1}^{4} \bar{g}_{hk} \frac{d\bar{x}^{(k)}}{d\sigma} \frac{d\bar{x}^{(l)}}{d\sigma}, \quad h = 1, 2, 3, 4.$$

The things that were stated in subsection **2** can immediately verify the covariant form of (11.a) here, and just like (11), that will lead to a simple connection between the proper acceleration vector and the magnitude of the proper acceleration and the generalized principal normal and the first curvature of the worldline.

Namely, the first differential quotient $dx^{(h)} / ds$ is contravariant of degree one in the x (i.e., the components of a vector: the tangent). That is, if one introduces coordinates $\bar{x}$ in place of the x that make:

$$d\sigma^2 = -\sum_{i,k=1}^{4} g_{ik}\, dx^{(i)} dx^{(k)} = -\sum_{i,k=1}^{4} \bar{g}_{ik} d\bar{x}^{(i)} d\bar{x}^{(k)}$$

then one will see immediately that:

$$\frac{d\bar{x}^{(h)}}{d\sigma} = \sum_{i=1}^{4} \frac{d\bar{x}^{(i)}}{d\sigma} \frac{\partial \bar{x}^{(h)}}{\partial x^{(i)}} .$$

In order to express that, we write:

$$\frac{dx^{(h)}}{d\sigma} = x^{(h)}_{/\sigma} .$$

That is no longer true for the second differential quotients $d^2 x^{(h)} / ds^2$; they *do not* define a vector. However, one can define such a thing as follows (this is an *extension* of the definition that **Ricci** and **Levi-Civita** gave):

$$x^{(h)}_{/\sigma} = \frac{d^2 x^{(h)}}{d\sigma^2} + \sum_{k.l=1}^{4} \begin{Bmatrix} k\ l \\ h \end{Bmatrix} \frac{dx^{(k)}}{d\sigma} \frac{dx^{(l)}}{d\sigma} .$$

In this, one has:

([1]) In real units of arc-length (σ): the longest.
([2]) **L. Bianchi**, *Differentialgeometrie*, *first* edition, pp. 569, equation (A), Leipzig 1899.

$$\begin{Bmatrix} k\ l \\ h \end{Bmatrix} = \sum_{p=1}^{4} g^{(hp)} \begin{bmatrix} k\ l \\ h \end{bmatrix} = \sum_{p=1}^{4} g^{(hp)} \left(\frac{\partial g_{kp}}{\partial x^{(l)}} + \frac{\partial g_{pl}}{\partial x^{(k)}} - \frac{\partial g_{kl}}{\partial x^{(p)}} \right),$$

and $g^{(kp)} = g^{(pk)}$ is the form reciprocal to g_{hp} such that:

$$\sum_{p=1}^{4} g^{(hp)} g_{pq} = \begin{cases} 1 & h = q \\ 0 & h \neq q \end{cases} .$$

The magnitude of that vector is the "first curvature":

$$\frac{1}{R_1} = \sqrt{\sum_{k,l=1}^{4} g_{kl}\, x_{/\sigma\sigma}^{(k)}\, x_{/\sigma\sigma}^{(l)}} ,$$

and the direction of the "principal normal" (**L. Bianchi**, *loc. cit.*, pp. 604) will then have an intrinsic connection with the generalized proper acceleration. Furthermore, it can be shown that:

$$\sum_{k=1}^{4} g_{hk}\, x_{/\sigma\sigma}^{(k)} = \frac{d}{d\sigma} \left(\sum_{k=1}^{4} g_{hk} \frac{dx^{(k)}}{d\sigma} \right) - \frac{1}{2} \sum_{k,l=1}^{4} g_{kl} \frac{dx^{(k)}}{d\sigma} \frac{dx^{(l)}}{d\sigma},$$

with which, the form (11.a) is shown to be the covariant of a vector.

It should be pointed out that in the **Landsberg** formulas for the moving 4-frame that were given in (I, Appendix 1), the dx / dt, d^2x / dt^2, d^3x / dt^3, etc., must merely be replaced with $x_{/t}^{(h)}$, $x_{/tt}^{(h)}$, $x_{/ttt}^{(h)}$, etc., in order to give the **Frenet** formulas for a general arc-length element. In that, t is an arbitrary parameter that does not need to be the arc-length.

From the standpoint of the transformation to generalized coordinates, it is immediately clear that *force-free mass-points must also be "unaccelerated" in the new coordinates.*

Now, since the integrals of the equations of motion (11.a) for $\bar{x}^{(1)}$, $\bar{x}^{(2)}$, $\bar{x}^{(3)}$ as functions of $\bar{x}^{(4)}$ are not generally (1) a geodetic lines of the spatial manifold $\bar{x}^{(4)}$ = const., we must alter the **Galilean** law of inertia:

Rather, force-free mass-points generally describe *curvilinear* paths.

4. – For **Minkowski**, the equations of motion for the non-force-free mass-point read:

(12)
$$E_0 \frac{d^2 x^{(h)}}{d\sigma^2} = K^{(h)}, \quad h = 1, 2, 3, 4,$$

in which E_0 is the internal energy. The transformation implies that:

(12.a)
$$E_0 \left[\frac{d}{d\sigma} \left(\sum_{k=1}^{4} \bar{g}_{hk} \frac{dx^{(k)}}{d\sigma} \right) - \tfrac{1}{2} \sum_{k,l=1}^{4} \frac{\partial \bar{g}_{kl}}{\partial \bar{x}^{(h)}} \frac{d\bar{x}^{(k)}}{d\sigma} \frac{d\bar{x}^{(k)}}{d\sigma} \right] = \bar{K}_h, \quad h = 1, 2, 3, 4,$$

(1) However, cf., § **6**, subsection 3.

in which the internal energy is assumed to be invariant according to equations (12). (The choice of mass instead of energy would demand the introduction of proper time τ, which is not an invariant.) In that way, the force fulfills the known orthogonality condition:

$$\sum_{h=1}^{4} \bar{K}_h \frac{d\bar{x}^{(h)}}{d\sigma} = 0 .$$

5. – In order to apply that to our case, we must exhibit the arc-length element. (Cf., I, Appendix 4) In order to derive it again, we have:

$$X = x - \mathfrak{Z}' \, c_2 + \mathfrak{X}' \, c_3 + \mathfrak{Y}' \, c_4 ,$$

since it is obvious that the "true" coordinates must be employed. It then follows from the **Frenet** formulas (I, Appendix 1) that:

$$dX = i\, c\, d\,\mathfrak{T}' \cdot (1 + \mathfrak{Z}' / R_1) \cdot c_1 - d\,\mathfrak{Z}' \cdot c_2 + d\,\mathfrak{X}' \cdot c_3 + d\,\mathfrak{Y}' \cdot c_4 ,$$

so

(13) $$dS^2 = (d\mathfrak{X}')^2 + (d\mathfrak{Y}')^2 + (d\mathfrak{Z}')^2 - c^2 \left(\frac{b+\mathfrak{Z}'}{b^2}\right)^2 (d\mathfrak{T}')^2 .$$

Since that expression was derived from the arc-length element of **Minkowski** space, it is *Euclidian*, i.e., the Riemann symbols vanish ([1]).

The equations of motion assume the form:

(14) $$\left\{\begin{aligned} &\frac{d^2\mathfrak{X}'}{d\sigma^2} = 0, \\ &\frac{d^2\mathfrak{Y}'}{d\sigma^2} = 0, \\ &\frac{d^2\mathfrak{Z}'}{d\sigma^2} + c^2 \frac{b+\mathfrak{Z}'}{b^2}\left(\frac{d\mathfrak{T}'}{d\sigma}\right)^2 = 0, \\ &\frac{d}{d\sigma}\left[c^2 \left(\frac{b+\mathfrak{Z}'}{b^2}\right)^2 \frac{d\mathfrak{T}'}{d\sigma}\right] = 0. \end{aligned}\right.$$

In this, we have:

$$\left(\frac{d\mathfrak{X}'}{d\sigma}\right)^2 + \left(\frac{d\mathfrak{Y}'}{d\sigma}\right)^2 + \left(\frac{d\mathfrak{Z}'}{d\sigma}\right)^2 - c^2 \left(\frac{b+\mathfrak{Z}'}{b^2}\right)^2 \left(\frac{d\mathfrak{T}'}{d\sigma}\right)^2 = -1 .$$

The integration of the equations of motion is easy and naturally implies the result (7.a) [(8), resp.] of § **3**.

In accordance with the standpoint that was described in subsection **2**, we call a mass-point that moves according to (14) "acceleration-free." That contradicts the usage of that term in

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([1]) **L. Bianchi**, pp. 571.

Lagrangian mechanics, but it conforms to the postulate that only vectorial constructions should be considered, which *ordinary* acceleration is *not*. It is only with the **Minkowski** arc-length element that the *ordinary* acceleration vanishes at the same time as the **Minkowski** one. The analogy is *no longer true here.*

Of course, that raises the question of how the observer can come to that intuition. It is naturally assumed in this that *he is already in possession of the true coordinates, so the arc-length* (13). If that is not true then he will operate with the *apparent coordinates* and a *Minkowski arc-length element*, and from (10), he will then find *apparent accelerations*, since he certainly establishes the *Galilean law of inertia*. However, if he comes to know about the *curvature of light rays*, so the arc-length (13), then he will see that the *simplest explanation* for the falling motion that he observes is not the assumption of forces, but the *assumption of the Galilean law of inertia*.

Of course, it might seem that this is only the replacement of one terminology with another and that one still does not capture the essence of things. However, since we are not treating ontological questions here, we would like to establish it here, *since only that conception of things is compatible with the formalism of differential invariants.* The law of inertia is then given by the arc-length element.

6. – In an analogous way, while always keeping the standpoint of the transformation to generalized coordinates in mind, one can always convert the remaining formulas of **Minkowski-Laue** dynamics, as well as the **Maxwell** equations, etc., to the arc-length element (13) with the help of differential invariants.

Naturally, that will imply minimal geodetics for the light rays, so equations of the form (14), but in which s is replaced with a parameter u and one demands the minimal property:

$$\left(\frac{d\mathfrak{X}'}{d\sigma}\right)^2+\left(\frac{d\mathfrak{Y}'}{d\sigma}\right)^2+\left(\frac{d\mathfrak{Z}'}{d\sigma}\right)^2-c^2\left(\frac{b+\mathfrak{Z}'}{b^2}\right)^2\left(\frac{d\mathfrak{T}'}{d\sigma}\right)^2=0\,.$$

One will again obtain the formulas of § **3** with $c^2 = q^2$ as a result.

§ 6. – Measurements and hyperbolic geometry.

1. – In the previous paragraphs, the observer was always in possession of the true coordinates, so the arc-length (13) was assumed instead of the **Minkowski** one. That raises the question of how that is possible for the observer. Obviously, he must arrive at some knowledge of the curvature of light rays at some point. The simplest experiment in that regard would be the following one: Assume that the observer at B (Fig. 4) would like to measure the distance from him to a second observer at $\mathfrak{P}'$ that he initially sees at P'. One way of doing that would be to stretch a measuring tape of sufficient length between them and to take the *shortest* distance $B\,\mathfrak{P}'$. In that way, since space should certainly be Euclidian, which is expressed in (13), he will come to the ordinary line $\overrightarrow{B\mathfrak{P}'}$, so the observer will immediately see the curvature of the light ray that comes to him from P', while the measuring tape will give the direction $\overrightarrow{B\mathfrak{P}'}$. However, that way of doing things is not generally accessible, since the curvature of the light rays, which indeed depends upon c^2 / g, where g is the apparent acceleration, is not large enough that it can be revealed over such short distances, while stretching a measuring tape is almost impracticable. The measurements, in fact, happen as a result of vision, so by way of *light*.

2. – We shall exhibit the character of those measurements by considering the following idealized experiment: The observer at B might calculate the distance $B\,\mathfrak{P}'$ from the time $B\,\mathfrak{P}'$ that it takes light to traverse it, where he can naturally base that upon a constant c. In that way, the standpoint of vision, which determines the line by way of light, will be taken into account.

A light ray needs a time of:

$$\int \frac{\sqrt{(d\mathfrak{X}')^2 + (d\mathfrak{Y}')^2 + (d\mathfrak{Z}')^2}}{c'} = \int d\mathfrak{T}'$$

to traverse the line segment $B\,\mathfrak{P}'$, since as a result of (13):

$$c' = c\,\frac{b + \mathfrak{Z}'}{b}\,.$$

From a light signal that leaves B at the time $\mathfrak{T}'_0$ and arrives at $\mathfrak{P}'$ at time $\mathfrak{T}'_1$ and immediately returns to B, which it once more reaches at time $\mathfrak{T}'_2$, where one has:

$$\mathfrak{T}'_1 - \mathfrak{T}'_0 = \mathfrak{T}'_2 - \mathfrak{T}'_1\,,$$

the observer will conclude that the distance is:

$$c\,\frac{\mathfrak{T}'_2 - \mathfrak{T}'_0}{2} = c\,\Delta\mathfrak{T}'\,,$$

and will naturally locate $\mathfrak{P}'$ in the direction P'. However, he will not come to, say, the *apparent* coordinates in that way. The distance that is thus determined is *different* from the apparent distance.

In order to see that, we calculate the apparent distance as a function of that third distance (which is equal to c times the true time) from formulas (2) and (4). It will be:

$$Z = (b + \mathfrak{Z}')\cosh\mathfrak{u} = (b + Z')\cosh\mathfrak{u} - \sqrt{X'^2 + Y'^2 + Z'^2}\,\sinh\mathfrak{u},$$
$$cT = (b + \mathfrak{Z}')\sinh\mathfrak{u} = (b + Z')\sinh\mathfrak{u} - \sqrt{X'^2 + Y'^2 + Z'^2}\,\cosh\mathfrak{u}.$$

The light-time is obviously given by:

$$c\,\Delta T' = b\,(u - \mathfrak{u}),$$

while the apparent distance is given by $\sqrt{X'^2 + Y'^2 + Z'^2}$. Now, one has:

$$(b + \mathfrak{Z}') \cosh (u - \mathfrak{u}) = b + Z',$$
$$(b+\mathfrak{Z}')\sinh(u-\mathfrak{u}) = \sqrt{X'^2 + Y'^2 + Z'^2},$$

which will make:

$$\tanh(u-\mathfrak{u}) = \tanh\frac{c\,\Delta\mathfrak{T}'}{b} = \frac{\sqrt{X'^2+Y'^2+Z'^2}}{b+Z'}.$$

Therefore, the *apparent distance* and the *light-time distance* will be different. Naturally, in the limit of $b = \infty$, both of them will be equal to the true distance. We will then find ourselves in the **Minkowski** world of rectilinear light rays.

3. – We then see the possibility that the observer does not at all come to the arc-length (13), *because he determines the line only by way of light rays.* Can that viewpoint be taken into account with no contradictions? That is, in fact, *possible*. It is possible to imagine that the light rays are the "lines" in space and that the speed of light c is regarded as constant. In order to see that, we consider the *conformal* map:

(15) $$d\bar{S}^2 = \left(\frac{b}{b+\mathfrak{Z}'}\right)^2 dS^2 = \left(\frac{b}{b+\mathfrak{Z}'}\right)^2 (d\mathfrak{X}'^2 + d\mathfrak{Y}'^2 + d\mathfrak{Z}'^2) - c^2 d\mathfrak{T}'^2.$$

We see immediately that the observer's line-of-sight remains unchanged, since the map is conformal. If we then imagine that a polar coordinate system is about B whose radii are light rays, whose distances are light-time distances, and whose angles are line-of-sight angles then we will have to prove the conservation of the latter, so we must prove the "rectilinearity" of the radii and the conservation of the distance.

From (11.a), the differential equations of the geodetics of (15) are:

(16) $$\begin{cases} \dfrac{d}{d\bar{\sigma}}\left[\left(\dfrac{b}{b+\mathfrak{Z}'}\right)^2 \dfrac{d\mathfrak{X}'}{d\bar{\sigma}}\right] = 0, \\[2ex] \dfrac{d}{d\bar{\sigma}}\left[\left(\dfrac{b}{b+\mathfrak{Z}'}\right)^2 \dfrac{d\mathfrak{Y}'}{d\bar{\sigma}}\right] = 0, \\[2ex] \dfrac{d}{d\bar{\sigma}}\left[\left(\dfrac{b}{b+\mathfrak{Z}'}\right)^2 \dfrac{d\mathfrak{Z}'}{d\bar{\sigma}}\right] + \dfrac{b^2}{(b+\mathfrak{Z}')^2}\left[\left(\dfrac{d\mathfrak{X}'}{d\bar{\sigma}}\right)^2 + \left(\dfrac{d\mathfrak{Y}'}{d\bar{\sigma}}\right)^2 + \left(\dfrac{d\mathfrak{Z}'}{d\bar{\sigma}}\right)^2\right] = 0, \\[2ex] \dfrac{d}{d\bar{\sigma}}\left(c^2 \dfrac{d\mathfrak{T}'}{d\bar{\sigma}}\right) = 0, \end{cases}$$

in which $\bar{\sigma}$ means a parameter for which:

$$\left(\frac{d\bar{S}}{d\bar{\sigma}}\right)^2 = -\left(\frac{b}{b+\mathfrak{Z}'}\right)^2 \left[\left(\frac{d\mathfrak{X}'}{d\bar{\sigma}}\right)^2 + \left(\frac{d\mathfrak{Y}'}{d\bar{\sigma}}\right)^2 + \left(\frac{d\mathfrak{Z}'}{d\bar{\sigma}}\right)^2\right] + c^2 \left(\frac{d\mathfrak{T}'}{d\bar{\sigma}}\right)^2$$

$$= h^2 = \begin{cases} 1 & \text{for timelike worldlines} \\ 0 & \text{for lightlike worldlines.} \end{cases}$$

It follows from the last equation in (16) that:

$$k\,d\bar{\sigma} = d\mathfrak{T}',$$

where *k* is a constant. The third equation can be written:

$$\frac{d}{d\bar{\sigma}}\left[\left(\frac{b}{b+\mathfrak{Z}'}\right)^2 \frac{d\mathfrak{Z}'}{d\bar{\sigma}}\right] + \frac{1}{b+\mathfrak{Z}'}\left[c^2\left(\frac{d\mathfrak{T}'}{d\bar{\sigma}}\right)^2 - h^2\right] = 0$$

or

$$\frac{d^2}{d\bar{\sigma}^2}\left(\frac{b}{b+\mathfrak{Z}'}\right) = \frac{1}{b+\mathfrak{Z}'}\frac{c^2k^2 - h^2}{b^2},$$

from which, the integrals of (16) will emerge in the form:

$$(17)\qquad \begin{cases} \mathfrak{X}' = \mathfrak{X}'_1 + (b+\mathfrak{Z}'_1)\dfrac{\alpha}{\sqrt{\alpha^2+\beta^2}}\tanh\left[\dfrac{\sqrt{c^2 - h^2/k^2}}{b}(\mathfrak{T}' - \mathfrak{T}'_1)\right], \\ \mathfrak{Y}' = \mathfrak{Y}'_1 + (b+\mathfrak{Z}'_1)\dfrac{\beta}{\sqrt{\alpha^2+\beta^2}}\tanh\left[\dfrac{\sqrt{c^2 - h^2/k^2}}{b}(\mathfrak{T}' - \mathfrak{T}'_1)\right], \\ b + \mathfrak{Z}' = (b+\mathfrak{Z}'_1)\dfrac{1}{\cosh\left[\dfrac{\sqrt{c^2 - h^2/k^2}}{b}(\mathfrak{T}' - \mathfrak{T}'_1)\right]}, \end{cases}$$

with *a*, *b*, $\mathfrak{Z}'_1$, $\mathfrak{T}'_1$ as integration constants, where $\mathfrak{T}'_1$ is the particular time-point of vertical reversal. By a comparison of that with (8) in § **3**, one will confirm that for light rays, so for:

$$c^2 = q^2 \qquad (h^2 = 0, \text{ resp.}),$$

the conformal map will include the minimal geodetics, so the light rays, which is indeed self-explanatory. Points that *B* then sees on *one* light ray (i.e., a "radius" in his polar system) will then also remain on *one* and *the same* light ray for the arc-length (15). As mentioned before, the polar angles will then remain the same. However, (and this is essential), the radii will be "lines" in the "space,"

That is implied by the differential equations of the geodetics in "space," whose arc-length element is indeed:

(18) $$d\bar{s}^2 = \left(\frac{b}{b+\mathfrak{Z}'}\right)^2 (d\mathfrak{X}'^2 + d\mathfrak{Y}'^2 + d\mathfrak{Z}'^2),$$

from (15). Those differential equations will be *identical* with the first three in (16) when one simply replaces $\bar{\sigma}$ with $\bar{s}$ in them. Now, since $d\bar{\sigma}/d\mathfrak{T}' = k$, one has:

$$\left(\frac{d\bar{S}}{d\bar{\sigma}}\right)^2 = \left(\frac{d\bar{s}}{d\bar{\sigma}}\right)^2 - c^2k^2 = -h^2,$$

so

$$d\bar{s} = \sqrt{c^2 - h^2/k^2}\, d\mathfrak{T}'.$$

The "lines" in "space" are then given by that:

(17.a) $$\begin{cases} \mathfrak{X}' = \mathfrak{X}'_1 + (b+\mathfrak{Z}'_1)\dfrac{\alpha}{\sqrt{\alpha^2+\beta^2}}\tanh\dfrac{\bar{s}-\bar{s}_1}{b}, \\ \mathfrak{Y}' = \mathfrak{Y}'_1 + (b+\mathfrak{Z}'_1)\dfrac{\beta}{\sqrt{\alpha^2+\beta^2}}\tanh\dfrac{\bar{s}-\bar{s}_1}{b}, \\ b+\mathfrak{Z}' = (b+\mathfrak{Z}'_1)\dfrac{1}{\cosh\dfrac{\bar{s}-\bar{s}_1}{b}}, \end{cases}$$

which shows that the curves (17) are "lines" that are traversed with uniform speed:

$$\frac{d\bar{s}}{d\mathfrak{T}'} = \sqrt{c^2 - h^2/k^2}.$$

The Galilean law of inertia is valid then.

In particular, it follows for light rays ($h = 0$) that:

For the arc-length element (15), *the light rays of the arc-length* (13) *are "lines" in "space" and are traversed with constant uniform speed c.*

The viewpoint of the observer will then be free of contradictions as long as the Euclidian space is replaced with another one.

4. – Now, what is that invariant for a "space" whose arc-length is given by (18)? It is nothing but the representation of a hyperbolic space whose curvature is:

$$K = -\frac{1}{R^2} = -\frac{1}{b^2}$$

in limiting sphere coordinates ([1]) that is well-known from non-Euclidian geometry.

The radius of curvature R is then equal to the radius of curvature R_1 of the worldline of the observer. If one writes:

$$d\bar{s}^2 = \left(\frac{b}{b+\mathfrak{Z}'}\right)^2 \{d\mathfrak{X}'^2 + d\mathfrak{Y}'^2 + d\mathfrak{Z}'^2\} = \left(\frac{b}{b+\mathfrak{Z}'}\right)^2 ds^2$$

then one will have the well-known **Poincaré-Klein** conformal map ([2]) of hyperbolic space to the Euclidian half-space $b + \mathfrak{Z}' > 0$. The "lines" in hyperbolic space then map to the (semi-)circles that intersect the limiting plane $b + \mathfrak{Z}' = 0$ perpendicularly. Their equations as functions of the arc-length s (i.e., the geodetic distance) are given by just (17.a) ([3]). The planes $b + \mathfrak{Z}'$ = const. are just the images of the "limiting spheres" – i.e., "spheres" whose centers lie at infinity and have zero constant curvature, so they can be developed onto the Euclidian plane. The "lines" $\mathfrak{X}'$ = const., $\mathfrak{Y}'$ = const., which are also given by lines in the Euclidian image, are the orthogonal trajectories of geodetically-parallel concentric limiting spheres $b + \mathfrak{Z}'$ = const., which converge to the infinitely-distant center of those spheres.

The points at infinity map to the points of the limiting plane $b + \mathfrak{Z}' = 0$. We know from § **2**, point **2**, that this is also a limiting plane at infinity for the accelerated reference system. Naturally, the light rays, as semi-circles that intersect the limiting plane perpendicularly (Fig. 3), are "lines," etc., as we saw before in (17.a) and (8). We then see that *hyperbolic space* realizes the intuition of the observer *that light in empty space propagates "rectilinearly" and uniformly* with no contradictions.

The possibility exists that the observer can learn about that hyperbolic character of the space that he has constructed with the help of "light lines" from experience (parallel angles!). However, since the nature of space is a *fact of experience*, it is possible that he can also establish the arc-length element (15) instead of (13) ([4]). We would therefore like to also study the phenomena of falling (8) from the viewpoint of (15).

§ 7. – Falling motion in the hyperbolic conception of space.

1. – It is clear from the outset that the falling motion (8) cannot follow from the law of inertia for (15), or in other words, that the worldlines (7.a) of falling motion (8) are *not geodesics* for $c^2 > q^2$. That is because the conformal map of (13) to (15) preserves merely the minimal geodetics

([1]) **L. Bianchi**, *loc. cit.*, pp. 581 or **H. Liebmann**, *Nichteuklidische Geometrie*, Leipzig, 1905, pp. 56.
([2]) **L. Bianchi**, *loc. cit.*, pp. 419, *et seq.*, or pp. 583, *et seq.*
([3]) **L. Bianchi**, pp. 584.
([4]) **H. Poincaré** (*La Science et l'Hypothèse*, Flammarion, Chap. V, 3, pp. 93), who had such a case in mind, was of a different opinion. **Poincaré**'s viewpoint, which emphasized the epistemological equivalence of both conceptions of space, which indeed also exists between (13) and (15) in our case, is contrary to the law of mathematical economy. A theory of physics in which the light rays are "rectilinear" and propagate uniformly and in which force-free mass-points move "rectilinearly" and uniformly is certainly mathematically simpler to handle.

$c^2 = q^2$. Ones see that from the map to the Euclidian half-space $b + \mathfrak{Z}' > 0$. Here, from Fig. 5, the paths of falling motion are *ellipses* that intersect the limiting plane perpendicularly, while they must be *circles* in order for them to be inertial paths as a result of (15).

2. – We must then look for *forces* that generate falling motion. One finds them as follows: Equations (14) are indeed fulfilled for falling motion. One then must merely rewrite them with the help of:

$$d\bar{\sigma} = \frac{b}{b+\mathfrak{Z}'}\,d\sigma$$

in such a way that the left-hand sides of (16). The desired forces then appear on the right-hand side. One then finds the following equations for falling motion in the hyperbolic space picture:

$$\text{(19)}\quad \left\{\begin{aligned}
&E_0\frac{d}{d\bar{\sigma}}\left[\left(\frac{b}{b+\mathfrak{Z}'}\right)^2\frac{d\mathfrak{X}'}{d\bar{\sigma}}\right] = -E_0\left(\frac{b}{b+\mathfrak{Z}'}\right)^2\frac{d\mathfrak{Z}'}{d\bar{\sigma}}\frac{d\mathfrak{X}'}{d\bar{\sigma}},\\
&E_0\frac{d}{d\bar{\sigma}}\left[\left(\frac{b}{b+\mathfrak{Z}'}\right)^2\frac{d\mathfrak{Y}'}{d\bar{\sigma}}\right] = -E_0\left(\frac{b}{b+\mathfrak{Z}'}\right)^2\frac{d\mathfrak{Z}'}{d\bar{\sigma}}\frac{d\mathfrak{Y}'}{d\bar{\sigma}},\\
&E_0\frac{d}{d\bar{\sigma}}\left[\left(\frac{b}{b+\mathfrak{Z}'}\right)^2\frac{d\mathfrak{Y}'}{d\bar{\sigma}}\right] + \frac{E_0}{b+\mathfrak{Z}'}\left[c^2\left(\frac{d\mathfrak{T}'}{d\bar{\sigma}}\right)^2 - 1\right] = -E_0\left(\frac{b}{b+\mathfrak{Z}'}\right)^2\frac{d\mathfrak{Z}'}{d\bar{\sigma}}\frac{d\mathfrak{Y}'}{d\bar{\sigma}} - E_0\frac{1}{b+\mathfrak{Z}'},\\
&E_0\frac{d}{d\bar{\sigma}}\left(c^2\frac{d\mathfrak{T}'}{d\bar{\sigma}}\right) = -E_0\frac{c^2}{b+\mathfrak{Z}'}\frac{d\mathfrak{Z}'}{d\bar{\sigma}}\frac{d\mathfrak{Z}'}{d\bar{\sigma}}.
\end{aligned}\right.$$

In that way, the third equation in (16) will be transformed with the help of the relation:

$$c^2\left(\frac{d\mathfrak{T}'}{d\bar{\sigma}}\right)^2 - \left(\frac{b}{b+\mathfrak{Z}'}\right)^2\left\{\left(\frac{d\mathfrak{X}'}{d\bar{\sigma}}\right)^2 + \left(\frac{d\mathfrak{Y}'}{d\bar{\sigma}}\right)^2 + \left(\frac{d\mathfrak{Z}'}{d\bar{\sigma}}\right)^2\right\} = 1.$$

One will then have for the (covariant) forces:

$$\text{(20)}\quad \left\{\begin{aligned}
&\mathfrak{K}_{\mathfrak{X}'} = K_1 = -\frac{E_0}{b+\mathfrak{Z}'}\frac{d\mathfrak{Z}'}{d\bar{\sigma}}\cdot\left(\frac{b}{b+\mathfrak{Z}'}\right)^2\frac{d\mathfrak{X}'}{d\bar{\sigma}},\\
&\mathfrak{K}_{\mathfrak{Y}'} = K_2 = -\frac{E_0}{b+\mathfrak{Z}'}\frac{d\mathfrak{Z}'}{d\bar{\sigma}}\cdot\left(\frac{b}{b+\mathfrak{Z}'}\right)^2\frac{d\mathfrak{Y}'}{d\bar{\sigma}},\\
&\mathfrak{K}_{\mathfrak{Z}'} = K_3 = -\frac{E_0}{b+\mathfrak{Z}'}\frac{d\mathfrak{Z}'}{d\bar{\sigma}}\cdot\left(\frac{b}{b+\mathfrak{Z}'}\right)^2\frac{d\mathfrak{Z}'}{d\bar{\sigma}} - \frac{E_0}{b+\mathfrak{Z}'},\\
&\mathfrak{K}_{\mathfrak{T}'} = \frac{c}{i}K_4 = -\frac{E_0}{b+\mathfrak{Z}'}\frac{d\mathfrak{Z}'}{d\bar{\sigma}}c^2\frac{d\mathfrak{T}'}{d\bar{\sigma}}.
\end{aligned}\right.$$

It will follow that one fulfills the orthogonality condition:

$$\sum_{h=1}^{4} K_h \frac{dx'^{(h)}}{d\bar{\sigma}} = \mathfrak{K}_{\mathfrak{X}'} \frac{d\mathfrak{X}'}{d\bar{\sigma}} + \mathfrak{K}_{\mathfrak{Y}'} \frac{d\mathfrak{Y}'}{d\bar{\sigma}} + \mathfrak{K}_{\mathfrak{Z}'} \frac{d\mathfrak{Z}'}{d\bar{\sigma}} - \mathfrak{K}_{\mathfrak{T}'} \frac{d\mathfrak{T}'}{d\bar{\sigma}} = 0 .$$

3. – Formulas (20) suggest their own application to the scalar theory. It is known that the orthogonality condition is not fulfilled by them *eo ipso* ([1]). With **Minkowski**, an supplementary force must be added that **Abraham** ([2]) is known to have attributed to the fact that the mass-point are supplied with non-mechanical energy (e.g., heat), which **Nordström** ([3]) interpreted in the context of gravitation in the sense of a change in the inertial mass under motion in that field.

We take the potential to be:

$$\Phi = \frac{c^2}{b}(b + \mathfrak{Z}'), \tag{21}$$

which is always positive, and in that expression, we recognize the potential of the apparent acceleration (§ **4**, subsection 4). According to (15), one has that the impulse of the mass-point in covariant form is

$$\mathfrak{G}_{\mathfrak{X}'} = G_1 = \frac{E_0}{c^2}\left(\frac{b}{b + \mathfrak{Z}'}\right)^2 \frac{d\mathfrak{X}'}{d\tau},$$

$$\mathfrak{G}_{\mathfrak{Y}'} = G_2 = \frac{E_0}{c^2}\left(\frac{b}{b + \mathfrak{Z}'}\right)^2 \frac{d\mathfrak{Y}'}{d\tau},$$

$$\mathfrak{G}_{\mathfrak{Z}'} = G_3 = \frac{E_0}{c^2}\left(\frac{b}{b + \mathfrak{Z}'}\right)^2 \frac{d\mathfrak{Z}'}{d\tau},$$

$$\mathfrak{G}_{\mathfrak{T}'} = \frac{c}{i} G_4 = \frac{E_0}{c^2} c^2 \frac{d\mathfrak{T}'}{d\tau},$$

with $d\bar{\sigma} = c\, dt$. It will then follow from (20) that:

$$K = -\frac{1}{\Phi}\frac{d\Phi}{d\tau} \cdot G - E_0 \cdot \frac{1}{\Phi} \cdot \mathrm{Grad}\,\Phi ,$$

and when:

$$M_g = \frac{E_0}{c^2}$$

is the mass constant ([4]), it will have a formal similarity with **Nordström**'s second theory. (19) can be abbreviated by means of E_0 / c^2. Naturally, the spirit of **Nordström**'s theory is something

([1]) **M. Abraham**, Phys. Zeit. **13** (1912), pp. 1.
([2]) **M. Abraham**, *loc. cit.*, **11** (1910), pp. 527.
([3]) **G. Nordström**, *loc. cit.* **13** (1912), pp. 1126.
([4]) **G. Nordström**, Ann. Phys. (Leipzig) **42** (1913), pp. 533. For **Nordström**, the symbol E_0 means something different from what it means here: For him, E_0 is the energy under transformation to a state of rest, while here, it means the energy in a state of *continual* rest, so outside of all fields.

different, as one sees upon comparing his arc-length element ([1]) – i.e., the arc-length element that the **Nordström** equations imply as *geodetic* equations (for inertial motions) – with (15).

4. – Here, since the conformal image (§ **4**, point 3) is an ellipse that intersects the limiting plane perpendicularly, the ballistic trajectory is a conic section in the hyperbolic plane, as one shows in **Weierstrass** coordinates ([2]).

As far as the potential (21) is concerned, it *no longer* fulfills the **Laplace** equation. Rather, the **Newton-Coulomb** potential that is associated with (15) would be ([3]):

$$c^2 \left(\frac{b + \mathfrak{z}'}{b} \right)^2 .$$

Despite that discrepancy, there is a geometric similarity between this and the **Newton-Coulomb** potential that belongs to (15). The equipotential surfaces are concentric spheres for both of them, while the lines of force are ∞^2 parallels that converge to the center at infinity. (The convergence of the parallels is a peculiarity of hyperbolic geometry.)

5. – Finally, we might refer to a certain similarity with a theory that **A. Byk** developed ([4]), in which he showed that *quantum theory* finds its simplest explanation in the assumption of a (strongly) hyperbolic curvature for the internal space of matter. He postulated Euclidian geometry for external space. *Here*, when we infer the processes in a gravitational field from the fact of an accelerated system by using **Einstein**'s equivalence hypothesis, we find that the *simplest explanation* for the processes in the gravitational field, at least at an infinite distance from matter (which we have to imagine is at the aforementioned infinitely-distant center of the limiting sphere), that preserve a constant speed of light and the form of the **Galilean** law of inertia, can be given by assigning a (weak) hyperbolic curvature to space, whereby the radius of curvature is inversely proportional to the constant acceleration of the gravitational field. If one would like to connect that with **Byk**'s theory then one would have to assume a gradual subsidence of the curvature in external space, from which one can suspect the possibility of a complete theory of matter that rests upon electromagnetic foundations.

6. – **Einstein**'s equivalence hypothesis does not suffice to exhibit such a thing. From the trajectories of a one-parameter group of orthogonal transformations (which was applicable to it, according to I), the hyperbolic motion will, in fact, yield a light-speed field that merely varies with *position*, while the remaining types of motion will imply a field that varies with the *direction* of the ray, as well ([5]). However, the scalar, non-tensorial nature of the light-speed field would be required if one were to regard the atomic forces as central forces.

([1]) **A. Einstein** and **A. D. Fokker**, Ann. Phys. (Leipzig) **44** (1914), pp. 44.

([2]) **H. Leibmann**, *loc. cit.*, pp. 182.

([3]) **H. Leibmann**, *loc. cit.*, pp. 227, in which one must now make the transformation:

$$b\, e^{\zeta / b} = b + \mathfrak{z}'.$$

([4]) **A. Byk**, Ann. Phys. (Leipzig) **42** (1913), pp. 1417, *et seq.*

([5]) As will be shown in another place, the extension of **Einstein**'s equivalence hypothesis that was given by **P. Ehrenfest** [Amsterdam Proceedings 30 V (1913)] and **Ch. H. van Os** [*ibidem*, 3 IX (1913)], in which hyperbolic

A new hypothesis must then be posed. The static field of an atom, which one regards most simply (along with **J. J. Thomson**) as a uniformly-charged positive ball, is close to the Ansatz that:

$$dS^2 = dx^2 + dy^2 + dz^2 - C^2\, dt^2 ,$$

in the spirit of **Einstein**'s theory, under the assumption that *the speed of light C should be proportional to the* ***Newton-Coulomb*** *potential*. At an infinite distance from the atom, one will actually get (13), while in the interior, if r is the distance from the center C_0 and R is an atomic constant then:

$$C = C_0 \frac{R^2 - r^2}{2R^2},$$

such that a conformal map that is analogous to the one in § **6** :

$$d\bar{S}^2 = \left(\frac{2R^2}{R^2 - r^2}\right)^2 dS^2 = \left(\frac{2R^2}{R^2 - r^2}\right)^2 (dx^2 + dy^2 + dz^2)\, C_0^2\, dt^2,$$

will imply a constant speed of light and *constant hyperbolic curvature* for the internal space of the atom, as in **Byk**. One shows that by introducing polar coordinates for which the Euclidian distance r and the hyperbolic one ρ are coupled as follows:

$$r = R \tanh \frac{\rho}{2R}.$$

Since it is not **Lagrange**'s equations of motion that we are dealing with here, but the **Einstein-Grossmann** ones, there will be (cf., Appendix 4) some differences with **Byk**'s mechanics: E.g., **Nordström**'s variability of mass, the replacement of **Byk**'s elastic potential with a similar one:

$$\left(\text{proportional to } \frac{1}{\cosh^2 \frac{\rho}{2R}}\right), \text{etc.}$$

In order to arrive at a useful quantum model, one might perhaps need to add Bohr's conception of things. (For example, the absence of radiation under uniform rotation; cf., I, § **5**.)

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motion appeared as a special case of a whole class of motions, certain paths are based upon a one-parameter *conformal* group and an extension of the Lorentz transformation that **H. Bateman** gave.