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Newton's law and metrics

By

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The opinion that *Newton's law of attraction is necessarily connected with the geometry of our space* was, and still is, often professed, especially by astronomers. The goal of the following study is to contradict that opinion. It will be shown how Newton's theory of gravitation must considered in a schema that is free of any metric, corresponding to the modern local-action or field physics, and how the metric will then be introduced into that schema by an *arbitrary convention.*

1.

Some instances in which the aforementioned opinion occurs in the literature might serve as an *introduction*.

In one of his early works "Gedanken von der wahren Schätzung der lebendigen Kräfte, etc. [†]" (1747), § 10 to 11, **I. Kant** remarked: "It is apparent that the three-fold measurement of space originates in the law by which the forces in substances act upon each other." Moreover: "The triple measurement thus seems to arise because the substances act upon each other in the existing world in such a way that the strengths of the effects vary inversely to the squares of their separations. Because of that, I believe that the substances in the existing world – of which, we are a part – have essential forces of such a kind that in conjunction with each other they will propagate their effects according to twice the inverse ratio of their distances. Secondly, I believe that the totality that emerges by means of that law has the property that it has three dimensions. Thirdly, I believe that this law is arbitrary, and that God allows one to choose another – for example, the inverse-cube ratio. Fourth and finally, I believe that an extension of other properties and measurements would flow from another law."

In 1824, **P. S. Laplace** ([1]) remarked that, due to its simplicity, generality, and agreement with physical experiments, Newton's law must be regarded as rigorous, and further remarked that its most important property is that, insofar as the measurements of

[†] Translation: "Thoughts on the true appraisal of *vis viva*, etc."

([1]) **P. S. Laplace**, "Exposition du système du monde," Oeuvres, v. VI, book V, chap. V, pp. 472 (cited in **R. Bonalo**, *Die nichteuklidische Geometrie*, Leipzig, 1908, pp. 56, *et seq.*)

the ratios of the reciprocal distances and velocities of all bodies in the universe would diminish, the celestial bodies would describe paths that would be exactly *similar* to the ones that they do describe, in such a way that the universe would always provide the observer with *the same view* when one progressively contracts it to the smallest-conceivable space. That implies the Euclidian character of astronomical space (viz., the existence of similar figures!).

Whereas **Kant**, in remarkable contradiction to his later *a priori* foundation of geometry ([1]), and with a certain premonition of Einstein's theory, then regarded geometry as a consequence of Newton's law, in such a way that it could be assumed to be arbitrary, **Laplace**, like most mathematicians from **Euclid** up to the his own era, decided upon the absolute validity of geometry, and consider Newton's law to be a consequence of it and certain simple physical axioms. For **Laplace**, the latter corresponded to the intuitions of the *theory of action-at-a-distance*; i.e., Newton's *integral law*, and not Laplace's *differential equation* (as in the *theory of local action*), is in the foreground of all consideration, such that the law in question would be obtained from simple assumptions about the so-called *central force* with the aid of astronomical experiment. Later, **J. Bertrand** ([2]) reinforced the inverse-square law with his well-known theorem: The paths that a material point describes under the influence of a central force that is only a function of the distance are *closed* only when the force is either proportional to the distance or inversely-proportional to the square of the distance. [As is known, **Newton** ([3]) himself justified the exponent 2 by the remark that deviations of that exponent from the value 2 would have precession of the perihelia of the planets as a consequence, and therefore paths that were not precisely closed.]

Laplace and his followers placed **Euclidian geometry** at the foundation of all of this. The conversion of Newton's law to *non-Euclidian* spaces ([4]) likewise came about from the *metric* viewpoint: From an external viewpoint, the attraction of different concentric "balls" that are endowed with equal homogeneous density to each other should be directly proportional to the masses that they were endowed with and inversely proportional to the area of their surfaces. (It is known that due to the smallness of the possible curvature of our space, the generalization of Kepler's laws that one obtains from this will imply no clue as to how one might resolve the question of its Euclidian or non-Euclidian character with the help of planetary astronomy.)

So much for the intuitions of the physics of *action-at-a-distance*, upon whose floor most astronomers still stand to this day when they deal with Newton's (integral) law or its generalizations. **J. Zenneck** ([5]) assumed the standpoint of *field physics*, with **Maxwell's** theory as its model. He based the exponent 2 in Newton's law on the fact that from the standpoint of the theory of field action, it is the only law that is consistent with the assumption of a general (i.e., external to matter) *source-free* distribution of field strengths, so it is only with the precise validity of that law that the concept of *lines of*

([1]) **I. Kant**, "Prolegomena zu einer jeden künftigen Metaphysik," (1738), § 38: "Here (i.e., in the law of attraction), we then have Nature, from which arises laws that the mind recognizes to be *a priori*, and indeed chiefly from general principles of the determination of space."

([2]) **J. Bertrand**, C. R. **77** (1873), pp. 846.

([3]) **I. Newton**, *Principia mathematica*, lib. I, sect. IX.

([4]) **E. Schering**, Gött. Nachr. (1870), pp. 311; (1873), pp. 149. – **W. Killing**, Crelle's Journal **98** (1885), in particular, pp. 7, *et seq.* and pp. 24, *et seq.*

([5]) **J. Zenneck**, Math. Encycl. **126** (1917), pp. 15.

force in a gravitational field will have any meaning. On just that basis, **J. Lense** ([1]) believed that Newton's law could be regarded as a natural consequence of the requirement that the *force flux* that flowed through a closed surface from the outside should have only the masses that existed inside of that surface to thank for its origin. Both authors employed the fact that Laplace's *differential equation*, which is fundamental for Newton's law, has the form of a *divergence*, which then defines the link to the methods of field physics ([2]). The connection between the latter and the metric of space remains unaffected.

2.

We now turn to the examination of the connection between field physics and metrics. The separation of the concepts of force flux and force proves to be fundamental in that, while the former belongs to field physics and the latter, to metrically-oriented mechanics.

Corresponding to the basic ideas of field physics, one must describe all processes by field quantities; they are, moreover, nothing but certain mathematical functions of position that characterize the presence of the field. Not they themselves, but their (mostly mechanical) effects define a *measurement*, and are therefore admissible in a *metric* picture.

A further consequence of the field physics picture is that the processes that take place inside of an arbitrary closed surface will be determined by the field that originates inside of it. Thus, field physics takes it must convenient starting point from the formulation of certain *integral laws* that couple an integral over the interior of a closed surface with an integral over the latter. The essence of the integrals must be found in nothing but *invariant* statements. It then happens that field physics is supported by the theory of *integral invariants* or *integral forms* ([3]).

The coefficients of an *integral form* define what one calls a *vector* of rank one, two, etc. in mathematical physics. That is the connection between field physics and vector analysis. *Vectors* are special cases of *tensors*, namely, *alternating tensors*, which change sign when one switches any two of their indices. In contrast to the analysis of vectors, the analysis of general, non-alternating tensors (viz., the absolute differential calculus) is necessarily coupled with a metric.

One now imagines an arbitrary, continuous, three-fold extended manifold M_3 and determines it points by any three numbers (i.e., coordinates) x_1, x_2, x_3 . Let the field be given at every point of this "space" by the second-rank vector with the components:

$$F_{23} = -F_{32}, \quad F_{31} = -F_{13}, \quad F_{12} = -F_{21} .$$

The *force flux through an infinitely-small surface* is determined by the integral form:

$$F_{23}\, dx_2\, dx_3 + F_{31}\, dx_3\, dx_1 + F_{12}\, dx_1\, dx_2 . \tag{1}$$

([1]) **J. Lense**, Wien Ber. **126** (1917), pp. 15.

([2]) Cf., also **H. Liebmann**, *Nichteuklidische Geometrie*, Leipzig 1905, pp. 224.

([3]) This theory goes back to **H. Poincaré**, *Nouvelles methods de la mécanique céleste III*, cf., **H. Poincaré**, "Analysis situs," Journal de l'école polytechique (1895). – **E. Goursat**, Journ. de math. **4** (1908), 331. – **F. Kottler**, Wiener Ber. **121** (1912), 1661, *et seq.*

The total force flux that goes through a two-sided, closed, nowhere-singular, outer surface is given by:

$$\iint F_{23}\, dx_2 dx_3 + F_{31}\, dx_3\, dx_1 + F_{12}\, dx_1\, dx_2\,, \tag{2}$$

in which the surface integral is taken over the closed outer surface. If one represents the points of it as functions of two parameters u, v then (2) will go to a double integral:

$$\iint \left\{ F_{23} \frac{\partial(x_2, x_3)}{\partial(u,v)} + F_{31} \frac{\partial(x_3, x_1)}{\partial(u,v)} + F_{12} \frac{\partial(x_1, x_2)}{\partial(u,v)} \right\} du\, dv\,, \tag{2a}$$

which is taken over the domain of u and v. One must pay attention to the sequence of differentials in the products $dx_2\, dx_3$, etc. in (2). Inverting that sequence will demand that one invert the sign of the term in question.

The vectorial nature of F finds its expression in the formulas that couple the transformed F' to the old F when one transforms x into the new coordinates x':

$$F_{23} = \sum_{\lambda,\mu=1}^{3} F'_{\lambda\mu} \frac{\partial x'_\lambda}{\partial x_2} \frac{\partial x'_\mu}{\partial x_3}\,, \qquad \text{etc.} \tag{3}$$

The force flux is then determined by a *covariant* vector of rank 2.

From the theorems of **Gauss** and **Green** (which are valid for any metric), (2) can be converted into a three-fold integral:

$$\iiint \left\{ \frac{\partial F_{23}}{\partial x_1} + \frac{\partial F_{31}}{\partial x_2} + \frac{\partial F_{12}}{\partial x_3} \right\} dx_1\, dx_2\, dx_3\,, \tag{4}$$

which is taken over the interior of the outer surface. The expression inside the integral is an integral form of rank 3:

$$F_{123}\, dx_1\, dx_2\, dx_3\,. \tag{5}$$

Its only coefficient is:

$$F_{123} = \frac{\partial}{\partial x_1} F_{23} - \frac{\partial}{\partial x_2} F_{31} + \frac{\partial}{\partial x_3} F_{12}\,, \tag{6}$$

whose defining law comes to light immediately, and which is the prototype for the well-known operation div in ordinary vector analysis, and its further consequences include Laplace’s differential expression.

One now addresses the presentation of the *integral laws* that were mentioned to begin with, which couple the internal phenomena with the field on the surface. Whether the former is the origin of the latter or vice versa is irrelevant from the standpoint of field physics. Newtonian physics, by contrast, assumes that the substance (i.e., mass) of the field that is found inside of the outer surface is the source of the effect. As is known, this one-sided *causal* picture is converted into a *functional* correlation for field physics.

One describes the internal processes by a three-fold integral:

$$\iiint \mu_{123}\, dx_1\, dx_2\, dx_3\,. \tag{7}$$

The third-rank vector μ_{123} characterizes the singular locations in the field, which one regards as its causes or sources in the older conception; i.e., the matter. Now, the field is determined by its singularities (or the latter are determined by the field) with the help of the *integral theorem:*

$$\iint F_{23}\, dx_2 dx_3 + F_{31}\, dx_3\, dx_1 + F_{12}\, dx_1\, dx_2 = \iiint \mu_{123}\, dx_1\, dx_2\, dx_3\,. \tag{8}$$

This is *Faraday's law of force flux*, which **Faraday** expressed by means of the well-known geometric concept of *lines of force.* With the help of the conversion (4), upon passing to the limit ([1]), it will imply the *differential equation:*

$$F_{123} = \frac{\partial F_{23}}{\partial x_1} + \frac{\partial F_{31}}{\partial x_2} + \frac{\partial F_{12}}{\partial x_3} = \mu_{123}\,. \tag{9}$$

This is the *nucleus of Laplace's differential equation* and the *origin of Newton's law of attraction.*

3.

The foregoing can be extended formally. Vectors are *geometric* quantities, so they are, in any event, subject to the law of *duality*. It then happens that one can express one and the same substratum by two different vectors, e.g., a vector of rank p and a contravariant vector of rank $n - p$ in an n-dimensional space. **Grassmann** called the one vector the *complement* of the other one.

In order to arrive at this, one considers the bundle of lines and planes at any point x of the three-fold extended manifold M_3 ; in the infinitesimal vicinity of it, one has linearity in space, and therefore projective geometry. A contravariant vector of rank one then determines a line of that bundle by the ratios of its components, etc. Instead of that bundle, one can also consider a projective plane E that one cuts the bundle with; we prefer the latter representation. The *contravariant* vectors of rank 1 (which are affixed to the point x of the original three-fold extended manifold M_3) then determine any system of triangular coordinates of the points in that plane by their components. However, one can determined a point in the plane E by means of any two lines that intersect at it, in addition to its point coordinates. One will then obtain the two-rowed sub-determinants from the matrix of line coordinates of the two lines, and they must obviously be the components of a *covariant* vector of rank 2 that belongs to the original *contravariant* vector of rank 1 by means of the law of duality.

([1]) The questions of the admissibility of the passage to the limit and the validity of the differential equation (9) inside of matter remain undisturbed here. **H. Weyl** made it clear in the 4th edition of *Raum-Zeit-Materie,* 275, *et seq.* that (9) is not applicable inside of matter. The integral law (8) remains untouched by that.

Up to now, the proportionality factor with which the homogeneous triangular coordinates are multiplied, which is left arbitrary in projective geometry, has not been considered.

Correspondingly, one must leave the proportionality factor on the definition of the *complement* (and thus, of the *dual vector*) arbitrary, if one is to avoid the introduction of metric viewpoints.

If one then seeks the contravariant vector that is complementary (or dual) to the covariant vector of the force flux with the components:

$$F_{23}, \quad F_{31}, \quad F_{12},$$

and whose components are denoted by:

$$\overset{*}{F}{}^1, \quad \overset{*}{F}{}^2, \quad \overset{*}{F}{}^3,$$

then they will be proportional to the former in the prescribed sequence, from the theorems of projective geometry that were just invoked. The *arbitrary* proportionality factor, which must naturally be the same for all vectors at the location x of the triply-extended manifold M_3, is not some scalar, but, as one easily sees, any covariant vector ε_{123} of rank 3. One then has ([1]):

$$F_{23} = \varepsilon_{123}\, \overset{*}{F}{}^1, \qquad F_{31} = \varepsilon_{123}\, \overset{*}{F}{}^2, \qquad F_{12} = \varepsilon_{123}\, \overset{*}{F}{}^3, \tag{10}$$

for the *complement* $\overset{*}{F}$ of F. One easily convinces oneself of the validity of the relations (10) by transforming to new coordinates x', in which one likewise postulates relations of the form (10) for the new F' ($\overset{*}{F}{}'$, resp.). In order for that to be true, ε_{123} must actually transform like a vector of rank 3:

$$\varepsilon_{123} = \varepsilon'_{123} \frac{\partial(x'_1, x'_2, x'_3)}{\partial(x_1, x_2, x_3)}. \tag{11}$$

$\overset{*}{F}{}^1$, $\overset{*}{F}{}^2$, $\overset{*}{F}{}^3$ then represents a contravariant vector of rank 1; i.e., one has:

([1]) If the complement of a covariant vector of rank p with the components $F_{i_1 i_2 \cdots i_p}$ is given by the relations:

$$F_{i_1 i_2 \cdots i_p} = \varepsilon_{12\ldots n}\, \overset{*}{F}{}^{i_{p+1} i_{p+2} \cdots i_n},$$

in which, $i_1\ i_2\ i_p\ i_{p+1} \ldots i_n$ must be a positive permutation of $1\ 2 \ldots n$. Analogous statements are true for the complement of its contravariant vector. The complement of the complement of a vector is equal to either the same vector or its opposite.

$$\overset{*}{F}{}^1 = \sum_{\lambda=1}^{3} \overset{*}{F}{}^\lambda \frac{\partial x_1}{\partial x'_\lambda}, \text{ etc.} \tag{12}$$

The force flux can then be determined in the same way from the covariant vector of rank 2 or its complement $\overset{*}{F}$. If one then sets:

$$\mu_{123} = \rho\, \varepsilon_{123}\,, \tag{13}$$

in which ρ is a scalar factor, then one can also write:

$$\frac{\partial}{\partial x_1}(\varepsilon_{123}\overset{*}{F}{}^1) + \frac{\partial}{\partial x_2}(\varepsilon_{123}\overset{*}{F}{}^2) + \frac{\partial}{\partial x_3}(\varepsilon_{123}\overset{*}{F}{}^3) = \rho\, \varepsilon_{123}\,, \tag{14}$$

in place of (9). One recognizes the prototype of the usual notation for the divergence (the Laplace differential expression, resp.) in this.

As we promised at the beginning of this paper, we have then brought Newton's theory of gravitation into the schema (9) [(14), resp.], *which is free of any metric* and is equivalent to the integral law (8). That is then the nucleus of Newton's theory from the standpoint of field physics, and is nothing but the mathematical expression for the concept of local action.

It remains for us to show how the metric will be introduced into this schema by an arbitrary convention.

4.

Force flux is a field concept, and as such, its measurement is not practicable. Only its mechanical effects – i.e., the force – can be measured.

It is know from Lagrangian mechanics how one can define (measure, resp.) force in arbitrary coordinates x_1, x_2, x_3. One measures force by the *work* that it can perform. In this, the work that infinitely-small displacement of a mass point from the location x to the location $x + dx$ performs will be given by a *linear differential form:*

$$dA = p_1\, dx_1 + p_2\, dx_2 + p_3\, dx_3\,. \tag{15}$$

With Lagrange, one calls the coefficients p of this differential form the *generalized force.* The form is then given by a *covariant* vector of rank 1.

Now, it is known that force is ordinarily represented by a line segment in mechanics. Thus, it shall be a contravariant vector of rank 1. This contradiction usually remains unnoticed, since one employs Cartesian orthogonal coordinates for the usual calculation of work in physics, as well as in engineering. Moreover, it is well-known that contravariance and covariance coincide in such coordinates.

In truth, the representation of a force by a line segment is based upon *the introduction of a metric*. Indeed, when one recalls the development of the mechanics and analysis in the context of the laws of levers, one can almost say that metric geometry, and in particular, the concept of orthogonality, arise from the work product (15) in mechanics.

From the previously-developed viewpoint of field physics, the train of thought that is implicit in this is, however, the following one: Originally, only the covariant vector F of rank 2 or the complementary contravariant vector $\overset{*}{F}$ of rank 1 have any meaning as field quantities (i.e., *force flux*). One needs to measure force flux by the work done by the force p, which is a covariant vector $\overset{*}{F}$ of rank 1. One must define a covariant vector p of equal rank from the contravariant vector $\overset{*}{F}$. One achieves that with the help of a polar correlation, namely, *generalized orthogonality*.

For this, one imagines any conic section as being drawn in the projective plane E that was treated in no. **3**. A line with the line coordinates p_1, p_2, p_3 will belong to the point of the plane E whose point coordinates are $\overset{*}{F}{}^1$, $\overset{*}{F}{}^2$, $\overset{*}{F}{}^3$ by means of the polar system of that conic, and that line will represent the desired covariant vector – i.e., the force that belongs to the force flux.

In order to express this in formulas, one must give the conic section by means of its quadratic form in the point coordinates ξ_1, ξ_2, ξ_3 of the plane E. Let:

$$\sum_{i,k} a_{ik}\xi_i\,\xi_k \qquad (a_{ik} = a_{ki}) \tag{16}$$

be the form. Naturally, the a_{ik} are, in general functions of the location x at which one consider the linear bundle (plane E, resp.); i.e., the metric that is introduced generally changes from point to point on the three-fold extended manifold M_3 .

With the help of this orthogonality, one has a covariant vector of rank 1 in:

$$\sum_{k=1}^{3} a_{1k}\,\overset{*}{F}{}^k\,, \qquad \sum_{k=1}^{3} a_{2k}\,\overset{*}{F}{}^k\,, \qquad \sum_{k=1}^{3} a_{3k}\,\overset{*}{F}{}^k\,,$$

which belongs to the contravariant vector $\overset{*}{F}$ of equal rank. A scalar proportionality factor still remains arbitrary. However, since the definition of force is still free of it, one can really set:

$$p_i = \sum_{k=1}^{3} a_{ik}\,\overset{*}{F}{}^k\,, \qquad i = 1, 2, 3. \tag{17}$$

By introducing the form that is reciprocal to (16), one will get:

$$\overset{*}{F}{}^k = \sum_{i=1}^{3} a^{ik}\,p_i \qquad k = 1, 2, 3, \tag{18}$$

in which the $a^{ik} = a^{ki}$ fulfill the known relations: $\sum_{i=1}^{3} a_{ik}a^{il} = \delta_k^l = \begin{cases} 1 & k = l \\ 0 & k \neq l. \end{cases}$

(19)

From (15) and (17), one will then have:

$$dA = \sum_{i,k=1}^{3} a_{ik} \overset{*}{F}_k \, dx_i \tag{20}$$

for the work that is done by the force flux F. One can glimpse the historical origin of the angle measure (orthogonality, projection, *et al.*) in this work product of mechanics. By the defective duality of the Euclidian metric, the measurement of length is independent of it. However, since nothing has been required of the a_{ik}, one can identify them with the coefficients of the quadratic form, which establish the arc length, and therefore, the angle measure in M_3 . Accordingly, let:

$$ds^2 = \sum_{i,k=1}^{3} a_{ik} dx_i \, dx_k \tag{21}$$

be that form. As we mentioned, we have a clue for this identification in the laws of levers.

We summarize them: First of all, the covariant vector F of rank 2 was the force flux, from which arises the complementary contravariant vector $\overset{*}{F}$ of rank 1, and from that arises the *polar reciprocal* (or briefly, the *reciprocal*) covariant vector p of rank 1 of force:

$$F_{23} = \varepsilon_{123} \, \overset{*}{F}{}^1 = \varepsilon_{123} \sum_{k=1}^{3} a^{1k} p_k \, , \qquad \text{etc.} \tag{22}$$

by means of orthogonality. In this, we have set:

$$\mu_{123} = \rho \, \varepsilon_{123} \, .$$

It is unnecessary to base the arbitrary tensor ε_{123} on the metric (21) that we introduced just now. When one establishes that for $\rho = 1$ the integral:

$$\iiint \mu_{123} \, dx_1 \, dx_2 \, dx_3 \, ,$$

when taken over a closed domain, shall give its volume, that will come about in the same way as when it is measured on the basis of (21). It follows from this that:

$$\varepsilon_{123} = \sqrt{a} \, , \tag{23}$$

in which a is the always-positive [for a positive-definite form (21)] determinant:

$$a = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} .$$

Upon transforming to new coordinates x', $\sqrt{a}$ actually behaves like a vector of rank 3, as long as the functional determinant satisfies:

$$\frac{\partial(x_1', x_2', x_3')}{\partial(x_1, x_2, x_3)} > 0.$$

In the opposite case, one must set $\varepsilon_{123}' = -\sqrt{a'}$ in order to remain in agreement with (11). One should never lose sight of this rule when one defines the Grassmann complement with the help of a vector of rank 3 that is defined by (23), as is customary.

We finally have:

$$F_{23} = \sqrt{a}\,\overset{*}{F}{}^1 = \sqrt{a}\sum_{k=1}^{3} a^{1k} p_k, \qquad \text{etc.}$$

$$\mu_{123} = \sqrt{a}\,\rho.$$

According to the chosen normalization (23), ρ means the (volume) density of matter, so the integral:

$$\iiint \mu_{123}\, dx_1\, dx_2\, dx_3$$

will give the total *mass* that is found in the domain ([1]).

Finally, Laplace's equation (14) will be:

$$\frac{\partial}{\partial x_1}\left(\sqrt{a}\sum_{k=1}^{3} a^{1k} p_k\right) + \frac{\partial}{\partial x_2}\left(\sqrt{a}\sum_{k=1}^{3} a^{2k} p_k\right) + \frac{\partial}{\partial x_3}\left(\sqrt{a}\sum_{k=1}^{3} a^{3k} p_k\right) = \sqrt{a}\,\rho. \qquad (24)$$

In the case where (15) is a complete differential, the force p will be conservative and representable by a potential φ:

$$p_k = \frac{\partial \varphi}{\partial x_k}, \qquad k = 1, 2, 3.$$

One will then obtain the familiar form:

([1]) **H. Weyl**, *loc. cit.*, pp. 98, referred to vectors like F as (linear) tensors, in order to distinguish them from the notation $\sqrt{a}\,\overset{*}{F}$, which he referred to as (linear) tensor *densities*, and "believed that the difference between quantity and intensity (to the extent that it has any physical meaning) has been captured more rigorously." From what we have done here, this difference is very well present, but must be grasped in a completely different way. F or $\overset{*}{F}$ are intensities, while the p are quantities.

$$\frac{1}{\sqrt{a}}\sum_{i=1}^{3}\frac{\partial}{\partial x_i}\left(\sqrt{a}\sum_{k=1}^{3}a^{ik}\frac{\partial\varphi}{\partial x_k}\right)=\rho, \tag{25}$$

from (24); i.e., the second-order Beltrami differential parameter, which will reduce to the classical **Laplace** form:

$$\frac{\partial^2\varphi}{\partial x_1^2}+\frac{\partial^2\varphi}{\partial x_2^2}+\frac{\partial^2\varphi}{\partial x_3^2}=\rho \tag{25a}$$

for a Euclidian ds^2 (21) in Cartesian coordinates.

5.

If one reviews what we have done up to now then one will see directly that the a_{ik} or the form (21) can still remain completely open. The metric cannot be derived from a mechanical viewpoint then; it is, moreover, a result of *an arbitrary convention*, as **Poincaré** has emphasized so often.

One can choose any metric, so the associated form of Newton's law would then drop out of (25), which would then naturally not have the simple form of the inverse-square of the distance, and one could then seek to compare the result with experiment. If the chosen metric is not in glaring contradiction to the approximate Euclidian nature of space then one will certainly be able to arrive at agreement with experiment to within the error in observation. The fact that space is approximately Euclidian does not need to amaze one; any manifold is Euclidian in an infinitesimal region. One can then glimpse in that fact at most an indication of the immensity of space, or perhaps the smallness of Man.

It is not our problem here to discuss how the choice of our metric actually comes about. It has been known since **Helmholtz**, **Riemann**, and **Lie** that the assumption of the *unchanging mobility of the fixed bodies* (viz., the requirement of homogeneity and isotropy in space) reduces the choice to the Euclidian or non-Euclidian space forms of constant curvature. The close decision between Euclidian and non-Euclidian space forms is, however, subject to only more open possibilities, and as is known, cannot be resolved experimentally to this day.

From the foregoing then, pure field physics is in no way connected with the metric of space. That seems to contradict Einstein's theory, which, as is known, derives gravitation from the metric on space and time. The task of clarifying that contradiction will have to be addressed in a later work. For **Einstein**, it is not the *field* that brings in the metric, but *light*, and only by its help can we comprehend our environment.

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