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Rotating reference systems in a Minkowski space

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The measurement of spacetime in rotating reference systems in the context of the general theory of relativity has frequently been the subject of recent discussions ([1]). That inspired me to return to my investigations of a generalization of the Lorentz transformation for certain (namely, "uniform") accelerated equations that I had already carried out before the general theory of relativity ([2]). The classes of motions that I studied includes the uniform rotations, in particular, to which I shall confine myself in what follows.

1. – Let a uniformly rotation reference system whose own mass can be neglected be given in the space of the fixed stars. According to the general theory of relativity (when one imagines that there is no solar system and overlooks cosmological hypotheses), that system will then be found in a **Minkowski** space, since all masses are found at an infinite distance from it. Neglecting the mass of the system is allowed in the same way that the mass of a test body that is brought into a gravitational field can be neglected.

Let X, Y, Z, T be coordinates referred to a system that is at rest with respect to the fixed stars. By assumption:

$$ds^2 = dX^2 + dY^2 + dZ^2 - c^2\, dT^2$$

is the metric of the given space. One is then dealing with the problem of finding coordinates x', y', z', t' that can be employed for the measurement of spacetime in a rotating system. In particular, x', y', z' must have *fixed* values for every co-rotating point of the system. It is self-explanatory that the x', y', z', t' must be represented as single-valued functions of the X, Y, Z, T, and conversely. At the moment, the general theory of relativity knows of no rule for distinguishing one of the possible ways of determining x', y', z', t'. It considers any and all of the X, Y, Z, T to be equally justified and demands only that the associated metrics should imply an equally-large ds^2. However, since we

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([1]) **A. Einstein**, "Die Grundlagen der allgemeinen Relativitätstheorie," Ann. Phys. (Leipzig) **49** (1916), § 3, and many authors since then.

([2]) **F. Kottler**, "Über die Raumzeitlinien der Minkowskischen Welt," Wiener Ber. IIa (1912). – "Relativitätsprinzip und beschleunigte Bewegung," Ann. Phys. (Leipzig) **44** (1914). – Cf., also "Über die physikalischen Grundlagen der Einsteinschen Gravitationstheorie," Ann. Phys. (Leipzig) **56** (1918), VIII.

obviously would like to make the *simplest* determination of x', y', z', t', we must try to make a choice such that ds^2 is represented as simply as possible in the new coordinates, in particular.

2. – Now, the most natural way of determining a spacetime measurement in a rotating system seems to be the one that **Ehrenfest** ([1]) chose a long time ago. In a rotating system, one imagines that observers are distributed throughout a continuum and that each of them are provided with certain yardsticks and a certain clock that should, however, be valid only in their immediate infinitesimal spatial neighborhood. Those yardsticks and that clock shall be assumed to possess the same velocity in direction and magnitude as the rotating observer in question at precisely the position considered and to coincide with it at the moment in time T considered, just as they would on the basis of the usual **Lorentz** transformation of a uniformly-moving observer. That is, yardsticks and clocks depend upon only the velocity and not also on the acceleration, at least in the stationary state (i.e., uniform acceleration). We would like to call these local yardsticks and clocks that are distributed everywhere in the rotating system the *natural* local yardsticks and clocks.

As a result, at certain places in the rest system, the natural yardsticks exhibit a **Lorentz** contraction that varies from place to place, while the natural clocks will likewise exhibit a corresponding dilatation. One measures the local "proper lengths" and "proper times" by means of those natural yardsticks and clocks. (**Einstein** used the word "natural" in a somewhat different sense.) From the assumption that was made, those yardsticks are obtained by optical means, just as they are in a rest system, in a known way: Transfer a well-defined wavelength of light to a yardstick following **Michelson** and infer the time measurement from the speed of light, perhaps by the method of rotating mirrors.

3. – Before we go on to discuss the question of whether a unified measurement of spacetime can be constructed on the basis of the space and time measurements that are chosen in that way, two historical remarks might find a place here. The are concerned with the so-called **Ehrenfest** paradox and something that I will call the **Einstein** paradox.

Ehrenfest (*loc. cit.*) argued as follows: Like the yardstick, the lengths in the rotating system as seen from the rest system must experience a contraction under some circumstances. If one then compares the periphery of a circular cylinder in the rest system and then in the state of uniform rotation around the axis of the cylinder then one will find that the periphery is equal to π times the diameter in the first case, while it is somewhat *less* than π times the diameter in the second case. Since that would contradict geometry in the rest system, with **Planck** ([2]), one must assume that the cylinder deforms as soon as it is put into rotation, in such a way that the element of a diameter before the rotation will no longer define the element of a diameter during the rotation. One must keep that in mind for the following **Einstein** paradox in regard to a rotating circular disc in order to not speak of inferring the rotating circular disc from the one at rest in the attempt.

([1]) **P. Ehrenfest**, "Gleichförmige Rotation starrer Körper und Relativitätstheorie," this Zeit. **10** (1909), 918.
([2]) **M. Planck**, Gleichförmige Rotation und **Lorentz** kontraktion," this Zeit. **11** (1910), 294.

Now, the aforementioned **Einstein** paradox ([1]) is the following one: One imagines a planar circular disc as already being in a state of uniform rotation. Peripheral lengths appear to be shortened in the rest system, but radial ones do not. Therefore, if the rotating observer measures the periphery (the diameter of the disc, resp.) then he must find that the periphery is *greater* than π times the diameter. The paradox is resolved by saying that the geometry of the rotating observer no longer needs to be Euclidian.

4. – I will now show that a unified spacetime measurement *cannot* be constructed on the basis of the natural space (time, resp.) measurements that were given.

The first thing that the rotating observer would have to do in that regard would be to make a spatial measurement of his system.

The first difficulty that one encounters will be whether he can appeal to his own yardstick. Therefore, he must assume that when the yardstick is brought to another location it will agree with the local yardstick that is there. That is in no way certain, since the yardstick must be accelerated during that transport when one goes, e.g., from smaller to larger rotational velocity or conversely. Our assumption of the independence of lengths from the stationary acceleration does not apply here since the acceleration that is carried out here is not stationary. We can get around that difficulty by prescribing that the observer must appeal to the local yardstick each time.

The second difficulty is the following one: Assume that the observer would like to measure the distance from a point *A* to a point *B*. How does he find the direction? That direction is defined mathematically by the straight line and that is once more defined to be the shortest connecting line *AB*. However, it is clear that a physical measurement cannot be based upon that mathematical definition. Otherwise, one would have to perform an infinitude of experiments in order to deduce what the shortest connecting line is from that. Rather, the physical definition of the straightest line in all geodetic and astronomical measurements is the path of a light ray. If the rotating observer applies that definition then he will clearly get a false direction from *A* to *B*, since light goes from *A* to *B* along a curvilinear path in the rotating system. Namely, on the basis of multiple aberrations, the "light ray" *AB* will suffer an angular rotation at *B* relative to its initial direction at *A*, which is equal to the angular velocity of the rotating system times the length of time that the light took along the segment *AB*, in the first approximation. Furthermore, the light ray *AB* is not identical to the light ray *BA*. The definition of a straight line with the help of light is therefore impracticable in the rotating system.

Of course, in practice, things mostly happen this way: Since the usual angular velocity is very small, the effect of the curvature of light can be neglected in the first approximation. The rotating observer then obtains a first approximation to the spatial measurement. Later, with increasing precision in his measurements, he must make improvements to it as soon as he sees that the light ray is curvilinear. On the basis of those improvements, we will again have to determine the precise path of the light ray, etc.; i.e., he will get what one calls a sequence of successive approximations in mathematics. The correct spatial measurement will then be the final term in that sequence. We cannot allow the observer to go down that path for our primary consideration, first of all, because

([1]) **A. Einstein**, *loc. cit.*

he will be led to make infinitely many experiments, and secondly, because arbitrarily large angular velocities are not permissible. Nothing will remain but for the observer to regard the direction from *A* to *B* as *fictitious*. However, that is the first point at which the principle of *natural* measurement breaks down. Only the natural spatial measurement can then be performed.

5. – We shall now turn to the natural measurement of time.

If one compares any two natural local clocks in the rest system then they will generally indicate different rates. The one that is further from the rotational axis will be found to be slower than the one that is closer (**Einstein**). Nonetheless, one can employ the natural local clocks for the measurement of time when one *fictitiously* calibrates them to have the same rate. That is the second point at which the principle of natural measurement breaks down.

However, that is still not enough. One must also be able to synchronize them for all time, and that is impossible, as one sees in the following way: If it were possible then all points of the rotating system would have to be "simultaneously" at rest at some point in time. However, from **Minkowski**'s principles, there is no measurement of time for which that would be possible. The exact proof of that will follow in no. **7**. That impossibility can also be grasped approximately with **Newtonian-Galilean** principles. In **Newtonian-Galilean** mechanics, all points of an arbitrary *translating* rigid body possess the same velocity, but never all points of a *rotating* one. Analogously, it is impossible in **Minkowskian** mechanics to "simultaneously" transform all points of a rotating system to "rest".

The natural measurement of time breaks down with that. Nothing remains but to give it up completely and introduce a fictitious measurement of time.

6. – However, with that, it is also already clear that a unified spacetime measurement on the basis of the natural spatial (temporal, resp.) measurement is impossible since the latter fails in that way. One then asks whether one can at most pair the natural measurement of space with any fictitious measurement of time in order to arrive at a unified spacetime measurement. However, that is impossible, as one will see in the following way:

Time was irrelevant to the spatial measurement in the rotating system, since one was dealing with lengths that were at rest in the rotating system. The measurement of such lengths that are relatively *at rest* by line segments must not be performed synchronously at all points of the segment. Things are different from the measurement of spacetime, since there one deals with processes that are variable in spacetime. The associated spatial lengths must not remotely be taken to be synchronous. However, they can therefore never coincide with the natural lengths, since otherwise a synchronization on the basis of the natural time measurement would not be possible.

It then follows from this that: The natural space and time measurement is unusable for a unified measurement of spacetime in the rotating system.

7. – Let us be permitted to give the mathematical basis for that unusability.

In **Minkowski**'s geometric representation, uniform rotation is represented by a bundle of worldlines that are coaxial helices with the same pitch (imaginary fourth coordinate!). The natural yardsticks (times, resp.) are the lengths that are taken in (normal to, resp.) space, which is normal to the tangent to the worldline that goes through it. However, only an infinitesimal piece of that of that space [the direction normal to it (viz., the proper time axis)] comes under consideration. For a neighboring world-point, an infinitesimal piece of a different space will be valid, etc. Now, when one arranges those infinitesimal spatial yardsticks one after the other, one will get what we have called the natural spatial lengths. If one considers, e.g., a material point A in the rotating system then it will correspond to a world-line A. Likewise, the points B (C, resp.) will correspond to worldlines B (C, resp.). In the Minkowski representation, the naturally-measured length AB will then correspond to a straight line that is just as long and intersects the worldlines A and B orthogonally, and similarly, the length AC will correspond to a line segment that intersects the worldlines A and C orthogonally. If one draws all possible segments AB, AC, … away from A at any point in time then they will obviously fill up a "planar" space that intersects the worldline A normally. However, e.g., the line segment BC will already no longer lie in that space in general. Generally speaking, the naturally-measured spatial lengths will not fill up any three-dimensional spatial manifold, whether "planar" or "curved," as they would have to if all of them were "simultaneous," because such a manifold would have to be intersected orthogonally by the worldlines everywhere. However, there is no three-dimensional manifold that possesses the bundle of coaxial helices as its orthogonal trajectories.

8. – Once we have convinced ourselves that a unified spacetime measurement cannot be achieved with the help of the *natural* yardsticks and clocks in a rotating system, we must look around for a *fictitious* way of obtaining such a spacetime measurement. We will then distribute yardsticks and clocks in some way such that they will no longer have to coincide with the natural ones. As seen from the rest system, they will no longer exhibit merely the **Lorentz** contraction (dilatation, resp.), but will be shorter or longer than the natural yardsticks and clocks. However, they must always be chosen such that they collectively make a spacetime measurement possible in the rotating system. Now, that is a multi-valued problem, since assuming that we have been given such a spacetime measurement x', y', z', t', we can obviously always derive another equally-possible spatial measurement x'', y'', z'', t'' by an arbitrary transformation of the x', y', z', etc. We have already remarked before that we have no other selection rule in regard to that multi-valuedness than that of greatest simplicity. In that way, we assume that the infinitude of experiments to which the rotating observer is led to do when he would like to move forward with the help of only the natural measurements would also most likely lead him to the metric that we referred to as the simplest one as the final result. With that likelihood assumption, the only thing that we can arrive at by introducing the fictitious metric is therefore an abbreviation of the process.

9. – We are now quite close to such a fictitious spacetime measurement. It is impossible to synchronize the natural local clocks. That will then suggest that we must single out *one clock*, say the clock at station A and impose its rate on the other stations by some fictitious means. It is clear from this that simultaneity is impossible in the rotating system. We are now very close to combining that distinguished local time at *A* with the local yardstick at *A*, in the hope that in that way we can arrive at a unified spacetime measurement in the rotating system. We will then have to distribute fictitious local clocks (yardsticks, resp.) about the rotating system in such a way that the angular velocity of *A* is what is appropriate, and not the angular velocity of the local station there. (Carrying out the introduction of such yardsticks and clocks in practice can only be achieved with the help of an observer at rest. However, it will be shown later on that one cannot achieve that distribution of yardsticks and clocks in practice, since one is dealing with an *opinion* on the natural order of things in that way.) Those yardsticks always appear (e.g., in the rest system) to be equally long in one and the same direction in which they are also found. One then sees immediately that the spatial measurement in the rotating system will be a Euclidian one with its help. Therefore, that satisfies the requirement that the spacetime measurement must have the greatest-possible simplicity.

There is only one question left to address: Do the coordinates x', y', z' of any station *B* in the rotating system as determined at *A* actually remain constant during the rotation? The answer is: Yes. It is a distinguished property of the uniform rotation and, generally speaking, of all "uniform" or stationary accelerations (and only those motions, when one adds a condition that will not discussed in detail here), as I showed in *loc. cit.*

Of course, we must recognize one objection: Previously, we have expressly excluded the transport of natural yardsticks from one place to another in the rotating system (cf., *supra*, no. **4**), because we do not know whether such a yardstick will assume the length of the indigenous natural local yardstick when it is brought to the other location. If we would like to nonetheless assume that the latter is correct then that would raise the question of how it can be combined with the fictitious metric, since the transported yardstick will not coincide with the fictitious local yardstick then. It could not be combined, either! We must *forbid* the transport of the yardstick with our metric. Now, we can come to think that our picture of spacetime will suffer a restriction on its basis in reality as a result of that ban. However, that is not true, because in reality we measure our earthly space at great distances, and we certainly *do not* measure stellar space by carrying yardsticks about, but indirectly exclude them on the basis of *visual* images when we base them upon certain laws of propagation of light. In that way, it is self-explanatory that we base them upon the local yardsticks and clocks at station *A*, which is precisely where we find ourselves. The single difference between that method and ours for the rotating system is only that we can no longer use the simple laws of propagation of light that we had in the rest system as a basis when we are in stellar space. However, having a knowledge of those laws is implicit in both types of measurements. As was emphasized before, the fact that we must anticipate that knowledge for the rotating observer, instead of acquiring it along the way, is the single "fictitious" moment in our spacetime measurement in the rotating system. On the other hand, as we see, banning the transport of yardsticks is not such a moment ([1]).

([1]) It is obvious that the measurement can also be made with the use of the respective local fictitious local yardsticks, except that they cannot be transported outside of their domain of validity.

10. – We shall now address the mathematical representation of the chosen spacetime measurement in the rotating system.

It is clear that we must perform a single ordinary **Lorentz** transformation that is appropriate to instantaneous rotational velocity of the distinguished station A if we are to determine x', y', z'. That is because we have already mentioned that it can be shown that those values of x', y', z' will prove to be the "comoving" points in any later position. Naturally, we require only one time-point for that **Lorentz** transformation. The observer at A rotates along a circle (radius a) in the plane $Z = b$ around the Z-axis with constant angular velocity ω. His instantaneous velocity will then have a magnitude of $a\,\omega$ and the same direction as the tangent to the circular path. Let x, y, z, t be the coordinates of A in the rest system X, Y, Z, T, so:

$$x = a \cos \omega t, \quad y = a \sin \omega t, \quad z = b \,.$$

We focus on a position t and perform the indicated **Lorentz** transformation in two steps:

First we displace and rotate the system X, Y, Z, T in such a way that the new origin falls at A in the position t and the new X_1-axis is radial, while the new Y_1 proves to be tangential:

$$\begin{aligned} X &= x + X_1 \cos \omega t - Y_1 \sin \omega t &&= (a + X_1) \cos \omega t - Y_1 \sin \omega t \,, \\ Y &= y + X_1 \cos \omega t + Y_1 \sin \omega t &&= (a + X_1) \sin \omega t + Y_1 \cos \omega t \,, \\ Z &+ z + Z_1 &&= b + Z_1 \,, \\ T &= t + T_1 \,. \end{aligned} \tag{1}$$

We then transform the velocity $a\,\omega$ to rest (parallel to the Y_1-axis). That is, we introduce a system x', y', z', t' in such a way that:

$$\begin{aligned} X_1 &= x', \\ Y_1 &= \frac{y' + a\,\omega T'}{\sqrt{1 - \frac{a^2\omega^2}{c^2}}}\,, \\ Z_1 &= z', \\ T_1 &= \frac{T' + a\,\omega / c^2\, y'}{\sqrt{1 - \frac{a^2\omega^2}{c^2}}}\,. \end{aligned} \tag{2}$$

As announced, here we have to take the single time-point $T = 0$ and combine the result with (1). That will give:

$$X = (a + x')\cos\omega t - \frac{y'}{\sqrt{1-\frac{a^2\omega^2}{c^2}}}\sin\omega t,$$

$$Y = (a + x')\sin\omega t + \frac{y'}{\sqrt{1-\frac{a^2\omega^2}{c^2}}}\cos\omega t,$$

$$Z = b + z', \tag{3}$$

$$T = t + \frac{a\,\omega / c^2\, y'}{\sqrt{1-\frac{a^2\omega^2}{c^2}}}.$$

Here, we can replace t with the proper time of station A :

$$t' = t\sqrt{1-\frac{a^2\omega^2}{c^2}},$$

but since it differs from t only by a constant factor, we shall refrain from doing so.

x', y', z', t' is our desired spacetime measurement for the rotating system. One proves that, in turn, when one shows that:

$$\left(\frac{dX}{dT}\right)_{x',y',z'=\text{const.}} = -\omega Y, \quad \left(\frac{dY}{dT}\right)_{x',y',z'=\text{const.}} = \omega Y, \quad \left(\frac{dZ}{dT}\right)_{x',y',z'=\text{const.}} = 0;$$

i.e., the material point x', y', z' = const. actually moves along a path that corresponds to a uniform rotation with angular velocity ω around the Z-axis.

The metric that belongs to (3) is calculated to be:

$$\begin{aligned} ds^2 &= dX^2 + dY^2 + dZ^2 - c^2\, dT^2 \\ &= dx'^2 + dy'^2 + dz'^2 + 2g_{14}\, dx'\, dt + 2g_{24}\, dy'\, dt + g_{44}\, dt^2, \end{aligned} \tag{4}$$

where:

$$g_{14} = -\frac{\omega\, y'}{\sqrt{1-\frac{a^2\omega^2}{c^2}}}, \quad g_{24} = +\frac{\omega\, x'}{\sqrt{1-\frac{a^2\omega^2}{c^2}}}, \quad g_{44} = -c^2 + (a + x')\,\omega^2 + \frac{y'^2\omega^2}{1-\frac{a^2\omega^2}{c^2}}.$$

As was predicted, it gives a *Euclidian spatial measurement* for $dt = 0$, so it is probably the *simplest possible spacetime measurement in the rotation system*.

I shall call the transformation (3) the *generalized Lorentz transformation for the uniformly-rotating system*.

11. – In conclusion, in order to get back to the natural lengths again from the fictitious ones dx', dy', dz', one observes that each natural spatial element is orthogonal to the worldline in question. A worldline is given by x', y', z' = const., so merely t will vary along it. One then finds the increases dx', dy', dz' that belong to the natural spatial element from:

$$g_{14}\, dx' + g_{24}\, dy' + g_{44}\, dt = 0\ .$$

One then comes to the length $d\sigma$ of the natural element when expressed in terms of the fictitious yardstick at station A :

$$d\sigma^2 = dx'^2 + dy'^2 + dz'^2 - \frac{1}{g_{44}}\,(g_{14}\, dx' + g_{24}\, dy')^2\ . \tag{5}$$

We would like to choose our viewpoint A to be on the rotational axis, say $a = b = 0$, in particular. Our generalized **Lorentz** transformation (3) will become:

$$\begin{aligned} X &= x' \cos \omega t - y' \sin \omega t\ , \\ Y &= x' \sin \omega t + y' \cos \omega t\ , \\ Z &= z', \\ T &= t \end{aligned} \tag{3.a}$$

in this case; i.e., the observer A is at rest, but only when referred to the rotating axes (which are axes that are fixed in the rotating system).

The expression for the natural spatial element in terms of its yardsticks will then be simply:

$$d\sigma^2 = dx'^2 + dy'^2 + dz'^2 + \frac{\omega^2}{c^2 - x'^2 + y'^2\omega^2}(-y'\, dx' + x'\, dy')^2, \tag{5.a}$$

or when one introduces polar coordinates r', φ' into the $x'y'$-plane:

$$d\sigma^2 = dr'^2 + \frac{r'^2}{1 - \dfrac{\omega^2 r'^2}{c^2}}\, d\varphi'^2 + dz'^2. \tag{5.b}$$

One confirms the **Einstein** paradox from that. The naturally-measured periphery of the circle r'= const., z'= const. will be:

$$\frac{2\pi r'}{\sqrt{1-\frac{\omega^2 r'^2}{c^2}}} > 2\pi r',$$

while the naturally-measured diameter is:

$$2r',$$

so their ratio is:

$$\frac{\pi}{\sqrt{1-\frac{\omega^2 r'^2}{c^2}}} > \pi.$$

One easily notes why conversely the naturally-measured spatial lengths are unusable for a unified spacetime measurement in the rotating system, since one has from (4) and (5) that:

$$ds^2 = d\sigma^2 + \frac{1}{g_{44}}(g_{14}\,dx' + g_{24}\,dy' + g_{44}\,dt)^2 .$$

However, there is no function $t(X, Y, Z, T)$ whose differential would be proportional to the linear differential expression in parentheses, such that one would have:

$$g_{14}\,dx' + g_{24}\,dy' + g_{44}\,dt = \lambda\,dt .$$

One confirms that easily by calculating the **Pfaff** integrability condition, which is never fulfilled for the values of g_{14}, g_{24}, g_{44} that are given in (4). Hence, $d\sigma^2$ cannot appear as a spacelike part of ds^2, either. The natural spacetime measurement is unusable for a unified spacetime measurement in a rotating system.

12. – This might be a good place to make a remark: One knows that **Einstein** applied the cited paradox in order to draw the conclusion that in general the gravitational field influences the spatial metric in such a way that the geometry is no longer Euclidian, because as a result of his equivalence hypothesis, the processes in the rotating system are equivalent to the ones in a certain gravitational field that I have called gravitational rotation field. However, **Einstein**'s aforementioned conclusion is correct only when the result of the natural spatial measurement is indicated as the geometry. As we have seen, **Einstein**'s conclusion is not true for the spatial part of the four-dimensional metric ds^2. The rotating system cannot give an example then of the influence on the spatial part of ds^2 by actual gravitational fields of the kind that appear, e.g., in the field of a point-mass (**Schwarzschild**).

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