"Wellenmechanik und halbzahlige Quantisierung," Zeit. Phys. **39** (1926), 828-840.

Wave mechanics and half-integer quantization

By **H. A. Kramers** in Utrecht

With 1 figure (Received on 9 September 1926)

Translated by D. H. Delphenich

This article treats an approximate method for solving **Schrödinger**'s eigenvalue and eigenfunction problem for an arbitrary system with one degree of freedom. In § 1, it will be shown that the so-called half-integer quantization represents a natural first approximation. § 2 contains approximation formulas for the graphical determination of the eigenfunctions. In § 3, the relationship between the method and the systematic process of approximations that was considered by **Brillouin** and **Wentzel** will be discussed. § 4 treats central motion and includes approximation formulas for spectral problems.

§ 1. The approximate solution of the wave equation. – Suppose that one has posed the problem of the quantization of a system with one degree of freedom whose motion has an oscillatory character. According to **Schrödinger**, the stationary states are determined by looking for the eigenvalues E_n for which the differential equation:

$$\varphi'' + \frac{y}{K^2}\varphi = 0, \tag{1}$$

where

$$y = 2m\,(E_n - V(x)) \tag{2}$$

($K = h / 2\pi$, m = mass, $V(x)$ = potential energy) has an everywhere-finite real solution φ_n. The quantum number n refers to the number of zeroes of φ_n between the two zeroes x_1 and x_2 of y, which coincide with the points of regression of the motion that is calculated in classical mechanics. For the case in which n is a large number, we can construct a function y that represents an approximate solution to (1) in the domain $x_1 < x < x_2$ by means of an elementary argument. From the wave-like character of φ in this domain and the argument that y varies only slightly in the domain of a wave length for large n, we will be led to make the following Ansatz for ψ:

$$\psi = g(x)\cos f(x), \tag{3}$$

in which $g(x)$ is imagined to be a "flat" function of the same kind as $y(x)$, while $f(x_2) - f(x_1)$ has the order of magnitude $n\pi$.

We will obtain an expression for $f(x)$ when we observe that the wave length would be equal to $2\pi K y^{-1/2}$ for constant y. That will, in fact, give the approximate condition for $f(x)$:

$$f(x + 2\pi K y^{-1/2}) - f(x) = 2\pi,$$

or

$$2\pi K y^{-1/2} f'(x) = 2\pi,$$

$$f(x) = \frac{1}{K}\int^{x} y^{1/2}\, dx\,. \tag{4}$$

We will get an expression for the function $g(x)$ that represents the dependency of the wave amplitude on x by considering the differential equation:

$$\varphi'' + \frac{y(x_0) + y'(x_0)(x - x_0)}{K^2}\varphi = 0, \tag{5}$$

which practically coincides with (1) in a region whose order of magnitude is one wave length. Upon neglecting small quantities that are proportional to the second and higher powers of y', a solution of (5) will be given by:

$$\varphi = \cos(K^{-1} y^{1/2}(x - x_0)) - \tfrac{1}{4} y^{-1} y'[(x - x_0)\cos K^{-1} y^{1/2}(x - x_0) + K^{-1} y^{1/2}(x - x_0)^2 \sin K^{-1} y^{1/2}(x - x_0)].$$

We deduce from this that the amplitude of the oscillating function that is represented by φ can be represented in the first approximation in a domain of order of magnitude one wave length by a factor of the form:

$$1 - \tfrac{1}{4} y^{-1} y'(x - x_0).$$

For that reason, we can then write:

$$g' g^{-1} = -\tfrac{1}{4} y' y^{-1}, \qquad g = y^{-1/4}$$

for the function $g(x)$. Our approximation for the eigenfunction will then assume the following form:

$$\psi = y^{-1/4} \cos \frac{1}{K}\int^{x} y^{1/2} dx\,. \tag{6}$$

We now come to the question of how the energy value E_n and the integration constant in (4) must be determined in order for ψ to actually approximate an eigenfunction of the problem. In order to answer it, it will not suffice to consider only the properties of ψ, since ψ is already infinite at x_1 and x_2 and will assume complex values for $x < x_1$ and $x > x_2$. For that reason, we would like to consider the solution of (1) in vicinity of x_1. When y' assumes the value α at x_1, and when we denote $x - x_1$ by ξ, (1) will assume the form:

$$\varphi'' + \frac{\alpha}{K^2}\xi\varphi = 0 \tag{7}$$

in the neighborhood of x_1 . The solution to this equation ([1]) can be represented in the form:

$$\varphi = \xi^{1/2} Z_{1/3}\left(\tfrac{2}{3}\sqrt{\alpha}/K\cdot\xi^{3/2}\right). \tag{8}$$

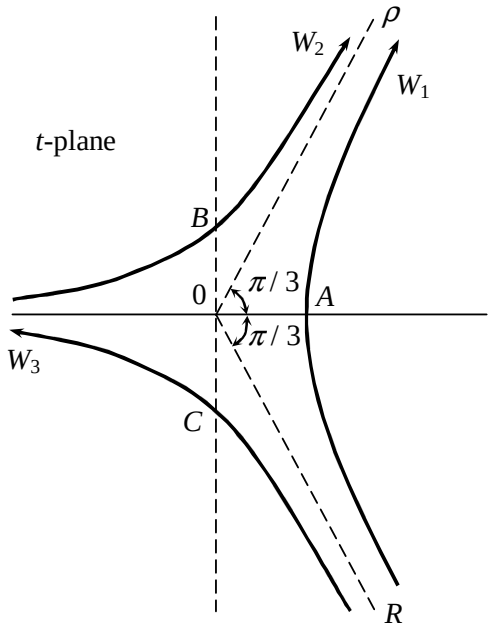

Figure 1.

However, the discussion will take on a simpler form when one writes it directly in the form of a definite integral:

$$\varphi = C\int \exp\left[\left(\frac{\alpha}{K^2}\right)^{1/2}\xi t + \frac{1}{3}t^3\right]dt \tag{9}$$

that lets the fact that φ is an entire function better emerge from the expression (8), among other things. The integral (9) represents a solution for all x when the path of integration asymptotically approaches the directions with the arguments $\pm\pi/3$, π (cf., Fig. 1). One further easily recognizes that for negative real ξ the integral that is extended along the curve W_1 goes to the value zero with increasing ξ, and thus corresponds to the particular solution of (1) that is desired in wave mechanics. If one takes the integral along a straight line from R to 0 and then from 0 to P then when one develops the integrand in powers of ξ and sets the factor $C = -i$, one will get:

$$\varphi = 3^{-1/6}\Gamma\left(\frac{1}{3}\right) + 3^{1/6}\Gamma\left(\frac{2}{3}\right)\left(\frac{\alpha}{K^2}\right)^{1/3}\xi + \cdots + \left(\frac{\alpha}{K^2}\right)\xi^3 + \ldots \tag{10}$$

([1]) For the literature on this equation and its solutions, cf., e.g., **Watson**, *Theory of Bessel Functions*, Cambridge, 1922, pp. 188, *et seq.*

The asymptotic developments for large values of the argument $\left(\frac{\alpha}{K^2}\right)^{1/3}\xi$ are important. One gets this simply with the help of the saddle-point method ([1]). For negative real ξ, we need the saddle point at A $\left[OA=\sqrt{\left(\frac{\alpha}{K^2}\right)^{1/3}|\xi|}\right]$, and the line of steepest descent, along which the integral is extended, will become a branch of a hyperbola W_1 whose asymptotes are *OP* and *OR*. One easily finds that:

$$\varphi=\sqrt{\pi}\left[\left(\frac{\alpha}{K^2}\right)^{1/3}|\xi|\right]^{-1/4}\exp\left[-\frac{2}{3}\left(\frac{\alpha}{K^2}\right)^{1/2}|\xi|^{3/2}\right]. \tag{11}$$

For positive real ξ, we need the saddle points *B* and *C*:

$$\left[OB=OC=\sqrt{\left(\frac{\alpha}{K^2}\right)^{1/3}\xi}\right].$$

The lines of steepest descent are the curve branches of degree three W_2 and W_3, and the integral is extended over W_2, as well as W_3. One then obtains the asymptotic expression:

$$\varphi=2\sqrt{\pi}\left[\left(\frac{\alpha}{K^2}\right)^{1/3}\xi\right]^{-1/4}\cos\left[\frac{2}{3}\left(\frac{\alpha}{K^2}\right)^{1/2}\xi^{3/2}-\pi/4\right], \tag{12}$$

which is different from (11) ([2]). We can now compare this expression directly with the expression that (6) goes to for values of α that are not too distant from x_1. One must then set $y=\alpha\xi/K^2$ and find immediately that:

$$\psi=\left(\frac{\alpha}{K^2}\right)^{-1/4}\xi^{-1/4}\cos\left[\frac{2}{3}\left(\frac{\alpha}{K^2}\right)^{1/2}\xi^{3/2}-\beta\right], \tag{13}$$

in which $-\beta$ initially represents the still-undetermined integration constant in the integral (4). A comparison of (12) and (13) implies immediately that it is only for $\beta=\pi/4$ that the function that is represented approximately by (6) will represent the particular solution of (1) that is required by wave mechanics; i.e., one must be able to write down ψ in the form:

$$\psi=y^{-1/4}\cos\left[\frac{1}{K}\int_{x_1}^{x}y^{1/2}dx-\pi/4\right]. \tag{14}$$

([1]) Cf., say, **Courant-Hilbert**, *Mathematische Physik*, pp. 435.

([2]) This is the so-called "**Stokes** phenomenon"; cf., **Watson**, *Bessel Functions*, pp. 435.

In this, we have introduced the permissible restriction that $\varphi(x_1)$ must be positive ([1]).

A consideration that is entirely similar to the one above can also be applied to the other points of regression, and one easily recognizes that in order for ψ to approximate an eigenfunction, one must likewise be able to write the expression (6) in the form:

$$\psi = y^{-1/4} \cos\left[\frac{1}{K}\int_{x_1}^{x_2} y^{1/2}dx - \pi/4 \pm \pi/2\right]. \tag{15}$$

The plus or minus sign is true according to whether $\varphi(x_2)$ is positive or negative, respectively.

The condition for (14) and (15) to represent the same function reads:

$$\frac{1}{K}\int_{x_1}^{x_2} y^{1/2}dx - \pi/4 = \frac{1}{K}\int_{x_1}^{x_2} y^{1/2}dx - \pi/4 \pm \pi/2 + m\,2\pi\,, \tag{16}$$

in which *m* is a whole number. This immediately implies that:

$$\frac{1}{K}\int_{x_1}^{x_2} y^{1/2}dx = K\,\pi(2m \pm \tfrac{1}{2}),$$

or, when one multiplies both sides by 2, introduces the classical phase integral, and replaces *K* with its value $h / 2\pi$:

$$2\int_{x_1}^{x_2} y^{1/2}dx = \oint p\,dx = (2m \pm \tfrac{1}{2})\,h. \tag{17}$$

We infer from this that our approximate representation (6) of the **Schrödinger** eigenfunction corresponds to exactly those energy values that would correspond to what one called a half-integer quantization, in the language of the old quantum theory. The relative precision of these energy values will have an order of magnitude that is higher than $1 / n$ (in general, probably $1 / n^2$), while the energy values that are calculated by whole-number quantization generally differ from the **Schrödinger** eigenvalues by quantities whose relative order is $1 / n$ ([2]). We have then arrived at a certain

---

([1]) The fact that this is, in fact, allowable for one to consider the simple equation (7) for the range of φ in the vicinity of x_1 follows as long as one can assume that $y'(x_1)(x_2 - x_1)$, $y''(x_1)(x_2 - x_1)$, etc., have the same order of magnitude as the values that *g* assumes between x_1 and x_2. A more precise examination will, in fact, show that when one assumes that *x* has, say, order of magnitude $(x_2 - x_1) / n^{2/5}$ – i.e., when (13) and (6) deviate from each other by the relative order of magnitude $n^{-2/5}$ – the deviations of the expressions (11) and (12) from the function (10) will likewise have the relative order of magnitude $n^{-2/5}$, and that the φ in (11) will already be small like $\exp(-n^{-2/5})$ for negative ξ, in addition.

([2]) In fact, if we assume that in the region $x_1 < x < x_2$, $y'(x)\,(x_2 - x_1)$ has the same order of magnitude that *y* has in that region then the distance between two successive zeroes of the function (6) will differ from the distance between the corresponding zeroes in the **Schrödinger** eigenfunction only by small quantities of order $(x_2 - x_1) / n^2$; i.e., they are small in comparison to the distance between successive zeroes. A relative change in the energy value of order $1 / n$ would provoke a perturbation of the location of the zero

understanding of the fact that is known from many examples that half-integer quantization proves to be a better approximation to the results of quantum mechanics than whole-number quantization.

§ 2. The practical calculation of the eigenfunctions. – Our arguments give us a simple means of determining the eigenfunctions approximately that can be of practical utility in the cases where they cannot be expressed by means of elementary functions. Namely, if we employ the function:

$$\varphi = y^{-1/4} \cos\left(\int_{x_1}^{x_2} \sqrt{y}\, dx - \pi/4\right) \qquad x_1 < x < x_2\,, \tag{18a}$$

in the domain $x_1 < x < x_2$, whose values can be ascertained graphically as long as y is known, then we can write [cf., (10), (12), and (13)]:

$$\varphi = \frac{1}{2\sqrt{\pi}}\left(\frac{\alpha_1}{K^2}\right)^{-1/6}\left[3^{-1/6}\Gamma\left(\frac{1}{3}\right) + 3^{1/6}\Gamma\left(\frac{2}{3}\right)\left(\frac{\alpha_1}{K^2}\right)^{1/3}(x-x_1) + \cdots\right], \quad x \text{ close to } x_1, \tag{18b}$$

$$\varphi = (-1)^n \frac{1}{2\sqrt{\pi}}\left(\frac{\alpha_2}{K^2}\right)^{-1/6}\left[3^{-1/6}\Gamma\left(\frac{1}{3}\right) - 3^{1/6}\Gamma\left(\frac{2}{3}\right)\left(\frac{\alpha_2}{K^2}\right)^{1/3}(x-x_2) + \cdots\right], \; x \text{ close to } x_2 \tag{18c}$$

in the close vicinity of the regression point itself. In this, one has $\alpha_1 = y'(x_1)$, $\alpha_2 = y'(x_2)$, n = number of zeroes between x_1 and x_2 . These formulas determine the tangents to the φ curve at the inflection points x_1 and x_2 .

The exponential decay of φ for $x < x_1$ and $x > x_2$ is given by the formulas:

$$\varphi = \frac{1}{2}\left(\frac{\alpha_1}{K^2}\right)^{-1/4}(x_1 - x)^{-1/4}\exp\left[-\frac{2}{3}\left(\frac{\alpha_1}{K^2}\right)^{1/2}(x-x_1)^{3/2}\right], \qquad x < x_1\,, \tag{18d}$$

$$\varphi = \frac{(-1)^n}{2}\left(\frac{\alpha_2}{K^2}\right)^{-1/4}(x - x_2)^{-1/4}\exp\left[-\frac{2}{3}\left(\frac{\alpha_2}{K^2}\right)^{1/2}(x-x_2)^{3/2}\right], \; x > x_2\,. \tag{18e}$$

However, instead of using formulas (18 b to e) explicitly, it is much simpler to evaluate the functional value of the integral:

$$\omega(x) = \frac{-i}{2\sqrt{\pi}}\int_{W_1}\exp\left(\xi t + \frac{1}{3}t^3\right)dt \tag{18f}$$

---

that would be incompatible with the properties of the eigenfunctions in the neighborhood of the two points of regression.

once and for all, for values of $x = -\infty$ up to, say, the first zero of $\omega(x)$ (cf., the function table on pp. 12) and employ the following expression for φ in the neighborhood of the regression point:

$$\varphi = \left(\frac{\alpha_1}{K^2}\right)^{-1/6} \omega\left[\left(\frac{\alpha_1}{K^2}\right)^{1/3}(x_1 - x)\right], \qquad \text{from } x = -\infty \text{ up to the first zero,} \tag{18g}$$

$$\varphi = (-1)^n \left(\frac{\alpha_2}{K^2}\right)^{-1/6} \omega\left[\left(\frac{\alpha_2}{K^2}\right)^{1/3}(x_2 - x)\right], \qquad \text{from the } n^{\text{th}} \text{ zero up to } x = \infty\,. \tag{18h}$$

I have applied the approximation formulas (18) to the case of the harmonic oscillator and found that for $n = 1$ and $n = 0$ they immediately put one into a position to construct the eigenfunctions with considerable precision.

§ 3. Relationship to the systematic method of approximation. – The function (6) that was derived by means of an elementary argument shows a close connection with the processes of successively calculating the eigenvalues and eigenfunctions that was treated by **Brillouin** ([1]) and **Wentzel** ([2]), in which the solution of **Hamilton**'s differential equation defines the first step. These authors made the Ansatz:

$$\varphi = \exp\left[\frac{i}{K}(S_0 + KS_1 + KS_2 + \cdots)\right], \tag{19}$$

and one easily finds that:

$$S_0' = \pm\sqrt{y}\,, \quad S_0 = \pm\int\sqrt{y}\,dx\,,$$

$$S_1' = \frac{i}{2}\frac{S_0''}{S_0'}, \quad S_1 = \frac{i}{2}\log S_0' = \frac{i}{4}\log y + \text{const.}$$

If one truncates at the second approximation then one will get:

$$\varphi = y^{-1/4}\exp\left(\pm\frac{i}{K}\int\sqrt{y}\,dx\right). \tag{20}$$

The plus and minus signs correspond to two particular solutions, and the function that is represented by (6) corresponds to just half the sum of these two solutions ([3]). Now, what

([1]) **L. Brillouin**, C. R. **183** (1926), 24.
([2]) **G. Wentzel**, Zeit. Phys. **38** (1926), 518.
([3]) In order to get (6) directly by means of the **Riccati** equation $\frac{h}{2\pi i}y' = p^2 - y^2$ that **Wentzel** appealed to explicitly (*loc. cit.*, pp. 518), one cannot set $y_0 = \pm p$ as a first approximation, but one must write:

is interesting about this is that the function (6) is not by any means a single-valued function of the same type as the eigenfunction that is approximating. It can reproduce the latter only in the domain $x_1 < x < x_2$, and indeed only when one determines the integration constant in the manner that was given in § 1. However, the regression points themselves are singular locations and the function does not return to its old value when the variable crosses them. One further infers from the formulas (18) that the eigenfunction that is approximated by (6) in the region outside the regression points by other approximate solutions of (1) will be approximated by, e.g.:

$$\varphi = \tfrac{1}{2}(-y)^{-1/4} \exp\left(\frac{-1}{K} \int_x^{x_1} \sqrt{-y}\, dx \right) \tag{21}$$

for $x < x_1$. Let it be pointed out here that there is no practical utility to preferring the simpler formula (18d) over the expression (21) for the calculation of the eigenfunction. Indeed, it obeys the differential equation (1) more precisely; however, due to the rapid exponential decay, the difference is not significant.

The essence of the multi-valuedness of the functions that one obtains from the approximation process (19) will become clear when one writes out the differential equations that the successive approximations obey. Hence, the functions (6) and (20) satisfy the differential equation:

$$\varphi'' + \left(\frac{y}{K^2} - \frac{5y'^2 - 4yy''}{16y^2} \right) \varphi = 0. \tag{22}$$

(22) has singular locations at the regression points. In a region that is finitely distant from them, the solutions of (22) will coincide with the solutions of (1) up to quantities of order K_2 , but in the vicinity of those points, the solutions will be completely different. It is known from mathematics that one should be careful when the solution to a differential equation is examined by means of a solution to an "approximate" differential equation ([1]). It is in just our present case that the solution to (22) at some distance from the regression points indeed represents at the same time an asymptotic expression for a solution of (1). However, the same particular solution of (1) will be approximated by various particular solutions of (22) in various parts of the x-plane. It then seems that the method that **Wentzel** had already employed for the calculation of eigenvalues by the consideration of the complex integral $\oint \varphi'/\varphi\, dx$ around the regression point, in which the successive approximations (19) for φ were employed, demands a more rigorous foundation. The examples that were worked out generally let one suspect that it is consistent. When one applies it to the approximate solution (20), one will immediately come to a prescription for half-integer quantization.

$$y_0 = i\, p\, \tan\left(\frac{2\pi}{h} \int p\, dx \right).$$

([1]) Cf., e.g., **Schlesinger**, *Differentialgleichungen*, pp. 199, *et seq.*

§ 4. Application to central motions. – According to Schrödinger, the problem of a central motion in space can be reduced to the determination of the eigenvalues and eigenfunctions of the differential equation:

$$\varphi'' + \frac{y}{K^2}\varphi = 0, \qquad y = 2m\,(E - V(x)) - \frac{K^2 k(k+1)}{r^2}, \tag{23}$$

in which the azimuthal quantum number k can assume the whole-number values 0, 1, 2, …, and $V(r)$ represents the potential energy of the central force.

When one has two regression points r_1 and r_2 for which $r \neq 0$ (that will be the case in the current problem when $k \neq 0$), the problem will be entirely analogous to the one that was treated in §§ 1 and 2, as long as the fact that the domain of the variable r extends from 0 to ∞ here necessitates no alteration of the approximation in question. However, in the problems that occur in atomic theory, in which the terms of the series or Röntgen levels can be described by a central field, one often encounters the case in which the assumptions upon which the goodness of the approximation that was described in §§ 1 and 2 was based are no longer fulfilled and the justification for a half-integer quantization of the radial phase integral seems questionable on first glance. Here, I shall envision the cases in which k is equal to 0, 1, or 2, and which correspond to non-hydrogenic terms in the so-called penetrating orbits (*eindringenden Bahnen*). In those cases, the function y has a very steep maximum quite close to the smallest regression point r_1 (for $k = 0$, where r_1 is equal to zero, it will become immediately infinite for $r = 0$), and the application of the differential equation (7) to a discussion of the eigenfunction in the neighborhood of that point would become illusory. However, one can give a simple approximate description for the behavior of the eigenfunction in the vicinity of $r = 0$ in these cases that will be adequate for most practical purposes, and which will show, at the same time, the meaning that the half-integer quantization can preserve. In order to arrive at this description, we remark that in the neighborhood of the smallest regression point r_1 and the maximum of y (for $k = 0$, that means close to $r = 0$), the force field is **Coulomb** to a good approximation, and can therefore be described by the potential:

$$V(r) = -\frac{Ne^2}{r} + a. \tag{24}$$

In this, N refers to an effective nuclear charge, while the constant a gives a measure of the so-called external shielding. It is usually small in comparison to the maximal value of $y / 2m$. Now, for the hydrogenic orbits, the eigenvalue E is also small in comparison with that maximal value, and we can therefore assert that for values of r that have order of magnitude r_1 or less, the y-function (23) can be given approximately by the formula:

$$\frac{y}{K^2} = \frac{2mNe^2}{K^2}\frac{1}{r} - \frac{k(k+1)}{r^2}. \tag{25}$$

If we take the radius K^2 / me^2 of the first "hydrogen orbit" as a unit of length then the differential equation (23) will assume the following form:

$$\varphi'' + \left(\frac{2N}{r} - \frac{k(k+1)}{r^2} \right) \varphi = 0. \tag{26}$$

It obviously corresponds to the parabolic paths in a **Coulomb** field. Those particular solutions of (26) that are not infinite for $r = 0$ can be expressed as follows by means of the **Bessel** functions of order $2k + 1$:

$$\varphi = \sqrt{r}\, J_{2k+1}\left(\sqrt{8Nr} \right). \tag{27}$$

When we make use of, e.g., the formulas that were given by **Jahnke-Emden** and consider only the first term in the asymptotic series $P_p(x)$ and $Q_p(x)$ (see pp. 90), we will get the asymptotic representation for (27):

$$\varphi = \pi^{-1/2} (2N)^{-1/4} r^{1/4} \cos \left[\sqrt{8Nr} + \frac{2k(k+1) + \frac{3}{8}}{\sqrt{8Nr}} - \pi\left(k + \tfrac{1}{2}\right) - \pi/4 \right]. \tag{28}$$

For $k = 0$ and $k = 1$, in particular, this formula will also be true with considerable accuracy for values of r for which the representation (25) is still suitable.

We shall now consider the values that the approximate solution (6) to the differential equation (1) would assume in the same domain of r – i.e., we take the step that would correspond to equation (13) in § 1. In order to do that, we set:

$$\frac{y}{K^2} = \frac{2N}{r} - \frac{l^2}{r^2}, \tag{29}$$

in which we temporarily undetermined and calculate the integral $\frac{1}{K}\int_{r_1}^{r_2} \sqrt{y}\, dr$, when we take the regression point $r_1 = l^2 / 2N$. One finds that:

$$\frac{1}{K}\int_{r_1}^{r_2} \sqrt{y}\, dr = 2\sqrt{2Nr - l^2} - 2l \arctan \sqrt{\frac{2Nr}{l^2} - 1} = \sqrt{8Nr} + \frac{2l^2}{\sqrt{8Nr}} - \pi l,$$

in which terms of relative order of magnitude $1 / r$ can be neglected in the last expression. With the same precision, the function (6) now assumes the form:

$$\psi = (2N)^{-1/4} r^{1/4} \cos \left(\sqrt{8Nx} + \frac{2l^2}{\sqrt{8Nx}} - \pi l - \beta \right), \tag{30}$$

in which $-\beta$ again means the undetermined integration constant in the integral (4). Now, a comparison between (28) and (30) teaches us that (6) actually does approximate the eigenfunction when one sets:

$$\beta = \pi / 4, \tag{31}$$

as in § 1, and makes the assumption that:

$$l = k + \tfrac{1}{2}, \tag{32}$$

in addition. In order to find an approximate solution of (23), we will be led to consider the mechanical problem for which the radial impulse p_r is given by:

$$p_r^2 = y = 2m\left(E - V(r) - \frac{K^2}{2m}\frac{\left(k+\frac{1}{2}\right)^2}{r^2}\right). \tag{33}$$

It is known that (33) corresponds to just the classical equation for the radial impulse in a central motion when one has quantized the angular impulse of the particle by half integers. Now, since the difference between the terms in (23) and (33) that include k is hardly significant, as long as one directs one's attention to values of r that lie at some distance from the maximum of the y-function, we would, in turn, like to regard the expression (18 a) as the approximate solution of (23), but in which y has been replaced with the function (33).

We can then once more apply the arguments of § 1 to the largest regression point r_2, and due to (15), we will be again led to equations (16) and (17). The half-integer quantization then proves to be the natural method for calculating the eigenvalues approximately, even for the smallest values of the azimuthal quantum number ([1]).

For the actual construction of the eigenfunctions of (23), one must now proceed as follows: For the very small values of r, one employs:

$$\varphi = \sqrt{\pi r}\, J_{2k+1}(\sqrt{8Nr}). \tag{34a}$$

One must connect this function with the function:

$$\varphi = y^{-1/4}\cos\left(\int_{r_1}^{r}\sqrt{y}\,dr - \pi/4\right), \tag{34b}$$

in which y is given by (33). At the second regression point and beyond it, one can again employ the old formulas (18 c) and (18 e) or (18 h).

The arguments in this paragraph can be of some assistance in the problem of calculating the transition probabilities in the Röntgen series and spectra. By means of the method that was applied by **Fues** and **Hartree**, one can in fact, construct a central field such that observed terms correspond to a half-integer quantization of the azimuthal and radial impulses. The associated eigenfunctions can be constructed to a certain approximation with the help of that field, and with the use of them one can once more calculate the characteristic oscillation amplitudes that are associated with the transition.

([1]) If one wishes to continue applying formula (6) to the expression (23) for y – i.e., if one sets $l^2 = k(k+1)$ in (29) – then one must change the quantization prescription in such a way that one sets $\frac{1}{h}\oint\sqrt{y}\,dr + \sqrt{k(k+1)} - \left(k+\frac{1}{2}\right) = n + \frac{1}{2}$ (n integer).

Appendix. – **M. van der Held** was kind enough to calculate the function $\omega(x)$ that was defined by equation (18 f). He found the following values:

ξ	ω	ξ	ω	ξ	ω	ξ	ω	ξ	ω
– 9.0	4.4×10^{-9}	– 1.9	0.071	– 0.8	0.299	0.3	0.761	1.4	0.871
– 5.0	0.00019	– 1.8	0.082	– 0.7	0.333	0.4	0.803	1.5	0.823
– 4.0	0.0017	– 1.7	0.096	– 0.6	0.369	0.5	0.843	1.6	0.761
– 3.5	0.0045	– 1.6	0.110	– 0.5	0.408	0.6	0.877	1.7	0.688
– 3.0	0.0121	– 1.5	0.127	– 0.4	0.451	0,7	0.904	1.8	0.603
– 2.5	0.028	– 1.4	0.144	– 0.3	0.493	0.8	0.931	1.9	0.507
– 2.4	0.034	– 1.3	0.164	– 0.2	0.538	0.9	0.944	2.0	0.403
– 2.3	0.039	– 1.2	0.186	– 0.1	0.584	1.0	0.950	2.1	0.290
– 2.2	0.045	– 1.1	0.211	0	0.629	1.1	0.947	2.2	0.172
– 2.1	0.054	– 1.0	0.239	+ 0.1	0.674	1.2	0.933	2.3	0.048
– 2.0	0.062	– 0.9	0.268	+ 0.2	0.719	1.3	0.907	2.4	– 0.079

The following integral is of interest for the normalization of the eigenfunctions:

$$\int_{-\infty}^{+2.3} \omega^2 d\xi = 1.54.$$