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CONTINUUM THEORY OF DISLOCATIONS AND PROPER STRESSES

BY

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WITH 39 FIGURES

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Foreword

Seldom has a new idea been so fruitful in its consequences as the discovery of G. I. Taylor, E. Orowan, and M. Polanyi in the year 1934 that the plastic forming of metals resulted with the help of dislocations. This ground-breaking insight has been common knowledge for a long time in mechanics and physics, and countless solid-body phenomena that were completely puzzling before can now be explained with the help of dislocations.

Already in his first work on dislocations, Taylor also recognized that dislocations always give rise to proper stresses, and that by this manner of representation, one can give an explanation for the observed hardening of formed metals that is still true today. In the course of developing a continuum theory, one is led to an extended view of the notion of dislocation, and today one has the law: The dislocation is the elementary source of proper stresses. The task of establishing and elaborating on this law assumes an important place in this report.

The appearance of individual dislocations is a physical phenomenon that can only be understood in terms of the atomic structure of the solid body. The collective action of many dislocations yields the macroscopically-observed plastic changes of form and proper stresses. One must describe this situation by a continuum theory. We would like to call it the "continuum theory of dislocations."

In Chapters I. and II, the standpoint of the ideal continuum will be taken. From this standpoint, the continuum theory of dislocations is an exact theory.

Real bodies are hardly ideal continua. The most important materials – e.g., all metals – are, at least in a small domain, crystalline in structure. In Chapter III, the body considered is therefore a crystal. The application of continuum-theoretical methods to real bodies always means a loss of precision. The fundamental flaws that thus appear in the majority of cases always exist essentially in the face of uncertainties that always appear on physical grounds.

Many problems of solid-body physics relate to the behavior and properties of the individual dislocation. Often, such questions may also be treated with the methods of continuum theory. Therefore, Chapters II and III contain much about the singular dislocation.

One can associate a body with dislocations with a natural state in a non-Riemannian space with torsion and take advantage of the extensive tools of differential geometry. This procedure seems, above all, appropriate when one is no longer concerned with a linear theory. The non-Riemannian geometry of dislocations will be discussed in Chapter IV. Finally, Chapter V gives some applications.

In this report, dynamical problems will only be touched upon lightly. Likewise, we shall do without a treatment of the thoroughly-examined special problems of phenomenological elasticity theory, as was given in, e.g., the recent textbooks of Hill [65] or Prager and Hodge [119].

Today, there are already many excellent papers on dislocations in crystals, and we cite the following works: The presentation of Read [121] that precedes all geometric questions, the more mathematical presentation of Nabarro [110], and the particularly

physical questions that were treated in the works of Cottrell [21], Seeger [134, 135], Friedel [55], Seitz, Koehler, and Orowan [144], Haasen and Leibfried [62]. All of the presentations start from the atomic standpoint. In contrast to them, the continuum aspect of dislocations will be put forth here. Many important results of atomic dislocation theory will thus be passed over or only touched upon. Nevertheless, we cannot, and would not like to, avoid speaking of atoms. The definition of the dislocation thus gains much from intuition. Above all, however, one also often cannot approach the consideration of atomic processes without the methods of continuum theory.

We believe that of the new results in solid-body mechanics precisely the continuum version of dislocations takes on a special meaning: It represents a bridge between the results of the atomic and phenomenological research in plasticity. It can therefore be the connecting link that brings together the two great communities that both endeavor to bring about progress in mechanics: On the one hand, the mechanicians and mathematicians, and, on the other, the solid-body physicists. R. Grammel [60] was recently responsible for bringing about a close cooperation between these two groups. I would also like to continue in that spirit in the present report.

Much support has been afforded me, of which I will only mention the most important examples: In the first place, Herrn Prof. U. Dehlinger must be graciously thanked for his ongoing help and his part in it; in particular, for the discussions of the systematics of the construction of this report. For some conversations relating to, inter alia, the understandability of the presentation, I am deeply grateful to Herrn Prof. R. Grammel. Furthermore, I would like to warmly thank Herrn Prof. E. Fues for his interest and inspiring criticism, Herrn Dr. A. Seeger for the fact that he placed his rich experience in dislocation theory at my disposal, my collaborator, Herrn Dr. G. Rieder, who played an essential part in the development of the theory, for numerous discussions and careful checking of the manuscript, and Herrn Dr. P. Haasen, for reading the corrections and some important suggestions. Furthermore, I would like to warmly thank Herrn Prof. Lösch for making this report possible for and his help with the editing. The writing of the report was supported by der Deutschen Forschungsgemeinschaft and the Max-Planck-Institut für Metallforschung in Stuttgart. Finally, I should mention the good cooperation with the publisher and printer.

Stuttgart, in December, 1957.

E. Kröner

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Introduction

Originally, continuum mechanics, as it was developed by Navier, Cauchy, Poisson, Stokes, et al., existed as the subdomains of elasticity theory and hydrodynamics. In the former, one was interested in the reactions – in particular, the stresses – that would be provoked in a solid body that was "externally" endowed with forces – and possibly also rotational moments. In the latter, one examined, above all, the motion of fluids.

Relatively early on, Duhamel and Neumann also included temperature stresses in elasticity theory. Thus, they always occupied a special position, since they were not subject to the Kirchhoff uniqueness theorem, which states that in the domain of linear elasticity theory the stresses of a simply connected body are uniquely determined by the external forces that act on it. The Kirchhoff theorem is valid only under the assumption that the St. Venant compatibility conditions for the elastic deformations are fulfilled in the entire body. In the case of temperature stresses these conditions are not actually valid, which clarifies the nature of their special position.

In the second half of the 19th Century, plastic behavior in continua was examined by Tresca, de St. Venant, Levy, et al. This "phenomenological" plasticity theory, which was further developed by v. Mises, Prandtl, Reuss, Prager, Hencky, Nadai, et al. later on, stood, to a certain extent, between elasticity theory and hydrodynamics. Thus, the resulting deformation (we also call it the total deformation $\boldsymbol{\varepsilon}^G$) of the plastic body includes an elastic part $\boldsymbol{\varepsilon}$ that gives rise to stresses as it would in ordinary elasticity theory, along with a second part that we refer to as the plastic deformation $\boldsymbol{\varepsilon}^P$, which certainly changes the form of the body, but produces no stresses. One has such deformations in a pure form in fluids. All together, one then has:

$$\boldsymbol{\varepsilon}^G = \boldsymbol{\varepsilon} + \boldsymbol{\varepsilon}^P. \tag{1}$$

Since at least one part of the stresses remains after plastic forming with no external forces, the elastic deformations $\boldsymbol{\varepsilon}$ obviously do not fulfill the compatibility conditions. Here, one sees a commonality between temperature stresses and the proper stresses after plastic forming.

It is essentially possible to single out a volume element (perhaps on the outer surface of the body) before the plastic forming and to measure the deformation that it suffers when compared to the initial state; this would be the deformation $\boldsymbol{\varepsilon}^G$. If one now cuts out the volume element and lets it relax then it will assume its original form, but will have preserved the plastic deformation. Now, just as at the onset, one again finds this element in its "natural" state, which has be employed since Cauchy, Green, et al., in elasticity theory. The element has changed its form, but not its state [1]. A function that says something about the body in question shall then be called a "state function" or a "state quantity" if one can measure it uniquely at a certain time without having to know the history of the body. The part $\boldsymbol{\varepsilon}^P$ of the total deformation is therefore not a state function, but the elastic deformation $\boldsymbol{\varepsilon}$ is. The difference between state functions and the state of unchanged functions is of deep significance and will thus be of interest to us very often.

[1] This statement is then strictly valid only when the plastic forming takes place without (plastic) change in volume; cf., § 2.

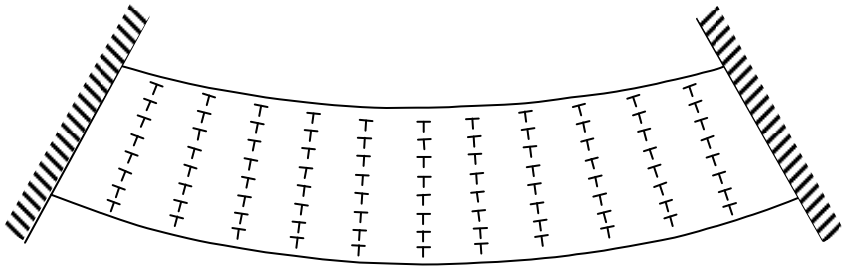

Fig. 1. The symbol ⊤ stands for the edge dislocations; it will first show up in § 23.

It first became obvious in recent years that continuum mechanics, with its three branches of elasticity theory, plasticity theory, and hydrodynamics, has not – at least, in its present scope – succeeded in describing all macroscopically measurable mechanical proper stresses of a body. A simple example may clarify this: Let a beam be attached to rigid walls, and then let it bent elastically or also piecewise plastically into the shape in Fig. 1. The two walls shall, moreover, remain in this shape and the beam warmed up. During the warming, the critical shear stress will be reduced (defined to be the shear stress at which a noticeable flow of the material sets in); i.e., a flow can take place inside of the beam, during which the elastic deformations will gradually be replaced with plastic ones. After sustaining the higher temperature for a sufficiently long time, the beam is again cooled to room temperature and the clamps are removed. One then observes practically no bending of the beam back to its original state, and the change of form will remain. One can cut out volume elements and establish that no (macroscopic) proper stresses are present [2]. Nevertheless, the beam reacts to, e.g., a renewed change of form in a different way from a beam of the same form "without memory." One can now excise an individual volume element of the beam and measure its critical shear stress. One will thus establish that the body is found in a completely determined state of hardening. One can thus characterize the change of state that takes place in another way that can be easily represented in continuum terms. Namely, if one irradiates the beam with Röntgen rays, or, in the event that it also transparent, with visible light, then one establishes diffraction (Beugung) effects that have their origins in a curvature of the original atomic lattice of the beam. An interpretation of the experiment yields that one can measure this curvature uniquely as a function of position, and thus characterize the stress-free curvature of the lattice planes (Netzebenen) for the state of the beam. Up to now, such curvatures were never described in continuum mechanics.

In order to grasp such geometric changes in the body, one must extend the three deformation tensors of eq. (1) by the addition of rotation tensors $\boldsymbol{\omega}^G$, $\boldsymbol{\omega}$, $\boldsymbol{\omega}^P$ to general asymmetric tensors of second rank $\boldsymbol{\beta}^G$, $\boldsymbol{\beta}$, $\boldsymbol{\beta}^P$, which we will refer to throughout as "distortion tensors."

Hardening can still not be correctly treated in continuum mechanics to this day. Moreover, the research of the last twenty years has shown that it has its origin in the

[2] That which are called lattice curvatures much later are linked with the proper stresses that change their signs in the microscopic realm, and thus cannot be established by the excision experiment that was described. These proper stresses, just like the macroscopic ones, lead to hardening (cf., infra).

proper stresses that come about during the plastic forming of the material. In principle, a precise knowledge of the proper stresses shall permit one to calculate the hardening of the material. Furthermore, it has shown that one can relate all of the proper stresses, as well as the lattice plane curvatures (and thus also hardening) to the same physical quantity: the dislocation. It is, moreover, not only definitive of the change of state of the body, but also the part of its total form that does not change state. This obviously permits one to arrive at a continuum theory of dislocations that closes one of the yawning gaps in continuum mechanics. This continuum theory of dislocations shall, above all, include a theory of proper stresses, and furthermore, a theory of stress-free lattice curvatures, as was first presented by Nye [113]. Moreover, it must therefore also describe the connection between the motion of dislocations and plastic forming. Thus, it already reverts to the phenomenological theory of plasticity. One then obtains separately and collectively the theories of elasticity, dislocations, and plasticity as part of a continuum mechanics of solid bodies that encompasses all mechanical phenomena in solid bodies.

It still remains to be said how one can incorporate temperature stresses and other stresses that go back to either external forces or plastic forming. (As a result, we call the stresses magnetostriction and electrostriction.) If one uniformly heats a body to high temperature then its points suffer displacements, without which restoring forces would be aroused; the same is also characteristic of plastic forming. This then suggests that we may regard the case of forming by means of temperature fields as a type of plastic forming; we would like to call it "quasi-plastic." One can then further attribute the temperature stresses to certain "quasi-dislocations" and thus obtain, with Kröner [82], a theory of temperature stresses that is, to a certain extent, a continuum theory of quasi-dislocations. This agreement is not only formally well-founded, but also physically, and it thus seems entirely natural to include the temperature stresses (and the other stresses mentioned above) in the continuum theory of dislocations. The treatment of the especially interesting problem in which one simultaneously has, e.g., temperature stresses and proper stresses after plastic forming will be simplified essentially in this way.

All of the continuum mechanics of solid bodies is now included in just a few equations. For the stationary state these equations are [3]:

$$\operatorname{Div} \boldsymbol{\sigma} + \mathfrak{F} = 0, \qquad \operatorname{Rot} \boldsymbol{\beta} = \boldsymbol{\alpha}, \tag{2}$$

where $\boldsymbol{\sigma}$ ($\boldsymbol{\beta}$, resp.) is the stress tensor (elastic distortion tensor, resp.), and $\mathfrak{F}$ ($\boldsymbol{\alpha}$, resp.) is the density of external forces (dislocation density, resp. – excluding quasi-dislocations). To these, we add the equation for the elastic energy density (= distortion energy function or elastic potential) [4]:

$$e = \frac{1}{2}\, \sigma_{ij}\, \varepsilon_{ij} \tag{3}$$

[3] We think of the boundary surface conditions as contained in these equations when we allow $\mathfrak{F}$ and $\boldsymbol{\alpha}$ to degenerate to superficial objects (and also linear or point-like ones). In the event that one allows external rotational moments, still more equations come about.

[4] We employ the summation convention throughout: Doubly-appearing indices will be summed from 1 to 3. The tensor symbolism that is employed is clarified in the Appendix.

and the material equation, which is generally taken to be Hooke's law for small deformations. The uniqueness theorem for the continuum mechanics of solid bodies may be proved for this case: The stresses and elastic distortions of a body are determined uniquely when one given the external forces $\mathfrak{F}$ and dislocations $\boldsymbol{\alpha}$. From this, it follows immediately: All proper stresses originate in dislocations. In the case of large distortions, this is, however, not true, as the example of the everted hemispherical membrane shows [160].

For the first twenty years, solid-body physicists concentrated their interest on crystalline structures, which most of our materials exhibit; in particular, metals do. In the realm of media that are at least 10^{-3} cm thick (corresponding to, say, 10^{15} atoms!) they are crystalline ("polycrystalline") in their composition. With methods that were developed at that time and have been greatly improved since then, it is now possible to grow almost arbitrarily large "unit crystals" of many materials. They are of great significance in experimental and theoretical research, and have thus found important application in technology – e.g., in the transistors of telecommunications.

The concept of dislocation was first invoked – although in a vague form – in 1928 by von Prandtl [108] for the clarification of inelastic phenomena in metals. By 1929, Dehlinger [29] could show, in the examination of the re-crystallization – i.e., the grain reconstruction (Kornneubilding) – that one observes in strongly plastic forming, which has its origin in the strong proper stresses that thus appear, that these proper stresses lead back to certain defective domains insider the otherwise regular distribution of atoms in crystals, and that these domains can be (meta-) stable. Dehlinger called his proper stress sources "hooks" (Verhakungen); they are nothing but two closely-neighboring dislocations of opposite signs. The manner in which proper stresses are possible at all in a crystalline medium was thus explained. Throughout these examinations, the attention was directed to the perturbation of regular atomic arrangements, especially. One calls such perturbations "lattice defects;" they play a decisive role in contemporary solid body physics.

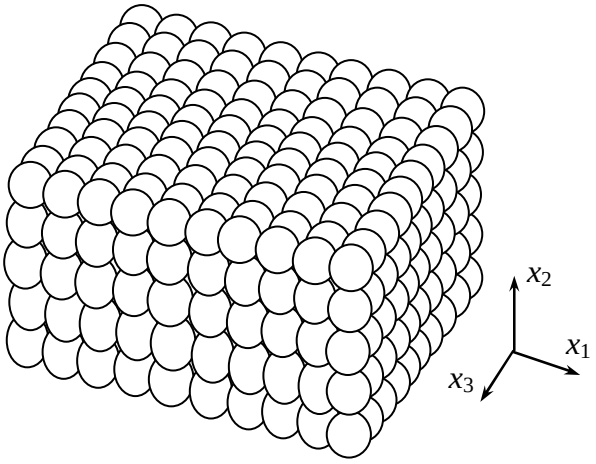

Fig. 2. Ideal crystal, cubic primitive lattice.

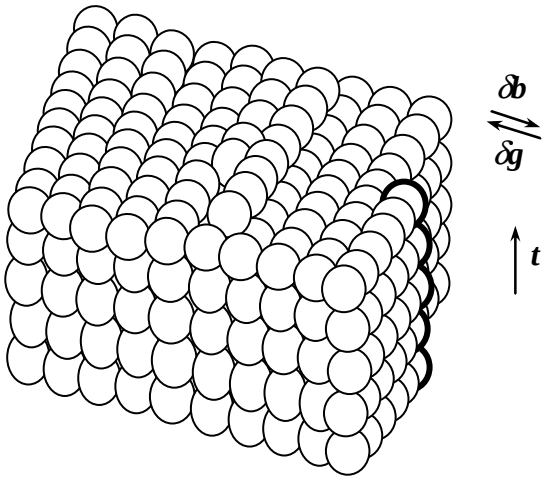

Fig. 3. Crystal of Fig. 2 after the immigration of an edge dislocation along the x_1 direction.

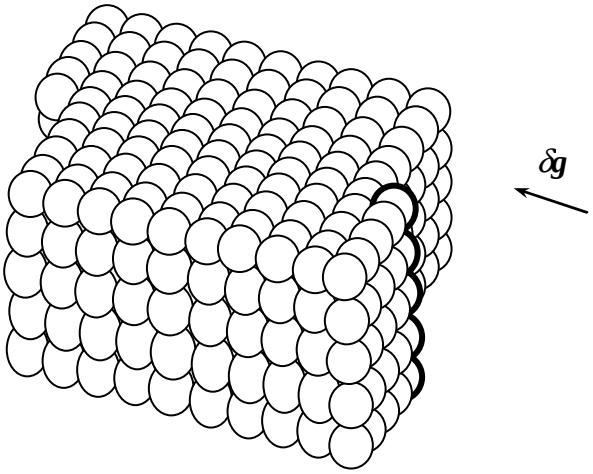

Fig. 4. The edge dislocation of Fig. 3 has migrated through the crystal in the x_1 direction.

In 1934, a lattice defect was described independently by Orowan [114], Polanyi [118], and Taylor [149] that we will explain by means of Figs. 2 and 3. Fig. 2 shows a completely regularly formed crystal called an "ideal crystal." Fig. 3 shows the same crystal after the immigration of a perturbation along the x_1 direction. The perturbation is characterized by saying that a lattice plane has adhered inside the crystal. One calls the boundary line of such an "extra lattice plane" an "edge dislocation line" or simply "edge dislocation." Fig. 4 shows the same crystal after the perturbation has migrated to the left. By means of the one-time drift of the dislocation through the crystal the upper and lower halves of the crystal will be displaced with respect to each other by an atomic distance. We call the vector that gives the relative displacement in the glide plane the "glide vector" $\boldsymbol{\delta g}$; it is perpendicular to the edge dislocation line. If the crystal had a shear stress σ_{31} imposed on it then work would be done during the migration. As a consequence, such a shear stress means a driving force for the dislocation. The cited authors now remarked that the drift of an edge dislocation must be possible for comparatively small

stresses. Fig. 3 gives a certain optical impression of the idea that the atoms bounding the dislocation shall move more easily that the remaining ones.

Already in 1926, Frenkel [54], with the help of an atomic model, had calculated that gliding, such as the transition from the crystal of Fig. 2 to that of Fig. 4 represents, then requires a shear stress of order of magnitude of the shear modulus G when the two lattice planes in question glide over each other rigidly through an atomic distance. Experimentally, a more than thousand-fold smaller critical shear stress was measured. The plasticity mechanism that was proposed by Orowan, Polanyi, and Taylor shall actually lead to a smaller critical shear stress [5].

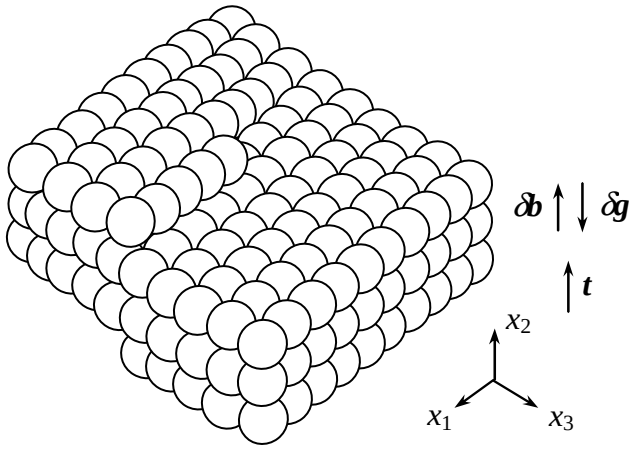

Fig. 5. The upper lattice planes of the crystal in Fig. 2 after the migration of a screw dislocation along x_1 .

In 1939, Burgers [12] had described another lattice defect, which had the effect of saying that the original lattice planes are now connected into a type of screw surface (Fig. 5). The screw axis is now called the "screw dislocation" (s-line). One sees that this screw dislocation can move relatively easily. One can think that it migrates along – say – the x_1 direction, as is Fig. 5. Fig. 6 and 7 show the crystal after the migration of the screw dislocation of Fig. 5 in the $-x_1$ (x_3, resp.) direction. Ultimately, certain parts of the crystal have glided over each other. Here, however, the glide vector is parallel to the screw dislocation line. Burgers has shown further that there are also dislocations whose glide vector is diagonal to the direction of the dislocation line. One often suitably regards

[5] For Dehlinger [31], these purely mechanical considerations were not sufficient for him to prove that a rigid gliding between two lattice planes could not actually occur. One must apply statistical thermodynamical considerations, in addition – in particular, the theorem that in solid bodies only transitions of the lowest order of reaction can take place. Applied to our case, this says: It is extremely improbable that by temperature fluctuations alone the atoms of a lattice plane would simultaneously have such an elevated energy that they then simultaneously make the glide step that a rigid gliding of the lattice plane in question would amount to. Such considerations are indispensable when one would now like to compute the critical shear stress theoretically under the assumption of the dislocation mechanism. Seeger [137] has shown that by ignoring the temperature fluctuations the critical shear stress that one would compute purely mechanically often comes out to be more than 100% too large.

Due to its importance in such problems, let us further mention a new paper of Donth [164], which showed that by a statistical treatment of dislocations one would come to the Kolmogoroff equations for statistical processes since the assumptions for the application of an Arrhenius equation are not true in the case of dislocations.

such dislocations as the combination of a screw and an edge dislocation along the same line, such that dislocations of this type imply nothing fundamentally new.

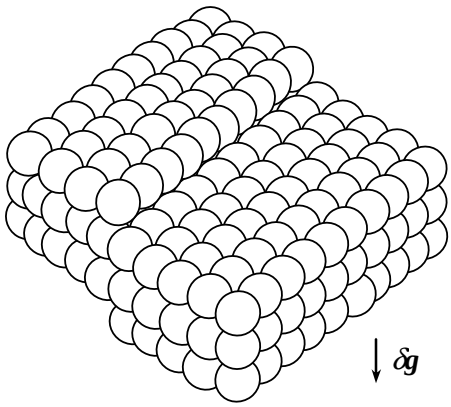

Fig. 6. The screw dislocation of Fig. 5 has migrated through the crystal in the $-x_1$ direction.

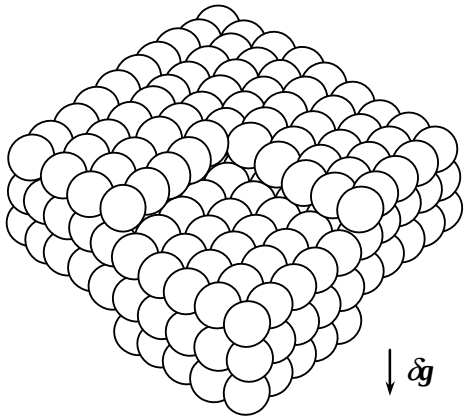

Fig. 7. The screw dislocation of Fig. 5 has migrated through the crystal in the x_3 direction

However, the possibility of motion for dislocations has still not been completely discussed. There still remains the essential possibility of considering a motion of the dislocation in Fig. 3 in the x_3 direction. This means an enlargement of the extra lattice plane, which is only so imaginable in practice, such that atoms in the vicinity of the dislocation will come back to it by diffusion. The change in position of an atom in a crystal always leads to an energy threshold with an order of magnitude of 1 eV ($= 1.63 \times 10^{-18}$ mkg) that cannot be overcome by means of externally applied stresses [6]. Moreover, the temperature fluctuations must first have at their disposal the required "activation energy." As a result, such diffusion can first occur, to a large extent, only at high temperatures. The dislocation motions that play out in this way are called "climbs," as opposed to the aforementioned "glides." Any atom that lies on the extra lattice plane leaves behind a so-called "lattice gap." One must account for these lattice gaps macroscopically in the volume being measured; i.e., the volume of the body changes while the dislocation climbs. This type of motion was thus called "non-conservative" (relative to the volume) by Nabarro [108], while the gliding motion was also called "conservative."

If a dislocation climbs in – e.g. – the x_3 direction completely through the crystal of Fig. 2 then this means that a new lattice plane is formed and thus the crystal will be lengthened in the x_1 direction. Thus, a pure tension (Zugspannung) ($\sigma_{11} > 0$) exerts a pressure on the dislocation to climb in the x_3 direction. On the other hand, a compression (Druckspannung) might break down the extra lattice plane, which is, however, possible only as long as all of the lattice gaps are filled up in the vicinity of the dislocation with atoms of the extra lattice plane. One sees that the volume of a body can also be changed plastically; the theory to be developed in Chapter I includes this possibility.

[6] Macroscopically, 1 eV is a minute energy. However, it must be localized in a space of only some 10^{-24} cm, and this is obviously not possible with externally applied stresses.

Certainly, for many solid-body phenomena, the climbing of the dislocations plays an important role for temperatures that are not too much less than the melting temperature, and we cite re-crystallization and the existence of casting stresses.

A glance at Figs. 2 to 7 shows that one should expect proper stresses in the states of Figs. 3 and 5, while the crystal is found in the natural state in the remaining figures [7]. In § 1, we will show the close relationship between these proper stresses and those of Volterra distortions. Building on the work of Volterra, in 1939 Burgers [12] had created an elasticity theory of singular dislocations in which the proper stresses, which originate in the dislocations, can be calculated. The numerous particular elasticity-theoretic calculations for dislocations have been based on this fundamental work.

Therefore, the following picture of the sequence of events in the plastic forming of metal can be given: Under the influence of externally applied stresses, the dislocations that are already present in the crystal define an additional large set of new dislocations. They correspondingly migrate around under the forces exerted, while they provoke the macroscopically observed changes of form. Thus, due to their increasing number, the dislocations bring about proper stresses of increasing measure that endeavor to inhibit the motion of the dislocations, as was first suggested by Taylor [149]; this effect leads to the hardening of materials.

[7] One distinguishes – e.g. – the states in Figs. 2 and 4 precisely by saying that as a result of the formation of edges the crystal of Fig. 3 has an altered outer surface stress. For our purposes, we do not need to address this; one can confer, e.g., the discussion in Nabarro [110], pp. 332.

Chapter I

Dislocations in a continuum: geometry

§ 1. Dislocations and Volterra distortions

At the beginning of this chapter the close connection between dislocations and Volterra distortions will be clarified.

Let f be a surface that is at least piecewise planar [1] in the interior of a simply connected medium with the (dimensionless) unit normal vector $\boldsymbol{n}(\boldsymbol{x})$ at the position $\boldsymbol{x}$. Let $\boldsymbol{t}(\boldsymbol{x})$ be the unit tangent vector of the boundary line of f, which is oriented by the right-hand screw rule. One thinks of the body in the stress-free initial state as being cut along the surface f such that the positive edge of the cut defined by f is bestowed with the infinitesimal plastic displacement $\delta\boldsymbol{g}(\boldsymbol{x})$ relative to the negative one. We would like to carry out the displacement $\delta\boldsymbol{g}(\boldsymbol{x})$ in two steps by decomposing it into two components: ($\delta\boldsymbol{g}^{\parallel}$), which is parallel to f, and ($\delta\boldsymbol{g}^{\perp}$), which is perpendicular to it. After the parallel displacement $\delta\boldsymbol{g}^{\parallel}$ both sides of the surface f will have no gap between them.

For the displacement $\delta\boldsymbol{g}^{\perp}$ that now follows, one must distinguish two cases: 1. The two sides of f are displaced with respect to each other. For this case, we have at our disposal the possibility that the resulting cavity is then filled with matter of the same type and density as the original body. 2. In the case for which $\delta\boldsymbol{g}^{\perp}$ means a displacement of the two edges of the cut with respect to each other, exactly enough matter from one of the two edges shall be carried along that this displacement is possible. After carrying out this operation, we think of everything as being deformed and the forces that produced the displacement are removed in such a way that once more we have a unitary, simply connected body in which proper stresses naturally remain. They are determined by not only the material and form of the body, but also the position of the surface f – i.e., by $\boldsymbol{n}$ – as well as the resulting "impressed" or "plastic" relative displacement $\delta\boldsymbol{g}$.

We further remark that from a well-known theorem of Colonetti [18] the volume of the body in the final state differs from that of the initial state by the volume of the new compressed (removed, resp.) matter, hence, by $\delta V = \iint_f \boldsymbol{n}\cdot\delta\boldsymbol{g}\,df$. This theorem is valid only in the realm of linear elasticity theory, and therefore only for homogeneous bodies, as well (hence, e.g., not for bodies that consist of two homogeneous parts with different elastic constants).

In general, along the surface f the deformations and rotations of the volume elements of the body change discontinuously, which was first examined by Weingarten [157], and later more thoroughly by Somigliana [147] [2]. Thus, if both cut edges due to a piece Δf of a surface f are merely impressed with a rigid displacement then the deformations are

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[1] The restriction to planar surfaces simplifies the presentation, although it is not necessary. One easily sees that the essential results of this paragraph – in particular, the definition of the dislocation – are also valid in the case of curved surfaces f.

[2] The older results of proper stress theory were referenced by Nemenyi [111]. This paper also contains much that is still worth reading today.

continuous across Δf. That the rotations are also continuous across Δf is, in addition, necessary in order to have $\boldsymbol{\delta g}$ = const across Δf.

Should the body be again simply connected at the end of the operations – hence, nowhere exhibiting a tear – then the cut edges of the entire surface f cannot be rigidly displaced with respect to each other. One comes to the dislocations by the following prescription: Let $\boldsymbol{\delta g}$ be almost constant across the entire surface f, except that on the boundary of f it goes quickly to zero. Fig. 8 shows the vanishing of $|\,\boldsymbol{\delta g}\,|$ over a (planar) surface f, which, for the sake of simplicity, is assumed to have a circular boundary. We now define the dislocation line to be the boundary line of the surface f, or, more precisely, the dyadic product $-\,\boldsymbol{t}\,\boldsymbol{\delta g} \equiv -\,(t_i\,\delta g_j)$, where $\boldsymbol{\delta g}$ shall mean the constant displacement over the greater part of the surface [1].

Still more precisely, one must say: Here, there is not a singular line $-\,\boldsymbol{t}\,\boldsymbol{\delta g}$, but a quasi-singular strip of very small width 2ζ (Fig. 8). We thus extend the above definition by adding that this shall be valid in the limit as $\zeta \to 0$ [2].

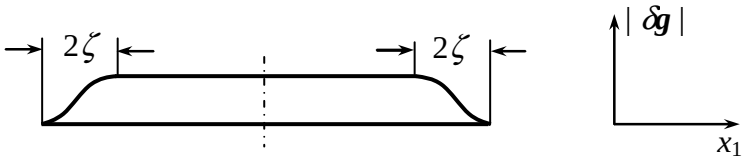


Fig. 8. The production of a dislocation in a continuum.

A second prescription leads to the Volterra distortions: One establishes that from now on (or, at the latest, after making the cut along f) the boundary line of f is given by a hollow torus of radius $\zeta > 0$. The body is then generally no longer simply connected, and the surface f has its entire boundary in the bounding surface of the body. Thus, a rigid displacement of the cut edges of the entire surface f is possible. If one sets $\boldsymbol{\delta g}$ = const. then one obtains a so-called Volterra distortion state of the first kind, which, upon the bulk removal of the hollow torus, one obviously cannot distinguish from the state produced by a dislocation (principle of St. Venant). The Burgers elasticity-theoretic treatment of dislocations is based upon this knowledge.

We will come to speak of the Volterra distortion state of the second kind, for which the rigid relative displacement is a rigid rotation of the cut edge, in § 7. From our standpoint, it does not have the same meaning as the state characterized by $\boldsymbol{\delta g} = 0$.

From the definition of dislocation, it follows that:

1. The dislocation, as the boundary line of a surface, can end only on the bounding surface of the body.

2. Since the deformation and the rotation of the surface f happen continuously, they can no longer be experimentally determined, from the definition of the dislocation lines. All surfaces that are bounded by $\boldsymbol{t}$ can therefore serve as cut surfaces to produce the dislocation (distortion, resp.) state. That is, this is already completely determined by the boundary line $\boldsymbol{t}$ and the relative displacement $\boldsymbol{\delta g}$.

---

[1] The minus sign is chosen by convention, in order to be in agreement with the most-used sign convention of Frank [47] (§ 21).

[2] The function $\boldsymbol{\delta g}$ then takes on the character of a Heaviside step function in the plane in which f lies.

Assume that during the operation of relative displacement a stress that arises from an external force was imposed upon the body. This stress could then do work under the displacement. As a result, stresses exert a force on the body, in the sense of the creation and spreading out of dislocations. In particular, if there is a transverse stress in a plane then there exists a tendency towards the conservative formation and spreading out of a dislocation (i.e., $\delta\boldsymbol{g} \parallel$ to the surface), while a normal stress that is perpendicular to the plane means a tendency towards non-conservative formation and spreading out of a dislocation ($\delta\boldsymbol{g} \perp$ the surface). Whether such behavior would actually be implied by the application of external stresses on the body alone would depend on the cohesive forces of the matter. In particular, a diffusion of matter would be necessary under the non-conservative formation and spreading out of a dislocation. In the introduction it was described how this behavior was the basic mechanism for the plastic forming of real bodies. We therefore also assume this for our ideal continua.

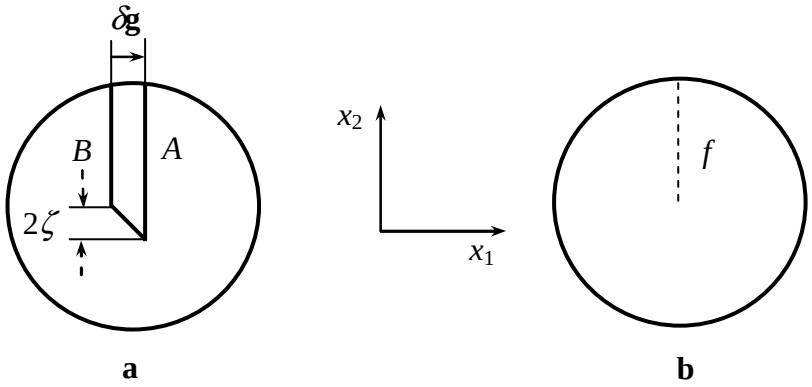


Fig. 9. On the production of a straight edge dislocation in a medium. One thinks of the slit in **a** as arising from the removal of matter from a solid cylinder.

In imitation of the introduction, we call the conservative spreading out of a dislocation a *glide* and the associated plane f the *glide plane*. We call the non-conservative spreading out a *climb* and the associated plane f the *climb plane*. Collectively, we speak of the *drift* of a dislocation along its *drift surface*. We further say that a dislocation has an *edge* character where $\boldsymbol{t} \perp \delta\boldsymbol{g}$ and a *screw* character where $\boldsymbol{t} \parallel \delta\boldsymbol{g}$. Figs. 9 and 10 show the production of pure edge and screw dislocations. Obviously, the purely non-conservatively formed dislocation is an edge dislocation. This corresponds to the assertion that we encountered in the introduction that in a crystal only edge dislocations climb. The conservatively formed dislocation has, in general, a mixed character. This corresponds to the finding that in a crystal screw dislocations, as well as edge dislocations, glide. These processes show clearly that the notion of dislocation that is used here is nothing but a conversion of the notion of dislocation from crystals to continua.

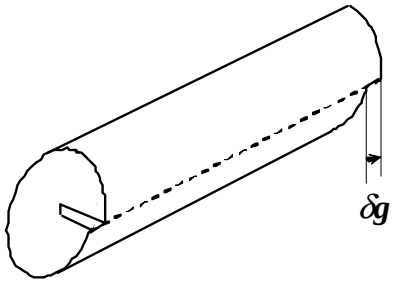
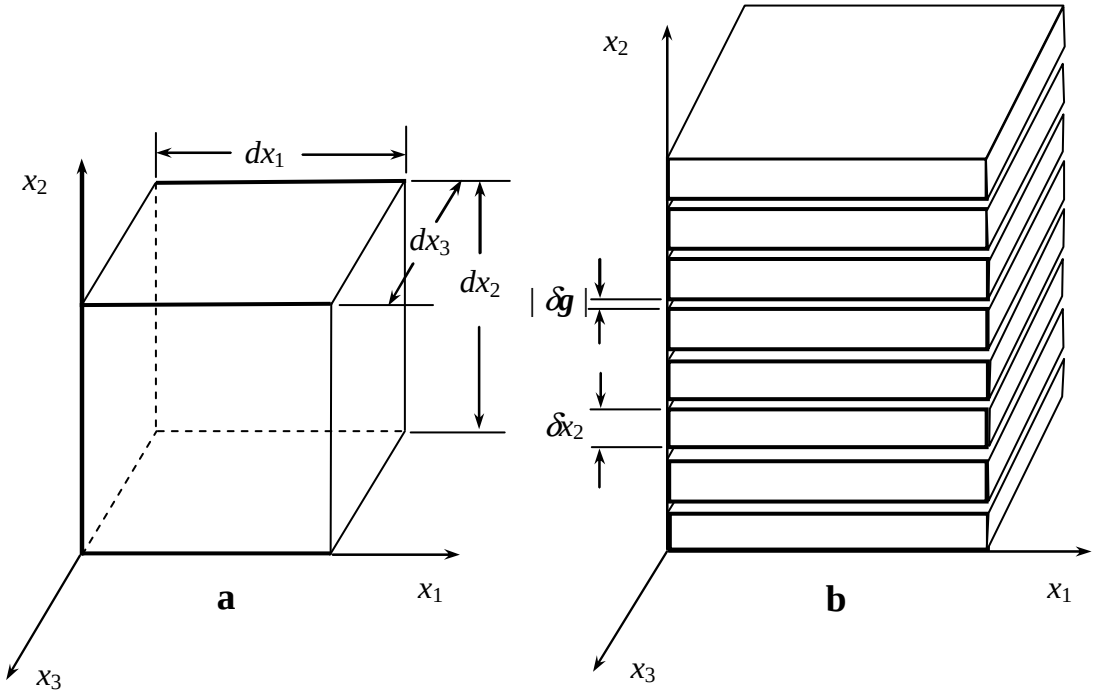


Fig. 10. On the production of a screw dislocation in a continuum

§ 2. Plastic and elastic distortion

First, a remark on the ideal continuum: For the sake of simplicity, let it be assumed that in the initial state it is homogeneous. By contrast, it would be a reduction of the fundamental significance if it were also assumed that it is isotropic. Here, we do not mean the elastic isotropy; that would be completely inessential for the geometrical examination of this chapter. One must, moreover, consider the possibility that the medium is geometrically anisotropic. This means that at each point of the medium three linearly independent distinguished directions exist for which it is assumed that their angles with respect to any three normal directions in space can be measured. This sort of geometrical structure must be required because real bodies, to which the continuum theory will be applied later on, have this structure. One knows this from – e.g. – Röntgenography.

We assume that this structure is a property of the individual volume elements in the continuum. The initial state will be defined to be the stress-free state of the medium, in which the distinguished directions of all volume elements are parallel to each other. In the final state, one then has a certain orientation distribution, which is evidenced by the rotation of volume elements (cf., *infra*) that occurs. For the sake of simplicity, we assume that the distinguished directions are orthogonal to each other in the initial state. The reader who wishes to think in terms of crystals may imagine the continuum to be, perhaps, a cubic primitive crystal with vanishing lattice constants.



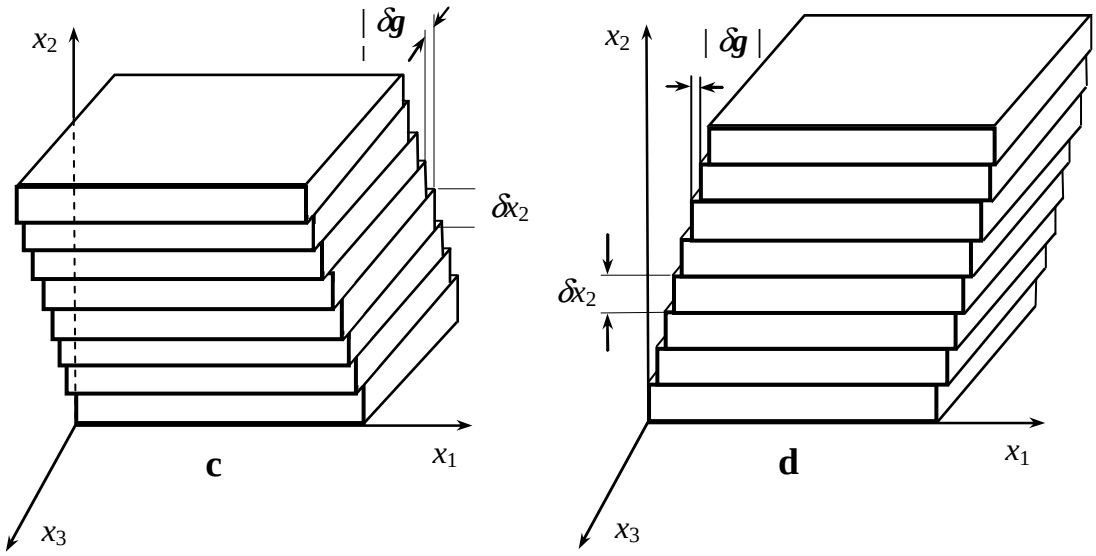


Fig. 11. On the definition of the macroscopic tensor of plastic distortions.

One can now find the operations described in the last paragraph taking place on very many surfaces f. If they are infinitely dense and the associated relative displacement $\delta \boldsymbol{g}$ is distributed continuously then one can, in this way, realize continuously-distributed purely plastic – or also mixed plastic-elastic – changes of form of the body. The first process may be envisioned as in Fig. 11. This shows an isolated volume element dV in the initial state (*a*). It shall be cut from the surfaces df at a distance δx_2 that is perpendicular to the x_2-direction, and then each two neighboring layers will be given the relative displacement $\delta \boldsymbol{g}$. One imagines that the passage to the limit $\delta x_2 \to 0$, $\delta \boldsymbol{g} \to 0$ has been carried out while keeping $\delta \boldsymbol{g} / \delta x_2$ constant. In the case of Fig. 11b, the gaps are filled with matter of the same type as the volume element in such a way that the density distribution remains homogeneous in this way. At the end of this, everything is again deformed. In total, the volume element is then stretched homogeneously and plastically in Fig. 11b (and thus "thinned"), and sheared homogeneously and plastically in Fig. 11c and d.

We generally let dg_j denote the (plastic) relative displacement of the bounding surfaces of the volume element on the + x_i-face with respect to the one on the – x_i-face and define the asymmetric tensor of plastic distortions $\boldsymbol{\beta}^P \equiv (\beta_{ij}^P)$ by the relation:

$$dg_j = \beta_{ij}^P \, dx_i \,, \tag{I.1}$$

where dx_i characterizes the relative distance of the bounding surfaces in question, and will relate to the initial state. The plastic distortions, which correspond to Fig. 11b to d, are then denoted by $\beta_{22}^P, \beta_{23}^P, \beta_{21}^P$. The diagonal components of the plastic distortion tensor β_{ij}^P are then plastic dilatations and the remaining components are plastic shears, where the first index gives the glide plane and the second one, the glide direction.

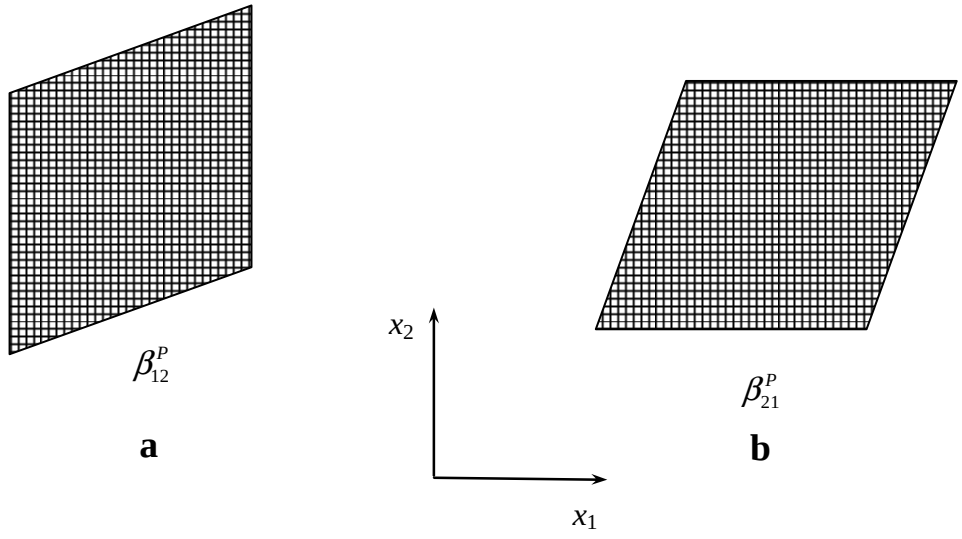
It is now particularly important to remark that under plastic distortion the volume element does not change. One concludes this from the way in which the distortions in Fig. 11 come about [1]. Thus, the difference between the shears β_{21}^P and β_{12}^P is not, perhaps, a rigid rotation, but a "plastic rotation" of the volume element, with preservation of its orientation (Fig. 12a, b). This statement is true for small distortions. The symmetric part of β_{ij}^P then describes a pure plastic deformation ε_{ij}^P, and the anti-symmetric part, a pure plastic rotation ω_{ij}^P, both of which preserve the orientation. The decomposition of the distortions into deformation and rotation [2]:

$$\beta_{ij}^P = \varepsilon_{ij}^P + \omega_{ij}^P \tag{I.2}$$

is also true for large distortions. Here, one must, however, understand ω_{ij}^P to mean the well-known asymmetric tensor for large rotations (versor) [34] [3].

The forming of the volume element that was considered up till now comes about in a stress-free way. We then come to the case of elastic forming. Let da_i be the elastic relative displacement of the bounding surfaces as before. We then define the asymmetric tensor of elastic distortions $\boldsymbol{\beta} \equiv (\beta_{ij})$ by the equation:

$$da_j = \beta_{ij}\, dx_i \,. \tag{I.3}$$



[1] One thinks of the volume element in Fig. 11a as – say – an infinitely densely-packed family of material lines that run in the x_2-direction. The operations that lead from Fig. 11b to d obviously do not change the direction of this line. One also simply postulates the preservation of orientation, because real bodies, to which the theory will later be applied, exhibit this property.

[2] The additive combination of deformation and rotation is valid for large distortions only when dx_i in eq. (I.1) relates to the initial state. cf., § 10.

[3] Bd. I, pp. 78.

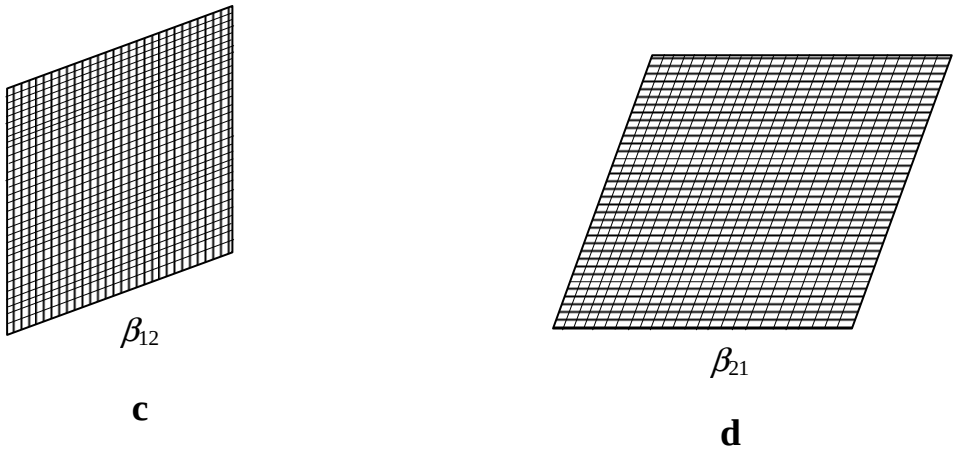


Fig. 12. Under plastic distortion, the orientation remains preserved (**a**, **b**). Under elastic distortions, it will generally be distorted and rotated (**c**, **d**).

The β_{ij} describe the same changes of form and the position of the volume element as the β_{ij}^{P}, although there exists an essential difference: Under elastic shears, the original right angle between the preferred directions in question will change into the shear angle. As a result, the difference between β_{21} and β_{12} for small distortions is now a rigid rotation of the volume element (Fig. 12c, d). If one again decomposes β_{ij} into symmetric and anti-symmetric parts:

$$\beta_{ij} = \varepsilon_{ij} + \omega_{ij}, \tag{I.4}$$

then ε_{ij} is the ordinary deformation tensor of elasticity theory and ω_{ij} is the tensor that describes the (rigid) rotation of the volume element. In the case of large distortions, they are the same as before.

It creates no fundamental difficulty to measure the elastic deformation of a volume element in the final state if one cuts it out and lets it relax. The preferred directions are thus again orthogonal to each other, and one can, in addition, measure the orientation of the element with respect to a normal orientation. If one does this for all elements then one can give the rotations that occur in them the same constant rotation with respect to one of them. This means that the elastic deformation is a state function, while the same is not true for the rotations, but only for their positional derivatives. This obviously describes a curvature of the structure. Since the elastic deformations and structure curvatures follow uniquely from the elastic distortion tensor, this characterizes the state of the medium after forming uniquely. By contrast, it is impossible to measure the plastic distortions, deformations, or rotations that occur in terms of just the final state. This is due to the fact that the state of a volume element will not change under a pure plastic distortion as in Fig. 11. One can also confer the introduction.

In the general case, a volume element will be simultaneously plastically and elastically distorted. Let:

$$ds_j^G = da_j + dg_j \tag{I.5}$$

be the total relative displacement of the bounding surfaces of the volume element, as before. We then define the tensor of total distortions $\boldsymbol{\beta}^G \equiv (\beta_{ij}^G)$ by the equation:

$$ds_j^G = \beta_{ij}^G dx_i \,. \tag{I.6}$$

It is sufficiently characterized by the relation:

$$\beta_{ij}^G = \beta_{ij} + \beta_{ij}^P \,. \tag{I.7}$$

Eq. (I.7) is also correct in the case of large distortions if one always refers dx_i to the initial state (§ 10).

§ 3. The fundamental geometric equation of continuum mechanics for solid bodies

We describe a Gedanken experiment that must serve as the basic experiment of the continuum theory of dislocations [1].

If one externally imposes a sufficiently large stress on a plastic medium then dislocations can be created and drift, and thus produce plastic distortions of the volume element of the body. These dislocations can come out – e.g., on the bounding surface of the body – while there can also be dislocations of the opposite sign inside the body that are annihilated, as well as dislocations that remain in place in the medium during their resulting drift and define a "dislocation density." We assume that these dislocations stay in place between the volume elements, not inside them. Since the magnitude of the volume element shall go to zero, we also then obtain, in any case, macroscopically continuous distribution functions of the dislocations if the externally imposed stresses are continuous. As one also assumes in continuum mechanics, the distortions shall be noticeably homogeneous in the domain of many volume elements dV, which means that the dislocations inside of such domains move rectilinearly.

After imposing the stresses upon the body, each volume element will now suffer a completely determined dislocation drift, and one can describe this, at least numerically, as a function of the position of the volume element, relative to – say – the initial state (§ 4). From now on, we imagine that the body is in its initial state and has been cut up into its volume elements so that each element is independent of the other elements in the dislocation drift that follows for it. In other words: We imprint each element with a plastic distortion $\boldsymbol{\beta}^P(\boldsymbol{x})$. Then the elements are, in any case, free of stresses, and their orientations remain preserved. There are now two possibilities:

1. The volume elements fit together with no gaps under the plastic distortion; there is no rip anywhere. One can then think of them as all deforming again without pressure and obtain the body in the state that it was also given in, if one did not cut them apart

[1] Experiments of this type, in which the deformations alone (but not the rotations) are considered, are described many times in the literature. Cf. Föppl [44], Reissner [122], v. Laue [87].

before the dislocation drift. In particular, the body is also free of proper stresses and structure curvatures. Thus, the state of the body has not changed during the dislocation drift, but only the form.

2. The volume elements no longer fit together after forming. Fig. 13 shows an example in which the connection of the elements was perturbed in such a way that dislocations that stay in place between the volume elements are extended perpendicular to the plane of the paper and migrate from above and left, such that the upper elements meet up with more dislocations and would thus be sheared more strongly than the lower ones, while, on the same grounds, the elements to the left would be more strongly stretched than the ones to the right. If one would unite the elements into a compact whole then one would have to distort it elastically in such a way that they again fit together without gaps. In general, elastic deformations and rotations are necessary for this. Under the former, stresses build up, while the latter give rise to rotations of the orientation. One now thinks of everything as again deforming and the forces that have brought about the elastic deformations are taken away. A relaxation of the body into a state of lowest possible elastic energy will then occur, in general. Thus, the stresses can vanish completely when nothing but rigid rotations of the volume elements have sufficed to again produce the connection that was disturbed by the plastic distortion. At the end of it, one naturally has arrived at the same state as one also obtains when one does not cut the body apart before the dislocation drift.

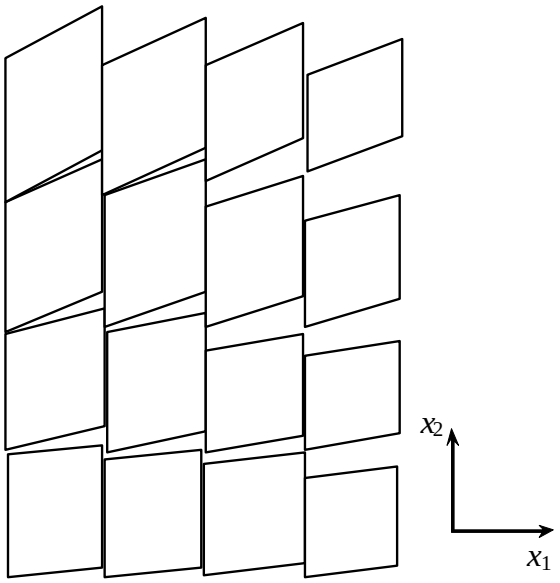


Fig. 13. A plastic distortion that does not undergo simultaneous elastic distortion generally disturbs the connectivity of the body.

This Gedanken experiment must now be evaluated quantitatively. Both possibilities have in common that the body shall be compact and show no rips in its final state. This means that in any case the total distortion $\boldsymbol{\beta}^G$ is a function of position such that the connectivity of the volume elements is preserved. This requirement means a restriction for the additional functions $\boldsymbol{\beta}^G$ that the functions $\boldsymbol{\beta}^P$ are not, however, subject to, at least in case 1, but not case 2. We now show that:

$$\left(\varepsilon_{ijk} \frac{\partial}{\partial x_j} \beta_{kl}^G \right) \equiv \text{Rot}\, \boldsymbol{\beta}^G = 0 \tag{I.8}$$

is a necessary condition for the connectivity relationships between the volume elements to be unchanged. Fig. 14a shows two elements in the initial state. The connection between the two then remains preserved when the right-hand boundary surface of the left-hand one and the left-hand boundary surface of the right-hand one suffer precisely

the same displacement. This means that the components β^G_{2j} and β^G_{3j} in both elements must be the same, while the components β^G_{1j} may vary. As an example, Fig. 14b and c show how it appears when the elements suffer a different distortion β^G_{21} (β^G_{12}, resp.). Thus, it follows that $\partial\beta_{2j} / \partial x_1 = \partial\beta_{3j} / \partial x_1 = 0$ is necessary for the preservation of the connectivity [1]. One immediately concludes the necessary condition (I.8) from this. The ds^G_j in eq. (I.8) is then obviously a complete differential; i.e., there exists a function $\boldsymbol{s}^G$ that measures the total displacement of the points of the body (up to a rigid translation). One then has:

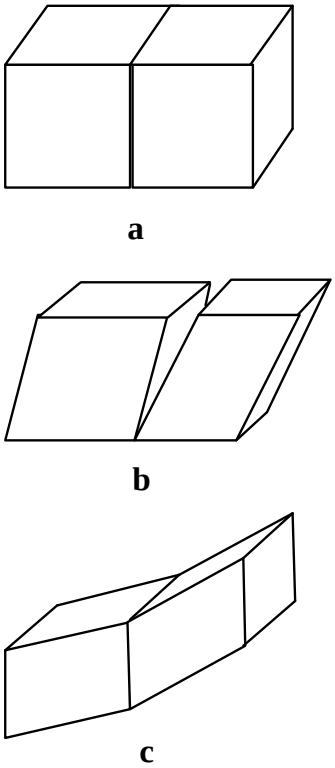


Fig. 14. (Coordinate system as in Fig. 13.)

$$\beta^G_{ij} = \partial s^G_j / \partial x_i \equiv (\text{Grad}\, \boldsymbol{s}^G)_{ij} \,. \qquad \text{(I.9)}$$

In the case 1 described above, since $\boldsymbol{\beta} = 0$, one has $\boldsymbol{\beta}^P = \boldsymbol{\beta}^G$; hence, the dg_j in eq. (I.1) will become a complete integral and:

$$\boldsymbol{\beta}^P = \text{Grad}\, \boldsymbol{g} \equiv \text{Grad}\, \boldsymbol{s}^P. \qquad \text{(I.10)}$$

In this case, one thus obtains a pure plastic displacement $\boldsymbol{s}^P$ of the points of the continuum under which this state will not be changed. This case is of considerable practical meaning for plastic forming; we shall come back to it later.

Henceforth, we heuristically define the asymmetric tensor density $\boldsymbol{\alpha} \equiv (\alpha_{ij})$ by the expression:

$$\boldsymbol{\alpha} \equiv - \text{Rot}\, \boldsymbol{\beta}^P, \qquad \text{(I.11)}$$

and then show, first of all, that this definition is consistent with that of the individual dislocation in § 1.

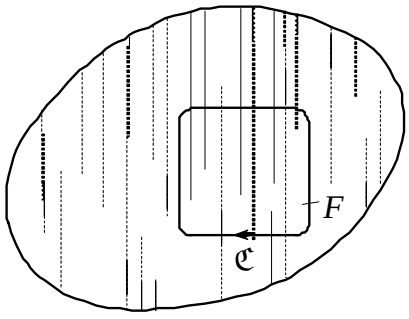


Fig. 15

Fig. 15 shows a body in which a small number of dislocations drift about, of which, we assume that they move perpendicular to the plane of the paper. The drift surface of the dislocations is drawn, and indeed, removed (ausziehen), if the dislocation that sits at the end of each drift surface runs through the surface F with the arbitrarily-oriented boundary line $\mathfrak{E}$; in the other cases, it is sketched

[1] It is then sufficient when one assumes continuous total displacements; we will not go further into this. One can refer to a non-vanishing function Rot $\boldsymbol{\beta}^G$ as a "rip density." One obtains such a thing, e.g., during the rolling of a sample of metal when it was chosen too big.

with dashed lines. The drift surface is oriented in such a way that it will point in the positive direction of $\mathfrak{C}$. Now, we go around $\mathfrak{C}$ once and add to each drift surface the relative displacement $\delta\boldsymbol{g}$ of its positive side with respect to its negative one that originates in the dislocation drift. For the sake of simplicity, we assume that $\delta\boldsymbol{g}$ is equal for all drift surfaces [1]. One immediately sees that the dashed surface does not contribute to the sum, since it produces two equal and opposite contributions. As a result:

$$\boldsymbol{b} \equiv -\sum_{\mathfrak{C}} \delta\boldsymbol{g} \tag{I.12}$$

is a direct measure of the number and type of the dislocations that run through the surface F. We refer to $\boldsymbol{b}$ as the "total Burgers vector" of these dislocations [2]. In the case where the circuit $\mathfrak{C}$ encircles only one dislocation $\boldsymbol{b} = -\delta\boldsymbol{g}$ is the Burgers vector of this individual dislocation.

We showed in § 1 that the state of a medium with one dislocation is determined completely by being given the line of motion $\boldsymbol{t}$ and the glide vector $\delta\boldsymbol{g}$. We now see that one can also give the Burgers vector in place of the glide vector. One observes the essential difference between these two vectors: $\delta\boldsymbol{g}$ assumes that for a dislocation that drifts along a surface the positive side of that surface will be displaced relative to the negative one by $\delta\boldsymbol{g}$. When no dislocations are drifting one has, by contrast, $\delta\boldsymbol{g} = 0$. $\delta\boldsymbol{g}$ is then a vector that is bound to the drift surface of the dislocation, and, in particular, also preserves its meaning when the dislocation leaves the medium, hence, no longer exists, at all. On the contrary, $\boldsymbol{b}$ is defined only in conjunction with the circuit $\mathfrak{C}$ (the surface F encircled, resp.) and thus says something about the distribution of the dislocations in the body.

In the case of sufficiently densely distributed dislocations, one can replace the sum in eq. (I.12) by an integral:

$$\boldsymbol{b} \equiv -\oint_{\mathfrak{C}} \delta\boldsymbol{g} \,. \tag{I.13}$$

If we are dealing with an infinitesimal surface ΔF [3] then we call the resulting Burgers vector $\Delta\boldsymbol{b}$. If one knows this for each arbitrary oriented surface element ΔF at each point of the medium then one obviously knows how many dislocations of each type run through each point of the medium. The expression:

$$\alpha_{ij} = \Delta b_j / \Delta F_i \tag{I.14}$$

shall thus be defined as the "tensor of dislocation density" or briefly, as the "dislocation tensor." Since the dislocation density is a tensor field, it is sufficient to know the Burgers vector at each point for three surface elements that are oriented perpendicular the

[1] This implies no restriction of generality in our considerations, as the following calculation confirms.

[2] In honor of J. M. Burgers, who introduced the circuit vector $\boldsymbol{b}$ to characterize a dislocation in the groundbreaking paper [12].

[3] ΔF must be essentially larger than than dF in order for a dislocation density to be defined by ΔF. If one first lets dF go to zero then one can, in addition, also take the limit $\Delta F \to 0$.

Cartesian coordinate axes. If one measures a Burgers vector – e.g., for the surface ΔF_1 – and it has the single component ΔB_1, while ΔF_2 and ΔF_3 give no contribution, then one obviously has the Burgers vector parallel to the line direction, and from § 1 the diagonal components of α_{ij} thus represent screw dislocations in the i (= j) direction. Correspondingly, one recognizes that the remaining components of α_{ij} mean edge dislocations in the i-direction with their Burgers vector in the j-direction. In total, the first index of α_{ij} gives the direction of the line and the second one, the direction of the Burgers vector. We also call the total Burgers vector of all the dislocations that go through an arbitrary surface F the "dislocation flux" through F; it obviously results from eq. (I.14) that:

$$\boldsymbol{b} = \iint_F d\boldsymbol{F} \cdot \boldsymbol{\alpha} \,. \tag{I.15}$$

On the other hand, we compute it from eq. (I.13) to be:

$$\boldsymbol{b} = -\oint_{\mathfrak{C}} \delta \boldsymbol{g} = -\oint_{\mathfrak{C}} d\boldsymbol{g} = -\oint_{\mathfrak{C}} d\boldsymbol{x} \cdot \boldsymbol{\beta}^P = -\iint_F d\boldsymbol{F} \cdot \mathrm{Rot}\, \boldsymbol{\beta}^P \,. \tag{I.16}$$

In this, we have used the fact that $\int \delta \boldsymbol{g}$, when integrated over a path dx_i, naturally yields $d\boldsymbol{g}$ (Fig. 11); this is substituted in eq. (I.1) and finally Stokes's theorem is applied. Since the surface F was arbitrary, one concludes the comparison with eq. (I.15) directly from eq. (I.11).

From eq. (I.7), (I.8), and (I.11), what immediately follows is the "basic geometric equation of continuum mechanics" [1]:

$$\mathrm{Rot}\, \boldsymbol{\beta} = \boldsymbol{\alpha}. \tag{I.17}$$

From the previous statements, this is to be understood as follows: If a dislocation drift (plastic distortion $\boldsymbol{\beta}^P$, resp.) takes place in such a way that the dislocations remain in place with a density $\boldsymbol{\alpha}$ in the medium then the distortion $\boldsymbol{\beta}^P$, when it happens by itself, will disturb the connectivity of the body. The cohesive forces of the medium oppose this and simultaneously build up a elastic distortion $\boldsymbol{\beta}$ in such a way that the body remains compact. Eq. (I.17) is also true for large distortions, as long as one refers $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ to the initial state, as well as performing the differentiations in the initial state. Cf., § 10.

From eq. (I.17), what follows immediately is the relation that first given by Nye [13]:

$$(\partial \alpha_{ij} / \partial x_i) \equiv \mathrm{Div}\, \boldsymbol{\alpha} = 0. \tag{I.18}$$

---

[1] Eq. (I.17) or, equivalent formulations, were given independently by Kondo [73, 74], Bilby, Bullough, and Smith [3, 4, 5], and Kröner [81, 82, 84]. The first-mentioned author gave formulations that were also valid for large distortions from the outset (cf., § 26 to 28), while, for the present author such distortions were included later on. The derivation of eq. (I.17) that was given here is due to Kröner and Rieder, which reads, in Cartesian Coordinates:

$$\partial \beta_{31} / \partial x_2 - \partial \beta_{21} / \partial x_3 = \alpha_{11} \,,\ \partial \beta_{32} / \partial x_2 - \partial \beta_{22} / \partial x_3 = \alpha_{12} \,,\ \partial \beta_{33} / \partial x_2 - \partial \beta_{23} / \partial x_3 = \alpha_{13} \,, \text{ etc.}$$

Since the first index of α_{ij} gives the direction of the line, this equation obviously means nothing but the statement that the dislocations cannot end inside of a medium. We have already brought this up in § 1.

From eq. (I.5), it follows that $d\boldsymbol{s}^G$ is a complete differential, as well as the relation $\oint d\boldsymbol{a} = -\oint d\boldsymbol{g}$ for an arbitrary closed circuit. Thus, it also follows from eq. (I.13) that:

$$\boldsymbol{b} = \oint_{\mathfrak{C}} d\boldsymbol{a} = \oint_{C} d\boldsymbol{x} \cdot \boldsymbol{\beta} \,. \tag{I.19}$$

Burgers [12] introduced the circuit vector $\boldsymbol{b}$ in this form.

Finally, we further mention the far-reaching analogy that exists with the theory of magnetic fields of stationary currents, and which has been of valuable service to the discovery of the basic geometric equation [81]. The analogous quantities are:

Electric current strength i and Burgers vector $\boldsymbol{b}$,

Current density $\boldsymbol{j}$ and dislocation density $\boldsymbol{\alpha}$,

Magnetic field $\boldsymbol{H}$ and distortion field $\boldsymbol{\beta}$.

For later purposes, we add:

Magnetic induction $\boldsymbol{B}$ and stress field $\boldsymbol{\sigma}$.

The equations that are analogous to eq. (I.15), (I.17), and (I.18) are $i = \iint d\boldsymbol{F} \cdot \boldsymbol{j}$, rot $\boldsymbol{H} = \boldsymbol{j}$, div $\boldsymbol{j} = 0$.

§ 4. Dislocation drift and plastic distortion

The basic geometrical equation (I.17) includes only state quantities and is therefore suitable for the description of the state after plastic forming. Thus, what is still missing is a quantitative description of the connection between the dislocation drift and the aforementioned plastic distortion. Obviously, no state quantities will appear in the equation that defines this.

One can think of the distortions of Fig. 11 as being produced by dislocations whose line direction was the x_3-direction, and which drift in the x_1-direction. The direction of the associated Burgers vector in Fig. 11b, c, d was then x_2, x_3, x_1. (We ignore the sign, for the moment). A complete examination must consider the nine independent dislocation components and the three independent drift directions. Thus, there are 27 different dislocation drifts to examine.

We describe a general dislocation drift by giving 27 quantities N_{ijk} at each position $\boldsymbol{x}$, which means the number of α_{jk}-dislocations at $\boldsymbol{x}$ that drift in the i-direction (per unit length perpendicular to the line and measured in the drift direction). Thus, for the sake of simplicity, we assume that all of the dislocations have the same magnitude b for the Burgers vector $\mathbf{b}$, although it is not more difficult to treat the case of differing Burgers vectors.

The indices of N_{ijk} thus mean:

1. Index of the drift direction
2. Index of the line direction
3. Index of the Burgers vector direction

} for the dislocations.

$j = k$ are screw dislocations, $j \neq k$ are edge dislocations. Then, as would follow from the considerations at the end of § 1, one also has the correspondences:

1. $i \neq j = k$ the glides of the screw dislocations
2. $k = i = j$ the glides of the edge dislocations
3. $i \neq j \neq k, k \neq i$ the climbs of the edge dislocations.

$i = j$ is a drift of the dislocations in the direction of their lines and yields no distortion. We thus do not need to consider this drift.

Thereby, all 27 components of N_{ijk} are accounted for. From the vector property of the drift direction and the tensor property of the dislocations, it follows that the N_{ijk} are the components of a tensor of rank three: It shall be called the "dislocation drift tensor."

Furthermore, we write $1 \equiv -1$, etc. Then, e.g., an N_{123} drift yields the same distortion as an $N_{\bar{1}\bar{2}3}$, $N_{1\bar{2}\bar{3}}$, $N_{\bar{1}2\bar{3}}$ drift. The free choice of the positive sides of the drift surface leads us to make them the $+x_i$-sides in this paragraph. The drifts that the plastic distortions provoked in Fig. 11b to d are then:

b: $N_{1\bar{3}2}$ ($N_{\bar{1}32}$, resp.) or also N_{312} ($N_{3\bar{1}2}$, resp.)
c: $N_{1\bar{3}3}$ ($N_{\bar{1}33}$, resp.) or also N_{313} ($N_{3\bar{1}3}$, resp.)
d: $N_{1\bar{3}1}$ ($N_{\bar{1}31}$, resp.) or also N_{311} ($N_{3\bar{1}1}$, resp.).

The givens are complete, and the reader is encouraged to check them on the basis of the sign convention [1] of § 1. If one gives the edge of the volume element the length 1 then $N_{ijk}\,\delta g$ is numerically equal to the value of the total glide vector that is provoked by the dislocations that drift through the volume element and thus also numerically equal to the distortion that the drift N_{ijk} provokes. Henceforth, we would like to denote this distortion by β^P_{ijk}.

One removes our examples of dislocation drifts that are anti-symmetric with respect to the first two indices and yield the same distortion. Since, as we remarked before, no distortion belongs to the drift N_{ijk}, one now obtains (Kröner and Rieder [84]) ($\delta g = -b$):

$$\beta^P_{ijk} = -(N_{ijk} - N_{jik})\, b \tag{I.20}$$

as an invariantly formulated connection between dislocation drift and plastic distortion. As an anti-symmetric tensor of rank three, β^P_{ijk} has none independent components and thus may be replaced by a tensor of rank two in the usual way:

[1] Line direction of the dislocation || right-hand screw boundary line of the drift surface (after the drift).

$$\beta^P_{ijk} = \varepsilon_{ijk}\beta^P_{lk}, \quad \beta^P_{kl} = \tfrac{1}{2}\varepsilon_{ijk}\beta^P_{ijl}. \tag{I.21}$$

One easily shows the validity of this notation when one substitutes β^P_{ijk} from eq. (I.20) in the second of eq. (I.21), and applies the resulting equation:

$$\beta^P_{kl} = -\varepsilon_{ijk} N_{ijl} b \tag{I.22}$$

to the examples *b* to *d*.

We now assume that a dislocation drift N_{ijk} varies in such a way that it remains in place with a constant dislocation density. The decrease in – e.g. – a width dx_2 for dislocations drifting in the x_1-direction through a distance dx_1 is naturally equal to the number of dislocations that pass through a surface element $dF = dx_1\, dx_2$ as a result of the drift. I. e., it is [84]:

$$b\frac{\partial N_{ijk}}{\partial x_i} = \alpha_{jk}, \tag{I.23}$$

and since $\partial N_{ijk} / \partial x_i = 0$ (i.e., the drifting dislocations do not also stop inside the body), it follows from eq. (I.20) that:

$$\frac{\partial \beta^P_{ijk}}{\partial x_i} = \alpha_{jk}. \tag{I.24}$$

In words: The plastic distortion varies in the drift direction when the family of drifting dislocations remains in place with a density α_{jk} ; Fig. 13 was an example of this. If one replaces β^P_{ijk} in eq. (I.24) with β^P_{ij} then this immediately yields eq. (I.11).

The dislocation drift tensor is more closely linked with the actual process of plastic forming than the previously-appearing quantities; therein lies all of its fundamental significance. One can, perhaps, differentiate it by time and thus define a dislocation velocity vector, which might possibly represent a suitable starting point for a later theory of dislocation dynamics. E. g., it is closely related to the problem of defining a law of friction for the dislocation drift such that the force of friction (which ultimately amounts to the driving force of equilibrium and thus bring about a constant velocity) is proportional to the dislocation velocity tensor.

§ 5. The invariant components of the distortion field

In this paragraph, we will assume an infinitely extended medium. Let the distortions be continuous and twice-differentiably distributed, and they may vanish at infinity. The decompositions:

$$\boldsymbol{\beta}^P = \text{Grad}\, \boldsymbol{s}^P + \text{Rot}\, \boldsymbol{\zeta}^P, \tag{I.25}$$

$$\boldsymbol{\beta} = \text{Grad}\, \boldsymbol{s} + \text{Rot}\, \boldsymbol{\zeta} \tag{I.26}$$

are unique. $\boldsymbol{\zeta}^P \equiv (\zeta^P_{ij})$ and $\boldsymbol{\zeta} \equiv (\zeta_{ij})$ are asymmetric tensors.

From § 3, a distortion, whether plastic or elastic, takes a compact body into another compact body when it is derivable from a displacement field by a gradient map. A plastic distortion Grad $\boldsymbol{s}^P$ then requires no additional elastic distortion in order to maintain the connectivity of the body and it results that it is stress-free, while the orientation remains preserved.

The total torsion $\boldsymbol{\beta}^G = \boldsymbol{\beta} + \boldsymbol{\beta}^P$ must fundamentally be a gradient tensor (eq. (I.9)), from which it follows that:

$$\text{Rot } \boldsymbol{\zeta} = -\text{ Rot } \boldsymbol{\zeta}^P. \tag{I.27}$$

This means that the perturbation of the connectivity that comes about by way of a plastic distortion Rot $\boldsymbol{\zeta}^P$ will be again canceled precisely by the elastic distortion Rot $\boldsymbol{\zeta}$.

Therefore, $\boldsymbol{s}^P$ seems to be completely independent of the functions $\boldsymbol{\zeta}^P$, $\boldsymbol{\zeta}$, $\boldsymbol{s}$. However, this comes from the fact that our considerations are still not complete. In reality, there exists a coupling between $\boldsymbol{s}^P$ and $\boldsymbol{\zeta}^P$ – at least, in real bodies, and thus we also assume this for our continuum; perhaps the coupling is of a sort that the number of dislocations that remain in place under the dislocation drift is a function of the number of drifting dislocations, which can naturally be independent of position. Such a relation between α_{ij} and N_{ijk} means a reduction of the additional dislocation drifts as a function of position and comes about from the part (Grad $\boldsymbol{s}^P$) of the total distortion:

$$\boldsymbol{\beta}^G = \text{Grad}(\boldsymbol{s} + \boldsymbol{s}^P) \tag{I.28}$$

that splits off with no change of state. The meaning of $\boldsymbol{s}^P$ is naturally the plastic displacement of the point of the medium that belongs to the part Grad $\boldsymbol{s}^P$ of $\boldsymbol{\beta}^P$.

The tensors $\boldsymbol{\zeta}$ and $\boldsymbol{\zeta}^P$ were not given an intuitive interpretation, up to now. They are a type of potential from which the distortions can be derived. By contrast, $\boldsymbol{s}$ is naturally an elastic displacement field. Namely, if one cancels the plastic distortion Rot $\boldsymbol{\zeta}^P$ with the elastic one by eq. (I.27), and then takes away the forces that the latter have produced then one finds a piecewise relaxation in the state of lowest elastic energy, under which the points suffer the displacement $\boldsymbol{s}$. Thus, one shows the manner in which the total displacement $\boldsymbol{s}^G$ is composed of an elastic displacement and a plastic one.

In an Appendix, it is shown that by a further decomposition of Rot $\boldsymbol{\zeta}$ one obtains the formula:

$$\beta_{ij} = \nabla_i s'_j - \varepsilon_{ijk}\,\varepsilon_{jmn}\nabla_k\,\nabla_m\,\imath_{ln} + \vartheta_{ij} \tag{I.29}$$

for $\boldsymbol{\beta}$. In this, $s'_j \equiv s_j + u_j$, where u_j is a vector field with div $\boldsymbol{u} = 0$; $\imath_{ln}$ is a symmetric tensor field and ϑ_{ij} is an anti-symmetric one. Indeed, when one defines the vector field ϑ_k by the equations:

$$\vartheta_{ij} = \varepsilon_{ijk}\,\vartheta_k, \qquad\qquad \vartheta_k = \tfrac{1}{2}\,\varepsilon_{ijk}\,\vartheta_{ij}\,, \tag{I.30}$$

one has:

$$\vartheta_k = \varepsilon_{ijk}\nabla_i\,u_j + \nabla_i\lambda, \tag{I.31}$$

where λ is a scalar field. We now generally define the incompatibility (Ink) of a tensor field of rank two by the identity:

$$\text{Ink } \boldsymbol{\xi} \equiv (-\varepsilon_{ikl}\, \varepsilon_{jmn} \nabla_k \nabla_m \xi_{ln}) \equiv \nabla \times \boldsymbol{\xi} \times \nabla. \tag{I.32}$$

The name is thus derived [77] from the fact that:

$$\text{Ink } \boldsymbol{\varepsilon} = 0 \tag{I.33}$$

is the compatibility condition for the (small) elastic deformations $\boldsymbol{\varepsilon}$ [86, 34]. (The deformations are compatible when their incompatibility vanishes.) One then easily computes that the incompatibility of a symmetric tensor yields another such tensor, and correspondingly for the asymmetric tensor.

Thus, one can also write eq. (I.29) as:

$$\boldsymbol{\beta} = \text{Grad } \boldsymbol{s}' + \text{Ink } \boldsymbol{\imath} + \boldsymbol{\vartheta}, \tag{I.34}$$

and one may naturally also write $\boldsymbol{\beta}^P$ in the same form:

$$\boldsymbol{\beta}^P = \text{Grad } \boldsymbol{s}'^P + \text{Ink } \boldsymbol{\imath}^P + \boldsymbol{\vartheta}^P. \tag{I.35}$$

Now, in order for the total distortion $\boldsymbol{\beta}^G = \boldsymbol{\beta} + \boldsymbol{\beta}^P$ to be a gradient tensor, one must have:

$$\text{Ink } \boldsymbol{\imath} = -\text{Ink } \boldsymbol{\imath}^P, \qquad \boldsymbol{\vartheta} = -\boldsymbol{\vartheta}^P. \tag{I.36}$$

If one equates this with eq. (I.27) then one must observe that:

$$\text{Rot } \boldsymbol{\zeta}^P = \text{Ink } \boldsymbol{\imath}^P + \boldsymbol{\vartheta}^P + \text{Grad } \boldsymbol{u}^P. \tag{I.37}$$

It thus suffices to just cancel the part Ink $\boldsymbol{\imath}^P + \boldsymbol{\vartheta}^P$ of Rot $\boldsymbol{\zeta}^P$, since the tensor Grad $\boldsymbol{u}^P$ is of no importance for the connectivity phenomena.

If we write ([52], v. I, pp. 97):

$$\text{Def } \boldsymbol{s}' \equiv \tfrac{1}{2}(\nabla_i s'_j + \nabla_j s'_i) \tag{I.38}$$

(read: "deformation of") then the symmetric part of eq. (I.34) can be represented in the form [1]:

$$\boldsymbol{\varepsilon} = \text{Def}(\boldsymbol{s} + \boldsymbol{u}) + \text{Ink } \boldsymbol{\imath}, \tag{I.39}$$

and the anti-symmetric part in the form:

$$\omega_{ij} = \tfrac{1}{2}[\nabla_i (s + u)_j - \nabla_j (s + u)_i] + \vartheta_{ij}\,, \tag{I.40}$$

which, from eq. (I.30, 31) may also be written:

[1] There is the theorem that in an infinite space one can uniquely decompose any tensor field that vanishes at infinity by an equation like (I.39).

$$\omega_{ij} = \tfrac{1}{2}[\nabla_i(s-u)_j - \nabla_j(s-u)_i] + \varepsilon_{ijk}\,\nabla_i\,\lambda\,. \tag{I.41}$$

From the easily-verified identity relations:

$$\left.\begin{aligned} &\text{Div Ink} \equiv 0 \\ &\text{Ink Def} \equiv 0 \end{aligned}\right\} \tag{I.42}$$

it follows that (I.39) represents the decomposition of the elastic deformation field into its compatible and incompatible part. Correspondingly, if eq. (I.41) is the decomposition of the rotation field into a compatible and incompatible part then one knows that only the part with λ remains when one replaces ξ_{ij} with ω_{ij} in eq. (I.32) [1]. The incompatible rotation field thus has the form $\omega_k^{ink} = (\text{grad}\,\lambda)_k$ [2]. One especially notes: The compatible deformations and rotations are no longer coupled as in classical elasticity theory, so there are possibly states in which $\boldsymbol{u} + \boldsymbol{s}$ is equal to an arbitrary vector $\boldsymbol{p}$ and $\boldsymbol{u} - \boldsymbol{s}$ is equal to an almost [3] arbitrary vector $\boldsymbol{q}$.

One may, in no event, interpret $\boldsymbol{u}$ as a displacement field. $\boldsymbol{u}$ is, precisely like λ, a type of potential from which the rotations are derived, which is not easy to understand intuitively. One also observes that by means of eq. (I.41) it is not the rotation of the individual volume elements, but the rotation field; i.e., eq. (I.41) includes statements about the manner in which the rotations of elements to other elements proceeds. At the location $\boldsymbol{x}$, ω_{ij} is defined to be a rigid rotation of the volume element $dV(\boldsymbol{x})$, as in § 2.

Eq. (I.34) is important here for the decomposition of the distortion that is responsible for the restoration of connectivity into its symmetric and anti-symmetric parts in a simple manner. If one substitutes the β_{ij} of eq. (I.34) in eq. (I.17) then one obtains:

$$\text{Rot}(\text{Ink}\;\boldsymbol{\iota} + \boldsymbol{\vartheta}) = \boldsymbol{\alpha} \tag{I.43}$$

as the basic equation in the form that was first derived by the author [81].

We shall treat the phenomena that are connected with the distortions Ink $\boldsymbol{\iota}$ and $\boldsymbol{\vartheta}$ in the next two paragraphs. Here, we would like to only enumerate the number of degrees of freedom that the plastic and elastic distortions are associated with. In all, there are twelve of them, namely: three in Grad $\boldsymbol{s}^P$ and Grad $\boldsymbol{s}$, and six in Rot $\boldsymbol{\zeta}^P$ (Rot $\boldsymbol{\zeta}^1$, resp.). Of the latter six, three of them go to the incompatible deformations Ink $\boldsymbol{\iota}^P$ (Ink $\boldsymbol{\iota}^1$, resp.) and three of the go to the rotations $\boldsymbol{\vartheta}^P$ ($\boldsymbol{\vartheta}$, resp.).

---

[1] Any tensor of the form $\varepsilon_{ijk}\,\nabla_k\lambda$ may be written as an anti-symmetric incompatibility tensor (Appendix).

[2] Another standpoint is: The rotations $\omega_{ij} - \vartheta_{ij}$ in eq. (I.40) are "agreeable" with the deformations Def($\boldsymbol{s} + \boldsymbol{u}$) in eq. (I.39), so one can also refer to the ϑ_{ij} as incompatible rotations [81].

[3] Since div $\boldsymbol{u} = 0$, one cannot simultaneously prescribe the gradient parts of $\boldsymbol{p}$ and $\boldsymbol{q}$ arbitrarily. The gradient part of $\boldsymbol{q}$ does not contribute to ω_{ij}, anyway.

§ 6. The geometric origin of the temperature stresses, magnetic stresses, and concentration stresses

The essentials of plastic forming, from the macroscopic standpoint, can be once more briefly summarized: One thinks of the body as being cut apart into its volume elements and each element as being imprinted with the desired plastic (stress-free) distortion $\boldsymbol{\beta}^P$ with the help of a dislocation drift. The volume elements then no longer fit together with no gaps, so elastic distortions (Ink $\boldsymbol{\imath} + \boldsymbol{\vartheta}$) are necessary in order to once more make them fit together. One then thinks of everything as deforming and the forces that produced the elastic deformation are removed. A relaxation (Grad $\boldsymbol{s}'$) into the state of lowest energy then occurs. At the conclusion, one establishes a dislocation density $\boldsymbol{\alpha} \equiv -$ Rot $\boldsymbol{\beta}^P$.

One can alter the examination in such a way that the volume elements that have been cut away from each other are stamped, not with a plastic distortion through dislocation drift, but a "quasi-plastic" distortion – e.g., by raising the temperature. It is well-known that for the volume element at the position $\boldsymbol{x}$ this is [79]:

$$\beta_{ij}^Q = \gamma \delta_{ij} T, \tag{I.44}$$

if γ is the thermal expansion coefficient and the relative temperature is set to zero. Along with $T(\boldsymbol{x})$, $\boldsymbol{\beta}^Q(\boldsymbol{x})$ is also a continuous function of the position of the volume element. Furthermore, as a spherical tensor, $\boldsymbol{\beta}^Q$ is naturally symmetric – hence, a pure deformation – such that we can also write $\boldsymbol{\varepsilon}^Q$ instead of $\boldsymbol{\beta}^Q$. We call $\boldsymbol{\beta}^Q$ quasi-plastic because such a distortion will arouse no restoring forces. Now, the equation:

$$\delta_{kl} \nabla_k \nabla_l \Phi \equiv \Delta \Phi = \gamma T \tag{I.45}$$

always has a solution. Therefore, we can, with the help of eq. (A.2), also write $\boldsymbol{\beta}^Q$ in the form:

$$\beta_{ij}^Q = \delta_{ij}\, \delta_{kl} \nabla_k \nabla_l \Phi = \nabla_i \nabla_j \Phi - \varepsilon_{ikm}\, \varepsilon_{jlm} \nabla_k \nabla_l \Phi, \tag{I.46}$$

or, from eq. (I.32) [$\boldsymbol{I} \equiv (\delta_{ij})$]:

$$\boldsymbol{\beta}^Q = \text{Def}(\text{grad } \Phi) + \text{Ink}(\Phi\, \boldsymbol{I}), \tag{I.47}$$

where Def can also be replaced with Grad. The second summand has the effect that the distortion $\boldsymbol{\beta}^Q$ results under perturbation of the connectivity, where the connectivity can be reproduced by an elastic distortion of the form Ink $\boldsymbol{\imath} = -$ Ink($\Phi\, \boldsymbol{I}$). One can now define a quasi-dislocation density by the equation:

$$\boldsymbol{\alpha}^Q \equiv -\text{ Rot } \boldsymbol{\beta}^Q. \tag{I.48}$$

The elastic state of distortion that belongs to $\boldsymbol{\alpha}^Q$ is then the same as the one that was produced by the dislocation drift, by which the dislocations were stuck with a density $\boldsymbol{\alpha} = \boldsymbol{\alpha}^Q$. In a continuum that is endowed with positional temperature fluctuations the

associated elastic distortions can thus be eliminated when they are associated with dislocations in it that has a density:

$$\boldsymbol{\alpha} = -\boldsymbol{\alpha}^{Q} = \gamma \mathrm{Rot}(T\,\boldsymbol{I}). \tag{I.49}$$

This process certainly plays an important role for strong temperature stresses, as would exist in – e.g. – cast metals when they are cooling. The ease by which the association of dislocations described here can be computed is an impressive example of the practical use of the conception of temperature stresses as being produced by dislocations.

If one brings a completely demagnetized probe of a ferromagnetic material into a sufficiently strong magnetic field then all of the elementary magnetic dipoles point in the field direction. Thus, in many cases, a quasi-plastic dilatation of the probe comes about in the direction of the magnetization field, where the volume element remains piecewise conserved. If the magnetization direction rotates in the body from point to point then one can again employ the Gedanken experiment above, and the quasi-plastic distortion of the volume element will then be a (symmetric) deviator, due to conservation of volume. Likewise, one can, with the help of eq. (I.48), define a quasi-dislocation density and deduce how the dislocations will be distributed in a magnetostrictively tensed medium in order for the elastic energy to a small as possible. Such examinations play an essential role in the currently ongoing attempts to understand the technical magnetization curves of ferromagnetic metals. On this, cf. [11, 155, 124].

To this also belongs the case of a crystal of atoms of type *A* being dissolved in atoms of type *B* in a macroscopically fluctuating concentration $C(\boldsymbol{x})$. One thinks of the pure crystal as being cut into its volume elements, and then in each of them a set of *B* atoms is dissolved, which amounts to a quasi-plastic distortion as above. Everything after that takes place as in the last two examples.

It scarcely needs to be remarked that the methods (to be justified in Chapter II) for computing the stresses that are produced by a dislocation density $\boldsymbol{\alpha}$ are also valid for the aforementioned cases with a quasi-dislocation density $\boldsymbol{\alpha}^{Q}$. For the computation of these stresses, one is well-served to work with a further geometric quantity that we shall now treat. It is the so-called incompatibility tensor $\boldsymbol{H} = (H_{ij})$ which, in many ways, plays a similar role to that of $\boldsymbol{\alpha}$. We define it by:

$$\boldsymbol{H} \equiv -\,\mathrm{Ink}\,\boldsymbol{\beta}^{Q}, \tag{I.50}$$

so due to (I.11) it is:

$$\boldsymbol{H} = \boldsymbol{\alpha} \times \nabla \equiv (\varepsilon_{ijk}\,\nabla_k\,\alpha_{ij}). \tag{I.51}$$

If one substitutes $\boldsymbol{\alpha}$ in eq. (I.17) then one obtains:

$$\mathrm{Ink}\,\boldsymbol{\beta} = \boldsymbol{H}, \tag{I.52}$$

and the symmetric part of this tensor equations reads, with $\boldsymbol{\eta}$ = symmetric part of $\boldsymbol{H}$ [1]:

$$\mathrm{Ink}\,\boldsymbol{\varepsilon} = \boldsymbol{\eta}. \tag{I.53}$$

[1] We temporarily assume small distortions.

For $\boldsymbol{\eta} = 0$, these are the compatibility conditions of de St. Venant. For the case of the temperature fields, one obtains – e.g., from eq. (I.51) and (I.49) [79]:

$$\boldsymbol{\eta} = \gamma \text{Ink}\,(T\,\boldsymbol{I}); \tag{I.52}$$

i.e., the incompatibility field $\boldsymbol{\eta}$ that belongs to a temperature field is very simple to compute. However, with the knowledge of $\boldsymbol{\eta}$, the associated stresses are ascertained relatively well (§ 13).

The meaning of eq. (I.52) in elasticity theory may perhaps be best characterized as follows: Since it arises from eq. (I.17) by taking the Rot and symmetrizing, it must still include one part of the statement of this equation, while another part must go away. From the relation Ink Def ≡ 0, one deduces the statement that in the case of $\boldsymbol{\eta} = 0$ the deformation $\boldsymbol{\varepsilon}$ can be derived from an elastic displacement field $\boldsymbol{s}$. If, as has always been done in elasticity theory up to now, one simultaneously lets ϑ go to zero in eq. (I.40) [1], then the elastic rotations $\frac{1}{2}(\nabla \boldsymbol{s} - \boldsymbol{s}\nabla)$ follow from the same displacement field. They are then, as is well known, determined up to a rigid rotation of the entire body. In this case, the equations (I.52) are equivalent to the equations Rot $\boldsymbol{\beta} = 0$. The latter automatically include the statement that $\vartheta = \text{const.}$, as will be shown in the next paragraph. One loses the validity of this statement precisely when one derives the equation Ink $\boldsymbol{\varepsilon} = 0$ from the equation Rot $\boldsymbol{\beta} = 0$. The classical elasticity theory is then to be defined by the equation Rot $\boldsymbol{\beta} = 0$, or, equivalently, by Ink $\boldsymbol{\varepsilon} = 0$, $\vartheta = 0$.

For $\boldsymbol{\eta} \neq 0$, the plastic deformation field has the form Def $\boldsymbol{s}^P$ + Ink $\boldsymbol{t}^P$. The last part always has the consequence that the plastic or quasi-plastic distortion does not maintain the connectivity of the body and thus gives rise to elastic deformations and then to proper stresses. The existence of an incompatibility field is therefore (at least, for simply connected bodies) an assumption for the appearance of proper stresses. It may easily be shown that in the realm of linear elasticity theory the totality of possible stresses in a body are uniquely determined by the given of an external force acting on it and the incompatibilities (§ 14).

§ 7. The stress-free structural curvatures.

The fact that dislocations suffer rotations while they drift has served as the clarification of important phenomena in metal physics. Thus, Burgers [12] and Bragg [10] first found that the grain boundaries between two crystallites (grains) whose orientations do not differ all that much will be defined by surface associations of dislocations in these grain boundaries.

We consider, say, the volumes in Fig. 16a. A family of α_{31}-edge dislocations might move along the x_1-direction and be stuck with a constant density along the surfaces in question. Had one previously cut apart these surfaces then one would produce the plastic distortion of Fig. 16b. If one rotated the individual strata through the angle $\delta\theta$ then one could again produce the perturbed connectivity. Between any two strata that are

[1] With $\boldsymbol{\theta} = 0$, from eq. (I.31), one has $\boldsymbol{u} = \text{const.}$ (since div $\boldsymbol{u} = 0$).

separated by a dislocation drift then there exists an angular difference of $\delta\theta$ in the orientation. Fig. 16c shows this for a drift of $-(\alpha_{22} + \alpha_{33})$-screw dislocations in the x_1-direction when $\alpha_{22} = \alpha_{33}$. The problem is now to relate the rotations to the density of the dislocations that they are stuck with.

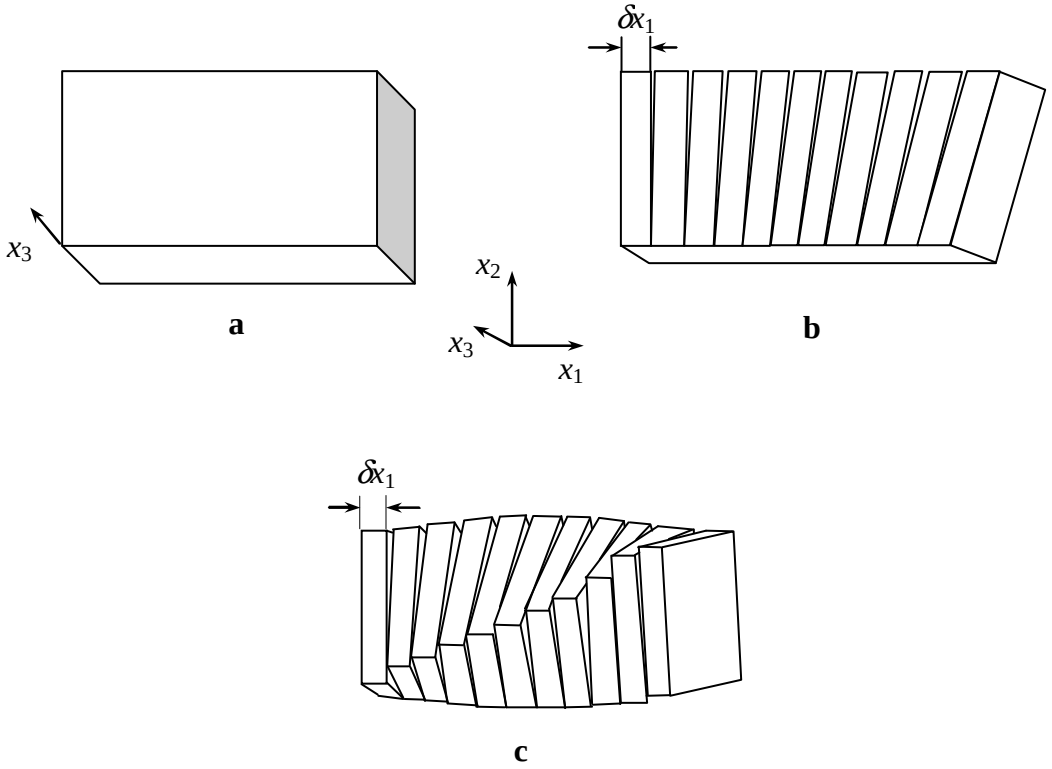


Fig. 16. In **b** and **c** one has a dislocation drift of constant strength between any two strata δx_1 . The dislocation drifts proceed from right to left

We can first restrict ourselves to the case in which the dislocation distribution is homogeneous. From eq. (I.51), the incompatibility tensor then vanishes, and if no external forces act upon the volume either then it is completely free of elastic deformations; in § 14, this will established exactly. This statement is valid only for small distortions (dislocation densities, resp.), to which we shall first restrict ourselves. Thus, in our case one has $\beta_{ij} = \vartheta_{ij}$, where ϑ_{ij} are the elastic (= rigid) rotations of the volume element dV, through which the disturbed connectivity of Fig. 16b, c will again be produced. For the Burgers circuit, one thus first obtains [(cf., eq. (I.15) and (I.19)]:

$$\oint_{\mathfrak{C}} da_j = \oint_{\mathfrak{C}} dx_i \beta_i = \iint_F dF_k \alpha_{kj} , \tag{I.54}$$

and for the other one, however, one has:

$$\oint_{\mathfrak{C}} dx_i \beta_{ij} = \oint_{\mathfrak{C}} dx_i \vartheta_{ij} = - \oint_{\mathfrak{C}} x_i d\vartheta_{ij} \tag{I.55}$$

since $\oint d(x_i \vartheta_{ij}) = 0$ [1]. If we set $d\vartheta_{ij} = \varepsilon_{ijk}\, d\vartheta_k$ then we obtain for the right-hand of eq. (I.55):

$$\left(-\varepsilon_{ijk} \oint_{\mathfrak{C}} x_i d\vartheta_{ij} \right) \equiv \oint_{\mathfrak{C}} \boldsymbol{x} \times d\boldsymbol{\vartheta}\,, \tag{I.56}$$

where $d\vartheta_k$ is the rotation angle between two neighboring volume elements. We now define, like Nye [113], the (macroscopic) curvature tensor $\boldsymbol{K} \equiv (K_{ij})$ by way of the equation:

$$d\vartheta_k = K_{kl}\, dx_l\,. \tag{I.57}$$

The diagonal components of K_{kl} are twists (screwing motions) of the x_l-surfaces, while the remaining components refer to bends of the x_l-surfaces in the k-direction, as one easily clarifies; e.g., with Fig. 16. If one substitutes eq. (I.57) in (I.56) then one obtains (Stokes's theorem):

$$\begin{aligned} -\,\varepsilon_{ijk} \oint_{\mathfrak{C}} x_i K_{kl} dx_l &= -\,\varepsilon_{ijk}\varepsilon_{lmn} \iint_F dF_m \nabla_n (x_i K_{kl}) \\ &= -\,\varepsilon_{ijk}\varepsilon_{lmi} \iint_F dF_m K_{kl}\,; \end{aligned} \tag{I.58}$$

the latter is true because for constant dislocation density K_{kl} is also constant and $\nabla_n\, x_i = \delta_{ni}$. By comparing with eq. (I.54), and with the decomposition formula (A.2), what results is the relation between dislocation density and structural curvature that was first derived by Nye [113] in another way:

$$\alpha_{ij} = \delta_{ij} K_{kk} - K_{ij} \tag{I.59}$$

with the inverse:

$$K_{ij} = \tfrac{1}{2}\, \delta_{ij}\, \alpha_{kk} - \alpha_{ij}\,. \tag{I.60}$$

This equation is then valid for small dislocation densities (curvatures, resp.); i.e., the change in orientation $d\vartheta_k$ in the direction dx_l must be small compared to 1.

For the further considerations, we assume a variable dislocation density and now call the relative rotation angle between two volume elements, on the same obvious grounds, $d\mathfrak{d}_k$.

If one now, in analogy to the Burgers circuit, makes a closed circuit $\mathfrak{C}$ around which one picks up the rotations $\delta\mathfrak{d}_i$ ($d\mathfrak{d}_i$, resp.) then one obtains:

$$D_i = \int_{\mathfrak{C}} d\mathfrak{d}_i = \int_{\mathfrak{C}} dx_j K_{ij}\,, \tag{I.61}$$

and with Stokes's theorem:

$$\boldsymbol{D} = -\iint_F (\boldsymbol{K} \times \nabla) \cdot d\boldsymbol{F}\,. \tag{I.62}$$

[1] For the proof, one assumes that ϑ_{ij} is a linear function of x_l.

On the other hand, it follows from eqs. (I.60), (I.18), and (I.51), which shall not be justified in detail here, that:

$$\boldsymbol{K} \times \nabla \equiv (\varepsilon_{ijk}\ \nabla_k\ K_{ij}) = -\ \boldsymbol{\eta}. \tag{I.63}$$

Thus, one has for infinitesimal surfaces:

$$\Delta D_i = \eta_{ij}\ \Delta F_j\ , \tag{I.64}$$

an equation that we can also regard, as the analogue to eq. (I.14), as the defining equation for $\boldsymbol{\eta}$.

From eq. (I.63), the $d\vartheta_i$ in eq. (I.57) is a total differential only when $\boldsymbol{\eta} = 0$, while for dg_i the condition reads $\boldsymbol{\alpha} = 0$. In the case $\boldsymbol{\eta} = 0$ there thus exists a continuous vector field $\mathfrak{d}_i$ that describes the portion of the structural rotations ("particle boundary contribution")[1] that are in the immediate vicinity of the dislocations and (for $\boldsymbol{\eta} = 0$) is identical with the ϑ_i in eq. (I.31). One recognizes the latter in our foregoing treatment of the problem: $d\vartheta_i$ were rotations that were further produced by the the perturbation of the connectivity due to the dislocation drift. The same is true for $d\mathfrak{d}_i$. The associated structural curvatures are stress-free (§ 14), due to the absence of external forces and incompatibilities.

The tensor $\boldsymbol{K}$ obviously does not include the curvatures described by the elastic rotations $(\nabla_i s_j - \nabla_j s_i)/2$. One can describe the effectively observed structural curvatures with a further curvature tensor that is defined by:

$$d\omega_i = \chi_{ij}\ dx_j \tag{I.65}$$

instead of eq. (I.57). Now, for continuously varying dislocation densities $\boldsymbol{\alpha}$, as well as $\boldsymbol{\beta}$, and therefore $\boldsymbol{\omega}$, are continuous functions of position (because $\boldsymbol{\beta}$ – at least, in simply-connected bodies – must be single-valued), hence, $d\omega_i$ is a complete differential. Thus, this is true only for small rotations; on this, cf., Bilby and Smith [5]. In the absence of elastic deformations K_{ij} becomes identical with ξ_{ij} (indeed, one then has $\nabla_i s_j - \nabla_j s_i = 0$).

Eq. (I.64) says: That which was the Burgers vector $\boldsymbol{b}$ for the dislocations is the rotation vector $\boldsymbol{D}$ for the incompatibilities; one once more confers Fig. 15. That which were originally the drift planes of the dislocations may now be the dislocation walls of constant surface density of the sort one finds in Fig. 16. Under a circuit of the boundary $\mathfrak{C}$ of the surface F that is run through by the dislocation walls one thus always adds the relative rotation of the two volume elements. The neighboring dislocation walls then obviously contribute nothing, since they will be run through twice, and in the opposite directions. Were one to cut out the surface, regarded in isolation, along the dislocation walls then the two edges of the cut would, at that point, gape apart by the rotation angle $d\mathfrak{d}_i$. In this way, one comes directly to a prescription for measuring the incompatibility of a stress state: One cuts out a closed ring at the position $\boldsymbol{x}$ that is as thin as possible, and which defines the boundary of a (macroscopic) surface element ΔF_j . One then removes

[1] For this terminology, see also § 23.

this ring and measures the resulting rigid relative rotation of the cut surfaces after relaxation. The rotation vector is ΔD_i , from which, with eq. (I.64), η_{ij} follows.

The ring that one thus removes must be thin, on account of the fact that only then is the associated surface ΔF_j defined with sufficient precision. For thick rings, there is an additional deformation of the cut edges that corrupts the measurement. In practical cases, one would scarcely ever measure a body in this manner. By comparison, one can provide an overview of the "cut incompatibilities," and therefore its state of proper stress (see Chapter II), when one carries out the previous measurements on some small macroscopic surface F [1].

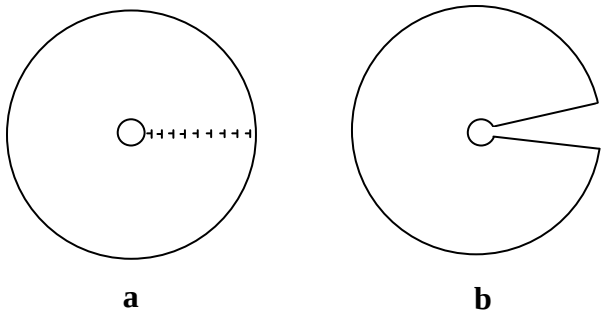


Fig. 17. On the production of a Volterra distortion of the second kind

With these considerations, we come very close to the Volterra distortions of the second kind (§ 1). Fig. 17a shows a cylinder in which only one dislocation wall has been cut, as in Fig. 15. Around the boundary line of this wall, let a hollow torus be spared. One then finds the hollow cylinder in a Volterra distortion state of the second kind, and by cutting along the dislocation wall, or also along any other surface, the two cut edges suffer the well-known rotation jump. I.e., the state in Fig. 17a can be generated from that of Fig. 17b without stresses by bending together and welding. In contrast to the previous view, the singular surface of the rotation jump can be experimentally determined, in any event. In case the body (*b*) is a unit crystal, it is evident that naturally one can immediately (and often more simply) establish the orientation jump in *a* Röntgenographically. One also succeeds in doing this for polycrystals, but with somewhat more trouble [2].

A complete description of the Volterra distortion state of the second kind is obtained from being given a singular surface, which one can verify experimentally, anyhow. The oft-used term "elementary distortion" thus relates only to the state of the first kind in our way of looking at things, which then agrees with the fact that one can produce any state of the second kind by certain accumulations of dislocations [3, 4].

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[1] The problem of measuring the proper stresses in the interior of a body has not yet been solved satisfactorily. It is worth mentioning that in many cases one can apply the magnetic method; on this, cf. Reimer [175].

[2] On this, also confer the discussion of Nabarro [110], pp. 349.

[3] As is well-known, Volterra used the word "distortion" in a somewhat different sense from ours. In our language, the statement above would read: The elementary state is the one created by a single dislocation.

[4] The results of this paragraph that are concerned with incompatibilities were discovered independently of the author, first by Moriguti [103], and later by Eshelby [41].

§ 8. The boundary surface conditions for the distortions.

Up to now, no sort of boundary surfaces for the test body have been considered, and we shall now rectify that. One naturally obtains the appropriate boundary surface conditions immediately when one uses the surface rotation in place of the Rot operation in eq. (I.11) and (I.17) and takes the surface dislocation density $\bar{\boldsymbol{\alpha}}$ instead of the spatial dislocation density $\boldsymbol{\alpha}$. If one denotes the two sides of the boundary surface by I and II, and if $\boldsymbol{n} \equiv (n_i)$ is the (dimensionless) unit normal vector of the boundary surface in the direction from I to II then from eqs. (I.11) and (I.17) one has [1]:

$$\boldsymbol{n} \times \boldsymbol{\beta}^P|_{\mathrm{II}} - \boldsymbol{n} \times \boldsymbol{\beta}^P|_{\mathrm{I}} = -\ \bar{\boldsymbol{\alpha}} \tag{I.66}$$

$$\boldsymbol{n} \times \boldsymbol{\beta}|_{\mathrm{II}} - \boldsymbol{n} \times \boldsymbol{\beta}|_{\mathrm{I}} = \bar{\boldsymbol{\alpha}}\,, \tag{I.67}$$

where one can, if one desires, consider the first equation to be the defining equation for $\bar{\boldsymbol{\alpha}}$, as before. For many purposes, it is thus advantageous to define the dislocation surface density $\bar{\alpha}_{ij}$ in the sense of the Schwartz distributional calculus [131] by the equation:

$$\alpha_{ij} = \bar{\alpha}_{ij}\delta(n)\,, \tag{I.68}$$

where the parameter n characterizes a family of surfaces in such a way that $n = 0$ becomes the boundary surface. $\delta(n)$ is everywhere 0 except for $n = 0$, where it becomes infinite in such a way that $\int_{-\infty}^{\infty} \delta(n)dn = 1$; $\bar{\alpha}_{ij}$ no longer depends upon n. From eq. (I.68), it follows that:

$$\int_{-\infty}^{\infty} \alpha_{ij}dn \ = \int_{-\infty}^{\infty} \bar{\alpha}_{ij}\delta(n)dn = \bar{\alpha}_{ij}\ . \tag{I.69}$$

Henceforth, we consider an infinitely extended body in a stress-free initial state. By means of suitable external forces, dislocations might form in it and drift. Thus, three groups shall be distinguished:

The first group of dislocations shall, at the end of the (continuously distributed) drift, no longer be there, thus, they shall be somehow annihilated [2]. The second group shall likewise remain in place as the superficial density $\bar{\boldsymbol{\alpha}}$, by means of which, two regions I and II of the body will be bounded, moreover.

At the conclusion, the connectivity of the body shall also be preserved along the boundary surface. This implies the boundary condition:

$$\boldsymbol{s}^G|_{\mathrm{II}} - \boldsymbol{s}^G|_{\mathrm{I}} = 0; \tag{I.70}$$

[1] Eq. (I.67) was first of all derived by Bilby, Bullough, and Smith [3]. These authors regarded multiple superficial dislocation distributions as totalities and then spoke of "surface dislocations," in contrast to the ordinary line dislocations. In later papers, which are cited on pp. ? (113 in the present reference), these authors gave various applications of the theory of surface dislocations.

[2] In infinite bodies, this annihilation can come about only by the combination of dislocations of opposite signs.

the $\boldsymbol{s}^G$ are the total displacements, as in § 2. One can differentiate these equations at the boundary surface and thus lose only an uninteresting constant. The equation:

$$\boldsymbol{n} \times \nabla \boldsymbol{s}^G|_{\text{II}} - \boldsymbol{n} \times \nabla \boldsymbol{s}^G|_{\text{I}} = 0 \tag{I.71}$$

are thus practically equivalent to eq. (I.70). Instead of (I.71), if we write, from eq. (I.9); e.g.:

$$\boldsymbol{n} \times \boldsymbol{\beta}^G|_{\text{II}} - \boldsymbol{n} \times \boldsymbol{\beta}^G|_{\text{I}} = 0 \tag{I.72}$$

then we find that at a place where $\boldsymbol{n}$ points in the x_1-direction the $\beta_{21}^G, \beta_{22}^G, \beta_{23}^G$, and $\beta_{31}^G, \beta_{32}^G, \beta_{33}^G$ must be equal on both sides of the boundary surface [1]. Eq. (I.72) is the sum of eq. (I.66) and (I.67); formally, it also follows naturally from eq. (I.8).

We once more consider the three groups of dislocations and assume that they drift apart from each other. The first group has naturally produced a plastic distortion Grad $\boldsymbol{s}_1^P$ that is continuous over the entire body, while the second group likewise produces a distortion Grad $\boldsymbol{s}_2^P$ + Rot $\boldsymbol{\zeta}_2^P$ that is continuous over the entire body (the fact that a part Grad $\boldsymbol{s}_2^P$ is present follows from the fact that distortions generated by the second group depend upon the drift path of the dislocations).

The distortions described by group 3 are discontinuous on the boundary surface, but continuous in the sub-bodies I and II. Which form do they have? These dislocations are either annihilated in the body, or remain in place. One will thus be reminded of the well-known fact that the decomposition of the tensor field:

$$\boldsymbol{\beta}_3^P = \text{Grad}\, \boldsymbol{s}_3^P + \text{Rot}\, \boldsymbol{\zeta}_3^P \tag{I.73}$$

in a medium with one or more boundary surfaces is not unique, but there is a distortion that can be represented as a vector gradient, as well as a rotor tensor. If one writes it as $\boldsymbol{s}_3^P$ then it follows, since it shall likewise be a rotor tensor, that:

$$\text{Div Grad}\, \boldsymbol{s}_3^P \equiv \Delta \boldsymbol{s}_3^P = 0. \tag{I.74}$$

Conversely, any gradient of a harmonic vector field may also be represented as a rotor tensor.

From these considerations, one can suspect [2] that the boundary conditions can be written in the form:

$$\boldsymbol{n} \times \nabla \boldsymbol{s}_3^P\Big|_{\text{II}} - \boldsymbol{n} \times \nabla \boldsymbol{s}_3^P\Big|_{\text{I}} = -\bar{\boldsymbol{\alpha}}, \tag{I.75}$$

or, when one sensibly writes:

$$\boldsymbol{g} \equiv \boldsymbol{s}_3^P\Big|_{\text{II}} - \boldsymbol{s}_3^P\Big|_{\text{I}}, \tag{I.76}$$

---

[1] The same considerations had previously led us to eq. (I.8).

[2] A stronger proof may be obtained in connection with § 4, perhaps.

for the jump in the plastic displacement at the boundary surface, also in the form:

$$\boldsymbol{n} \times \nabla \boldsymbol{g} = - \bar{\boldsymbol{\alpha}} . \tag{I.77}$$

This equation says how dislocations must be distributed in a surface that exhibits a non-constant displacement jump $\boldsymbol{g}$; it has great significance in the applications. Like eq. (I.17), it is also true for large distortions, as long as one refers all quantities to the initial state (§ 10).

The boundary conditions (I.66) and (I.67) become especially simple when one of the two sub-bodies becomes infinitely soft (air) or infinitely hard (rigid). In the first case, the boundary conditions are satisfied identically, as one best recognizes from their form itself (I.70). In the second case, one of the two summands in eq. (I.66) or eq. (I.67) drops out since the distortion is null in a rigid medium (naturally, no dislocations can drift in rigid media, either). Physically, there is indeed no such thing as a rigid medium, but it often happens that – e.g. – a metal contains an inclusion that is a hard metal whose forming cannot be neglected. In this case, the boundary-value problem to be solved is simplified considerably. One observes that the behavior of the well-known boundary-value conditions for the stresses is precisely the opposite. (They are fulfilled identically at a boundary surface with a rigid body.)

§ 9. The boundary surface conditions for the deformations, superficial incompatibility distributions

In § 8, it was assumed that the spatial density $\boldsymbol{\alpha}$ of dislocations embedded in it was a continuous function over the entire body. We shall now lift this restriction, by allowing an additional jump in $\boldsymbol{\alpha}$ across the boundary surface. One easily sees that this also leads to a jump in the plastic displacement. For example, in I, edge dislocations can climb under an enlargement of the volume of I, while in II a dislocation drift without volume change takes place. The associated displacement jump would be sensibly denoted by:

$$\boldsymbol{g}_2 \equiv \boldsymbol{s}_2^P\big|_{\mathrm{II}} - \boldsymbol{s}_2^P\big|_{\mathrm{I}} . \tag{I.78}$$

In § 7, the importance of the incompatibilities in the state of proper stresses was emphasized. One may thus expect that superficial distributions of incompatibilities might also play a role. Not only do the surface dislocations contribute to them, but also the jump in $\boldsymbol{\alpha}$.

In order to study this, we consider the case in which the plastic distortions $\boldsymbol{\beta}_{\mathrm{I}}^P$ and $\boldsymbol{\beta}_{\mathrm{II}}^P$ are continuously distributed, but the transition from I to II can remain discontinuous. We think of the functions $\boldsymbol{\beta}_{\mathrm{I}}^P$ as continuous and twice differentiable in II, and likewise for the $\boldsymbol{\beta}_{\mathrm{II}}^P$ in I. We write the distortions collectively for the whole region I and II:

$$\boldsymbol{\beta}^P = \boldsymbol{\beta}_{\mathrm{I}}^P + (\boldsymbol{\beta}_{\mathrm{II}}^P - \boldsymbol{\beta}_{\mathrm{I}}^P) \, \delta^{\mathrm{II}} , \tag{I.79}$$

where $\delta^0(n)$ shall be the Heaviside step function; i.e., $\delta^0(n) = 0$ in I and 1 in II. We shall employ the following rules of calculation [131]: $\frac{\partial}{\partial n}\delta^0(n) = \delta^1(n)$, $\frac{\partial}{\partial n}\delta^1(n) = \delta^2(n)$, where δ^1 is the Dirac delta function and δ^2 is the distribution that describes a double covering. Since all of the δ's depend only upon n, one has, moreover, $\nabla\delta^0 = \boldsymbol{n}\delta^1$ and $\nabla\delta^1 = \boldsymbol{n}\delta^2$. Ultimately, when f is a continuous function of $\boldsymbol{n}$ and – should the occasion arise – two more coordinates, one has $\frac{\partial}{\partial n}(f\delta^1) = f\delta^2$. Thus, we define the asymmetric incompatibility:

$$\boldsymbol{H} \equiv -\nabla\times\boldsymbol{\beta}^P\times\nabla. \tag{I.80}$$

One then has [1]:

$$-\boldsymbol{\alpha} = -\nabla\times\boldsymbol{\beta}^P = \nabla\times\boldsymbol{\beta}^P_{\rm I} + \nabla\times(\overset{\downarrow}{\boldsymbol{\beta}}{}^P_{\rm II} - \overset{\downarrow}{\boldsymbol{\beta}}{}^P_{\rm I})\delta^0 + \boldsymbol{n}\times(\boldsymbol{\beta}^P_{\rm II} - \boldsymbol{\beta}^P_{\rm I})\delta^1. \tag{I.81}$$

The first two summands represent the spatial density and the last one, from § 8, the surface density of dislocations. For the further differentiation of the last summand, we employ the decomposition into pieces that are differentiated perpendicular and parallel to the surface n:

$$\nabla = \boldsymbol{n}\frac{\partial}{\partial u} + \overset{n}{\nabla}, \qquad \overset{n}{\nabla} \equiv -\boldsymbol{n}\times(\boldsymbol{n}\times\nabla). \tag{I.82}$$

One thus obtains:

$$\left.\begin{aligned}\boldsymbol{H} \equiv \alpha\times\nabla = &-\nabla\times\boldsymbol{\beta}^P_{\rm I}\times\nabla - \nabla\times(\boldsymbol{\beta}^P_{\rm II} - \boldsymbol{\beta}^P_{\rm I})\times\nabla\delta^0 \\ &-[\nabla\times(\overset{\downarrow}{\boldsymbol{\beta}}{}^P_{\rm II} - \overset{\downarrow}{\boldsymbol{\beta}}{}^P_{\rm I})\times\boldsymbol{n} + \overset{\downarrow}{\boldsymbol{n}}\times(\overset{\downarrow}{\boldsymbol{\beta}}{}^P_{\rm II} - \overset{\downarrow}{\boldsymbol{\beta}}{}^P_{\rm I})\times\overset{n}{\nabla}]\delta^1 - \boldsymbol{n}\times(\boldsymbol{\beta}^P_{\rm II} - \boldsymbol{\beta}^P_{\rm I})\times\boldsymbol{n}\,\delta^2.\end{aligned}\right\} \tag{I.83}$$

The first two summands represent the spatial incompatibility density and the next two, a simple superficial one, and indeed, the first part of the latter describes the jump in the dislocation density $\boldsymbol{\alpha}$, while the second one follows from the surface density $\bar{\boldsymbol{\alpha}}$. Finally, the last summand corresponds to a superficial double covering of the incompatibilities.

Had we, instead of eq. (I.81), first carried out the operation on the right then this would yield as a factor of δ^1 in eq. (I.83):

$$\boldsymbol{n}\times(\overset{\downarrow}{\beta}{}^P_{\rm II} - \overset{\downarrow}{\beta}{}^P_{\rm I})\times\nabla - \overset{n}{\nabla}\times(\overset{\downarrow}{\beta}{}^P_{\rm II} - \overset{\downarrow}{\beta}{}^P_{\rm I})\times\overset{\downarrow}{\boldsymbol{n}}; \tag{I.84}$$

otherwise, eq. (I.83) would remain the same. One can show that this expression is identical with the stated one of (I.83), as it also should be. (Namely, one has $\overset{\downarrow}{\boldsymbol{n}}\times\beta\times\overset{n}{\nabla} = \overset{n}{\nabla}\times\beta\times\overset{\downarrow}{\boldsymbol{n}}$, as one easily shows, in which one writes $\overset{n}{\nabla} = \boldsymbol{v}\partial/\partial v + \boldsymbol{w}\partial/\partial w$, perhaps, where $\boldsymbol{v}$, $\boldsymbol{w}$ are the principal curvatures. One then has $\partial\boldsymbol{n}/\partial v \sim \boldsymbol{v}$, $\partial\boldsymbol{n}/\partial w \sim \boldsymbol{w}$.)

For the part of $\boldsymbol{H}$ that lies on the boundary surface, we further obtain the expression:

[1] In the doubtful cases an arrow is used to denote the differentiation.

$$\bar{\boldsymbol{H}}\,\delta^1 + \overline{\overline{\boldsymbol{H}}}\,\delta^2, \tag{I.85}$$

with:

$$\bar{\boldsymbol{H}} \equiv -\,\overline{\mathrm{Ink}}\left(\boldsymbol{\beta}_{\mathrm{II}}^{P} - \boldsymbol{\beta}_{\mathrm{I}}^{P}\right), \qquad \overline{\overline{\boldsymbol{H}}} \equiv -\,\overline{\overline{\mathrm{Ink}}}\left(\boldsymbol{\beta}_{\mathrm{II}}^{P} - \boldsymbol{\beta}_{\mathrm{I}}^{P}\right), \tag{I.86}$$

where the operations $\overline{\mathrm{Ink}}$ and $\overline{\overline{\mathrm{Ink}}}$ are defined by:

$$\overline{\mathrm{Ink}}\,\boldsymbol{\beta} \equiv \nabla \times \overset{\downarrow}{\boldsymbol{\beta}} \times \boldsymbol{n} + \overset{\downarrow}{\boldsymbol{n}} \times \overset{\downarrow}{\boldsymbol{\beta}} \times \overset{n}{\nabla} \tag{I.87}$$

$$\overline{\overline{\mathrm{Ink}}}\,\boldsymbol{\beta} \equiv \boldsymbol{n} \times \boldsymbol{b} \times \boldsymbol{n}. \tag{I.88}$$

Thus, it is very plausible (cf., eq. (I.51)) that:

$$\bar{\boldsymbol{H}} = \boldsymbol{\alpha} \times \boldsymbol{n}\,|_{\mathrm{II}} - \boldsymbol{\alpha} \times \boldsymbol{n}\,|_{\mathrm{I}} + \bar{\boldsymbol{\alpha}} \times \overset{n}{\nabla}, \qquad \overline{\overline{\boldsymbol{H}}} = \bar{\boldsymbol{\alpha}} \times \boldsymbol{n}. \tag{I.88a}$$

Naturally, if $\overline{\mathrm{Ink}}\left(\boldsymbol{\beta}_{\mathrm{II}}^{G} - \boldsymbol{\beta}_{\mathrm{I}}^{G}\right) = 0$ and $\overline{\overline{\mathrm{Ink}}}\left(\boldsymbol{\beta}_{\mathrm{II}}^{G} - \boldsymbol{\beta}_{\mathrm{I}}^{G}\right) = 0$ then the boundary surface conditions for the elastic distortions may be written thus:

$$\overline{\mathrm{Ink}}\,(\boldsymbol{\beta}_{\mathrm{II}} - \boldsymbol{\beta}_{\mathrm{I}}) = \bar{\boldsymbol{H}}, \qquad \overline{\overline{\mathrm{Ink}}}\,(\boldsymbol{\beta}_{\mathrm{II}} - \boldsymbol{\beta}_{\mathrm{I}}) = \overline{\overline{\boldsymbol{H}}}. \tag{I.88b}$$

One can easily show that:

$$\left(\overline{\mathrm{Ink}}\,\boldsymbol{\beta}\right)^{S} = \overline{\mathrm{Ink}}\,\boldsymbol{\beta}^{S}, \qquad \left(\overline{\overline{\mathrm{Ink}}}\boldsymbol{\beta}\right)^{S} = \overline{\overline{\mathrm{Ink}}}\,\boldsymbol{\beta}^{S}. \tag{I.88c}$$

With $\boldsymbol{\beta}^{S} \equiv \boldsymbol{\varepsilon}$ and $\boldsymbol{H}^{S} \equiv \boldsymbol{\eta}$ one then has:

$$\overline{\mathrm{Ink}}\left(\boldsymbol{\varepsilon}_{\mathrm{II}} - \boldsymbol{\varepsilon}_{\mathrm{I}}\right) = \bar{\boldsymbol{\eta}}, \qquad \overline{\overline{\mathrm{Ink}}}\left(\boldsymbol{\varepsilon}_{\mathrm{II}} - \boldsymbol{\varepsilon}_{\mathrm{I}}\right) = \overline{\overline{\boldsymbol{\eta}}}. \tag{I.89}$$

These are the boundary surface conditions for the deformations. From eq. (I.88a), one deduces the following relations:

$$\bar{\boldsymbol{\eta}} = (\boldsymbol{\alpha} \times \boldsymbol{n}|_{\mathrm{II}} - \boldsymbol{\alpha} \times \boldsymbol{n}|_{\mathrm{I}})^{S} + \left(\bar{\boldsymbol{\alpha}} \times \overset{n}{\nabla}\right)^{S}, \qquad \overline{\overline{\boldsymbol{\eta}}} = (\bar{\boldsymbol{\alpha}} \times \boldsymbol{n})^{S}, \tag{I.90}$$

which are of practical importance.

Eqs. (I.90) and (I.51) allow one to recognize whether a body does or does not have proper stresses for given conditions (e.g., given dislocation densities or impressed distortions) with the help of simple calculations. One can easily show that in the realm of linear elasticity the proper stresses are determined uniquely for given $\bar{\boldsymbol{\eta}}$, $\overline{\overline{\boldsymbol{\eta}}}$, and $\boldsymbol{\eta}$ (§ 14). In particular, the proper stresses vanish simultaneously with the vanishing of $\bar{\boldsymbol{\eta}}$, $\overline{\overline{\boldsymbol{\eta}}}$,

and $\boldsymbol{\eta}$. It is relatively likely that in the future one will be able to compute the proper stresses associated with given incompatibilities $\bar{\boldsymbol{\eta}}$, $\bar{\bar{\boldsymbol{\eta}}}$, and $\boldsymbol{\eta}$ (§ 13-15). For that reason, here we shall only mention that it is quite simple to put the first boundary value problem of elasticity (given boundary value displacements) into the form "given $\bar{\boldsymbol{\eta}}$, $\bar{\bar{\boldsymbol{\eta}}}$." One then proposes that the boundary displacements will be maintained in such a way that the body is attached to a rigid neighborhood and one can thus interpret the boundary displacement $\boldsymbol{s}$ as a jump displacement – $\boldsymbol{g}$ as in § 8. From $\boldsymbol{g}$, what then results in a simple way is something that one thinks of as a surface dislocation density – $\boldsymbol{n} \times \nabla \boldsymbol{g}$, and, from (I.90), the associated $\bar{\boldsymbol{\eta}}$ and $\bar{\bar{\boldsymbol{\eta}}}$. For the solution of the problem "given $\bar{\boldsymbol{\eta}}$, $\bar{\bar{\boldsymbol{\eta}}}$," cf., § 15.

As the only application, we consider the case in which the temperature of two substances undergoes a jump $\Delta T = T_2 - T_1$ along the planar boundary surface between them, and we assume that the temperature is constant in each of them. One then only has to substitute the $\boldsymbol{\beta}^Q$ of eq. (I.44) in eq. (I.86) and obtain, by a simple calculation (due to the symmetry of $\boldsymbol{\beta}^Q$) the surface compatibilities $\bar{\bar{\boldsymbol{\eta}}}$, while $\bar{\boldsymbol{\eta}}$ vanishes from the constancy of $\boldsymbol{\beta}^Q$. An intuitive interpretation for the double covering incompatibility $\bar{\bar{\boldsymbol{\eta}}}$ follows in § 23.

§ 10. Some comments on large distortions

We already emphasized that the derivation of the basic geometric equation was valid for arbitrarily large distortions when one referred α_{ij} and β_{ij} to the initial state. One then perhaps thinks of there being some sort of inhibition that first prevents the body from being distorted by the formation of dislocations considered and their drift. One can naturally refer the distortions that the volume element then undergoes after the removal of the inhibition to the initial state entirely. Only when it is thus interpreted does the basic geometric equation have the simple form (I.17). One immediately sees that distortions are additive when taken in the initial state and perhaps defined by the equation:

$$da_j = \beta_{ij}^{\nu} dx_i \tag{I.91}$$

(but not their deformations and rotations). Namely, if one sums over a number of successive distortions β_{ij} then one obtains:

$$dA_j \equiv \sum_{\nu} da_j^{\nu} = \sum_{\nu} \left(\beta_{ij}^{\nu} dx_i^{\nu} \right) = \left(\sum_{\nu} \beta_{ij}^{\nu} \right) dx_i \,, \tag{I.92}$$

where the latter equation is valid only when one always understands dx_i to mean the (constant) distance between the points considered in the initial state. In this case, one also composes the distortions additively from the deformations and rotations; i.e.:

$$\beta_{ij} = \varepsilon_{ij} + \omega_j \,, \tag{I.93}$$

where ω_{ij} has the well-known form [34][1], [86]:

$$\omega_{ij} = (1 - \cos q)\,(k_i\,k_j - \delta_{ij}) + \sin q\;\varepsilon_{ijk}\,k_k\,, \tag{I.94}$$

when k_i is a unit vector in the direction of the rotational axis and q is the magnitude of the rotation angle. Since ε_{ij} is symmetric and ω_{ij} is already completely determined by its anti-symmetric part a given distortion can be very easily decomposed using eq. (I.93). Eq. (I.93) is thus not a division of the tensor β_{ij} into its symmetric and anti-symmetric parts. It is thus clear that all equations in which ε_{ij} is assumed to be the symmetric part of β_{ij} or ω_{ij} is assumed to be the anti-symmetric part are valid only for small distortions. Thus, in particular, the symmetric incompatibility equations are concerned with $\bar{\boldsymbol{\eta}}$, $\overline{\overline{\boldsymbol{\eta}}}$, and $\boldsymbol{\eta}$, while the asymmetric compatibility equations, like, e.g., (I.51), are also true for large distortions; its meaning is generally still not satisfactorily clarified. If one refers all quantities to the final state then the basic geometric equation assumes a complicated form that was derived in Chapter IV. Now, on the other hand, the basic static equation (= equilibrium conditions for the forces) in its well-known form is simple in the final state, while it becomes complicated in the initial state. This means that, in general, one cannot simultaneously employ the simple form of the basic geometric and static equations.

There is an important exception: The rotations ω_{ij} (and indeed, its grain boundary piece ϑ_{ij}) are large, but the deformation ε_{ij} is small. For that reason, this case is of such great practical meaning, because the rotations ϑ_{ij} come about without stresses [2], hence, they consume no energy, and dislocations in crystals are thus preferentially distributed in such a manner that ε_{ij} is as small as possible, while ω_{ij} can, by all means, become large.

Now, in the case of pure rotations ϑ_{ij} the total distortion $\vartheta_{ij}^{P} + \vartheta_{ij} = 0$ (cf., § 5); i.e., all of the volume elements remain in place, while only the lattice orientation will be rotated [3]. If small deformations ε_{ij} are simultaneously present then the volume elements also experience only small shifts, and one does not need to distinguish the positions in the initial and final states; i.e., the equilibrium conditions also retain their simple form when they are expressed in the coordinates of the initial state. In the next paragraphs it will be shown how one can divide the total dislocation density into one part that is responsible for the rotations ϑ_{ij} and one that is responsible for the deformations ε_{ij}, by which one calculates the lattice rotations. The determination of the elastic deformations (the proper stresses, resp.) is then only a problem in linear elasticity theory.

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[1] Bd., pp. 78.

[2] The statement is valid in crystals only locally (§ 23), in contrast to the continuum.

[3] This case corresponds to the one Fig. 16c, but not Fig. 16b, where the layers δx_1 have experienced no pure (plastic) rotation, but were likewise deformed plastically, which gives rise to a total displacement of the layers. Thus, if one again produces the connectivity through elastic rotations then the body appears to be bent, although (in the case of small plastic distortions) it is still stress-free.

§ 11. Determination of the distortions of a body with dislocations

One of the most important problems in the theory of dislocations is to compute the elastic distortions, proper stresses, and lattice rotations (curvatures, resp.) that are associated with a given distribution of dislocations $\boldsymbol{\alpha}$, $\bar{\boldsymbol{\alpha}}$. $\boldsymbol{\alpha}$, $\bar{\boldsymbol{\alpha}}$ are not completely arbitrary functions of position, but they must satisfy the condition that dislocations can not stop in the interior of a body or its boundary surface. Thus, a dislocation $\bar{\boldsymbol{\alpha}}$ that runs around the boundary surface can leave it and thus contribute a spatial density $\boldsymbol{\alpha}$. Eq. (I.18):

$$\nabla_i \, \alpha_{ij} = 0 \tag{I.95}$$

was necessary and sufficient for the dislocations $\boldsymbol{\alpha}$ to not stop. Thus, for each boundary surface one has an equation [1]:

$$\overset{n}{\nabla}_i \, \bar{\alpha}_{ij} + n_i \left(\alpha_{ij}\big|_{\mathrm{II}} - \alpha_{ij}\big|_{\mathrm{I}} \right) = 0, \tag{I.96}$$

which says that anywhere the dislocations have divergences the dislocations $\boldsymbol{\alpha}$ run into the boundary surface and bend around it. If one sets $\bar{\boldsymbol{\alpha}} = \boldsymbol{n} \times (\boldsymbol{\beta}_{\mathrm{II}} - \boldsymbol{\beta}_{\mathrm{I}})$ and $\boldsymbol{\alpha} = \nabla \times \boldsymbol{\beta}$ in this equation then it is fulfilled identically, which one can regard as the proof of eq. (I.96). The given dislocation distribution shall thus always fulfill the conditions (I.95) and (I.96).

The simplest problem in which $\boldsymbol{\alpha}$ and $\bar{\boldsymbol{\alpha}}$ play a role is that of a rigid neighborhood stuck to a body; we would like to first restrict ourselves to it. (For a free boundary, one will have $\bar{\boldsymbol{\alpha}} = 0$.)

At this time, the aforementioned problem can be solved for small distortions at best when one first computes the stresses (and thus the deformations) and then the rotations. Thus, one must compute the incompatibilities $\boldsymbol{\eta}, \bar{\boldsymbol{\eta}}, \bar{\bar{\boldsymbol{\eta}}}$ from $\boldsymbol{\alpha}$, $\bar{\boldsymbol{\alpha}}$, which, from § 9, is very simple. After computing the stresses that belong to $\boldsymbol{\eta}$, $\bar{\boldsymbol{\eta}}$, $\bar{\bar{\boldsymbol{\eta}}}$ (§ 13, 15), one obtains the elastic deformations $\boldsymbol{\varepsilon}$ by means of Hooke's law. If one now writes the basic equation (I.17) in the form:

$$\mathrm{Rot}\, \boldsymbol{\omega} = \boldsymbol{\alpha} - \mathrm{Rot}\, \boldsymbol{\varepsilon}, \tag{I.97}$$

then the right-hand is now known. From eq. (I.97), after an easy calculation, one now has [2]:

$$\delta_{kl} \, \nabla_m \, \omega_m - \nabla_l \, \omega_k = (\boldsymbol{\alpha} - \mathrm{Rot}\, \boldsymbol{\varepsilon})_{kl} \,, \tag{I.98}$$

and by contraction:

$$2 \, \nabla_m \, \omega_m = (\boldsymbol{\alpha} - \mathrm{Rot}\, \boldsymbol{\varepsilon})_{mm} \,. \tag{I.99}$$

Substituting this in eq. (I.98), one obtains:

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[1] For the definition of $\overset{n}{\nabla}$, cf., eq. (I.82).

[2] For Rot $\boldsymbol{\omega}$, one writes $\varepsilon_{kji} \, \nabla_j \, \omega_l = \varepsilon_{kji} \, \varepsilon_{ilm} \nabla_j \, \omega_m$. With the decomposition formula (A 2), (I.98) follows.

$$\nabla_l \, \omega_k = \tfrac{1}{2} \delta_{kl} \, (\boldsymbol{\alpha} - \text{Rot } \boldsymbol{\varepsilon})_{mm} \; - (\boldsymbol{\alpha} - \text{Rot } \boldsymbol{\varepsilon})_{kl} \, , \tag{I.100}$$

where the right-hand side is still a known function. The structure curvatures follow from this by an ordinary integration, up to a constant.

It is particularly noteworthy that the boundary value problem to be solved for the determination of the stresses appears in the form "given $\bar{\boldsymbol{\eta}}$, $\bar{\bar{\boldsymbol{\eta}}}$" from the outset; cf., § 9.

Now, let us describe a method of calculating the structural rotations before the determination of the stresses, which, at present, has been worked out only for infinitely extended media [81], but which can be extended to finite media with none of the all-to-great difficulties of finite media. The starting point is the basic equation in the form (I.43):

$$\text{Rot}(\text{Ink } \boldsymbol{\imath} + \boldsymbol{\vartheta}) = \boldsymbol{\alpha}. \tag{I.101}$$

One easily shows that $(\text{Rot Ink } \boldsymbol{\imath})_{ii}$ vanishes identically, because Ink $\boldsymbol{\imath}$ is a symmetric tensor. Thus, similar to the above, it follows that:

$$(\text{Rot Ink } \boldsymbol{\imath})_{kl} - \nabla_l \, \vartheta_k = \alpha_{kl} - \tfrac{1}{2} \alpha_{mm} \, \delta_{kl} \, . \tag{I.102}$$

The left-hand side is (for small distortion), from eq. (I.60), equal to minus the curvature tensor K_{kl} . By applying the rotation on the left, we previously got $\boldsymbol{\eta}$; cf., eq. (I.63). Now, we apply the divergence on the right, so the first term drops out (since Ink $\boldsymbol{\imath} \equiv \nabla \times \boldsymbol{\imath} \times \nabla$):

$$\Delta \vartheta_k = \nabla_l \left(\frac{1}{2} \delta_{kl} \alpha_{mm} - \alpha_{kl} \right). \tag{I.103}$$

Here, the right-hand side is known. ϑ_k follows by integration, although it is determined only up to a harmonic vector. The indeterminacy corresponds to the fact that the decomposition of $\boldsymbol{\beta}$ into Grad $\boldsymbol{s}$ + Rot $\boldsymbol{\zeta}$ is not unique in a finite medium, and that one can, if one desires, write the part of the distortion that is described by one of the surface densities $\bar{\boldsymbol{\alpha}}$ in the form Grad $\boldsymbol{s}$ or Rot $\boldsymbol{\zeta}$. On the other hand, in an infinite medium ϑ_k is determined uniquely by eq. (I.103) (assuming that $\boldsymbol{\alpha}$ vanishes at infinity).

As we already remarked in connection with eq. (I.94), a rotation tensor is already determined by its anti-symmetric part. The integration of eq. (I.103) thus delivers, after a brief additional calculation, the rotation tensor with $\boldsymbol{\vartheta}$ as the anti-symmetric part; we call it $\tilde{\boldsymbol{\vartheta}}$. This gives the grain boundary part of the structural rotations. Now:

$$\boldsymbol{\varepsilon}^{ink} = \text{Ink } \boldsymbol{\imath} - \boldsymbol{\vartheta} - \tilde{\boldsymbol{\vartheta}}, \tag{I.104}$$

where $\boldsymbol{\varepsilon}^{ink}$ is obviously the incompatible part of the deformation. The basic equation thus takes on the form:

$$\text{Rot } \boldsymbol{\varepsilon}^{ink} = \boldsymbol{\alpha} - \text{Rot } \tilde{\boldsymbol{\vartheta}}, \tag{I.105}$$

where the right-hand side is known. By further applying the rotor map on the right, one can thus calculate the incompatibility tensor $\boldsymbol{\eta}$ that belongs to the elastic deformation $\boldsymbol{\varepsilon}$, and which, in the case of small $\boldsymbol{\varepsilon}$, gives the determination of the stresses in a relatively simple way. This method allows one to calculate them from the dislocation density, at least in the case of infinite media, and for large rotations and small deformations. In these calculations, one no longer needs to distinguish between initial and final state.

The case of large rotations and small deformations has not been treated by the first method up to now. Namely, the symmetric part of the rotation tensor contributes to the quantities $\boldsymbol{\eta}$, $\bar{\boldsymbol{\eta}}$, $\overline{\overline{\boldsymbol{\eta}}}$ that are calculated from $\boldsymbol{\alpha}$, $\bar{\boldsymbol{\alpha}}$ by means of the formulas of § 9, and one cannot neglect that part for rotations that are large compared to the deformations. In eq. (I.105), however, the mostly worked out (primär ausgerechnete) part of the rotation tensor will be considered exactly.

The special case that was just treated is certainly at least as important as the case that has not been treated at all up to now, that of stresses in conjunction with large distortions. Indeed, in metalworking one frequently has plastic forming between 10 and 100%, but it must have the form Grad $\boldsymbol{s}^P$, for the most part, since the forming Rot $\boldsymbol{\zeta}^P$ means a simultaneous elastic distortion Rot $\boldsymbol{\zeta}$, whose symmetric part gives rise to stresses. With the relatively weak forces with which one forms plastically, however, elastic formings of 10 and 100% can never be produced; i.e., one can at most regard the deformation part of Rot $\boldsymbol{\zeta}$ as small and thus also those ($\boldsymbol{\varepsilon}$) of $\boldsymbol{\beta}$.

Chapter II

Dislocations in a continuum: statics

Preliminary remarks

Statics is the study of the forces that act on matter, and the problems that are treated in it include, in particular, the computation of internal forces (stresses) in a body that will be aroused by any external influences. For us, these influences will be, above all, dislocations, as well as quasi-dislocations, as in § 6. In the literature up to now one finds almost only computations involving singular dislocation lines, at best, also computations involving surface distributions of dislocations. One can treat this problem quite thoroughly by the methods of classical elasticity theory. This is due to the fact that outside of dislocations the elastic deformation field still has the simple form $\varepsilon_{ij} = (\nabla_i s_j + \nabla_j s_i)/2$ that it has in classical elasticity theory. In the case of dislocations that are distributed over the entire body, by comparison, the elastic deformation can no longer be derived from a displacement field, and a fundamentally new method is necessary if one is to – e.g. – compute the proper stresses σ_{ij} that belong to a dislocation distribution α_{ij} . Now, the equilibrium conditions for elastostatics must also be naturally fulfilled in a body with dislocations, which can be written in the form:

$$\text{Div } \boldsymbol{\sigma} = 0 \tag{II.1}$$

in the absence of external forces. These equations state that the proper stress tensor $\boldsymbol{\sigma}$ is a special tensor, namely, an incompatibility tensor:

$$\boldsymbol{\sigma} = \text{Ink } \boldsymbol{\chi}, \tag{II.2}$$

which follows immediately from eq. (I.42) [1]. The symmetric tensor $\boldsymbol{\chi} \equiv (\chi_{ij})$ is called the "second order tensor of stress functions," since its components are the Maxwell and Morera stress functions [2]. In contrast to the previous observations, the stress functions are also suitable tools for spatial problems in elasticity theory. In the case of three-dimensional distributions of dislocations, where the method of the displacement field breaks down, the stress functions are indeed indispensible. One must then regard them as not only convenient aids to computation since their place in the continuum theory of dislocations is of fundamental significance. This will be best illuminated by the remark that the tensor of stress functions represents the analogue of the much-used vector potential **A** of electrodynamics, with whose help the Maxwell equation div **B** = 0 is satisfied identically in a manner that corresponds to the equilibrium conditions (II.1) in terms of the stress function tensor.

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[1] Eq. (II.2) was first written out by Beltrami [161], but not pursued further. Cf., also Finzi [43] on eq. (II.2).

[2] The additional phrase "second order" shall imply that one must differentiate the stress functions twice in order to obtain the stresses. One uses it often to distinguish these stress functions from each other; see below.

§ 12. The stress function tensor

We define the stress tensor $\boldsymbol{\sigma}$ in the usual way by the differential form:

$$dp_j = \sigma_{ij}\, dF_i\,, \tag{II.3}$$

where dp_j is the force applied to a surface slice dF_i when no displacement of the points of the body comes about as a result of the slice [1].

As far as the state of the body is concerned, the time-independent continuum mechanics of solid bodies will now be controlled by the equations:

$$\text{Rot}\ \boldsymbol{\beta} = \boldsymbol{\alpha} \tag{II.4}$$
$$\text{Div}\ \boldsymbol{\sigma} = -\,\mathfrak{F}, \tag{II.5}$$

to which the equation for energy density:

$$e = \tfrac{1}{2}\,\sigma_{ij}\,\varepsilon_{ij} \tag{II.6}$$

must be added. By comparison, the plastic forming Grad $\boldsymbol{s}^P$, which does not change the state of the body, will not be fixed by these equations. In this section, we are interested only in the state of the body after forming. Along with equations (II.4) to (II.6), comes the material equation, which we will always assume to be Hooke's law:

$$\sigma_{ij} = c_{ijkl}\,\varepsilon_{kl}\,. \tag{II.7}$$

As remarked in § 11, one may, in general, compute with it for large plastic formings on the basis of its validity for metals.

c_{ijkl} is the Hooke tensor of elastic moduli with the symmetry properties:

$$c_{ijkl} = c_{jikl} = c_{ijlk} = c_{klij}\,. \tag{II.8}$$

In the case of elastic isotropy, it is:

$$c_{ijkl} = \lambda\,\delta_{ij}\,\delta_{kl} + \mu\,(\delta_{ik}\,\delta_{jl} + \delta_{il}\,\delta_{jk}), \tag{II.9}$$

where λ, μ are the Lamé constants. We define the tensor s_{ijkl} of elastic coefficients that is reciprocal to c_{ijkl} by:

$$c_{ijkl}\,s_{ijkl} = \tfrac{1}{2}(\delta_{ik}\,\delta_{jl} + \delta_{il}\,\delta_{jk}). \tag{II.10}$$

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[1] In contrast to isotropic bodies, asymmetric stress tensors also play a certain role in crystals, which we will consider only in § 19. In the remaining paragraphs we will assume that the stress tensor is symmetric in order to not complicate the presentation unnecessarily. Thus, it is good to also establish the index notation that appeared in eq. (II.3). (The first index of σ_{ij} characterizes the surface element, and the second index, the applied force.)

For isotropy, one then has:

$$s_{ijkl} = \lambda' \, \delta_{ij} \, \delta_{kl} + \mu'(\delta_{ik} \, \delta_{jl} + \delta_{il} \, \delta_{jk}) \tag{II.11}$$

with:

$$\lambda' = -\frac{1/2G}{m+1}, \qquad \mu' = 1/4 \, G, \tag{II.12}$$

where $G = \mu$ is the shear modulus and m is the transverse contraction number (Querkontraktionszahl). Hooke's law then takes on the forms:

$$2 \, G \, \varepsilon_{ij} = \sigma_{ij} - \frac{1}{m+1} \sigma_{kk} \, \delta_{ij} \, , \qquad \sigma_{ij} = 2G\left(\varepsilon_{ij} + \frac{1}{m-2} \varepsilon_{kk} \delta_{ij} \right). \tag{II.13}$$

In § 6, it was shown that the incompatibility tensor associated with a distribution of dislocations can be calculated in a simple way. For that reason, instead of eq. (II.4) we shall consider equation (I.52):

$$\text{Ink } \boldsymbol{\varepsilon} = \boldsymbol{\eta}, \tag{II.14}$$

from which, as in § 6, the structure rotations are excluded. The equilibrium conditions will now be satisfied identically with the stress function Ansatz (II.2), and no longer need to be considered in further investigations. In Cartesian coordinates, eq. (II.2) is written:

$$\left.\begin{aligned} \sigma_{11} &= -\frac{\partial^2 \chi_{22}}{\partial x_3^2} - \frac{\partial^2 \chi_{33}}{\partial x_2^2} + 2\frac{\partial^2 \chi_{23}}{\partial x_2 \partial x_3} \\ \sigma_{12} &= -\frac{\partial}{\partial x_3}\left(-\frac{\partial \chi_{12}}{\partial x_3} + \frac{\partial \chi_{23}}{\partial x_1} + \frac{\partial \chi_{31}}{\partial x_2} \right) + \frac{\partial^2 \chi_{33}}{\partial x_1 \partial x_2}, \end{aligned}\right\} \tag{II.15}$$

from which four more equations follow by cyclic permutation of the indices. If one sets $\chi_{11} = \chi_{22} = \chi_{33} = 0$ then one has the well-known Ansatz of Morera [102] before one, and with $\chi_{12} = \chi_{23} = \chi_{31} = 0$, obtains the Maxwell Ansatz [99]. If one assumes that $\partial/\partial x_3 = 0$ – i.e., a state of planar stress – then what remains of eqs. (II.15) are:

$$\sigma_{11} = -\frac{\partial^2 \chi_{33}}{\partial x_2^2} \, , \qquad \sigma_{22} = -\frac{\partial^2 \chi_{33}}{\partial x_1^2} \, , \qquad \sigma_{12} = \frac{\partial^2 \chi_{33}}{\partial x_1 \, \partial x_2} \tag{II.16}$$

$$\sigma_{23} = -\frac{\partial}{\partial x_1}\left(-\frac{\partial \chi_{23}}{\partial x_1} + \frac{\partial \chi_{31}}{\partial x_2} \right), \qquad \sigma_{31} = \frac{\partial}{\partial x_2}\left(-\frac{\partial \chi_{23}}{\partial x_1} + \frac{\partial \chi_{31}}{\partial x_2} \right) \tag{II.16′}$$

$$\sigma_{33} = -\frac{\partial^2 \chi_{11}}{\partial x_2^2} - \frac{\partial^2 \chi_{22}}{\partial x_1^2} + 2\frac{\partial^2 \chi_{12}}{\partial x_1 \, \partial x_2} \, . \tag{II.16″}$$

Here, eq. (II.16) represent precisely the Airy stress function Ansatz for the planar stress problem [1]. If one sets the bracketed expressions in eq. (II.16′) equal to a function Φ then it is the well-known stress function of torsion [2]. One observes that each stress function χ_{ij} appears in only one of the three rows in (II.16), which means that the associated stress states are independent of each other, at least as far as the equilibrium conditions are concerned.

Maxwell [99] and Morera [102] have shown that one can describe each stress state with Div $\boldsymbol{\sigma} = 0$ using their three functions. The fact that the symmetric tensor $\boldsymbol{\chi}$ then includes only three degrees of freedom thus rests on the fact that, from eq. (I.42), an expression $\boldsymbol{\chi}^0 = \text{Def}\, \boldsymbol{q}$ does not contribute to $\boldsymbol{\sigma}$. A stress function tensor of the form Def $\boldsymbol{q}$ is thus also called a "tensor of null stress functions" [126]. For that reason, one can subject the tensor $\boldsymbol{\chi}$ to certain supplementary conditions; the Maxwell ones are $\chi_{12} = \chi_{23} = \chi_{31} = 0$, and the Morera ones are $\chi_{11} = \chi_{22} = \chi_{33} = 0$. In all cases, however, one must prove that these supplementary conditions are actually "supplementary." We say that the supplementary conditions are extra when one can describe any arbitrary stress state that satisfies the equation Div $\boldsymbol{\sigma} = 0$ with the totality of all stress functions that are restricted by the supplementary conditions.

The equilibrium conditions are fulfilled by the stress function Ansatz. The extended compatibility conditions (II.14) impose additional restrictions on the stress functions. One arrives at them when one introduces the $\boldsymbol{\sigma}$ of eq. (II.2) into eq. (II.14) with the help of Hooke's law (II.7):

$$\text{Ink}\,[s_{ijkl}\,(\text{Ink}\,\boldsymbol{\chi})_{kl}] = \boldsymbol{\eta}\,. \tag{II.17}$$

It is not worthwhile to write out these equations completely. They are already very complicated in the Maxwell and Morera cases, and for that reason these functions are practically never used.

For the further treatment of eq. (II.17) we define – when we restrict ourselves to elastic isotropy – the symmetric tensors χ'_{ij}, η'_{ij} by the equations:

$$2G\chi'_{ij} = \chi_{ij} - \frac{1}{m+2}\,\chi_{kk}\,\delta_{ij}, \qquad \chi_{ij} = 2G\left(\chi'_{ij} + \frac{1}{m-1}\chi'_{kk}\delta_{ij}\right) \tag{II.18}$$

$$\eta'_{ij} = 2G\left(\eta_{ij} + \frac{1}{m-1}\eta_{kk}\delta_{ij}\right), \qquad 2G\eta_{ij} = \eta'_{ij} - \frac{1}{m+2}\eta'_{kk}\delta_{ij}\,. \tag{II.18′}$$

With the supplementary conditions:

$$\nabla_i \chi'_{ij} = 0 \tag{II.19}$$

eq. (II.17) then assumes the simpler form [77]:

$$\Delta\Delta\chi'_{ij} = \eta_{ij} \tag{II.20}$$

or, at the same time:

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[1] Mostly, $\chi = -\chi_{33}$ is defined to be the Airy stress function.
[2] See, e.g., Love [95] or Biezeno-Grammel [1].

$$\Delta\Delta\chi'_{ij} = \eta'_{ij}\,, \tag{II.20′}$$

as we will now show.

First, with the decomposition formula (A.1) one has:

$$\begin{aligned}\eta_{ij} = (\mathrm{Ink}\ \varepsilon)_{ij} &= -\,\varepsilon_{ijk}\,\varepsilon_{jmn}\,\nabla_k\nabla_m\,\varepsilon_{ln} \\ &= \Delta\varepsilon_{ij} - (\nabla_i\nabla_k\,\varepsilon_{kj} + \nabla_j\nabla_k\,\varepsilon_{ki}) + \nabla_k\nabla_l\,\varepsilon_{ij} + \nabla_i\nabla_j\,\varepsilon_{kk} - \Delta\varepsilon_{kk}\,\delta_{ij}\,,\end{aligned} \tag{II.21}$$

and with Hooke's law (II.13) and the equilibrium conditions $\nabla_i\sigma_{ij} = 0$ it easily follows that:

$$\Delta\sigma_{ij} + \frac{m}{m+1}(\nabla_i\nabla_j\,\sigma_{kk} - \Delta\sigma_{kk}\,\delta_{ij}) = 2\,G\,\eta_{ij}\,. \tag{II.22}$$

In the case of $\eta_{ij} = 0$ (from which, it follows that $\Delta\sigma_{kk} = 0$) these equations are known as Beltrami's equations. We now set $\sigma_{ij} = (\mathrm{Ink}\ \chi)_{ij}$, think of these equations as being written in the form corresponding to eq. (II.21) and introduce χ'_{ij} into eq. (II.22). It then results, upon consideration of (II.19), that:

$$\sigma_{ij}/2\,G = \Delta\chi'_{ij} + \frac{m}{m-1}(\nabla_i\nabla_j\chi'_{kk} - \Delta\chi'_{kk}\delta_{ij})\,. \tag{II.23}$$

When this is substituted in eq. (II.22) it yields eq. (II.20) directly, as one easily confirms.

The supplementary conditions (II.19) are sufficient, but not necessary, so eq. (II.17) goes over to eq. (II.20). One obtains the necessary and sufficient conditions when one introduces the σ_{ij} into eq. (II.22) without the assumption (II.19). One then has:

$$\Delta\Delta\chi'_{ij} - \Delta(\nabla_i\nabla_k\chi'_{kj} + \nabla_j\nabla_k\chi'_{ki}) + \frac{m}{m+1}\nabla_i\nabla_j\nabla_k\nabla_l\chi'_{kl} + \frac{1}{m+1}\Delta\nabla_k\nabla_l\chi'_{kl}\delta_{ij} = \eta_{ij}\,, \tag{II.24}$$

and the necessary and sufficient conditions obviously read:

$$-\Delta(\nabla_i\nabla_k\chi'_{kj} + \nabla_j\nabla_k\chi'_{ki}) + \frac{m}{m+1}\nabla_i\nabla_j\nabla_k\nabla_l\chi'_{kl} + \frac{1}{m+1}\Delta\nabla_k\nabla_l\chi'_{kl}\delta_{ij} = 0, \tag{II.25}$$

which is identical with:

$$\mathrm{Def}[(m+1)\Delta\boldsymbol{p} - m\,\nabla\nabla\cdot\boldsymbol{p}] - \Delta\nabla\cdot\boldsymbol{p}\,\boldsymbol{I} = 0, \quad \boldsymbol{p} \equiv \nabla\cdot\boldsymbol{\chi}'. \tag{II.25}$$

They are fulfilled identically by the stronger conditions (II.19). For the proof that the conditions (II.19) are extra, which naturally includes that of the conditions (II.25), we refer to the original work [77], in which the author first posed the conditions (II.19); they were also found independently by Marguerre [98].

§ 13. Solution of the summation problem for proper stresses

The first problem that was solved by the three-dimensional stress functions related to infinitely extended bodies. Let the volume be *V*. One then has a summation problem before one, but not a boundary value problem.

The stress function tensor $\boldsymbol{\chi}$ of the problem in question must fulfill the necessary and sufficient conditions (II.7). We had replaced them with the sufficient conditions (II.20) [(II.20′, resp.] and (II.19). The former of these equations will be satisfied by the expressions [1]:

$$\chi_{ij}(\boldsymbol{x}) = -\frac{1}{8\pi}\iiint_V \eta'_{ij}(\boldsymbol{x}')\,|\,\boldsymbol{x}-\boldsymbol{x}'\,|\,dV' \qquad \text{(II.26)}$$

$$\chi'_{ij}(\boldsymbol{x}) = -\frac{1}{8\pi}\iiint_V \eta_{ij}(\boldsymbol{x}')\,|\,\boldsymbol{x}-\boldsymbol{x}'\,|\,dV', \qquad \text{(II.26′)}$$

as is known from the theory of the bipotential equation. That the supplementary conditions (II.19) are also fulfilled by these equations follows easily from the identity $\nabla_i \eta_{ij} = 0$.

In the examples of the Maxwell and Morera functions we have seen that the tensor Ink$\boldsymbol{\chi}$ has, in reality, three degrees of freedom and not six. That is true for any incompatibility tensor, hence, also for $\boldsymbol{\eta}$ and $\boldsymbol{\eta}'$. Thus, the six integrations may be reduced to three as follows:

As remarked above, the $\boldsymbol{\chi}'$ of eq. (II.26′) is an incompatibility tensor (Div $\boldsymbol{\chi}'$ = 0) because $\boldsymbol{\eta}$ is one. Likewise, one easily shows that the $\boldsymbol{\chi}$ in eq. (II.26) becomes a deformator when one substitutes such a tensor for $\boldsymbol{\eta}'$. However, since $\boldsymbol{\sigma}$ = Ink $\boldsymbol{\chi}$, according to eq. (I.42), a deformator does not contribute to the stresses. We thus obviously obtain the same stresses when we add an arbitrary deformator to the actual $\boldsymbol{\eta}'$ and substitute the resulting tensor ($\boldsymbol{\eta}''$) in place of $\boldsymbol{\eta}'$ in (II.26). However, we can choose $\boldsymbol{\eta}''$ such that, e.g., $\eta''_{11} = \eta''_{22} = \eta''_{33} = 0$ or $\eta''_{12} = \eta''_{23} = \eta''_{31} = 0$. We thus obtain a representation in the Morera (Maxwell, resp.) functions, which proves to be a particularly convenient tool for ascertaining the proper stress states.

The computation of $\boldsymbol{\eta}''$ is now quite simple. If we set:

$$\boldsymbol{\eta}'' = \mathrm{Def}\,\boldsymbol{a} + \boldsymbol{\eta}' \qquad \text{(II.27)}$$

then we shall have, e.g.:

$$\partial a_1 \,|\, \partial x_1 = -\eta'_{11}, \qquad \partial a_2 \,|\, \partial x_2 = -\eta'_{22}, \qquad \partial a_3 \,|\, \partial x_3 = -\eta'_{33}, \qquad \text{(II.28)}$$

[1] If one substitutes eq. (II.26) in (II.2) then, after performing the differentiations one obtains the stresses as functions of the incompatibility distribution. These formulas were first given by Moriguti [103] (without the use of the stress function tensor) and proved by direct verification. I warmly thank Herrn Dr. J. D. Eshelby for the fact that he brought the works of Moriguti to my attention (in March 1957).

from which one obtains suitable functions a_1, a_2, a_3 by ordinary integration. If one substitutes them in eq. (II.27) then one obtains $\boldsymbol{\eta}'' = (\eta''_{ij})$ with $i \neq j$. The Morera stress functions of the proper stress state follow from this:

$$\chi_{ij}(\boldsymbol{x}) = -\frac{1}{8\pi}\iiint_V \eta''_{ij}(\boldsymbol{x}')\,|\,\boldsymbol{x}-\boldsymbol{x}'\,|\,dV', \qquad i \neq j. \tag{II.29}$$

The simple Morera formulas are then valid for the stresses:

$$\left.\begin{aligned} \sigma_{11} &= 2\frac{\partial^2 \chi_{23}}{\partial x_2\,\partial x_3}, \text{etc.} \\ \sigma_{23} &= -\frac{\partial}{\partial x_1}\left(-\frac{\partial \chi_{23}}{\partial x_1}+\frac{\partial \chi_{31}}{\partial x_2}+\frac{\partial \chi_{12}}{\partial x_3}\right), \text{etc.} \end{aligned}\right\} \tag{II.30}$$

It is only somewhat simpler for one to establish values for a_1, a_2, a_3 such that:

$$\frac{1}{2}\left(\frac{\partial a_1}{\partial x_2}+\frac{\partial a_2}{\partial x_1}\right) = -\eta'_{12}, \text{ etc.} \tag{II.31}$$

and obtain the Maxwell stress functions of the proper stress states with eq. (II.27):

$$\chi_{ij}(\boldsymbol{x}) = -\frac{1}{8\pi}\iiint_V \eta''_{ij}(\boldsymbol{x})\,|\,\boldsymbol{x}-\boldsymbol{x}'\,|\,dV', \qquad i = j. \tag{II.32}$$

The stresses follow from this according to:

$$\left.\begin{aligned} \sigma_{11} &= -\frac{\partial \chi_{22}}{\partial x_3^2}-\frac{\partial \chi_{33}}{\partial x_2^2}, \text{etc.} \\ \sigma_{23} &= \frac{\partial^2 \chi_{11}}{\partial x_2\,\partial x_3}, \text{etc.} \end{aligned}\right\} \tag{II.33}$$

For finite media, the summation problem is then linked with a boundary value problem. Before we treat it, we must first see whether we can also apply to finite media the method of determining the particular integrals of the differential equations (II.17) that we worked out for infinite media.

One easily shows that in finite media the $\boldsymbol{\chi}'$ of eq. (II.26′) does not generally fulfill the required supplementary condition Div $\boldsymbol{\chi}' = 0$. Thus, it is not clear at this point whether this $\boldsymbol{\chi}'$ represents a solution of eq. (II.17). Since Div $\boldsymbol{\chi}' = 0$ was, however, a supplementary condition, it must give solutions of $\Delta\Delta\boldsymbol{\chi}' = \boldsymbol{\eta}$ for which Div $\boldsymbol{\chi}' = 0$, hence, the eq. (II.17) are also fulfilled.

In order to obtain such a solution one seeks an extension of the function $\boldsymbol{\eta}'$ into the exterior of the medium such that across the boundary surface $\boldsymbol{\eta}'$ is continuous and differentiable and vanishes at infinity sufficiently strongly. Such an extension is not difficult to obtain. We call the function thus established that agrees with $\boldsymbol{\eta}'$ in the body $\boldsymbol{\eta}'_f$. We substitute this function for $\boldsymbol{\eta}'$ in eq. (II.26) and integrate over the infinite space. We thus obtain a stress function field that fulfills the differential equations (II.20′) and the supplementary conditions (III.19) everywhere in the body and thus represents the desired particular solution of eq. (II.17).

We would now like to add to $\boldsymbol{\eta}'_f$, as above, a deformator Def $\boldsymbol{a}$ such that we come to either the Maxwell or the Morera functions. If the method is successful then this deformator might contribute nothing to the stresses. That this is also the case is relatively simple to show, so we skip the proof. Therefore, one also computes a tensor $\boldsymbol{\eta}''_f = \boldsymbol{\eta}'_f +$ Def $\boldsymbol{a}$, where $\boldsymbol{\eta}''_f$ has only three components different from zero. One then has $\Delta\Delta\boldsymbol{\chi} = \boldsymbol{\eta}''_f$, so eq. (II.29) and (II.32), with $\boldsymbol{\eta}''_f$ instead of $\boldsymbol{\eta}''$, give particular integrals to these equations that likewise satisfy eq. (II.17).

It must be remarked that in the case of a given dislocation (incompatibility distribution, resp.) the methods given in this paragraph are practically the only ones for the solution of summation problem [1]. In the case of quasi-dislocations (§ 6), where primarily the impressed deformation $\boldsymbol{\varepsilon}^G$ is given, there are, however, well-known methods in the older theory of temperature stresses of Duhamel [33] and Neumann [112] for computing the associated "quasi-forces" $\mathfrak{F}_f^Q$ with the help of the expression:

$$\nabla_i c_{ijkl} \varepsilon_{kl}^Q = \mathfrak{F}_f^Q , \tag{II.34}$$

as well as the displacement field associated with them that is calculated by the usual methods, from which, the total deformation $\boldsymbol{\varepsilon}^G$ then follows by definition of the deformator. The elastic deformation is then $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^G - \boldsymbol{\varepsilon}^Q$, from which the stresses follow by Hooke's law. These methods may not differ essentially in their use from those of stress functions.

Finally, we introduce another method that is, at present, applicable only to infinite media. In place of (II.2) [2], one sets:

$$\boldsymbol{\sigma} = \text{Rot } \boldsymbol{\varphi}, \tag{II.35}$$

where $\boldsymbol{\varphi} \equiv (\varphi_{ij})$ is the asymmetric stress function tensor of order one (because it is differentiated only once in order to arrive at the stresses). One thus obviously has $\boldsymbol{\varphi} = \boldsymbol{\chi} \times \nabla$, from which it follows that:

$$\varphi_{ii} = 0, \qquad \nabla_j \varphi_{ij} = 0. \tag{II.36}$$

[1] Cf., on this, also Eshelby [41], pp. 91 et seq.

[2] Günther [61] and Schaefer [126] employ a stress function tensor $\gamma_{jk} = \varepsilon_{jil} \varphi_{lk}$ for a different purpose.

Taking the rotation on the right further converts the supplementary conditions (II.19), when written in terms of χ with the help of (II.18), into:

$$\nabla_i \, \varphi_{ij} = 0, \tag{II.37}$$

as one easily checks. Since $\boldsymbol{\chi}$, with the restriction (II.19), has three degrees of freedom, $\boldsymbol{\varphi}$, when restricted by the conditions (II.36) and (II.37), likewise has three. With Hooke's law, it follows from eq. (II.35) that:

$$\varepsilon_{ij} = s_{ijkl} \, \varepsilon_{kmn} \, \nabla_m \, \varphi_{nl} \, , \tag{II.38}$$

and taking the rotation on the left gives:

$$\varepsilon_{ghi} \, \nabla_h \, \varepsilon_{ij} \equiv \alpha'_{gj} = s_{ijkl} \, \varepsilon_{ghi} \, \varepsilon_{kmn} \, \nabla_h \, \nabla_m \, \varphi_{nl} \, , \tag{II.39}$$

which in the case of elastic isotropy will become, from eq. (II.11):

$$\alpha'_{gj} = \lambda' \, \varepsilon_{ghi} \, \varepsilon_{kmn} \, \nabla_h \, \nabla_m \, \varphi_{nl} + \mu' (\varepsilon_{ghk} \, \varepsilon_{kmn} \, \nabla_h \, \nabla_m \, \varphi_{nj} + \varepsilon_{ghl} \, \varepsilon_{jmn} \, \nabla_h \, \nabla_m \, \varphi_{nl}). \tag{II.40}$$

Multiplication by ε_{fgj} gives, with eq. (A.2, 3) and (II.36):

$$\varepsilon_{fgj} \, \alpha'_{gj} = -\, 2(\lambda' + \mu') \, \varepsilon_{lmn} \, \nabla_h \, \nabla_m \, \varphi_{nl} \, . \tag{II.41}$$

We substitute this in eq. (II.40) after replacing the indices *f*, *g*, *j* with *h*, *p*, *q* and finally obtain, when we apply the decomposition formula (A.2) and consider (II.36, 37) [1]:

$$\Delta \varphi_{ij} = -\, \frac{2G}{m-1} (m \alpha'_{ij} - \alpha'_{ji}) \, . \tag{II.42}$$

What is α'_{ij} then? We assume the decomposition:

$$\boldsymbol{\varepsilon} = \mathrm{Def} \, \boldsymbol{s} + \mathrm{Ink} \, \boldsymbol{\iota}, \tag{II.43}$$

which is unique in an infinite medium, and express the equation $\mathrm{Div} \, \boldsymbol{\sigma} = 0$ with the help of Hooke's law in terms of $\boldsymbol{s}$ and $\boldsymbol{\iota}$ [2]. With (I.42), one easily obtains:

$$\Delta s_i + \frac{m}{m-2} \nabla_i \nabla_j s_j + \frac{2}{m-2} \nabla_i (\mathrm{Ink} \, \boldsymbol{\iota})_{jj} = 0 \tag{II.44}$$

and by taking the rotation, it follows that:

$$\Delta \, \mathrm{rot} \, \boldsymbol{s} = 0. \tag{II.45}$$

---

[1] Due to (II.36, 37), one has $\varepsilon_{ghl} \, \varepsilon_{jmn} \, \nabla_h \, \nabla_m \, \varphi_{nl} = -\, \Delta \varphi_{gj}$, which follows easily from (A.1).

[2] We thus restrict ourselves to small deformations and rotations.

In infinite media, one thus has rot $\boldsymbol{s}$ = const. It then follows that:

$$\text{Rot Def } \boldsymbol{s} \equiv \tfrac{1}{2} \nabla \times (\nabla \boldsymbol{s} + \boldsymbol{s} \nabla) = \tfrac{1}{2} (\text{rot } \boldsymbol{s}) \nabla = 0. \tag{II.46}$$

and one has:

$$\boldsymbol{\alpha}' \equiv \text{Rot } \boldsymbol{\varepsilon} = \text{Rot Ink } \boldsymbol{\iota}. \tag{II.47}$$

That is, $\boldsymbol{\alpha}'$ is, from § 5, that part of the total dislocation density that is responsible for the stresses. In the case of quasi-dislocations $\boldsymbol{\alpha}^Q$ of § 6, $\boldsymbol{\alpha}'$ if often directly equal to $\boldsymbol{\alpha}^Q$. In § 11, we showed that, above all, one can compute $\boldsymbol{\alpha}'$ when one computes the rotations ($\boldsymbol{\alpha}'$ is equal to the right-hand side of eq. (I.105)). Therefore, we can henceforth regard the $\boldsymbol{\alpha}'$ as given functions. From eq. (II.47), it easily follows that $\boldsymbol{\alpha}'$ submits to the same conditions (II.36) and (II.37) as $\boldsymbol{\varphi}$. This means that the particular integral of eq. (II.42):

$$\varphi_{ij}(\boldsymbol{x}) = \frac{G/2\pi}{m-1} \iiint_V [m \alpha'_{ij}(\bar{\boldsymbol{x}}') - \alpha'_{ji}(\bar{\boldsymbol{x}}')] / |\boldsymbol{x} - \boldsymbol{x}'| \, dV' \tag{II.48}$$

also fulfills these conditions, and thus gives the correct stress function tensor that goes with α'_{ij}, from which, eq. (II.35) gives the stresses.

One can easily reduce the nine integrations in eq. (II.48) to six, and one can similarly reduce the six integrations (II.26) to three. Whether the integrations (II.48) can generally be reduced to three must remain open. Up to now, the stress functions φ_{ij} have not been explored at all, so we believe, on deeper grounds, that such an exploration would be worthwhile. Namely, if we heuristically write $\boldsymbol{n} \times \boldsymbol{\varepsilon} = -\bar{\boldsymbol{\alpha}}'$ for the body bounded by a rigid neighborhood then, from Div $\boldsymbol{\sigma} = 0$, the elastic energy of this body is represented in the form:

$$E = \frac{1}{2} \iiint_V \alpha'_{ij} \varphi_{ij} \, dV + \frac{1}{2} \iint_F \bar{\alpha}'_{ij} \varphi_{ij} \, dF, \tag{II.49}$$

or also:

$$E = \frac{1}{2} \iiint_V \alpha_{ij} \varphi_{ij} \, dV + \frac{1}{2} \iint_F \bar{\alpha}_{ij} \varphi_{ij} \, dF, \tag{II.50}$$

as will be shown in the next paragraph. E is then expressed by means of the dislocations that generate the proper stresses.

If, e.g., φ^1_{ij} and φ^2_{ij} is the stress function field that is produced by two isolated dislocations α^1_{ij} and α^2_{ij} in an infinite medium then the energy may be written:

$$E = \frac{1}{2} \iiint_V \alpha^1_{ij} \varphi^1_{ij} \, dV + \frac{1}{2} \iiint_V \alpha^2_{ij} \varphi^2_{ij} \, dV + \frac{1}{2} \iiint_V \alpha^1_{ij} \varphi^2_{ij} + \frac{1}{2} \iiint_V \alpha^2_{ij} \varphi^1_{ij} \, dV. \tag{II.51}$$

Obviously, the third and fourth integrals mean the potential energy of the dislocation 1 in the field of the dislocation 2, and conversely. One thus arrives at an interpretation for the stress functions φ_{ij}: They represent a potential for the dislocations. (The situation is

analogous to that of electrodynamics, where one has an energy $E = \frac{1}{2}\iiint_V \rho U \, dV$, if ρ is the charge density. U will then denote a potential (for charges)).

The stress functions of second order are correspondingly an elastic potential for incompatibilities.

§ 14. Elastic energy and the variational problem for a medium with proper stresses

We would now like to compute the expression for the elastic energy of a medium with proper stresses, as expressed in terms of stress functions and incompatibilities. The starting point is the formula:

$$E = \frac{1}{2}\iiint_V \sigma_{ij}\varepsilon_{ij} dV \, , \tag{II.51}$$

which, from eq. (II.2), can be written:

$$E = -\frac{1}{2}\varepsilon_{ijk}\varepsilon_{lmn}\iiint_V \varepsilon_{ij}\nabla_j\nabla_m\chi_{kn} dV \, . \tag{II.52}$$

Partial integration gives:

$$E = -\frac{1}{2}\varepsilon_{ijk}\varepsilon_{lmn}\left[\iiint_V n_j\varepsilon_{il}\nabla_m\chi_{kn} dF - \iiint_V (\nabla_j\varepsilon_{il})\nabla_m\chi_{kn} dV\right], \tag{II.53}$$

which is identical to eq. (II.49) [1]. Since one can write the distortions β_{ij} in place of the deformations in the starting equation (II.51) (due to the symmetry of σ_{ij}), eq. (II.50) is thus also proved.

After partial integration of the volume integral in (II.53) this yields:

$$E = -\frac{1}{2}\varepsilon_{ijk}\varepsilon_{lmn}\left[\iint_F n_j\varepsilon_{il}\nabla_m\chi_{kn} dF - \iint_F (\nabla_j\varepsilon_{il})n_m\chi_{kn} dF\right] + \frac{1}{2}\iiint_V \chi_{kn}\eta_{kn} dV \, , \tag{II.54}$$

where the relation (II.21) was used. Here, we decompose the ∇_m in the first integral, with the help of eq. (A.2), into the easily verified formula:

$$\nabla_m = n_m\nabla_p \overset{\downarrow}{n}_p + \varepsilon_{mpq}\varepsilon_{rsq}\nabla_r \overset{\downarrow}{n}_p \, , \tag{II.55}$$

[1] One has $\varepsilon_{lmn}\nabla_m\chi_{kn} = -\varphi_{kl}$, $\varepsilon_{ijk}\nabla_j\varepsilon_{il} = -\alpha'_{kl}$, $\varepsilon_{ijk}n_j\varepsilon_{il} = \bar{\alpha}'_{kl}$.

where the arrow shall mean that in addition to the function upon which ∇_m acts, n_p will also be differentiated. The integral defined in the second summand of (II.55) will be once again partially integrated by applying Stokes's theorem. The associated line integral vanishes since F is a closed surface. With the abbreviation $\overset{n}{\nabla}_m \equiv \varepsilon_{mpq}\, \varepsilon_{rsq}\, n_s\, n_p\, \nabla_m$ of eq. (II.82), what remains is:

$$E = -\frac{1}{2}\varepsilon_{ijk}\varepsilon_{lmn}\left[\iint_F n_j \varepsilon_{il} n_m \nabla_p (n_p \chi_{kn})\, dF - \iint_F \chi_{kn}\left\{n_m \nabla_j \varepsilon_{il} + \overset{n}{\nabla}_m (n_j \varepsilon_{il})\right\} dF\right] + \frac{1}{2}\iiint_V \chi_{kn}\eta_{kn}\, dV\,. \tag{II.56}$$

Comparison with eqs. (I.87) and (I.90) gives, in the case of a body that is attached to a rigid neighborhood:

$$E = \frac{1}{2}\iiint_V \chi_{ij}\eta_{ij}\, dV + \frac{1}{2}\iint_F \chi_{ij}\bar{\eta}_{ij}\, dF + \frac{1}{2}\iint_F \nabla_k (n_k \chi_{ij})\, \overline{\overline{\eta}}_{ij}\, dF\,. \tag{II.57}$$

In this equation, the n_k are the Cartesian components of the unit normal vectors $\boldsymbol{n}$ of the family of surfaces n (§ 8), one of which is the boundary surface F. This interpretation of the n_k is necessary in order to be able to carry out the differentiations $\nabla_k\, n_k$ according to the rules if $\boldsymbol{n}$ also has meaning outside of F, as well (it is enough that $\boldsymbol{n}$ be defined in an infinitesimal neighborhood of F).

Eq. (II.57) says that in the absence of volume forces in a body that is fixed in a rigid neighborhood the elastic energy, and therefore the proper stresses, vanish when the incompatibility vanishes.

We now treat bodies bounded by air. From a well-known theorem of Colonetti [17], the elastic energy of a body that is subject to both external forces and proper stresses is additively composed of the elastic energies of the two parts; in our expression:

$$E(\boldsymbol{\eta}, \mathfrak{F}) = E(\boldsymbol{\eta}) + E(\mathfrak{F}), \tag{II.58}$$

where $\mathfrak{F}$ shall stand for volume and boundary forces [1]. The transition from eq. (II.51) to (II.52) is valid in the absence of volume forces. Thus, eqs. (II.52), et seq., include a part

[1] The theorem is also true for the medium embedded in a rigid region (one can then include the surface incompatibilities in $\boldsymbol{\eta}$, as well). Namely, if one sets $\boldsymbol{\sigma} = \boldsymbol{\sigma}^L + \boldsymbol{\sigma}^E$, $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^L + \boldsymbol{\varepsilon}^E$, where L suggests the load stresses, and E, the proper stresses, then one distinguishes the $E(\boldsymbol{\sigma})$ computed in eq. (II.51) from $E(\boldsymbol{\sigma}^L)$ + $E(\boldsymbol{\sigma}^E)$ by the interaction term $\iiint_V \sigma_{ij}^E \varepsilon_{ij}^L\, dV$ (Betti's theorem), and since $\nabla_i \sigma_{ij}^E = 0$ it may be brought into the form that corresponds to (II.57):

$$E^{LE} = \iiint_V \chi_{ij}^E \eta_{ij}^L\, dV + \iint_F \chi_{ij}^E \bar{\eta}_{ij}^L\, dF + \iint_F \nabla_k (n_k \chi_{ij}^E)\, \overline{\overline{\eta}}_{ij}^L\, dF\,.$$

due to the boundary forces. On the other hand, as already remarked in § 8, the surface incompatibilities are null on a free outer surface. One can now show [1] that the outer surface integral in eq. (II.54) vanishes in the absence of boundary forces when $n_i\ \eta_{ij} = 0$, such that all that remains is:

$$E = \frac{1}{2}\iiint_V \chi_{ij}\eta_{ij}\,dV \qquad \text{when} \qquad n_i\ \eta_{ij} = 0. \tag{II.59}$$

This equation includes the theorem that in simply connected bodies, and in the regime of linear elasticity theory all proper stresses will lead back to incompatibilities. Eq. (II.57) includes the same theorem for bodies that are embedded in a rigid region. However, it is also still true when one creates new boundary surfaces in which one allows η_{ij} to degenerate to surface quantities (or even linear quantities) in the interior of the body. Thus, the theorem that all proper stresses will be produced by incompatibilities is obviously valid in complete generality in the domain of the linear elasticity theory. Furthermore, the converse is true that all (symmetric) incompatibilities provoke proper stresses, which is self-explanatory from the meaning of the incompatibilities as derivatives of elastic deformations.

Finally, the question of the uniqueness of the solutions is still important. Thank to the Kirchhoff uniqueness theorem of classical elasticity theory and the theorem of Colonetti, it suffices to assert that in the absence of external forces the proper stresses (that follow from the stress functions) are uniquely determined by the givens of the incompatibilities. The proof is adapted to the infinite media immediately, since in that case eq. (II.26) is the unique solution to eq. (II.20) and (II.19) as long as no incompatibilities lie at infinity. By contrast, in a finite medium it must be shown that the boundary value problem that emerges is uniquely soluble. In the next paragraph, we must show that both of the well-known boundary-value problems of elasticity for which the uniqueness proof is indeed in the literature also yield proper stresses. Thus, one establishes in the regime of linear elasticity that the stresses in a body are uniquely determined by external forces and incompatibilities that act on them. In particular, all stress causes will lead back to these two influences.

The bodies considered up to now were simply connected. All of the theorems of this paragraph are also valid for multiply connected bodies when one allows the incompatibilities that are found there to also be found outside the body. (This situation is well-known from hydrodynamics. For the current around a body, one can think of sources or vortices as being present in the body!) In the regime of nonlinear elasticity theory, by comparison, not all stresses can lead back to external forces or proper stresses, as the example of the everted hemispherical shell shows [160].

That variational problem of proper stresses was already formulated by Colonetti [19]. In our notation, the variation of the energy expression:

Since $\bar{\eta}_{ij}^{L}$, $\bar{\bar{\eta}}_{ij}^{L}$, and η_{ij}^{L} vanish, $E^{LE} = 0$.

[1] Cf., Southwell [177] and Eshelby [41], pp. 93.

$$\iiint_V \left(\frac{1}{2} \varepsilon_{ij} \sigma_{ij} + \varepsilon_{ij}^P \sigma_{ij} \right) dV , \tag{II.60}$$

where ε_{ij}^P is the impressed (plastic or quasi-plastic) deformation. Colonetti called the second summation of eq. (II.60) the "potential of the impressed deformation." If we set $\varepsilon_{ij}^P = \varepsilon_{ij}^G - \varepsilon_{ij}$ in eq. (II.60), as in eq. (1), pp. ?, then it goes over to:

$$- \frac{1}{2} \iiint_V \varepsilon_{ij} \sigma_{ij} dV \tag{II.61}$$

since:

$$\iiint \varepsilon_{ij}^G \sigma_{ij} dV = 0, \tag{II.62}$$

which was first found for the case of a body bounded by air by Rieder [125]. In this case, one has:

$$\begin{aligned} \iiint_V \varepsilon_{ij}^G \sigma_{ij} dV &= \iiint_V (\nabla_i s_j^G) \sigma_{ij} dV \\ &= \iint_F n_i s_j^G \sigma_{ij} dF - \iiint_V s_j \nabla_i \sigma_{ij} dV = 0, \end{aligned} \tag{II.63}$$

since $n_i \sigma_{ij}$ and $\nabla_i \sigma_{ij}$ vanish in the absence of external forces. From Rieder [125], eqs. (II.17) are, moreover, the Euler-Lagrange equations of the variational problem associated with (II.61) (in the absence of being given external forces – i.e., incompatibilities).

In the case of rigid neighborhoods the left-hand side of eq. (II.62), when multiplied by 1/2, goes over to the expression (II.57), where η_{ij}, $\bar{\eta}_{ij}$, $\overline{\overline{\eta}}_{ij}$ are now the incompatibilities that belong to ε_{ij}^G. They vanish from the physical meaning of ε_{ij}^G ($\boldsymbol{\varepsilon}^G$ is a deformator, hence, $\eta_{ij}^G = 0$, and furthermore, s_i^G is null on the boundary of a body embedded in a rigid neighborhood, and therefore so are $\bar{\eta}_{ij}^G$, $\overline{\overline{\eta}}_{ij}^G$; cf., eqs. (I.87) to (I.90)). The variational problem that is associated with the energy expression (II.61) ((II.57), resp.) shall be derived from not only the differential equations (II.17), but also the boundary conditions (I.89), when expressed in terms of χ_{ij}, which was not verified up to now.

For the solution of the variational problem for the body bounded by air with the aid of direct methods, one must observe that there are stress functions for which the associated η_{ij} vanishes, but not, however, the associated boundary forces $n_j \sigma_{ij}$. Such stress functions contribute nothing to the integral (II.59), but only to the integral (II.61). In order to obtain the correct solution one must therefore make the integral (II.57) an extremum while $\boldsymbol{\eta}$ is held fixed [1]. For the case of a body embedded in a rigid region, by

[1] The fact that (II.61) shall be an extremum already appeared in Föppl [44]. The advance here consists in the additional condition "while $\boldsymbol{\eta}$ is held fixed."

comparison, the integral (II.57) is equivalent to that of (II.61). One of these two will be made an extremum while keeping η_{ij}, $\bar{\eta}_{ij}$, $\overline{\overline{\eta}}_{ij}$ constant.

We now briefly consider the case of the two boundary-value problems for the body bounded by air (suppose the summation problem has already been dealt with). The problem of "given boundary displacement" is, from § 8, identical with the problem of "given $\bar{\eta}_{ij}$, $\overline{\overline{\eta}}_{ij}$" for the bodies embedded in rigid regions (cf., the next paragraphs, as well).

If one makes the Ansatz that was first proposed by Schaefer [127] [1]:

$$\chi_{ij} = \omega_{ij} - \omega_{kk}\,\delta_{ij} + \Omega\,\delta_{ij} \tag{II.64}$$

with $\Delta\omega_{ij} = 0$ then, in addition to the equilibrium conditions, equation (II.17), with $\boldsymbol{\eta} = 0$, is automatically fulfilled, only if:

$$\Delta\Omega = \frac{m}{m-1}\nabla_i\nabla_j\omega_{ij}\,; \tag{II.65}$$

ω_{ij} like χ_{ij}, is a symmetric tensor of rank two. If we take our solution of (II.65) to be:

$$\Omega = \frac{m/2}{m-1}x_i\,\nabla_j\,\omega_{ij} + \omega_{kk} + v \tag{II.66}$$

then this gives:

$$\chi_{ij} = \omega_{ij} + H\delta_{ij}\,, \tag{II.67}$$

with:

$$H = v + \frac{m/2}{m-1}\left(x_1\frac{\partial\omega_{11}}{\partial x_1} + x_2\frac{\partial\omega_{22}}{\partial x_2} + x_3\frac{\partial\omega_{33}}{\partial x_3}\right). \tag{II.68}$$

Thus, $\Delta v = 0$. One can show [77] that one may set $v = \omega_{12} = \omega_{23} = \omega_{31} = 0$. With:

$$\chi_{ij} = \omega_{ij} + H\delta_{ij} \qquad\qquad i = j, \tag{II.69}$$

the Maxwell functions for $\boldsymbol{\eta} = 0$ then come down to three harmonic functions. One easily shows that the conditions (II.25), but not (II.19), are fulfilled identically. For the functions ω_{ij} one perhaps assumes them to be a series development in harmonic functions and determines the coefficients by the usual methods in such a way that the boundary conditions, when expressed in terms of the χ_{ij}, are fulfilled as much as is possible. If one adds, should the occasion arise, the stress functions thus obtained to the particular Maxwell functions obtained in § 13 then one obtains the resulting Maxwell stress functions of the stress state in question.

[1] Naturally, these ω_{ij} have nothing to do with the ones that were previous employed.

§ 15. The boundary-value problems that occur for proper stresses and their treatment by means of stress functions

The particular integrals (II.29) and (II.32) of the differential equations (II.17) that are definitive for proper stresses do not, in general, fulfill the boundary conditions for stresses $n_i\ \sigma_{ij} = 0$ in bodies bounded by air, whereas the boundary conditions for deformations (I.89) are not fulfilled in bodies embedded in rigid regions. Thus, in the former case there remains a boundary-value problem of the form:

$$n_i\ \sigma_{ij} = A_j\ , \tag{II.70}$$

and in the latter case, one of the form:

$$\overline{\mathrm{Ink}}\,\boldsymbol{\varepsilon} = \bar{\boldsymbol{\eta}}\ , \qquad \overline{\overline{\mathrm{Ink}}}\,\boldsymbol{\varepsilon} = \bar{\bar{\boldsymbol{\eta}}} \tag{II.71}$$

that remains to be solved. In both cases, the stress functions that are used in them must satisfy eq. (II.17) with $\boldsymbol{\eta} = 0$. From § 12, we replace them with the equations:

$$\Delta\Delta\boldsymbol{\chi}' = 0, \qquad \nabla_i \chi'_{ij} = 0. \tag{II.72}$$

In practice, problems appear in which, say, two sub-regions in a body that are separated by a dislocation wall have different elastic moduli. In such cases, boundary surface conditions must be considered for stresses and deformations simultaneously. The two boundary cases (elastic moduli = 0, ∞, resp., in the sub-volumes II) of these problems are characterized by (II.70) and (II.71), to which one is also always led back ultimately. These remarks shall clarify why we place such value on the bodies embedded in rigid regions.

We now show that one can always replace the boundary-value problem (II.71) with the problem "given boundary displacements." Let them be prescribed, perhaps, by η_{ij}, $\bar{\eta}_{ij}$, $\bar{\bar{\eta}}_{ij}$. One then collects the elastic deformation into two parts: The particular solution ε^O_{ij} and a second part ε^H_{ij}, which satisfies the homogeneous equations:

$$\mathrm{Ink}\ \boldsymbol{\varepsilon} = 0, \qquad \mathrm{Div}\ \boldsymbol{\sigma} = 0, \qquad \sigma_{ij} = c_{ijkl}\ \varepsilon_{kl}\ . \tag{II.73}$$

For $\boldsymbol{\varepsilon}^O$, as well as $\boldsymbol{\varepsilon}^H$, one can formally associate surface incompatibilities $\bar{\boldsymbol{\eta}}^O$, $\bar{\bar{\boldsymbol{\eta}}}^O$ and $\bar{\boldsymbol{\eta}}^H$, $\bar{\bar{\boldsymbol{\eta}}}^H$ according to eq. (II.71). Therefore, since $\boldsymbol{\varepsilon}^H = \boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^O$, one also has $\bar{\boldsymbol{\eta}}^H = \bar{\boldsymbol{\eta}} - \bar{\boldsymbol{\eta}}^O$, etc. The problem is thus to determine the deformations that satisfy eq. (II.72), as well as the boundary conditions (II.71), when written with the index *H*. For sake of simplicity, we henceforth omit the index *H* (in $\boldsymbol{\eta} = 0$, one has $\boldsymbol{\varepsilon}^H = \boldsymbol{\varepsilon}$, anyway).

Due to the first eq. (II.73), $\boldsymbol{\varepsilon}$ has the form Def $\boldsymbol{s}$. One can easily show, although we shall not do so here, that one can integrate the boundary conditions in the boundary surface and bring them into the form:

$$\boldsymbol{s} = \boldsymbol{g}, \tag{II.74}$$

where $\boldsymbol{g}$ follows from $\bar{\boldsymbol{\eta}}$ and $\bar{\bar{\boldsymbol{\eta}}}$. $\boldsymbol{g}$ is that displacement that the outer surface of the medium would undergo if one suddenly removed the pressure on the side of the rigid neighborhood.

Thus, it is clear that one also can also address the boundary-value problems that are known up to now by means of proper stresses. For them, there exist numerous well-known methods of solution, so for that reason we will only speak briefly about the possible application of the three-dimensional stress functions to the solutions of these problems.

The stress function tensor does not represent a single-valued system of functions, and thus includes a much larger manifold than perhaps the displacement vector, which is expressed in the additional supplementary conditions. One thus has the possibility of adapting given problems by a particular choice of supplementary conditions. Furthermore, the Airy stress function has proved itself so well for two-dimensional problems that one would like to arrive, at the least, at a similar method for three-dimensional problems, as well.

This objective has still not been achieved. Meanwhile, the simple form of the energy equation (II.57) leads one to expect that the boundary-value problem defined by eq. (II.71) ((II.74), resp.) can also be treated by stress functions successfully. Namely, if one begins with the classical method of Green then one must start with the Betti theorem, which, by comparison with eq. (II.57), is written in terms of stress functions as:

$$\begin{aligned}\iiint_V \overset{1}{\chi}_{ij}\,\overset{2}{\eta}_{ij}\,dV + \iint_F \overset{1}{\chi}_{ij}\,\overset{2}{\bar{\eta}}_{ij}\,dF + \iint_F \nabla_k(n_k\,\overset{1}{\chi}_{ij})\,\overset{2}{\eta}_{ij}\,dF \\ = \iiint_V \overset{1}{\chi}_{ij}\,\overset{1}{\eta}_{ij}\,dV + \iint_F \overset{2}{\chi}_{ij}\,\overset{1}{\bar{\eta}}_{ij}\,dF + \iint_F \nabla_k(n_k\,\overset{2}{\chi}_{ij})\,\overset{1}{\bar{\bar{\eta}}}_{ij}\,dF\,.\end{aligned} \tag{II.75}$$

In it, one then identifies η^1_{ij}, $\bar{\eta}^1_{ij}$, $\bar{\bar{\eta}}^1_{ij}$ with the given incompatibilities and χ^1_{ij} with the desired stress function tensor, while one has substituted the fundamental integral (principal solution) of the differential equations (II.17) for χ_{ij} for χ^2_{ij}. Naturally, one would therefore like to expect that these equations go over to the corresponding supplementary conditions in the form $\Delta\Delta\chi_{ij} = 0$, since the fundamental integral of this equation is well-known to be given in the form $|\boldsymbol{x} - \boldsymbol{x}'|/8\pi$. One can thus hope that the entire boundary-value problem somehow comes down to the determination of a biharmonic Green function in the domain in question. The difficulty is, *inter alia*, that one presently does not know how one can provide for the fulfillment of the supplementary conditions, which indeed first guarantee the satisfaction of the differential equations (II.17).

The situation for the second boundary-value problem is similar. Written in terms of χ_{ij}, the boundary conditions (II.70) read:

$$\varepsilon_{ijk}\,\varepsilon_{lmn}\,n_j\,\nabla_j\,\nabla_m\,\chi_{kn} = A_l\,, \tag{II.76}$$

which one must simultaneously fulfill along with eq. (II.17). If one substitutes them in eq. (II.72) then one must somehow guarantee the fulfillment of $\nabla_i\chi'_{ij} = 0$. One can do this in such a way that perhaps one prescribes:

$$\nabla_i \chi'_{ij} = 0, \qquad \frac{\partial}{\partial n}(\nabla_i \chi'_{ij}) = 0 \qquad \text{on the boundary,} \tag{II.77}$$

in addition to the boundary conditions (II.76). Then, along with (II.72), one also has:

$$\Delta\Delta(\nabla_i \chi'_{ij}) = 0, \tag{II.78}$$

from which it is well-known that it follows that with the boundary values (II.77), $\nabla_i \chi'_{ij}$ likewise vanishes in the entire volume. The biharmonic problem that is defined by the boundary conditions (II.76) and (II.77) has still not been treated up to now.

On the other hand, one finds a field χ_{ij} that satisfies the boundary conditions (II.76) and the differential equation $\Delta\Delta\chi_{ij} = 0$ relatively easily. For this, one again needs only to construct the biharmonic Green function. It does not seem out of the question that one will find completely, without the all-too-great difficulties, the field that one must add to these χ_{ij} so that, in addition, the supplementary conditions – and thus, also the differential equations (II.17) – are fulfilled.

§ 16. Extension to elastic anisotropy, force couples

The metallic crystals, to which dislocation theory will be applied, are, in many cases, strongly anisotropic elastically, a fact that one must always keep in mind. For that reason, Burgers [13] has already applied anisotropic elasticity theory to dislocations. We now summarize the most important formulas that allow one to not only treat dislocations by means of elastic anisotropy, but also to lay the foundations for the treatment of other important elastic singularities.

In the region outside the singularities, first let $\varepsilon_{ij} = \frac{1}{2}(\nabla_i s_j + \nabla_j s_i)$. If we substitute this into the equilibrium conditions (II.5), with the aid of Hooke's law, then we obtain:

$$D_{jl}\, s_l + \mathfrak{F}_j = 0, \qquad D_{jl}(\nabla) \equiv c_{ijkl}\, \nabla_i \nabla_k . \tag{II.79}$$

Let $f(\nabla)$ be the determinant $| D_{ik} |$, and let $D^*_{ij}(\nabla)$ be the symmetric tensor of sub-determinants of f; i.e., $D_{jl} D^*_{lk} = f\, \delta_{jk}$. With:

$$s_l = D^*_{kl} h_k , \tag{II.80}$$

from eq. (II.79), one will have:

$$f\, h_j + \mathfrak{F}_j = 0. \tag{II.81}$$

For the case of an isolated force P_j at the position $\boldsymbol{x}'$, if we write:

$$\mathfrak{F}_j(\boldsymbol{x}) = P_j\, \delta(\boldsymbol{x} - \boldsymbol{x}'), \quad \delta(\boldsymbol{x} - \boldsymbol{x}') \equiv \delta(x_1 - x'_1)\, \delta(x_2 - x'_2)\, \delta(x_3 - x'_3) \tag{II.82}$$

then we have:

$$f\, h_j + P_j\, (\boldsymbol{x}')\, \delta(\boldsymbol{x} - \boldsymbol{x}') = 0. \tag{II.83}$$

By means of the equation:

$$f\,U + \delta(\boldsymbol{x}) = 0, \tag{II.84}$$

one defines the fundamental integral (principal solution) $U(\boldsymbol{x})$ of the linear, sixth-order, homogeneous, differential equation $f\,u = 0$ in an infinite medium (and thus, it is unique up to an uninteresting entire function of degree five in $\boldsymbol{x}$).The knowledge of U likewise implies that of the particular integral of (II.81):

$$h_j(\boldsymbol{x}) = \iiint_V U(x)\mathfrak{F}_j(\boldsymbol{x}')\,dV', \qquad x \equiv |\,\boldsymbol{x} - \boldsymbol{x}'\,|, \tag{II.86}$$

and thus for the isolated force in an infinite medium, one has:

$$h_j(\boldsymbol{x}) = U(x)\,P_j(\boldsymbol{x}'). \tag{II.86}$$

The associated displacement field results from (II.80):

$$s_j(\boldsymbol{x}) = S_{ij}(x)\,P_i(\boldsymbol{x}'), \qquad S_{ij} = D^*_{ij}U\,. \tag{II.87}$$

The symmetric tensor S_{ij} is the fundamental integral of the elastic differential equations (II.79) for the displacements. With its help, the particular solution of eq. (II.79) reads:

$$s_j(\boldsymbol{x}) = \iiint_V S_{ij}(x)\mathfrak{F}_i(\boldsymbol{x}')\,dV'. \tag{II.88}$$

The physical meaning of the components of S_{ij} follows easily from eq. (II.87) when one assumes that $|\,P_i\,| = 1$. S_{ij} is then the j-component of the associated displacement.

One says, let:

$$s_k(\boldsymbol{x}) = P_i(\boldsymbol{x}')\,\nabla_i\,S_{jk}(x) \tag{II.89}$$

be the displacement at the position $\boldsymbol{x}$ that is provoked by a force couple P_{ij} at the position $\boldsymbol{x}'$. We also call the not-necessarily-symmetric tensor P_{ij} the "force dipole." Its first index gives the direction in which the two equal and opposite isolated forces, whose points of application lie l_i from each other and whose direction is described by the second index, point when they come together; i.e.:

$$P_{ij} = \lim_{l_i \to 0,\, P_j \to \infty} l_i P_j\,. \tag{II.90}$$

The diagonal components of P_{ij} are force couples without moments, and the remaining components have moments about an axis that is perpendicular to the i and j direction. The total rotational moment is included in the anti-symmetric part of P_{ij} . For further information on force couples, cf., Love [95].

The displacement:

$$s(\boldsymbol{x}) = P_{ijk}(\boldsymbol{x}')\,\nabla_i\,\nabla_j\,S_{kl}(x) \tag{II.91}$$

will be provoked by a force quadrupole P_{ijk} , and one can define higher multipoles in a corresponding way.

In the case of elastic isotropy, one computes, from eq. (II.9):

$$D_{ij} = (\lambda + \mu)\nabla_i \nabla_j + \mu \Delta\delta_{ij} \tag{II.92}$$

$$D^*_{ij} = [-\mu(\lambda + \mu)\nabla_i \nabla_j + \mu(\lambda + 2\mu) \Delta\delta_{ij}]\Delta \tag{II.93}$$

$$f = \mu^2(l + 2\mu) \Delta\Delta\Delta, \qquad U = \frac{1}{96\pi\mu^2(\lambda+2\mu)} x^3; \tag{II.94}$$

hence, with $\Delta x^3 = 12x$:

$$S_{ij} = \frac{1}{8\pi\mu}\left(-\frac{\lambda+\mu}{\lambda+2\pi}\nabla_i\nabla_j + \delta_{ij}\Delta\right)x\,. \tag{II.95}$$

Here, the components of S_{ij} thus become elementary functions of x. The same is still true only for hexagonal symmetry [76, 180].

We now briefly describe the method of stress functions for anisotropy [80], which again has the advantage of being applicable to continuously distributed dislocations. One sets:

$$\chi_{ij} = X_{ijkl}\ \psi_{kl} \qquad \nabla_i\ \psi_{ij} = 0, \tag{II.96}$$

where X_{ijkl} is a second-order differential operator that, up to now, has been given only for isotropy and cubic symmetry. It has the same symmetry as the Hooke tensor of the crystal in question, and is written in the case of isotropy:

$$X_{ijkl} = \mu^3\ [2\lambda\ \delta_{ij}\ \delta_{kl} + (\lambda + 2\mu)(\delta_{ik}\ \delta_{jl} + \delta_{il}\ \delta_{jk})]\ \Delta. \tag{II.97}$$

The ψ_{ij} satisfy the differential equation:

$$f\ \psi_{ij} = \eta_{ij}\,, \tag{II.98}$$

which is solved by:

$$\psi_{ij}(\boldsymbol{x}) = -\iiint_V \eta_{ij}(\boldsymbol{x}')U(x)\,dV'\,. \tag{II.99}$$

For the stress functions χ_{ij} one then has:

$$\chi_{ij}(\boldsymbol{x}) = -\iiint_V \eta_{ij}(\boldsymbol{x}')X_{ijkl}U(x)\,dV'\,. \tag{II.100}$$

For isotropy, with (II.97) and (II.94), one has:

$$X_{ijkl}\ U = \frac{\mu}{8\pi}\left[\frac{2\lambda}{\lambda+2\mu}\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right]x\,, \tag{II.101}$$

from which, eq. (II.100) goes to eq. (II.26).

For the application to straight dislocation lines, we are also interested in the representation of the planar problem. Let, say, $\partial/\partial x_3 = 0$. We then characterize the resulting functions by a prime. It is easy to show that the sixth-degree function $\bar{f}$ can be decomposed into a triple product of expressions that are homogeneous of degree two (for isotropic and hexagonal crystals this is already possible for three-dimensional *f*). For that reason, however, the characteristic equation of degree six that is associated with $\bar{f}$ has elementary roots, and the associated (two-dimensional) fundamental integral $\bar{U}$ will be an elementary function that can be presented relatively easily in all cases. We then assume that is known. This yields:

$$\bar{U}(\bar{x}) = \int_{-\infty}^{\infty} U(x)\, dx_3 \,, \tag{II.102}$$

which is sufficiently well-known from the theory of differential equations. Thus, we have laid the foundations for the static treatment of dislocations and other singularities rather completely.

§ 17. The elasticity-theoretic treatment of singular dislocations

The isolated dislocations play a very great role in applications. In Fig. 9a, b the production of edge dislocations was made intuitive. One thinks of the cylinder in Fig. 9a that was cut out as being pressed together with the cylinder of Fig. 9b in such a way that the relative displacement $\boldsymbol{g}$ of the surface *A* compared to *B* has only an x_1-component, so *A* and *B* shall deform into a surface *f*. Fig. 10 shows the production of a screw dislocation: One thinks of the cylinder in Fig. 10 as being produced from a defect-free cylinder by cutting and a relative displacement of the cut lips in the direction of the cylinder axis.

In the general case, a dislocation will run along an arbitrary space curve *L* with a unit tangent vector $\boldsymbol{t}$, to which the generating surface *f* is affixed with a unit normal vector $\boldsymbol{n}$. Let $\boldsymbol{q}$ be the unit vector $\boldsymbol{n} \times \boldsymbol{t}$. We assume that ζ is so small that one can locally regard the "dislocation strips" of width 2ζ as planar. Let *q* then be the shortest distance from a point lying on *f* in the vicinity of the line *L* to it.

The dislocation density inside the strip is, from eq. (I.77), $-\boldsymbol{n} \times \nabla \boldsymbol{g}$, which is equal to $\boldsymbol{t}\, \partial \boldsymbol{g}/\partial q$, since $\boldsymbol{g}$ varies only in the *q*-direction. The distribution of $\boldsymbol{g}(q)$ naturally remains open to us; in Fig. 9, 10 it is depicted as linear. In general, $\partial |\, \boldsymbol{g}\, |/\partial q$ can be any curve, which we write as $-\chi(q)\, |\, \boldsymbol{g}^0\, |$, if $\boldsymbol{g}^0$ is the constant displacement over the greater part of the surface *f*. From § 1, 2, $\boldsymbol{b} = -\boldsymbol{g}^0$ is then the Burgers vector of the dislocation. We thus obtain a surface dislocation density:

$$\bar{\alpha}_{il} = t_i\, b_l\, \chi(q). \tag{II.103}$$

With these preparations, we now come to the actual calculation. The starting point is eq. (II.26′), in which one has introduced η_{ij} according to eq. (I.51) [1]:

$$\eta_{ij} = -\,(\varepsilon_{jkl}\,\nabla_k\,\alpha_{il})^S\,. \tag{II.104}$$

By partial integration this yields, since the boundary surface integral vanishes, and with $\nabla'_k \equiv \partial/\partial x'_k$:

$$\chi'_{ij} = -\,\frac{1}{8\pi}\left(\varepsilon_{jkl}\iiint_V \alpha_{il}(x')\nabla'_k x\,dV'\right)^S, \qquad x \equiv |\,\boldsymbol{x}-\boldsymbol{x}'\,|. \tag{II.105}$$

Here, one obviously has to set $\alpha_{il}\,dV' = \bar{\alpha}_{il}\,dq'\,dL'$, where dL' is the contribution of the line element dL'_i to the line L. With (II.103), one then obtains:

$$\alpha_{il}\,dV' = t_i\,b_l\,\gamma(q')\,dq'\,dL' = b_l\,dL'_i\,\gamma(q')\,dq'\,. \tag{II.106}$$

If one substitutes this in eq. (II.105), and then sets $\gamma(q')$ equal to the delta function $\delta(q')$ then one can carry out the integration over q' and obtain the singular dislocation line:

$$\chi'_{ij} = -\,\frac{1}{8\pi}\left(\varepsilon_{jkl}b_l\oint_L\int_{-\zeta}^{\zeta}\nabla'_k x\,dL'_i\,\delta(q')\,dq'\right)^S = -\,\frac{1}{8\pi}\left(\varepsilon_{jkl}b_l\oint_L\nabla'_k x\,dL'_i\right)^S. \tag{II.106′}$$

If we then bring the ∇'_k out of the integral, while switching the sign, since we are now differentiating by x_i, we ultimately obtain the formula that was first given by the author [78]:

$$\chi'_{ij} = \frac{1}{8\pi}\left(\varepsilon_{jkl}b_l\nabla_k\oint_L x\,dL'_i\right)^S. \tag{II.107}$$

With this, the stress state, which originates in the dislocation that runs along the line L, will come down to the relatively simple integral $\oint x\,dL'$. It will be shown that the stresses diverge along the line itself, which naturally originates in the fact that one takes the limit $\gamma(q) \to \delta(q)$. Thus, if one is interested in the state in the immediate vicinity of the line L then one may not take that limit, but one must integrate the left-hand equation (II.106′) over $\gamma(q)$ [2].

In the event that the dislocation is straight and extends along the x_3-axis, dL_1 becomes dx_3. One easily verifies the formula:

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[1] S means "symmetric part of."

[2] At some distance from L the principle of St. Venant applies; it no longer applies to the distribution function $\gamma(q)$ precisely.

$$\int_{-L/2}^{L/2} x\, dx_3' = -\rho^2 \ln(\rho/L) + L^2/4, \qquad \rho^2 \equiv x_1^2 + x_2^2, \tag{II.108}$$

which is true everywhere that $\rho, x_3 \ll L$ [1]. L is, accordingly, the length of the dislocation, which shall now go to ∞. According to eq. (II.107) and (II.2), the expression (II.108) will then be differentiated three times until one obtains the stresses. Thus, $\rho^2 \ln L$ does not contribute to this, and we write, when we substitute (II.108) into (II.107):

$$\chi_{ij}' = -\frac{1/2}{8\pi}(\varepsilon_{jkl} + \varepsilon_{3kl}\,\delta_{3j})\, b_l\, \nabla_k\, \rho^2 \ln \rho\,. \tag{II.109}$$

From the superficial agreement of eq. (II.26′) and (II.99), one immediately concludes with the corresponding formulas for anisotropy, namely, eq. (II.107) corresponds to:

$$\psi_{ij} = \left(\varepsilon_{jkl}\, b_l\, \nabla_k \int_L U(x)\, dL_i' \right)^S \tag{II.110}$$

and eq. (II.109) to:

$$\psi_{3j} = -\frac{1}{2}(\varepsilon_{jkl} + \varepsilon_{3kl}\,\delta_{3j})\, b_l\, \nabla_k\, \bar{U}(\bar{x})\,, \tag{II.111}$$

from which, the stress functions can be computed using eq. (II.96).

The computation of X_{ijkl} that is necessary only once for each crystal did not seem very substantial in the two-dimensional case, but in three dimensions it is substantial [80]. One sees that in the case of straight dislocation lines the stresses always yield to elementary functions for anisotropy, as well (since $\bar{U}$, from § 16, is an elementary function). Eshelby [36] has shown how one can also obtain them in the complex plane $x_1 + i\,x_2$, starting from a complex displacement field $s_1 + i\,s_2$. Burgers [13] has already established that the anisotropic formulas in cubic crystals simplify considerably when the dislocation extends along a distinguished crystallographic direction. In particular, one obtains the same displacements as for isotropy for a screw dislocation in the <001>-direction.

We now distinguish two cases:

1. Straight edge dislocation in the x_3-direction. One then has $b_3 = \nabla_3 = 0$, and in eq. (II.109) one will have $j = 3$, i.e., only χ_{33}' is different from zero. If one chooses, say, $x_1 \parallel b_l$ (i.e., $l = 1$) then one has:

$$\chi_{33}' = \frac{1}{8\pi} b_1 \frac{\partial}{\partial x_2}(\rho^2 \ln \rho)\,. \tag{II.112}$$

[1] The exact expression for (II.108) reads:

$$\frac{\rho^2}{2}\ln\left[\left(\frac{L}{2}+x_3+\sqrt{+}\right)\left(\frac{L}{2}-x_3+\sqrt{-}\right)\Big/\rho^2\right]+\frac{1}{2}\left(\frac{L}{2}+x_3\right)\sqrt{+}+\frac{1}{2}\left(\frac{L}{2}-x_3\right)\sqrt{-} \qquad \sqrt{\pm} \equiv \sqrt{\rho^2+\left(\frac{L}{2}\pm x_3\right)^2}\,.$$

From eq. (II.16) and (II.18), the connection between the Airy stress function χ and χ'_{33} is given by $-\frac{2Gm}{m-1}\chi'_{33} = \chi$ [1], from which, we can also write eq. (II.112):

$$\chi = -\frac{A}{2}\frac{\partial}{\partial x_2}(\rho^2 \ln \rho), \qquad A \equiv \frac{b_1}{2\pi}\frac{mG}{m-1}. \tag{II.113}$$

This equation was first given by Koehler [111]. From the usual rules of elasticity theory, the stresses result from this:

$$\left.\begin{aligned} \sigma_{11} &= \partial^2\chi/\partial x_2^2 = -A\frac{x_2}{\rho^2}\frac{3x_1^2+x_2^2}{\rho^2}, \\ \sigma_{12} &= \partial^2\chi/\partial x_1\partial x_2 = A\frac{x_1}{\rho^2}\frac{x_1^2-x_2^2}{\rho^2}, \\ \sigma_{22} &= \partial^2\chi/\partial x_1^2 = A\frac{x_2}{\rho^2}\frac{x_1^2-x_2^2}{\rho^2}, \end{aligned}\right\} \tag{II.114}$$

along with the displacements [2]:

$$\left.\begin{aligned} s_1 &= \frac{b_1}{2\pi}\left[\left(\varphi-\frac{\pi}{2}\right)+\frac{m/2}{m-1}\frac{x_1x_2}{\rho^2}\right], \qquad \varphi = \operatorname{arc\,tg}\frac{x_2}{x_1} \\ s_2 &= -\frac{b_1}{4\pi(m-1)}\left[(m-2)\ln\frac{\rho}{\zeta}+m\frac{x_2^2}{\rho^2}\right]. \end{aligned}\right\} \tag{II.115}$$

The calculation of the displacement requires special consideration, since s_2 diverges logarithmically with ρ [71].

2. Screw dislocation in the x_3-direction. Eq. (II.109) will then become:

$$\chi'_{31} = -\frac{1/2}{8\pi}b_3\frac{\partial}{\partial x_2}(\rho^2\ln\rho), \qquad \chi'_{32} = \frac{1/2}{8\pi}b_3\frac{\partial}{\partial x_1}(\rho^2\ln\rho). \tag{II.116}$$

After multiplication by $2G$, these are likewise the χ_{ij} values. From eq. (II.16), the stress function of torsion is then:

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[1] One finds from eq. (II.18) that χ_{11} and χ_{22} are different from zero. With the help of (II.16), one easily shows that the relation $m\,\sigma_{33} = \sigma_{11} + \sigma_{22}$, which is known from the theory of planar distortion states, follows from this [1]. σ_{11}, σ_{22}, and σ_{33} are determined by χ alone, so we do not need to consider χ_{11} and χ_{22}.

[2] These equations were first derived by Burgers [12] in another way. Taylor [149] was the first to apply elasticity theory to crystal dislocations and, in particular, presented the connection with Volterra, in which the detailed calculations were not completely correct.

$$\Phi = -\frac{Gb_3}{8\pi}\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}\right)\rho^2 \ln\rho = -\frac{Gb_3}{2\pi}(\ln\rho+1), \qquad \text{(II.117)}$$

the stresses are:

$$\sigma_{31} = -\frac{Gb_3}{2\pi}\frac{x_2}{\rho^2}, \qquad \sigma_{32} = \frac{Gb_3}{2\pi}\frac{x_1}{\rho^2}, \qquad \text{(II.118)}$$

and the displacements are:

$$s_1 = s_2 = 0, \qquad s_3 = \frac{b_3}{2\pi}\varphi, \qquad \varphi = \text{arc tg}\,\frac{x_2}{x_1}. \qquad \text{(II.119)}$$

Here, the representation in terms of displacements is especially instructive since one reads off the transition from the x_3-plane into screw surfaces immediately from eq. (II.119) [1].

Although the integral (II.107) appears very simple, one arrives at that integration in closed form only in the simplest cases. This happens for dislocations that consist of piecewise straight lines, and also for dislocations that define plane curves of degree two. In the latter case, one obtains elliptic integrals.

Originally (by Burgers [12]), in place of formula (II.107), the displacement field of the general dislocation line was represented as a surface integral. By application of Green's method in the case $\boldsymbol{\eta} = 0$, one can express the dislocations in a body by the volume and boundary surface forces that produce it, as well as the boundary surface displacement. One obtains [2] (for the detailed calculation, cf., perhaps, Seeger [134]):

$$s_h(\boldsymbol{x}) = \iiint_V S_{ih}(x)\,\mathfrak{F}_i(\boldsymbol{x}')\,dV' + \iint_F S_{ih}(x)\,\overline{\mathfrak{F}}_i(\boldsymbol{x}')\,dF + \iint_F c_{ijkl}\,n_i\,s_j(\boldsymbol{x}')\frac{\partial}{\partial x_k}S_{lh}(x)\,dF'. \qquad \text{(II.120)}$$

One can apply this formula to, e.g., the incised cylinder in Fig. 9a, when one now bends it together (Fig. 9b) in order to next weld it. One then has, in any event, $\mathfrak{F}_i = 0$, but the second integral of eq. (II.120) also vanishes, since the boundary surface forces that bring about the bending together of A and B are equal and opposite at the same points of them. Thus, what remains of eq. (II.120), when we sensibly denote the displacement source $s_i^A - s_i^B$ by g_i (§ 8), is:

$$s_h(\boldsymbol{x}) = \iint_f c_{ijkl}\,n_i\,g_j\frac{\partial}{\partial x_k}S_{lh}(x)\,df' + \iint_F c_{ijkl}\,n_i\,s_j(\boldsymbol{x}')\frac{\partial}{\partial x_k}S_{lh}(x)\,dF', \qquad \text{(II.121)}$$

[1] Since $s_1 = s_2 = 0$, it is also very convenient to derive these equations by starting from the displacements s_3, as Burgers [12] originally did, and can be looked up in any book on dislocations today.

[2] These formulas were first given by Fredholm [57] and thoroughly discussed by Gebbia [58].

where f is the fused surface AB, with A as the positive side and F, the boundary surface of the body after welding. If one now foregoes a precise description in the immediate vicinity of the dislocation, which comes down to assuming that $\boldsymbol{g}$ is constant on f up to the line L, and one likewise restricts oneself to an infinite medium, then one obtains:

$$s_h(\boldsymbol{x}) = - b_j \iint_f c_{ijkl}\, n_i \frac{\partial}{\partial x_k} S_{lh}(x)\, df'. \tag{II.122}$$

From Burgers, this is a good approximation to the displacement field that arises from a dislocation that runs along an arbitrary line L. In the next approximation, it is precisely as small as eq. (II.107). In eq. (II.122), the glide vector is replaced with the Burgers vector.

Upon comparison with eq. (II.89), one concludes that $- c_{ijkl}\, n_i\, b_j$ is a surface density of force dipoles. They have obviously taken over the role of the external forces that one has bent the cylinder together with after welding. The possibility that a field produced by a dislocation can be thought of as either produced by the dislocation line (eq. (II.110)) or a surface density of force dipoles corresponds precisely to the well-known fact of magnetic field theory of the equivalence of current lines and magnetic double layers [1]. In general, the direct proof of the equivalence of eq. (II.110) and (II.122) is very tedious and, up to now, it has only been carried out for isotropy [83].

For isotropy, the integral (II.122) includes the part:

$$\frac{b_h}{4\pi}\Omega(\boldsymbol{x}) = \frac{b_h}{4\pi}\iint_f n_k \nabla_k \frac{1}{x}\, df', \tag{II.123}$$

as one easily verifies by, e.g., substitution in eq. (II.9) and (II.95). $\Omega(\boldsymbol{x})$ is then the spatial angle of vision (Sehwinkel), with which the dislocation line will be seen from the point $\boldsymbol{x}$. This part introduces a multi-valuedness into the dislocations [2]. The remaining part can be represented as a line integral, as in Burgers [12]. Peach and Koehler [115] have also represented Ω as a line integral and thus obtained the total displacement field as a line integral, so their formulas do not have the simplicity of our line integral (II.107). For anisotropy, the displacement field was first given by Liebfried [90] as a line integral. This representation has a definite meaning since the convertibility of eq. (II.122) into a line integral represents the mathematical proof of the fact that the position of surface f, upon which one thinks of the force dipoles as being distributed, is arbitrary, except that its boundary is the dislocation line. Naturally, this proof also follows from eq. (II.107).

The second integral in eq. (II.121) represents the displacement that one must add to that of (II.122) in order to fulfill the boundary conditions. Incidentally, eq. (II.122) is also true when $\boldsymbol{g}$ is an arbitrary function on the surface f. This is the case of the

[1] On this, see also Nabarro [109].

[2] That the displacement field is many-valued in the presence of a dislocation follows from the existence of the Burgers vector $\boldsymbol{b}$, which says that the displacement increases by $\boldsymbol{b}$ when one goes around the dislocation once. The dipole surface f is the branching surface. The corresponding behavior of the electric potential around a linear electric current is indeed well-known.

"Somigliana dislocation." In recent times, it was treated by Mann [97] and Bogdanoff [9].

§ 18. The elastic energy of the singular dislocation

We define the elastic (self-) energy of a dislocation to be the increase in internal energy in a medium that was originally found in the natural state that it experiences as a result of the immigration or formation of a dislocation. If the dislocation immigrates from the outside then one often has an edge in the boundary surface (Fig. 10) across which the boundary surface stress of the continuum will vary locally. This part of the change in internal energy is, for most purposes, trivial, and will thus no longer be considered. (Cf., Nabarro [110], pp. 332.)

We restrict ourselves to the infinitely extended medium. At the center of the singular dislocation the stresses will be, from eq. (II.114), infinite. Therefore, the energy of the dislocation per unit length (linear energy) diverges. This is the great difficulty for all energetic questions about dislocations. Real-world dislocations all have a certain finite "width" 2ζ, and for that reason, a finite self-energy. Fortunately, ζ enters into energy the formula only logarithmically, so for practical purposes not exactly all of the extent $\boldsymbol{g}(q)$ of the plastic displacement enters the dislocation strip 2ζ. From now on, we will speak of the dislocation "line" and emphasize this fact especially whenever its finite extent is of issue.

With Cottrell [22], we think of a dislocation as running along a line L^B in an infinite medium in the absence of external forces, and further define a second dislocation that intersects a surface f with a boundary line L^A, while the two lips of the cut are plastically displaced with the glide vector $g_i^A = -\,b_i^A$. The work done by this is [1]:

$$A = b_i^A \iint_f \left(\sigma_{ij}^B + \frac{1}{2}\sigma_{ij}^A \right) df_j \,, \tag{II.124}$$

if σ_{ij}^B is the stress function, which originates on the dislocation L^B, while σ_{ij}^A is the additional stress field that comes about during the process (and thus, the factor of 1/2) and originates on the line L^A. If we substitute the stresses in eq. (II.2) by means of the stress functions and apply Stokes's theorem then we obtain:

$$A = -\,b_i^A \varepsilon_{ijk} \oint_{L^A} \nabla_j \left(\chi_{kl}^B + \frac{1}{2}\chi_{kl}^A \right) dL_l^A \,, \tag{II.125}$$

or, from eq. (II.96), when expressed in terms of ψ_{ij} :

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[1] In a finite medium, for Bilby [2], a summand $\iint_F s_i \left(\sigma_{ij}^B + \frac{1}{2}\sigma_{ij}^A \right) dF_j$ appears in eq. (II.124) that means the same thing as the second integral in eq. (II.121).

$$A = - \, b_i^A \varepsilon_{ijk} \oint\limits_{L^A} \nabla_j X_{klmn} \left(\psi_{kl}^B + \frac{1}{2} \psi_{kl}^A \right) dL_l^A \, . \tag{II.126}$$

Here, we substitute ψ_{mn}^B from eq. (II.110) and obtain:

$$E^{AB} = b_i^A b_q^B M_{iq}^{AB} \tag{II.127}$$

$$\left. \begin{aligned} M_{iq}^{AB} &= -\varepsilon_{ijk} \varepsilon_{npq} \oint\limits_{L^A} \oint\limits_{L^B} \nabla_j \nabla_p X_{klmn} U(x) \, dL_m^B \, dL_l^A , \\ x &\equiv | \boldsymbol{x}^A - \boldsymbol{x}^B | \end{aligned} \right\} \tag{II.127′}$$

as the part of the work done by the formation of the dislocation line L^A that is attributed to the presence of the dislocation L^B. One calls it the "interaction energy" of the dislocations L^A and L^B. One has: $E^{AB} = E^{BA}$. In the case of elastic isotropy, with eq. (II.94) and eq. (II.97), one gets for eq. (II.127′):

$$M_{iq}^{AB} = - \frac{G}{8\pi} \varepsilon_{ijk} \varepsilon_{npq} \oint\limits_{L^A} \oint\limits_{L^B} (\nabla_j \nabla_p x) \left[\frac{2}{m-1} dL_n^B dL_k^A + dL_k^B dL_n^A + dL_l^B dL_l^A \delta_{nk} \right]. \tag{II.128}$$

The generally asymmetric tensor M_{iq}^{AB} defines the analogue of the well-known counter-inductivity in the theory of linear currents [1].

Now, one cannot also substitute ψ_{mn}^A from eq. (II.110) since one would then obtain a double line integral over the same line that diverges. One thus obtains no simple formula like (II.127′) for the self-energy of the dislocation. Starting with this one can thus approach the problem as follows: One now regards the dislocation strip 2ζ as consisting of nothing but infinitely dense "dislocation threads" with the infinitesimal glide vector $d\boldsymbol{g}(q) = - \boldsymbol{b}^A \, \gamma(q)$ on them (cf., § 17) and computes the interaction energy of all of these threads by means of eq. (II.127). One thus obtains (similarly written for isotropy) [2]:

$$E^{AA} = b_i^A b_q^A M_{iq}^{AA} \tag{II.129}$$

$$\begin{aligned} M_{iq}^{AA} = - \frac{G}{16\pi} \varepsilon_{ijk} \varepsilon_{npq} \int\limits_{-\zeta}^{\zeta} dq \, \gamma(q) \int\limits_{-\zeta}^{\zeta} dq' \, \gamma(q') \oint\!\!\oint (\nabla_i \nabla_j x) \\ \times \left[\frac{2}{m-1} dL_n' dL_k + dL_k' dL_n + dL_l' dL_l \delta_{nk} \right]. \end{aligned} \tag{II.129′}$$

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[1] Blin [7] has given a formulation that is completely equivalent to (II.128), and Stroh [148] gave a further expression for the case in which the two dislocations lie in a plane. Eq. (II.127′) was first given by the author. The formula (II.128) that was given in the same work contains a mistake in calculation. For special arrangements of dislocations the dislocation counter-inductivity can be directly equal to the magnetic inductivity that corresponds to the current arrangement, as Hart [170] has found.

[2] The notation A is omitted on the right-hand side of eq. (II.129). The integrations $\oint\!\!\oint$ extend over two dislocation threads.

Here, the interaction energy between each line element and all of the elements of all other threads is considered, but not, however, the interaction energy between the elements of those threads and the proper energy of the thread element. One recognizes in this that for an increasing number of dislocation threads the latter part no longer enters into the former one, and for infinitely dense threads it will be infinitely small when compared to the former. Thus, eq. (II.129) does, in fact, correctly yield the self-energy of the dislocation strip, and the symmetric tensor M_{iq}^{AA} is the analogue of self-inductivity.

With the help of the self-inductivity and counter-inductivity, the energy of an arrangement of many dislocations may be written in the form:

$$E = \sum_{A,B} b_i^A b_q^B M_{iq}^{AB} , \tag{II.130}$$

where A and B each run over each dislocation line.

We now present an application for the very important equation (II.128). One extends, e.g., a straight dislocation line L^B in the x_3-direction. The integration over L^B in eq. (II.128) then becomes elementary. We employ eq. (II.108) and write:

$$\int_{L^B} (\nabla_j \nabla_p x) dL_s^B = \nabla_j \nabla_p \int_{L^B} x \, dL_s^B = - \nabla_j \nabla_p \rho^2 \ln \frac{\rho}{L} i_s . \tag{II.131}$$

With this, one has:

$$M_{iq}^{AB} = - \frac{G}{8\pi} \varepsilon_{ijk} \varepsilon_{npq} \int_{L^A} \nabla_j \nabla_p \left(\rho^2 \ln \frac{\rho}{L} \right) \left[\frac{2}{m-1} dL_k^A \, i_n + dL_n^A \, i_k + dL_i^A \, i_l \delta_{nk} \right]. \tag{II.132}$$

One easily verifies the validity of the relation:

$$\nabla_j \nabla_p \rho^2 \ln \frac{\rho}{L} = 2 \left[\left(\ln \frac{\rho}{L} + \frac{1}{2} \right) \delta_{jp} + \frac{x_j x_p}{\rho^2} \right], \qquad j, p = 1, 2. \tag{II.133}$$

We further restrict ourselves to the case in which the dislocation L^A also runs in the plane $x_2 = 0$, and then obtain $2 \left[\left(\ln \frac{x_1}{L} + \frac{1}{2} \right) \delta_{jp} + \delta_{jp}^1 \right]$ for the right-hand side of eq. (II.133), where $j, p = 1, 2$ and $\delta_{jp}^1 = 1$ for $j = p = 1$, but vanishes otherwise. We substitute this in eq. (II.132) and likewise consider that one must set $i_n = i_k = i_l = i_3$ there, and that $dL_2^A = 0$, moreover. For $j = p = 1$, one obtains the part:

$$\frac{G}{8\pi} \left[\frac{4}{m-1} \varepsilon_{i13} \varepsilon_{31q} + 2 \varepsilon_{i13} \varepsilon_{31q} + 2 \varepsilon_{i1k} \varepsilon_{k1q} \right] \int_{L^A} \left(\ln \frac{x_1}{L} + \frac{3}{2} \right) dx_3 \tag{II.134}$$

of M^{AB}, and for $j = p = 2$, the part:

$$\frac{G}{8\pi}\left[\frac{4}{m-1}\varepsilon_{i2k}\varepsilon_{32q}+2\varepsilon_{i23}\varepsilon_{k2q}\right]\int_{L^A}\left(\ln\frac{x_1}{L}+\frac{3}{2}\right)dx_3+\frac{G}{4\pi}\varepsilon_{i2k}\varepsilon_{k2q}\int\left(\ln\frac{x_1}{L}+\frac{1}{2}\right)dx_3\,. \quad \text{(II.135)}$$

From this, it easily follows that:

$$\left.\begin{aligned}
M_{11}^{AB}&=-\frac{G}{2\pi}\frac{m}{m-1}\int_{L^A}\left(\ln\frac{x_1}{L}+\frac{1}{2}\right)dx_3\\
M_{22}^{AB}&=-\frac{G}{2\pi}\frac{m}{m-1}\int_{L^A}\left(\ln\frac{x_1}{L}+\frac{1}{2}\right)dx_3\\
M_{33}^{AB}&=-\frac{G}{2\pi}\int_{L^A}\left(\ln\frac{x_1}{L}+1\right)dx_3\\
M_{31}^{AB}&=\frac{G}{2\pi}\frac{1}{m-1}\int_{L^A}\left(\ln\frac{x_1}{L}+\frac{1}{2}\right)dx_1\\
M_{13}^{AB}&=\frac{G}{4\pi}\int_{L^A}\left(\ln\frac{x_1}{L}+\frac{1}{2}\right)dx_3\\
M_{23}^{AB}&=M_{32}^{AB}=M_{12}^{AB}=M_{21}^{AB}=0.
\end{aligned}\right\} \quad \text{(II.136)}$$

For the further integrations, one must substitute the curve equation $x_1(x_3)$ of the dislocation line L^A wherever dx_3 appears in the logarithm. In the event that L^A is likewise a straight line and parallel to L^B, one next has $M_{31}^{AB} = M_{13}^{AB} = 0$, since $dx_1 = 0$. Thus, all that remain are the diagonal components of M_{iq}^{AB}. That is, parallel dislocation lines whose Burgers vectors are perpendicular to each other have no effect in isotropic media (this conclusion follows from eq. (II.127)). We then write down the components of M_{iq}^{AB} for $x_1 = d$ (= parallel dislocations at a distance d):

$$\left.\begin{aligned}
M_{11}^{AB}&=-\frac{G}{2\pi}\frac{m}{m-1}L'\left(\ln\frac{d}{L}+\frac{1}{2}\right),\\
M_{22}^{AB}&=-\frac{G}{2\pi}\frac{m}{m-1}L'\left(\ln\frac{d}{L}+\frac{3}{2}\right),\\
M_{33}^{AB}&=-\frac{G}{2\pi}L'\left(\ln\frac{d}{L}+1\right).
\end{aligned}\right\} \quad \text{(II.137)}$$

We have performed the second integration with the limits $-L'/2 \ldots L'/2$, where $L' \ll L$ is assumed, since only then is the assumption for eq. (II.136) valid (viz., the validity of eq. (II.108)). In that regard, we shall show later on that eq. (II.137) will be modified only inessentially for exact computations for $L' \to L$ as long as $L \gg d$.

The interaction of second-degree parallel dislocations at a distance d, from which, the force of attraction (b^A, b^B anti-parallel) or repulsion (b^A, b^B parallel) follow by differentiation, was treated by various

authors, where the formulas for the interaction energy that follow from eq. (II.137) by multiplication with $b^A\, b^B$ were obtained in a maybe somewhat simpler way than by us, since one employs the simplifications that enter into the present special case throughout. We have arrived at our derivation, above all, in order to present eq. (II.136), which is relatively simple, but then already gives rise to a host of problems that are significant in practice [1]. We will describe an application of these equations in § 29.

Our formulas (II.137) differ from those of the usual authors in two ways [2]: Cottrell [22], who started with eq. (II.124), did not convert the surface integral into a line integral, and thus in the logarithm in his final formulas there was not our dislocation length L, but the dimension R of the medium that is perpendicular to the dislocation line, and which likewise goes to ∞. Eshelby (cf., [110], pp. 305-306) obtained the same result by approaching the problem in various ways; e.g., with the use of bipolar coordinates. In all cases, one obtains a logarithmic divergence of the interaction energy per unit length for parallel straight line dislocations, such that one is actually compelled to compute in finite bodies. We will see that the same is true for the self-energy of a straight-line dislocation. These complications will often be circumvented by replacing R (L, resp.) with the approximate value of the mean distance of the dislocations with opposite signs ("dislocation networks," § 29)(e.g., 10^{-4} cm in unformed metal). This procedure is, moreover, not completely adequate. In practice, the problems are often given (§ 29) in such a way that L (R, resp.) is either taken out (sich heraushebt) or is known from the outset, where due to the only logarithmic dependency of the energy L and R the approximate values of these quantities are mostly satisfied.

Secondly, our formulas differ from those of the cited authors in the summands that enter into $\ln(d/L)$. By assumption ($d \ll L$), this is small compared to $\ln(d/L)$; the difference originates in the different treatments of the dislocation centers. Whether one has R or L in the logarithm also plays a role.

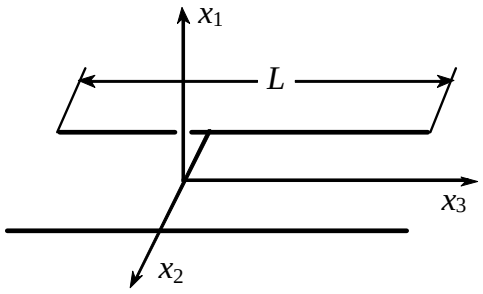

Fig. 18

Next, we apply eq. (II.128) to two straight dislocations running parallel to the x_3-direction at a distance ρ [3], both of which have the finite length L (Fig. 18), and indeed, one shall have $\rho \ll L$. Before the integrations, we carry out the differentiations $\nabla_j\, \nabla_p\, x$ and then neglect all terms in this expression that have a factor of $x_1^A - x_1^B$ or $x_2^A - x_2^B$ [4]. One easily sees that then only the differentiations $\partial^2/\partial x_1^2$ and $\partial^2/\partial x_2^2$ give any contribution, while all others vanish. Furthermore, we assume that the Burgers vectors of the two dislocations are equal and lie in the plane $x_2 = 0$. Let its angle with the line direction be b, so the components of $\boldsymbol{b}$ in the x_1-direction and x_3-direction are then $b \sin\beta$ ($b \cos\beta$, resp.) Then, one easily computes the interaction energy E^{AB} from eqs. (II.127) and (II.128):

[1] One also immediately finds in eq. (II.136) an essential part of the results on dislocations whose lines run perpendicular to each other that were discussed by Nabarro [11], pp. 309; L^A thus has the x_1-direction. Eq. (II.136) is then $dx_3 = 0$, and interaction exists only in the case M_1^{AB} (L^A and L^B screw dislocations) and in the case M_{31}^{AB} (L^A and L^B edge dislocations with Burger vectors currently parallel to the other line).

[2] Koehler [71] was the first to compute the energy of straight line dislocations and the interaction energy of two dislocations.

[3] This will lead us to a method that substantially abbreviates the computation of the self-energy.

[4] These terms are smaller than the remaining ones by a factor ρ/L, as a closer examination shows.

$$E^{AB} = \frac{Gb^2}{4\pi}\left(\frac{m}{m-1}\sin^2\beta + \cos^2\beta\right)\iint \frac{dx_3^A dx_3^B}{x} \quad . \tag{II.138}$$

We calculate the integral in a manner that is somewhat more general than is required here, namely, with the bounds:

$$\int_{-L/2}^{L/2} dx_3^A \left(\int_{-L/2}^{x_3-\varepsilon} dx_3^B \cdots + \int_{x_3+\varepsilon}^{L/2} dx_3^B \cdots \right), \tag{II.139}$$

and obtain exactly:

$$\iint \frac{dx_3^A dx_3^B}{x} \;=\; 2L \ln \frac{L+\sqrt{\rho^2+L^2}}{\varepsilon+\sqrt{\rho^2+\varepsilon^2}} - 2\sqrt{\rho^2+L^2} \; . \tag{II.140}$$

With $\varepsilon = 0$ this then yields, when we further neglect ρ^2 with respect to L^2 and divide by L:

$$T^{AB} = \frac{Gb^2}{4\pi}\left(\frac{m}{m-1}\sin^2\beta + \cos^2\beta\right)\left(\ln\frac{2L}{\rho} - 1\right). \tag{II.141}$$

This is the interaction energy per unit length of the dislocation lines. It differs only slightly from the results of eqs. (II.137) for large L, as one knows, when one sets $\ln(2L/\rho) = \ln(L/\rho) + \ln 2$ in eq. (II.141).

From now on, we assume that the two dislocations are threads of a dislocation line whose self-energy we would like to compute from eq. (II.129). We can then introduce the result we just obtained directly in eq. (II.129) (with $q = x_1$):

$$T^{AA} = \frac{G(b^A)^2}{4\pi}\left(\frac{m}{m-1}\sin^2\beta + \cos^2\beta\right)\int_{-\zeta}^{\zeta} dx_1\, \gamma(x_1) \int_{-\zeta}^{\zeta} dx_1'\, \gamma(x_1')\left(\ln\frac{2L}{|x_1 - x_1'|} - 1\right). \tag{II.142}$$

The simplest case is γ = const. = $1/2\ \zeta$, which corresponds to a linear increase of the relative dislocation in strips of 2ζ (Fig. 9). The integrations (II.14) are then elementary to carry out, and one obtains:

$$T^{AA} = \frac{G(b^A)^2}{4\pi}\left(\frac{m}{m-1}\sin^2\beta + \cos^2\beta\right)\left(\ln\frac{L}{\zeta} + \frac{1}{2}\right), \tag{II.143}$$

or also:

$$T^{AA} = \frac{G(b^A)^2}{4\pi}\left(\frac{m}{m-1}\sin^2\beta + \cos^2\beta\right)\left(\ln\frac{L}{\zeta/e^{3/2}} - 1\right), \tag{II.144}$$

as the linear energy of the dislocation. This formula is exact for $L \to \infty$, as long as Hooke's law is also valid in the strips 2ζ, which happens for sufficiently small b^A. From

the logarithmic dependency of ζ one can conclude that the energy is not very sensitive to small changes in ζ and $\gamma(x_1)$.

One can now obtain precisely the same formula (II.144) when one substitutes the integral (II.140) in eq. (II.138) with $\rho = 0$ and $\varepsilon = \zeta / e^{3/2}$ and multiplies the result by 1/2. That is, when one now substitutes:

$$M_{iq}^{AA} = -\frac{1}{2}\varepsilon_{ijk}\varepsilon_{npq}\oint_{L}\oint_{L_s'}(\nabla_j \nabla_p x)\left[\frac{2}{m-1}dL_n' dL_k + dL_k' dL_n + dL_l' dL_l \delta_{nk}\right] \qquad \text{(II.145)}$$

in eq. (II.129) in place of (II.129′), where L_s' means that the piece $x - \varepsilon \ldots x + \varepsilon$ is to be removed from the integral in question, one obtains the correct self-energy, at least in the case of the straight dislocation line. One can establish that this is also valid to a close approximation for curved dislocations [1]. However, in many cases the integral (II.145) cannot be evaluated, in which one can no longer cope with the integrations in (II.129′) with a reasonable expenditure of effort. Therein lies the practical meaning of the calculations that were carried out here.

One more remark on eq. (II.144): The energy per unit length T^{AA} of the dislocation line depends upon L only logarithmically. In many problems, one must consider, e.g., the bending of an originally straight dislocation line, under which, the length of the dislocation does not change very appreciably. One can then ignore the dependency of T^{AA} on L and β to a good approximation and find that the energy of the dislocation is proportional to its length. One thus often calls T^{AA} the "linear stress" of the dislocation, in analogy with the behavior of stressed strings. One can, e.g., present a differential equation for the oscillations of such a dislocation that has precisely the form of a string-oscillation equation. On this, cf., perhaps the work of Eshelby [35] and Koehler [72].

In the last two paragraphs, we considered dislocations in infinite media. In practice, one always has a bounded body before one, and in many cases the result for the infinitely extended medium certainly represents no good approximation to reality. (This is especially true for problems with rectilinearly moving dislocations.) One then must solve a boundary-value problem, in addition. Such problems were treated by Dietze [163] and Liebfried and Dietze [171] with noteworthy success. Seeger reported on them in an appendix; see [134], pp. 560. It dealt with dislocations in bodies that were bounded by planar or circular cylindrical outer surfaces. In this case, one obtains solutions in closed form in general. Thus, the reflection process that was applied by Liebfried and Dietze [171] is especially interesting. Furthermore, cf., also Eshelby and Stroh [167].

[1] The most important argument is: The main contribution of a line element lies in its far-reaching stress field. In (II.145), however, all of the interaction energy parts of an element are correctly included along with the not very close ones. The local part of the action, however, does not "notice" the curvature of the dislocation when the radius of curvature is large compared to ε. For straight dislocations, however, (II.145) is exact.

§ 19. The forces on dislocations and other elastic singularities. The dislocation as elementary source of proper stress.

The problem of the forces that a stress field exerts on elastic singularities – in particular, dislocations – is completely unknown in classical elasticity theory, but again has an analogy with electrodynamics, where one knows simple formulas for the forces on linear currents, magnetic dipoles, etc. Only slightly less elementary are the formulas that we will derive below.

The great significance of such consequences for the dislocations lies in the fact that, on the one hand, the motion of dislocations, and thus the plastic forming of materials, comes about as a result of the externally applied stresses. This influence must attain a certain measure in order to first make the formation of dislocations possible and secondly, to facilitate their migration.

After the early work of Mott and Nabarro [105] and Leibfried [88], Peach and Koehler [115] succeeded in presenting the general expression for the force that the line element of a dislocation at the position $\boldsymbol{x}$ experiences in the stress field $\boldsymbol{\sigma}(\boldsymbol{x})$. This much-used formula must represent a fundamental equation of dislocation theory. It is, to some extent, analogous to the formula for the Lorentz force on an electric current line element in a magnetic field.

One defines the force $d\boldsymbol{K} \equiv (dK_i)$ on a line element dL_i of a dislocation with a Burgers vector b_i as follows: Let $-dW^a$ be the work that the external forces do under a shift of the dislocation line element through dx_i , and let dW^i be the simultaneous increase in the elastic energy content of the bodies. $d\boldsymbol{K}$ will then be defined by the equation:

$$-(dW^a + dW^i) = d\boldsymbol{K} \cdot d\boldsymbol{x}. \tag{II.146}$$

We now think of the progress of the dislocation of $d\boldsymbol{L}$ as taking place as follows: The surface piece that will be swept out by $d\boldsymbol{L}$ is $d\boldsymbol{f} = d\boldsymbol{x} \times d\boldsymbol{L}$ ($d\boldsymbol{L}$ is then the right-hand screw boundary line of $d\boldsymbol{f}$ after the motion). We cut along $d\boldsymbol{f}$ and, as long as no dislocation is present, affix the force $d\boldsymbol{f} \cdot \boldsymbol{\sigma}$ ($-d\boldsymbol{f} \cdot \boldsymbol{\sigma}$, resp.) to the two lips of the cut. We now consider the two lips of the cut as parts of the outer surface of the body. (The rest of it is the original outer surface; i.e., the body is now doubly connected so the internal forces will now be external.)

Now, the relative shift of the two lips of the cut along $d\boldsymbol{L}$ shall result in a glide vector $\boldsymbol{g} = -\boldsymbol{b}$, which then means precisely the migration of $d\boldsymbol{L}$ through $d\boldsymbol{x}$. Insofar as we can regard this shift as infinitesimal and virtual, in the sense of the principle of virtual shifts, it will exert no work in total, since the body is found in equilibrium; i.e.:

$$dW^a + dW^i + d\boldsymbol{f} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{b} = 0. \tag{II.147}$$

From a comparison with eq. (II.146) it follows, when we then consider that $(d\boldsymbol{x} \times d\boldsymbol{L}) \cdot \boldsymbol{\sigma} = d\boldsymbol{x} \cdot (d\boldsymbol{L} \times \boldsymbol{\sigma})$, that:

$$d\boldsymbol{K} = d\boldsymbol{L} \times \boldsymbol{\sigma} \cdot \boldsymbol{b}, \tag{II.148}$$

or:

$$dK_k = \varepsilon_{ijk}\, dL_i\, b_l\, \sigma_{jl}\,. \tag{II.149}$$

This is the formula of Peach and Koehler. The single assumption that have used was that one may regard the shift through $-\boldsymbol{b}$ as infinitesimal and virtual in the sense of the principle of virtual shifts. For finite $|\boldsymbol{b}|$ this assumption is not strictly fulfilled, which shows that one must consider eq. (II.148) as an approximation (which generally suffices). σ_{jl} is the stress that one measures at the position $\boldsymbol{x}$ of the line element when one makes the known cut there. Thus σ_{jl} includes, not only the external forces and other stresses that come from proper stress sources, but also those stresses that all of the remaining line elements of the dislocation provoke at the position $\boldsymbol{x}$, and indeed the part of the stresses that the line element – perhaps, on the outer surface or other boundary surfaces in the body – exerts upon itself. Furthermore, eq. (II.148) is true for arbitrary inhomogeneity in the elastic constants of the body. Rieder [124] has, moreover, shown, by a method that is similar to that of Eshelby (cf., *infra*), that eq. (II.148) is also valid for quasi-dislocations.

The far-reaching validity of eq. (II.148) resides in the fact that it alone is a consequence of the very general principle of virtual shifts. Many times [134, 108], eq. (II.148) has been connected with the theorem of Colonetti that was mentioned in § 14, which says for this case that under special conditions (Hooke's Law, no further sources of proper stresses than the dislocation in question, elastic homogeneity) $dW^i = 0$. However, one thus overlooks the fact that the validity of Colonetti's theorem for the medium in question does not belong to the assumptions of eq. (II.148).

The derivation of eq. (II.148) does not assume the symmetry of the stress tensor. We show an important consequence of this fact. We ask: What sort of stresses can set in motion a planar distribution of crossed screw dislocations [1] as in Fig. 16c such that they are all perpendicular to their plane (e.g., $x_1 = 0$)? In this case, the dislocation tensor $dL_i\, b_l \equiv \alpha^{\circ}_{il}$ in eq. (II.148) has only the components $\alpha^{\circ}_{22} = \alpha^{\circ}_{33} \equiv \alpha^{\circ}$. One thus obtains:

$$dK_1 = \alpha^{\circ}_{22}\ \sigma_{32} - \alpha^{\circ}_{33}\ \sigma_{23} = \alpha^{\circ} \qquad (\sigma_{32} - \sigma_{23}). \tag{II.149}$$

That is, the migration of dislocations that corresponds to Fig. 16c can only come about from the asymmetry of the stress tensor. We now cite some examples for which asymmetric stress tensors actually appear in crystals.

In ferromagnetic crystals one has a spontaneous magnetization that lies in a preferred crystallographic direction because then the free energy of the crystals is especially low. An external magnetic field can rotate this magnetization into an energetically unfavorable direction. One of the magnetically preferred directions of the crystal may then rotate into the new magnetization direction, so the external magnetic field must exert a rotational moment on the volume element, which has asymmetric stresses as a consequence. From a remark of Rieder [124], it would be imaginable that in favorable cases the force that thus arises can actually bring about the migration of a grain boundary of the stated type.

Through current flux in a crystal, one may, for strong anisotropy of the conductance, likewise produce such rotational moments, which are also small. However, we would like to believe that through a strong difference in the free energy of the two crystallites separated by the grain boundary – as with, e.g., re-crystallization – phase drifts and similar processes can appear such that strongly asymmetric stresses will be generated, while the grain boundary in the crystallite drifts into the crystallite with the greater energy. Rieder [124] has briefly described how one can treat such asymmetric stresses by means of elasticity theory. Meanwhile, we would like to assume that the Cosserat deformation moment [20] that Rieder neglected can essentially influence such behavior in fact. Introducing their research into the context of the aforementioned processes in crystals seems to be a worthwhile problem.

[1] Such assignments of dislocations have, in practice, great stability since they imply a complete elimination of their far-reaching stress fields for the distributed dislocations (§ 23).

We now describe another important application of the Peach-Koehler formula. Let L be a small, planar, dislocation loop with Burgers vector b_i, not necessarily circular. In the domain of the loop let there be no volume forces, so $\nabla_i \sigma_{ij} = 0$. One obtains the total force on this loop in the stress field σ_{ij} by integrating eq. (II.148) along L. One then applies Stokes's theorem, develops σ_{ij} around the midpoint $\boldsymbol{x} = 0$ of L in a Taylor series, and easily obtains:

$$K_k = \nabla_k \sigma_{jl}\Big|_0 + \iint_f n_j b_l \, df + \nabla_k \nabla_m \sigma_{jl}\Big|_0 \iint_f x_m n_l \, df + \cdots \tag{II.150}$$

We now let $f \to 0$ while b_l increases in such a way that the integral in (II.150) remains finite. Thus, f will be practically pointlike, and $n_j \, b_l \, df$ means that the two points on the positive and negative sides of f undergo the relative displacement $- b_l = g_l$. We thus define the "displacement dipole" Q_{jl} by:

$$Q_{jl} \equiv \lim_{f \to 0} \iint_f n_j g_l \, df \tag{II.151}$$

and correspondingly, the displacement quadrupole Q_{mjl}, by:

$$Q_{mjl} \equiv \lim_{f \to 0} \iint_f x_m n_j g_l \, df \, . \tag{II.152}$$

The sign in eq. (II.151) is established such that, e.g., a positive dipole Q_{11} corresponds to a drawing apart of the cut lips. Symbolically, $\leftarrow \!\!-\!\!- \rightarrow$. Previously, we called a force dipole P_{11} positive when it emerges from two isolated forces $\leftarrow \!\!-\!\!- \rightarrow$ upon passing to the limit.

We would now like to compare eq. (II.151) with eq. (II.122). There, we recognized the expression $- c_{ijkl} \, n_i \, b_j \, df$ as an infinitesimal force dipole. From eq. (II.151), an infinitesimal dislocation dipole can be written $- n_j \, b_l \, df$. That is, between force dipoles and dislocation dipoles there exists the relation [1]:

$$P_{ij} = c_{ijkl} \, Q_{kl} \, . \tag{II.153}$$

Thus, one should note that this is true only when the elastic constants of the body are homogeneous in the neighborhood of the dipoles and directly at its position; this was an assumption in eq. (II.122).

From now on, we write the total force on a dislocation loop (II.150) [2]:

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[1] Collectively, one can refer to P_{ij} and Q_{ij} as "elastic dipoles."

[2] The form for this equation that was given by the author in the original paper [82]:

$$\boldsymbol{K} = \text{grad} \, (\sigma_{jl} \, Q_{jl} + \nabla_m \, \sigma_{jl} \, Q_{mjl} + \ldots)$$

was, to some extent, an unfortunate choice, as one must add in that paper that the differentiation grad of σ_{jl} must be carried out before carrying out the multiplications in the bracket. Also, this form for eq. (II.154) can lead to false conclusions about the physical meaning of the bracket expression (cf., *infra*).

$$K_k = Q_{ij}\,\nabla_k\,\sigma_{jl} + Q_{mjl}\,\nabla_k\,\nabla_m\,\sigma_{jl} + \ldots \tag{II.154}$$

We would like to no longer consider quadrupoles. The force on a dislocation dipole alone will thus be:

$$K_k = Q_{ij}\,\nabla_k\,\sigma_{jl}\,. \tag{II.155}$$

Such a force then also acts on anti-symmetric dislocation dipoles. We would thus like to no longer consider this, since it does not have the meaning of a symmetric dislocation dipole [82]. Furthermore, let Q_{jl} be a symmetric tensor then. When one now substitutes σ_{jl} with ε_{ij} in eq. (II.155) by means of Hooke's law and considers (II.153) then one obtains for the force on a force dipole:

$$K_k = P_{ij}\,\nabla_k\,\varepsilon_{ij}\,. \tag{II.156}$$

This equation is also valid only under the homogeneity of the elastic constants inside the domain of the dislocation loop that is equivalent to the dipole [1]. An example makes this clearer:

In a body II, let a small region I of another material be forced in such that one can associate the boundary surface f with unit normal vector $\boldsymbol{n}$ between I and II with a dislocation jump $\boldsymbol{g}$ using eq. (I.77). In this case, $n_i\,g_j$ is the associated density of the dislocation dipole in the boundary surface and $\iint_f n_i g_j\,df = Q_{ij}$ represents the total dislocation dipole. We let the volume of the inclusion go to 0 with a decrease in b_j such that Q_{ij} remains finite. The associated force dipole (we assume that the higher poles vanish) gives the forces that the included region I exerts on its surroundings. For a given dislocation dipole this will be greater for harder inclusions. Indeed, one can now write down an eq. (II.153), but the c_{ijkl} in it may be identified with the elastic moduli of either I or II, while first there is an elastic boundary-value problem relative to the boundary surface f in order to obtain the correct values for the c_{ijkl}.

In the practical applications of these consequences (§ 31), one is especially interested in the case in which elastic homogeneity is not present at the location of the inclusion. Eq. (II.156) then remains valid, but not eq. (II.155). One recognizes this best in connection with a method by which Eshelby [38] determined the force on elastic singularities in an elastic field.

Let a body with outer surface S_0 be endowed with outer surface forces $n_i\sigma_{ij}^a$ that provoke dislocations s_i^a and stresses σ_{ij}^a in it. Furthermore, it includes a singularity [2] at the location x' that gives rise to additional dislocations s_i^s and stresses σ_{ij}^s, and a second

[1] The homogeneity is not guaranteed when the relative dislocation g goes to infinity, since it corresponds to taking the limit (II.151). That is, eq. (II.156) is, at first, only true for dipoles of infinitesimal strength. Indeed, it is only for this case that the Peach-Koehler formula is also exact. The conclusion of the validity of eq. (II.156) follows for finite dipole strength (cf., *infra*).

[2] The notion of "singularity" is used rather broadly here, so one can also treat, e.g., a collection of singularities, or at least arbitrary currents that are localized to a sub-region of the body.

singularity with the corresponding quantities s_i^t and σ_{ij}^t. The force on the singularity will be defined by the relation:

$$K_l = -\left(dW^a / dx_l' + dW^i / dx_l'\right) \equiv K_i^a + K_l^b + K_l^t, \tag{II.157}$$

which is equivalent to eq. (II.146), and in which dW^i / dx_l' consists of two pieces: $-K_l^b \equiv dW^b / dx_l'$ is the so-called "image force (Bildkraft)" – i.e., the force that the singularity on the outer surface exerts on itself – while $-K_l^t \equiv dW^t / dx_l'$ is the force that results from the change in interaction energy of the two singularities. Furthermore, the validity of the last-mentioned theorem of Colonetti for the medium in question will be assumed. Next, one obviously has:

$$dW^a = dx_l' \iint_{s_0} \sigma_{ij}^a \frac{\partial s_i^s}{\partial x_l'} dS_j \tag{II.158}$$

$$K_i^a = \iint_{s_0} \sigma_{ij}^a \frac{\partial s_i^s}{\partial x_l'} dS_j \ . \tag{II.159}$$

What is definitive for this part is those additional dislocations that the outer surface of the body will experience under a shift of the singularity through dx_l'. The corresponding expression for K_l^b was derived by Eshelby to be:

$$K_l^b = \iint_{s_0} \sigma_{ij}^b \frac{\partial s_i^s}{\partial x_l'} dS_j \ . \tag{II.160}$$

This form is plausible since the image force does indeed act on the outer surface. Since the expressions (II.159) and (II.160) are completely independent of the existence of the second singularity, one can, without changing the value of integral, contract the integration surface S_0 to a closed surface S that now encloses only the moving singularities. After some conversions, one obtains (, l means differentiation by x_l):

$$K_l^a + K_l^b = \iint_s [(s_i^a + s_i^b)\sigma_{ij,l}^\infty - (\sigma_{ij}^a + \sigma_{ij}^b) s_{i,l}^\infty] dS_j, \tag{II.161}$$

where s_i^∞, σ_{ij}^∞ means the dislocations and stresses that the singularities would have in an infinite medium. Eshelby could now confirm that the force K_l^t can be represented in the same form such that it ultimately yields the total force:

$$K_l = \iint_s (s_i \sigma_{ij,l}^\infty - \sigma_{ij} s_{i,l}^\infty) dS_j, \tag{II.162}$$

where $s_i \equiv s_i^a + s_i^b + s_i^t$, corresponding to σ_{ij}. Eq. (II.162), with further conversions, may be written in the form:

$$K_l = \iint_S M_{jl}\, dS_j\,, \qquad M_{jl} \equiv -\,\sigma_{ij}\, s_{i,\,l} + \frac{1}{2}\,\sigma_{ik}\,\varepsilon_{ik}\,\delta_{jl}\,. \tag{II.163}$$

In analogy with the situation in electrodynamics, Eshelby calls the asymmetric tensor M_{jl} the "Maxwell tensor of elasticity."

Eqs. (II.162) and (II.163) are valid quite generally. Especially important is their application to pointlike singularities. By contrast, the line element of the dislocation will not be included, since one can draw a small closed surface S around it that lies completely within a region with no sources of internal stresses. For our purposes, the meaning of the Eshelby equations rests upon the fact that the singularity required in them is, to a certain extent, not defined at its position, but at some distance from it (through the dislocations that it provokes there when the medium is infinitely extended), where the same sort of behavior rules as at the center of the singularity. It thus makes no difference whether, e.g., the force dipole arises from a weak or strong inclusion (from eq. (II.89), this is indeed definitive for the displacement s_i^∞). All that matters is the magnitude of the displacement that it produces on the surface S. However, this is, from eq. (II.89), proportional to the force dipole. From this, one immediately concludes from the general validity of eq. (II.156) that when it is true for one case (no inhomogeneity at the location of the inclusion) it must also be true for arbitrary inhomogeneities.

One can now further develop σ_{ij} and s_i in eq. (II.162) in a Taylor series and come to our formula (II.156), after some conversions in the case of the dipole [83]. This is likewise the proof that eq. (II.156) is also valid for finite dipole strengths. Eshelby [38] carried this computation for the special case of the so-called dilatation center (cf., Love [95]) $P_{ij} \equiv P_{ii} > 0$, which is particularly interesting, since an atom of type B in a lattice of type A can often be described as such a center (§ 31).

One can now calculate the work that the force K_k in eq. (II.156) will do under a change of position for the dipole from a point with null deformation to a point with the deformation ε_{ij} and obtain:

$$\int_{\varepsilon_{ij}=0}^{\varepsilon_{ij}} K_k\, dx_k = \int P_{ij}\, \nabla_k\, \varepsilon_{ij}\, dx_k = P_{ij} \int \nabla_k\, \varepsilon_{ij}\, dx_k = P_{ij}\, \varepsilon_{ij}\,. \tag{II.164}$$

This work is obviously independent of the path. Thus, one can interpret the expression:

$$U = -\,P_{ij}\, \varepsilon_{ij} \tag{II.165}$$

as the potential energy of the dipole P_{ij} in the deformation field ε_{ij}. In the case where no other proper stress sources besides P_{ij} are present, $P_{ij}\, \varepsilon_{ij}$ is equal to change in potential energy of the boundary force plus the change in the self-energy of the dipole as a result of the changing reaction with itself that goes over the outer surface. The sign in eq. (II.165) says sensibly that, e.g., a compressive inclusion ($P_{ii} > 0$) in a compressed part of the body

($\varepsilon_{ii} < 0$) yields a positive energy. One sees further that in the case of volume forces no such simple formula like (II.156) can exist since such forces have no potential in general, and thus can yield no path-independent integral $\int K_k \, dx_k$.

From eq. (II.165), the formula for the rotational moment $\boldsymbol{L}$ on a symmetric force dipole P_{ij} follows easily. If we write this as $P_{ij} = l_i \, P_j$, corresponding to eq. (II.90), then the change of P_{ij} under a rotation through an angle $d\vartheta_k$, is, since $d\boldsymbol{l} = d\boldsymbol{\vartheta} \times \boldsymbol{l}$, $d\vec{\boldsymbol{P}} = d\boldsymbol{\vartheta} \times \vec{\boldsymbol{P}}$ [1]:

$$dP_{ij} = dl_i \, P_j + l_i \, dP_j = (\varepsilon_{ikl} \, l_l \, P_j + \varepsilon_{jkl} \, l_i \, P_l) \, d\vartheta_k$$
$$= (\varepsilon_{ikl} \, P_{lj} + \varepsilon_{jkl} \, P_{il}) \, d\vartheta_k \, . \qquad \text{(II.166)}$$

On the other hand, the change in potential energy of the dipole under an infinitesimal rotation is:

$$dU = - L_k \, d\vartheta_k = - \varepsilon_{ij} \, dP_{ij} = - \varepsilon_{ij} \, (\varepsilon_{ikl} \, P_{lj} + \varepsilon_{jkl} \, P_{il}) \, d\vartheta_k \, . \qquad \text{(II.167)}$$

Due to the symmetry of ε_{ij} and P_{ij} , we thus immediately conclude that:

$$L_k = 2 \, \varepsilon_{ikl} \, P_{lj} \, \varepsilon_{ij} \, . \qquad \text{(II.168)}$$

The domain of validity of this formula is the same as for eq. (II.156).

In the case of a homogeneous inclusion (this shall mean that elastic homogeneity is disturbed at the location of the inclusion), a further effect comes about that was first precisely examined by Eshelby [38] and Crussard [27], namely, the "polarization" of the singularity [2]. The simplest case is a stress-free body with an inclusion of another material that fits precisely into a cavity in II without pressure. Now, if the body were, e.g., endowed with outer surface forces, then a force dipole P_{ij}^{ind} would be induced into the inclusion, where eq. (II.156) is likewise to be applied, and Eshelby can then show that one again obtains an expression like (II.163). One then appropriately defines a polarizability R_{ijkl} by the equation:

$$P_{ij}^{ind} = R_{ijkl} \, \varepsilon_{kl} \, , \qquad \text{(II.169)}$$

whose determination is fundamentally, and also for the smallest inclusions, a boundary-value problem relative to the boundary surface to be solved. For the case of a spherical inclusion with elastic isotropy one obtains elementary solutions after Eshelby [38], [40] [3].

It is now clear that one can represent the elastic displacement of an arbitrary pointlike proper stress source in an infinite medium by the equation:

$$s_i = P_{jk} \, \nabla_j \, S_{ki} + P_{jkl} \, \nabla_j \, \nabla_k \, S_{ki} + \ldots , \qquad \text{(II.170)}$$

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[1] Here, it is assumed that the self-energy of the dipole does not change during the rotation.

[2] See also, the further work of Crussard [28], Eshelby [40], and Teltow [151].

[3] Eshelby [165] could recently show that the stress field induced in an ellipsoidal inclusion is homogeneous when the induced stress field is homogeneous at a large distance from the inclusion (which is also true for elastic anisotropy). A corresponding result for the polarization of an ellipsoidal dielectric is indeed well-known.

from which the proper stresses can be calculated in the usual way. Now, a quadrupole is nothing but two juxtaposed dipoles, and analogously for the higher poles. That is, one can describe each such proper stress source by a certain combination of force dipoles. They are, in their own right, again nothing but dislocation loops of infinite extension. Thus, one has the theorem that was already cited in the Foreword: All proper stresses in a continuum will be provoked by dislocations. In § 14, this read: All proper stresses originate from incompatibilities. This is still true now, and is in harmony with the theorem above. Then, from (II.51), the incompatibilities owe their origin to the dislocations. One can thus refer to the dislocation or the incompatibilities as elementary proper stress sources. However, a consequence is the statement, which is identical with eq. (II.17): The dislocations are the roots of the elastic distortions, just as the forces are the sources of stresses.

On the other hand, one can also describe continuous distributions of proper stress sources – hence, dislocations – by a spatial distribution of such pointlike sources. One can thus, with equal right, declare the force dipole to be elementary in proper stress theory. The situation is the same as in the (stationary) Maxwell theory. There, infinitesimal current loops and magnetic dipoles are equivalent. Nevertheless, one gives preference to the electric current. They alone enter into the actual Maxwell equations. In this sense, we have also the given preference to the dislocation and thus, as we believe, obtained an impressive representation of the continuum mechanics of solid bodies.

Chapter III

Dislocations in crystals

§ 20. Generalities

This chapter presents applications of the continuum theory of dislocations to real bodies, the most important of which are crystalline structures. Heretofore, physical problems of this type related primarily to individual crystals, while the carry-over of the results obtained to the polycrystals that are generally available in technology has still come only so far. Thus, we will first restrict ourselves almost completely to treating the problems of the unit crystal, but let it be expressly remarked that nothing fundamental stands in the way of an application of continuum theory of dislocations to polycrystalline bodies, so we shall likewise return to them once more.

An essential difference between a continuum and a body that consists of individual mass points – like crystals, for instance – is that in the latter no volume element is defined initially. In a continuum, the distortion of the volume element is the distinguished geometrical quantity, while relatively little is said about dislocations. However, in a point system one has primarily a displacement of points and seeks to construct everything from that. Moreover, that quickly shows that one generally does not succeed in this way, since the required number of degrees of freedom is not attained. Rather, one has to consider the relative displacement of any two neighboring atoms. From the conclusions of § 21, it easily follows that a distribution of such relative displacements has three times too many degrees of freedom, like a distribution of dislocations. One can then develop a theory of dislocations in crystals with complete precision (§ 21), where one generally does not obtain differential equations, but difference equations. Moreover, since the number of points (atoms) in a crystal is outrageously large one can, for many purposes, write these difference equations to a best approximation as differential equations, and one must also do this, in general, since otherwise the problem might no longer be numerically tractable.

This process is particularly sensible for the numerous “microscopic” problems of crystal physics in which one is perhaps interested in the behavior and properties of individual dislocations. However, for the macroscopic problems one must deal with the collective effects of very many dislocations, so in that case it is natural to introduce certain “physical” volume elements whose properties must emerge from the following considerations:

The assumption of the applicability of continuum mechanics to real bodies is, on the one hand, that the forming of the volume element can be measured as a continuous function of the position. Thus, the volume element shall be sufficiently small compared to the external dimensions of the body, since otherwise one could no longer formulate differential equations. On the other hand, actual glide and climb planes are discrete and microscopically fairly far apart from each other, where their distance and the content of the glide (climb, resp.) that follows from them are subject to statistical fluctuations. Should such a distortion change continuously from volume element to volume element, then one can only speak of a mean distortion; this will also vary continuously only when each volume element is endowed with sufficiently many dislocations. If the glide plane

separation is, e.g., 10^{-6} cm, and one regards 1000 glide planes as necessary for a mean of the fluctuations then a physical volume element must have a linear extension of at least 10^{-3} cm. In general, one can regard this as sufficiently small compared to the body dimensions. For essentially larger glide plane separations, as one finds in many special cases, and for likewise small test bodies, the volume elements computed with the previous prescription might also then be no longer small compared to the body dimensions, in which case, the application of continuum mechanics is no longer sensible. We see that the theory of dislocations in crystals is fundamentally "less precise" than the continuum theory of dislocations, but still sufficiently precise to justify and suggest their use. The inexactitude of these calculations naturally consists in the fact that one then assumes the physical volume element to be mathematically infinitesimal, which means that one henceforth regards all formulas of continuum theory as also being applicable to real bodies.

This process is very simple and corresponds to the conception of this book. Another standpoint is that one constructs the macroscopic equations obtained in continuum mechanics from the equations of the microscopic problems when one adds the mutual interaction of very many dislocations and takes the necessary means. During this process, one continually remains in crystals. Solid-body physics is achieved when one thinks in terms of crystals, and in it the crystal is an imaginary body, to a large degree. Thus, the transition from microscopic quantities to macroscopic ones will also be briefly presented (§ 22).

In polycrystals the deformations of one crystallite to another generally change discontinuously, in one case, due to elastic anisotropy, in another, due to the plastic anisotropy of the crystallite, which originates in the fact that in each crystallite there is only one discrete glide system (= glide plane family plus the associated possible glide directions), which then comes into play for a definite shear stress [1]. If one would like to have a deformation change continuously from volume element to volume element then one can again treat only a mean deformation, so one needs a physical volume element that consists of many crystallites.

More difficult to resolve is the question of the structure curvatures. Naturally, one can define the rigid (elastic) rotation of a volume element compared to the initial state of an "ideal crystal," so the concept of orientation in this polycrystal is no longer meaningful. The problem is now whether the rigid rotations of the volume elements (e.g., in the absence of elastic deformations) that are present in a polycrystal changes the state of the body. One must then verify this experimentally and macroscopically. The investigation of this question yields (§ 23) that, in fact, structure curvatures can also be confirmed in polycrystals. Thus, the continuum theory of dislocations, in its previously developed form, can obviously be applied to polycrystals [2].

[1] This leads to the fact that in general not all crystallites being to flow simultaneously. Greenough [168], among others, has successfully treated the interesting problems that come about from this.

[2] Herr Prof. U. Dehlinger brought this to my attention.

§ 21. The geometrical foundations in crystals: the microscopic theory

We begin with a definition of a dislocation in a crystal that goes back to Frank [47]. Fig. 19a shows a lattice plane of the ideal crystal in Fig. 2, and Fig. 19b shows the same thing for the "perturbed" or "real crystal" of Fig. 3. Let the difference between the position vectors of two neighboring atoms in Fig. 19a and b be $\delta\boldsymbol{x}$ ($\delta\boldsymbol{x}'$, resp.). One now forms the sum $\sum_{\mathfrak{C}} \delta\boldsymbol{x}'$ along an arbitrary closed path $\mathfrak{C}'$ in the real crystal. One thus starts from, perhaps, the point P', goes from atom to atom seven steps in the x_3-direction, then four steps in the x_1-direction, etc., along an arbitrary path back to P'. One then repeats the same program (hence, seven steps in the x_3-direction, four steps in the x_1-direction, etc.) starting from the point P in the ideal crystal that corresponds to P' (circuit $\mathfrak{C}$). Along $\mathfrak{C}$, when the circuit goes around a dislocation line, with the step by which one again arrives at P' along the circuit $\mathfrak{C}'$, one does not also come back to P correspondingly, as the figures show. We now determine that the path $\mathfrak{C}'$ shall encircle the dislocation line in the right-hand screw sense. The vector $\overrightarrow{QP} \equiv \boldsymbol{\delta b}$ from the endpoint Q of the path $\mathfrak{C}$ that corresponds to $\mathfrak{C}'$ to its starting point is then characteristic of the dislocation that encircles $\mathfrak{C}'$. One can thus define the dislocation with the help of this "Frank-Burgers circuit" [1]. $\boldsymbol{\delta b}$ will be called the Burgers vector of the crystal dislocation. Further considerations will yield that it corresponds to the Burgers vector in a continuum.

We can think of the dislocation in Fig. 19b as migrating from the right into the crystal. Thus, any two neighboring atoms, between which the dislocation wanders, experience a plastic relative displacement $\delta\boldsymbol{g} = -\boldsymbol{\delta b}$ [2]. If we then form the circuit sum $\sum_{\mathfrak{C}} \delta\boldsymbol{g}$ we obtain:

$$\sum_{\mathfrak{C}} \delta\boldsymbol{g} = -\boldsymbol{\delta b} \,. \tag{III.1}$$

This equation corresponds to eq. (I.12) of § 3 exactly. Thus, one must now observe the following: As long as the circuit was defined, $\boldsymbol{\delta b}$ will henceforth be a so-called lattice vector; i.e., a vector that points from one atom to another atom (in an ideal crystal). This is a physical requirement: The plastic relative displacement of the atoms under the migration must result in such a way that the regular arrangement of the atoms, except for the center of the dislocation, remains the same everywhere; an irregular atomic arrangement over surface-like or even three-dimensional regions would imply an

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[1] In the literature up to now, the Burgers vector of the isolated dislocation was mostly denoted by $\boldsymbol{b}$ (also in Fig. 19a). From the continuum standpoint, the notation $\boldsymbol{\delta b}$ (cf., *infra*) is more recommended. One should look for no deeper significance in this notation.

[2] It is not unnecessary to add that for all of the remaining pairs of atoms one must demand that $\delta\boldsymbol{g} = 0$. The minus sign corresponds to the convention that, on the one hand, the directional sense of the dislocation line is chosen such that $\mathfrak{C}$ ($\mathfrak{C}'$, resp.) will be a right-hand screw circuit, while, on the other hand, $\delta\boldsymbol{g}$ shall be the relative displacement of the atom on the positive side of $\mathfrak{C}$ compared to that on the negative side.

extraordinarily large increase in the internal energy of the crystal, and is therefore "forbidden."

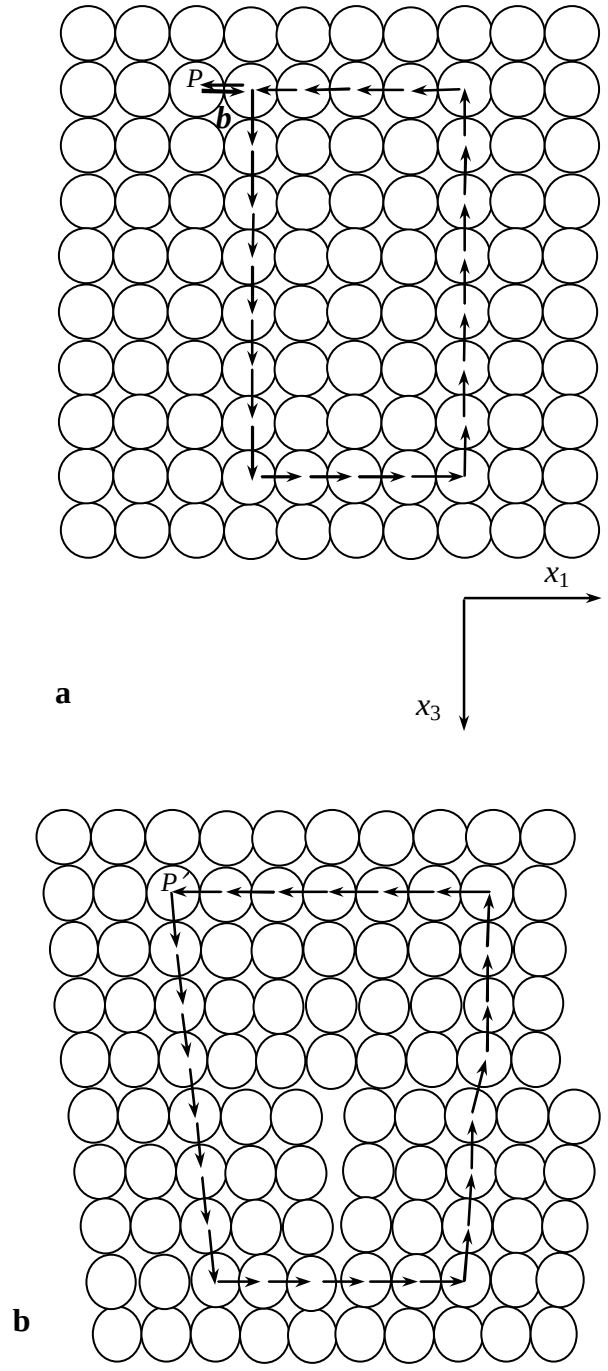

Fig. 19. On the explanation for the Frank-Burgers circuit. The symbol *P* in **a** refers to an atom that is further to the right.

On the other hand, one cannot exclude a certain spatial extension of the especially strongly perturbed region in the immediate neighborhood of the dislocation from the outset. That is, one can indeed assume that at a large distance from the dislocation center

the atoms affected by the dislocation have already experienced the entire relative displacement $\delta\boldsymbol{g}$, although for the atoms that are found close to the center one cannot necessarily say this; i.e., the transition of $\delta\boldsymbol{g}$ between the values $-\delta\boldsymbol{b}$ (in Fig. 19b) to the right of the dislocation and zero (to the left of the dislocation) does not need to result abruptly from one atom to another, but it can take place over a region of, e.g., two or three atomic spacings. This would correspond to our previous conception of a "dislocation width" 2ζ. In order to include this possibility we would again like to assert that the dislocation is composed of superficially distributed threads of infinitesimal strength $\partial\boldsymbol{b}$, where one must have $\int \partial\boldsymbol{b} = \delta\boldsymbol{g}$ for the dislocation. Each thread then gives a relative displacement through $-\partial\boldsymbol{b}$ [1] of the atoms between which it migrates.

Now, suppose we have a crystal in the ideal initial state. Let its atoms be numbered, say, consecutively, and let the relative position of any two neighboring atoms be denoted by δx_i . We let δu_j denote an, at first, only imaginary relative displacement of the atom on the positive side of δx_i compared to that of the negative one. Thus, let $\delta u_j(x_i)$ be given in the entire crystal, where x_i is the location of the atom that lies on the negative side of δx_i in the initial state. We would like to allow δu_j to be arbitrarily discontinuous, such that, in particular, after performing the relative displacement the crystal no longer needs to be connected. Only the possibility that two or more atoms might fall upon the same point afterwards shall be excluded.

The question is now: Is it possible at all to bring the crystal into such a state that, in fact, two atoms have always experienced the prescribed displacement difference of δu_j ? The answer reads: In general, this is not possible. One already sees the essence of this in a planar "crystal" that consists of only four atoms 1, 2, 3, 4. Namely, if one prescribes δu_j for the atom pairs 12, 23, 34 then the mutual configuration of the four atoms is already completely determined, and a possible choice for the last atom pair 41 is no longer arbitrary, but must depend upon the first three choices (Fig. 20).

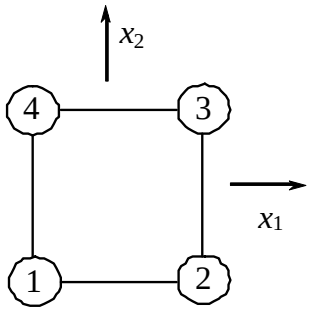

Fig. 20. Plane "crystal" consisting of four atoms. The atom pairs 13 and 24 are not neighboring atoms.

The next question is: What are the restricted conditions that the δu_j must be subject to in order for the production of a state described by the δu_j to be possible? One immediately sees that the sum of the δu_j taken along an arbitrary path from an atom a to an atom b must be independent of the path; i.e.:

$$\sum \delta u_j = 0 \qquad \text{(III.2)}$$

for an arbitrary closed path that is carried out in an ideal crystal. From eq. (III.2), there follows the existence of a function $u_j(x_i)$ that can be arbitrarily discontinuous. u_j is naturally the displacement of the atom, which is single-valued up to a rigid displacement of the crystal.

Let the "microscopic" distortion tensor $\boldsymbol{\gamma} \equiv (\gamma_{ij})$ be defined by the equation:

[1] For Frank, the circuit $\mathfrak{C}'$ is to be completed at a sufficient distance from the dislocation center, such that all dislocation threads lie inside of the circuit. We will not insist upon this in our investigations.

$$\delta u_j = \gamma_{ij}\, \delta x_i \,. \tag{III.3}$$

We explain this as follows: A certain atom can be found at the positions x_i . Its three neighbor atoms in the direction of the positive x_i-axis have the positions (in an ideal crystal) $x_i + \delta x_i$. These four atoms, which define the basic triad for the lattice, succeed in sensibly defining and clarifying a distortion at the point x_i . From eq. (III.3), e.g., the appearance of a distortion γ_{11} means that the two successive points separated by δx_1 experience a relative displacement δu_1 in the x_1-direction, i.e., a positive γ_{11} is a rotation of the triad in the x_1-direction (Fig. 21b). Correspondingly, one sees that, e.g., γ_{21} means a shear of the triad as in Fig. 21c. In the case of small distortions the symmetric part of γ_{ij} is a pure deformation and the anti-symmetric part is a pure rotation of the triad.

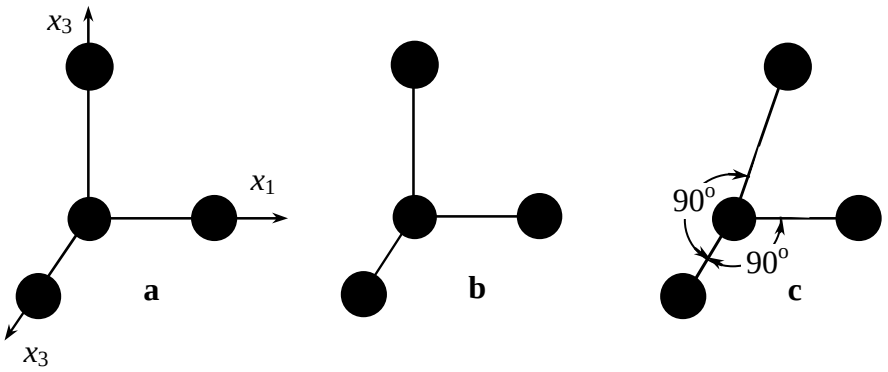

Fig. 21. On the definition of the microscopic distortion tensor.

If we substitute eq. (III.3) into eq. (III.2) then it follows, with Stokes's theorem, that [1]:

$$({}^{\delta}\mathrm{Rot}\ \boldsymbol{\gamma})_{il} \equiv \varepsilon_{ijk} \frac{\delta \gamma_{kl}}{\delta x_j} = 0. \tag{III.4}$$

Our distortions, which satisfy eq. (III.4), are actually possible in our Euclidian space [2].

The application of Stokes's theorem is also meaningful in the case of discretely distributed points and arbitrary discontinuous relative displacements δu_j, as one can already show in the example of four atoms in Fig. 20. All points shall remain in the plane $x_3 = 0$. The following distortions are then defined according to eq. (III.3). For point 1: β_{11} , β_{12} , β_{21} , β_{22} ; for point 2: β_{21} , β_{22} ; for point 3: none; for point 4: β_{11} , β_{12} . Formally, one has:

$$(\mathrm{Rot}\ \boldsymbol{\beta})_{31} = \frac{\delta \beta_{21}}{\delta x_1} - \frac{\delta \beta_{11}}{\delta x_2}, \qquad (\mathrm{Rot}\ \boldsymbol{\beta})_{32} = \frac{\delta \beta_{22}}{\delta x_1} - \frac{\delta \beta_{12}}{\delta x_2} \tag{III.5}$$

as both being non-zero, or, when written as a difference equation:

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[1] By ${}^{\delta}$Rot, it shall be implied that one is dealing with difference equations in atomic space.

[2] However, to each arbitrary distribution δu_j there is a non-Euclidian space in which this is possible. The x_i are then the coordinates of this space; e.g., a curved surface in the case of a two-dimensional crystal.

$$(\text{Rot}\,\boldsymbol{\beta})_{31} = \frac{\beta_{21}(2)-\beta_{21}(1)}{\delta x_1} - \frac{\beta_{11}(4)-\beta_{11}(1)}{\delta x_2};$$
$$(\text{Rot}\,\boldsymbol{\beta})_{32} = \frac{\beta_{22}(2)-\beta_{22}(1)}{\delta x_1} - \frac{\beta_{12}(4)-\beta_{12}(1)}{\delta x_2}. \qquad \text{(III.6)}$$

The same thing happens as before for the distortion components that were established by definition. After multiplication by $\delta x_1\, \delta x_2$, one obtains $\sum \delta x_i\, \beta_{ij}$. Thus, in total, when we write δF_k for $\delta x_i\, \delta x_j$, we have:

$$\delta F_k\, (\text{Rot}\,\boldsymbol{\beta})_{kj} = \sum \delta x_i\, \beta_{ij}, \qquad \text{(III.7)}$$

i.e., Stokes's theorem.

The previous considerations were of a purely geometrical nature. Whether the relative displacement of two atoms implied reaction forces was not mentioned. Henceforth, we consider the process of the immigration of a dislocation into the ideal crystal of Fig. 2, which will be then brought into the state of Fig. 3 or 5. If one makes a circuit $\mathfrak{C}$ in an ideal crystal and one meanwhile sums the aforementioned relative displacement δg_j between two atoms then one will have:

$$\sum_{\mathfrak{C}} \delta g_j = -\,\delta b_j, \qquad \text{(III.8)}$$

if the circuit $\mathfrak{C}'$ that corresponds to $\mathfrak{C}$ after the migration of the dislocation encircles the dislocation; in the other case, one has $\sum \delta g_j = 0$ [1]. From eq. (III.8), it follows that the plastic relative displacement δg_j in does not fulfill the conditions (III.2) in a crystal with dislocations. Now, Fig. 3 and 5 show intuitively that a dislocation in a crystal is always surrounded by a region of elastic deformation. We denote the elastic relative displacement of two neighbor atoms by δa_j. We would now like to call the total relative displacement δs^G and combine the elastic and plastic displacements:

$$\delta s_j^G = \delta a_j + \delta g_j, \qquad \text{(III.9)}$$

so one has:

$$\sum \delta s_j^G = 0 \qquad \text{(III.10)}$$

for each arbitrary closed path. From this, there follows the existence of a function $s_j^G(x_i)$ that gives the difference in position of a certain atom in the ideal state and the "dislocated" state up to a constant displacement that is common to all atoms. The existence of this function, which does not need to be continuous, is a result of the fact that the operation that takes the crystal from, e.g., the state of Fig. 2 into that of Fig. 3 or 5 is possible in Euclidian space.

[1] One can, in place of the ideal plus real crystals, also consider only one ideal crystal in which the dislocation migrates, as a result of the inhibition that at first no distortion is present. In this sense, dislocations can also be brought into ideal crystals. When regarded in this way, one also easily obtains equations for arbitrarily large distortions. Cf., § 10, the beginning.

Furthermore, we define the generally asymmetric tensors of (microscopic) total distortion $\boldsymbol{\beta}^G \equiv (\beta_{ij}^G)$, elastic distortions $\boldsymbol{\beta} \equiv (\beta_{ij})$, and plastic distortion $\boldsymbol{\beta}^P \equiv (\beta_{ij}^P)$ by the equations:

$$\delta s_j^G = \beta_{ij}^G \delta x_i, \qquad \delta a_j = \beta_{ij}\, \delta x_i, \qquad \delta g_j = \beta_{ij}^P \delta x_i. \tag{III.11}$$

We must then first explain to what extent one sensibly links the elastic and plastic distortion at all, since this, just like δa_j and δg_j, naturally represents precisely the same operations that are not, in and of themselves, possible in Euclidian space. One now remarks that any such operation δu_j, which only relates to a triad, can be performed in Euclidian space, so only after one adds more atoms – e.g., constructed from a cube of eight atoms – will the restriction (III.2) be effective. Obviously, this connected with the fact that a triad, to a certain extent, can have no dislocation in its interior, while, e.g., the aforementioned cube can (cf., *infra*).

If we substitute the third of eq. (III.11) in eq. (III.8) then it follows with Stokes's theorem that:

$$\sum_F \delta F_i ({}^{\delta}\mathrm{Rot}\, \boldsymbol{\beta}^P)_{ij} = -\, \delta b_j, \tag{III.12}$$

where F is a surface that spans $\mathfrak{C}$. This result is then sensible when the circuit contacts only four atoms. One then writes eq. (III.12):

$$({}^{\delta}\mathrm{Rot}\, \boldsymbol{\beta}^P)_{ij} = -\, \delta b_j / \delta F_i, \tag{III.13}$$

in the event that one assumes that all threads of the dislocation run through the surface element δF_i that is spanned by the four atoms [1]. $\delta b_j / \delta F_i$ is then obviously a mean surface density of the dislocation thread in the domain of δF_i. This statement is then also valid when some of the threads of the dislocation lie outside of δF_i, as long as one understands δb_j to simply mean the total Burgers vector of the threads that run through δF_i [2]. In other words, $\delta b_j / \delta F_i$ is the (microscopic) dislocation density, so we call it α_{ij} and write eq. (III.13) as:

$${}^{\delta}\mathrm{Rot}\, \boldsymbol{\beta}^P = -\, \boldsymbol{\alpha}. \tag{III.14}$$

In the case where many dislocations run through the surface F of eq. (III.12), it is the total Burgers vector of all dislocations that run through F that appears on the right-hand of this equation.

Due to eq. (III.10), one naturally has for the total distortion that:

$$\boldsymbol{\beta}^G = \boldsymbol{\beta} + \boldsymbol{\beta}^P, \tag{III.15}$$

and:

$${}^{\delta}\mathrm{Rot}\, \boldsymbol{\beta}^G = 0, \tag{III.16}$$

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[1] In the sense of footnote [1] of pp. ?.

[2] In § 25 it will be shown that one can calculate the distribution of dislocation threads in a dislocation approximately, the result being that by far most of the threads of a dislocation lie inside a cross-section of magnitude $|\, \delta F\, |$.

and we thus obtain the (microscopic) fundamental geometric equation of the crystal as:

$$ {}^{\delta}\mathrm{Rot}\,\boldsymbol{\beta} = \boldsymbol{\alpha}. \tag{III.17} $$

This equation states that the presence of dislocations in crystals is always linked with elastic distortions, since the plastic distortions that enter into migration or formation of dislocations are, by definition, in themselves not possible in Euclidian space [1].

§ 22. The fundamental geometric equation in a crystal: transition to the macroscopic theory

In this paragraph we would like to make the transition from microscopic to macroscopic quantities. Thus, we first define the macroscopic (= physical) volume element ΔV, which shall be the minimum of the quantities required in § 20 for the initial state of an ideal crystal. The term "element" expresses the idea that, on the other hand, the test body shall contain a very large number of such volume elements.

Here, we come to the dislocation tensor. It is clear that we now no longer need to concern ourselves with the thread structure of individual dislocations, since we can, as before, describe them by their unit tangent vector and Burgers vector; i.e., through $t_i\ \delta b_j$. Due to the enormous number of dislocations that appear in all crystals, one can naturally no longer give the trace and Burgers vector for each line.

We now ask ourselves how a state with very many dislocations running very close to each other can be described most simply and completely. Obviously, such a mean is sensible when we give at each point $\boldsymbol{x}$ of the crystal, the numbers (N^{ab}) of dislocations with the direction $\boldsymbol{t}^a$ and Burgers vector $\delta\boldsymbol{b}^b$ go through each directed surface element ΔF at $\boldsymbol{x}$, where the surface element shall indeed be so large that it will be filled with many dislocations; in other words, the numbers N^{ab} change from one surface element to another only slightly. One can then say that the dislocations run through the surface elements with a certain density.

We let $\Delta\boldsymbol{b}$ denote the total Burgers vector of all of the dislocations that run through $\Delta\boldsymbol{F}$. It is:

$$ \Delta\boldsymbol{b} = \sum_{a,b} N^{ab}\delta\boldsymbol{b}^b\ . \tag{III.18} $$

One observes that $\Delta\boldsymbol{b}$ does not change when one, e.g., doubles the numbers N^{ab} and simultaneously halves the associated Burgers vector. The transition from a continuous distribution is now to be performed in such a way that one always simultaneously lets the numbers N^{ab} increase while the Burgers vectors go to zero, such that the total Burgers vector $\Delta\boldsymbol{b}$ remains constant [2].

In a macroscopic theory, one now no longer envisions a discrete distribution, as will indeed be represented by the numbers N^{ab}. That is, one must restrict oneself to regarding the total Burgers vector $\Delta\boldsymbol{b}$ as given. For crystals this is an actual loss, which generally

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[1] I thank Herrn Prof. E. Fues for his critical remarks on my earlier work, which gave rise to the presentation that was given above.

[2] This passage to the limit was first carried out by Nye [113].

can again be corrected only by some additional crystallographic consideration. That comes from the fact that the Burgers vectors can assume only discrete values, and for that reason the dislocations actually take on an isolated character [1]. As a result, for many purposes we are also interested here in precisely those arrangements of dislocation lines of the particular type that was described in eq. (III.18).

When we know the total Burgers vectors of all of the dislocation lines that go through each surface element $\Delta \boldsymbol{F}$, we are then oriented towards the dislocation state with previous restrictions. We are, moreover, in a position to define the macroscopic tensor of dislocation density $\boldsymbol{\alpha} = (\alpha_{ij})$ with the help of the equation:

$$\Delta b_j = \alpha_{ij} \, \Delta F_i \, . \tag{III.19}$$

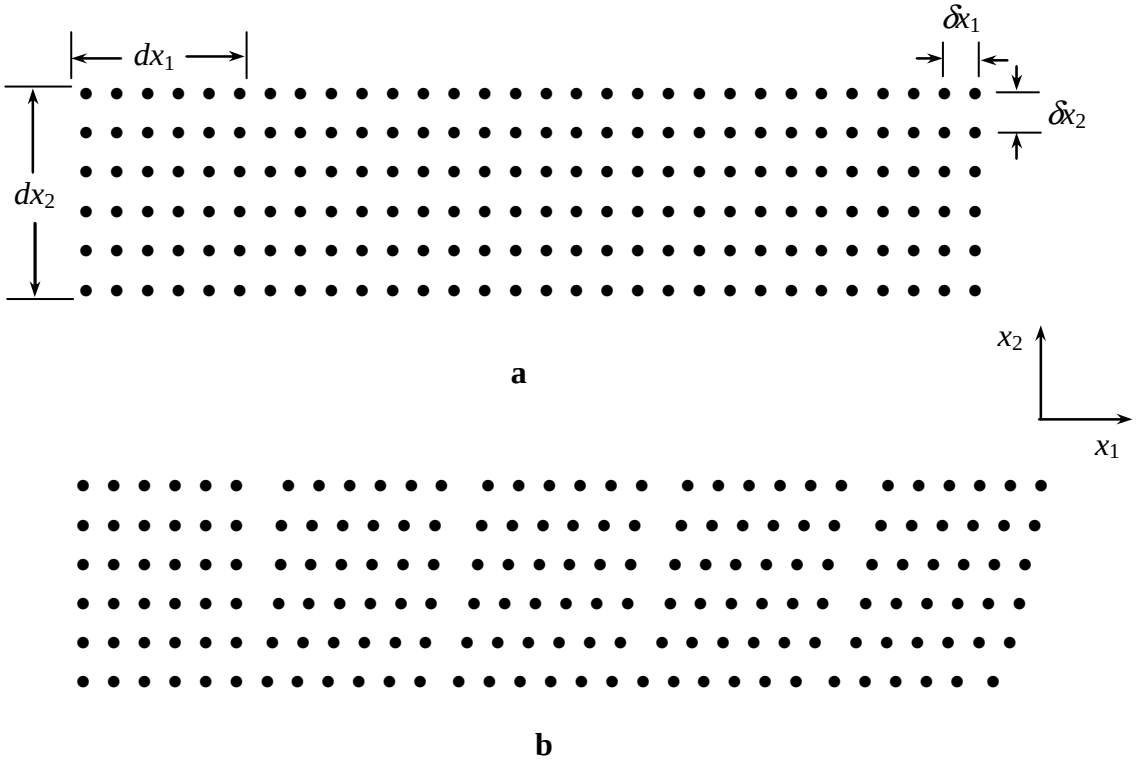
Since, by assumption, the path of the dislocation is essentially homogeneous in a neighborhood of the surface element, we can assume the dislocations to be straight inside the volume element ΔV. In addition, we would like to assume that the dislocations α_{ij} intersect the surface ΔF_i perpendicularly. Indeed, real dislocation lines do not generally do this, so we have indeed passed over from the distribution of total Burgers vectors to the individual dislocation lines of different types according to eq. (III.18) from crystallography. Our macroscopic dislocation lines α_{ij} thus run in the i-direction and all of their Burgers vectors are in the j-direction. As a consequence, the diagonal components of $\alpha_{ij}(\boldsymbol{x})$ at the position $\boldsymbol{x}$ represent the previous screw dislocations, while the remaining components represent edge dislocations.

The macroscopic dislocation density in a crystal is a very intuitive quantity, so it indeed directly represents a family of lines, as in Fig. 3 and 5. Its flux through an arbitrary surface F – i.e., the dislocation flux – is equal to the total Burgers vector of all dislocation lines that run through F, and from eq. (III.19):

$$\boldsymbol{b} = \iint_F \Delta \boldsymbol{F} \cdot \boldsymbol{\alpha} \, . \tag{III.20}$$

We come, moreover, to the connection between the microscopic and macroscopic distortions. For this, each pair of neighbor atoms in the crystal in the initial state can again be first thought of as having been subjected to a relative dislocation δu_j, as before. Now, let δu_j be homogeneously distributed inside of a volume element composed of very many atoms, which we would like to call dV (in order to distinguish it from the previously employed volume element ΔV; cf., *infra*), so it may change discontinuously from one element to another. Fig. 22 shows a simple example.

[1] With fewer assumptions, one has the theorem that the Burgers vector must be not only a lattice vector (§ 21), but the smallest possible lattice vector, in order for the perturbation at the dislocation center to not require too much energy (this is certainly true for $\boldsymbol{b}^2$, § 18). Three discrete Burgers vectors are then possible in a primitive cubic lattice; in a cubic face-centered lattice, there are six.



a

b

Fig. 22

We can characterize the geometric state of all atoms in Fig. 22b by giving the (microscopic) γ_{ij} that were defined by eq. (III.3) as functions of the positions of the atoms in the initial state. One then obtains γ_{21} and γ_{11} non-zero, where γ_{21} depends only upon x_1, while γ_{11} is non-zero only for atoms that bound the element dV on the $+x_1$-side, and there it depends only upon x_2 . The conditions (III.4), which naturally must be fulfilled everywhere, are then written:

$$\frac{\delta\gamma_{21}}{\delta x_1} - \frac{\delta\gamma_{11}}{\delta x_2} = 0. \qquad \text{(III.20)}$$

The function u_j can be obtained from eq. (III.3) for a known γ_{ij}, up to a rigid displacement of the crystal.

Another description of the state of Fig. 22b would be that one now always gives the distortion for the entire volume element dV, since, by assumption, it is constant inside of it. We can consider this to be a definition of the macroscopic distortion. The diagonal components will be functions of the position in the initial state that measure the dilatation ratio of the elements in the initial state, while the remaining components measure the tangents of the shear angles. In the interior of a homogeneously distorted volume element, each atomic triad naturally has numerically as many (microscopic) distortions as the element (macroscopically). For an element of Fig. 22b, only γ_{21} is (macroscopically) non-zero. Let $dx_i = n\ \delta x_i$ be, e.g., the distance from two neighboring elements dV in the initial state to the center of mass. $d\gamma_{21}$ is then numerically equal to value $\delta\gamma_{21}$ in the boundary surface, thus $d\gamma_{21} / dx_1$ is numerically equal to the value $\delta\gamma_{21} / n\ \delta x_1 = \delta\gamma_{11} / n$

δx_2 in the boundary surface. For given macroscopic $\gamma_{21}(x_1)$, one can thus microscopically regain γ_{11} on the boundary surface, up to a constant. It likewise follows that for macroscopic γ_{ij}, in general, the statement:

$$(\text{Rot } \boldsymbol{\gamma})_{il} \equiv \varepsilon_{ijk} \frac{\partial \gamma_{kl}}{\partial x_j} \neq 0 \tag{III.22}$$

is valid, in contradiction to the microscopic γ_{ij}. The condition for Rot $\boldsymbol{\gamma} = 0$ to be true obviously reads $\delta\gamma_{11} / \delta x_2$ in the previous example; in other words, there can be no variable source in the boundary surface of the volume element. We shall come back to this.

The previously assumed homogeneity of the distortion inside a volume element that is composed of very many atoms is not given for real processes of plastic forming. One can thus assume, for a sufficient large *dV*, that at least a mean homogeneity exists, which might perhaps be known empirically, and that after being cut out the outer surface of a distinguished volume element *dV* is not noticeably deformed under forming or also relaxation (in the other case, the assumptions for the applicability of the continuum theory stated in § 20 are not fulfilled).

Thus, we can identify the physical volume element *dV* with the mathematical volume element *dV* that was used in § 3, in which there is now precisely the inexactitude that we described in § 20. For these elements, we define, as before in § 2, the tensors of macroscopic total distortion, as well as the elastic and plastic distortions β_{ij}^G, β_{ij}, β_{ij}^P. Previously, we found that the equation:

$$\text{Rot } \boldsymbol{\beta}^G = 0 \tag{III.23}$$

is necessary when the body shows no rip after forming [1]. We see that this macroscopic equation says something entirely different from the microscopic equation (III.16):

$$^{\delta}\text{Rot } \boldsymbol{\beta}^G = 0. \tag{III.24}$$

This insures the possibility of the operation in Euclidian space. One thus does not speak of the formation of a rip, and in fact, the concept of "rip" is not even defined in a system composed of discrete points. In contrast to this, the way in which the distortions of the macroscopic theory were defined thenceforth insures the possibility of the operation in Euclidian space [2], and indeed this true not only for the total distortion, but also for the elastic and plastic distortions. Additionally, one now comes to the restriction (III.23) that hinders the formation of a rip.

---

[1] The function $\boldsymbol{\beta}^G$, when restricted by eq. (III.23), is naturally also of such a type that one never expects that parts of two neighboring elements will be found at the same place. If one refers to the piece of space in which (as is only hypothetically possible) parts of two volume elements are both found (i.e., they overlap) as the "negative rip" then in this case one does not always need to expressly mention the normal "positive rip."

[2] Otherwise expressed: A forming that can be described with the help of a macroscopic distortion tensor field can be carried out primarily in Euclidian space.

One observes that the macroscopic plastic distortion is also an operation that can be performed only in Euclidian space alone, so generally the connectivity of the space is not disturbed, since Rot $\boldsymbol{\beta}^P \neq 0$. From the remarks connected with eq. (III.22), it follows that wherever one has Rot $\boldsymbol{\beta}^P \neq 0$ there exists a linearly varying plastic displacement source between the volume elements, and from § 8 this is always the case wherever dislocations remain in place between volume elements in a constant density. If one performs a circuit around a surface element ΔF in the initial state that is composed of many elements dF then one obtains, as in § 3, the total Burgers vector of the conserved dislocations:

$$\Delta b_j = -\int_{\mathfrak{C}} dx_i\, \beta_{ij}^P = -\iint_{\Delta F} dF_k\, (\mathrm{Rot}\, \boldsymbol{\beta}^P)_{kj} = -\Delta F_k (\mathrm{Rot}\, \boldsymbol{\beta}^P)_{kj}\,, \tag{III.25}$$

when one assumes a homogeneous distribution of Rot $\boldsymbol{\beta}^P$ in the domain of ΔF. After comparing with eq. (III.19), one again obtains:

$$\mathrm{Rot}\, \boldsymbol{\beta}^P = -\boldsymbol{\alpha}, \tag{III.26}$$

and after combination with eq. (III.23), the fundamental macroscopic equation:

$$\mathrm{Rot}\, \boldsymbol{\beta} = \boldsymbol{\alpha}\,. \tag{III.27}$$

Here, we calculated in such a way as to suggest that the dislocations remain in place between volume elements. In reality, they remain in place spatially, as well as superficially. If we then pass to the limit $dV \to 0$, the superficial arrangement between the volume elements will indeed become a spatial arrangement.

We spoke of the case of linearly varying plastic displacement sources, but not of the case of constant displacement sources. For such entities, the volume elements will be macroscopically dislocated opposite to each other; i.e., the displacement of the points of the medium will be macroscopically discontinuous. This case does not seem to suggest any particular practical interpretation, so we would like to consider it no further

§ 23. Planar configurations of dislocations in crystals

In this paragraph, we treat an application of the boundary surface equations of § 8 and 9 to crystalline bodies. One calls the boundary surface between two crystallites, which are distinguished only by their orientations, the "grain boundary." Such grain boundaries arise, e.g., during the growth of crystals from the process of melting. Growth is found to come about from any statistically defined seeds. When two neighboring seeds have different orientations and grow they ultimately merge together, and along the boundary surface there exists a region of atomic disorder. When the difference in orientation amounts to no more than, say, 20^o, one can distinguish individual dislocations in the grain boundary that mediate the transition from one orientation to the other. A simple example is shown in Fig. 23. The associated configuration of dislocations is indicated

schematically next to it [1]. Here, one treats a wall of edge dislocations whose Burgers vector is perpendicular to the plane of the wall. However, a wall of edge dislocations with a Burgers vector in the wall gives no difference in orientation (Fig. 24, next page). Such dislocation walls appear in phase boundaries (§ 32) with non-constant density as a build-up in glide planes [2] when an obstacle blocks the further migration of the dislocations.

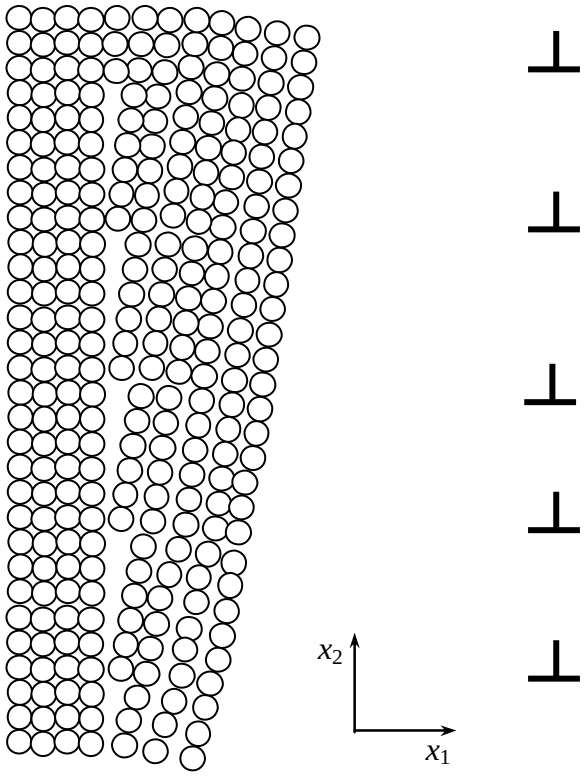


Fig. 23. Grain boundary of the first type. The difference in orientation between the colliding grains is $|\boldsymbol{g}| / d$, if d is the distance between the dislocations. This follows from eq. (III.32)

[1] A straight edge dislocation that runs perpendicular to the plane of the paper will often by denoted by the symbol ⊥ .The two lines of the symbol give the glide plane and inserted lattice planes of the dislocation in an easily understood way. The dislocation in Fig. 19b would thus be denoted by ⊥ .

[2] One calls them "glide zones." The calculation of the dislocation distribution in a glide zone is an interesting mathematical problem that was solved under various assumptions by Eshelby, Frank and Nabarro [166], and Leibfried [89]. Leibfried showed, *inter alia*, that one can employ a continuous distribution of dislocations in place of a discrete one to a good approximation if the glide zone contains only a few dislocations. One then obtains the equilibrium distribution of dislocations as it depends upon the applied external stresses from a linear integral equation. The closely connected problem of calculating the proper stresses that belong to this equilibrium distribution could, for Haasen and Leibfried [169], be solved in general by integration in the complex plane. The stresses then come about essentially by differentiation of the dislocation distribution. Leibfried [172] has further treated the build-up of circular screw dislocations in a glide plane. The application of these computations involves the hardening of metals.

We next treat grain boundaries. From § 7, we know that walls of circular screw dislocations can also serve as grain boundaries. Unfortunately, this is not shown very well. Two problems appear in practice:

1. Given the orientation difference between two grains, find the configuration of dislocations that defines the grain boundary.

2. Given the dislocation wall, find the orientation difference between the two bounding grains [1].

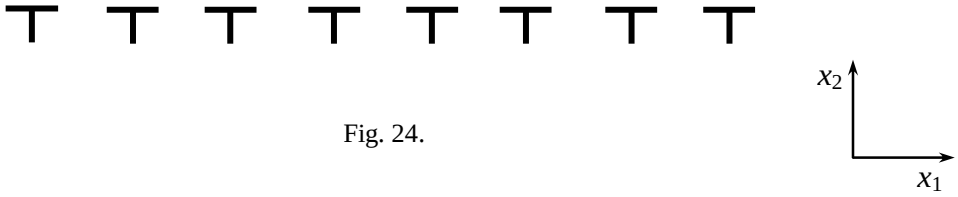


Fig. 24.

We decompose these problems into a continuum-theoretic part and a crystallographic part. In the former, one establishes the macroscopic dislocation density $\boldsymbol{\alpha}$, and in the latter, with the help of that density and the crystallographic condition, one establishes the microscopic arrangement.

We can give the solution of the continuum-theoretical part immediately. A grain boundary is, from the above, defined by the orientation difference between the grains alone. Arbitrary macroscopic deformations are thus not involved. (We do not need to consider microscopic elastic deformations that stem from the fact that the Burgers vectors have finite strengths in crystals, at least in the continuum part of the problem.) Thus, in our case, the elastic distortion β_{ij} is simply a rigid rotation ω_{ij} of the volume element, and from eq. (I.67), one will have:

$$\varepsilon_{ijk}\, n_i\, \omega_{jl}\,|_{\mathrm{II}} - \varepsilon_{ijk}\, n_i\, \omega_{jl}\,|_{\mathrm{I}} = \bar{\alpha}_{kl}\,. \tag{III.28}$$

In the case of large orientation differences, one must replace the ω_{jl} with the rotation tensor of eq. (I.94). For small orientation differences, its symmetric part is negligible, and one can take, in place of the anti-symmetric part, the associated rotation vector:

$$\omega_i = \varepsilon_{ijk}\, \omega_{jk}\,, \qquad\qquad \omega_{ij} = \varepsilon_{ijk}\, \omega_k\,. \tag{III.29}$$

From eq. (III.28), one then has:

$$\varepsilon_{ijk}\, \varepsilon_{jlm}\, n_i\, \omega_m\,|_{\mathrm{II}} - \varepsilon_{ijk}\, \varepsilon_{jlm}\, n_i\, \omega_m\,|_{\mathrm{I}} = \bar{\alpha}_{kl}\,, \tag{III.30}$$

and with the decomposition formula (A.2):

$$(\delta_{kl}\, n_i\, \omega_m - n_l\, \omega_k)|_{\mathrm{II}} - (\delta_{kl}\, n_i\, \omega_m - n_l\, \omega_k)|_{\mathrm{I}} = \bar{\alpha}_{kl}\,^2. \tag{III.31}$$

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[1] One thanks Frank [46] for the general solution of these problems.

[2] This is obviously the boundary surface form of eq. (I.59).

In Fig. 23, only the components n_1 and ω_3 were non-zero. Thus, the associated dislocation density $\bar{\alpha}_{ij}$ includes only the component ($n_1 = 1$):

$$\bar{\alpha}_{31} = -\omega_3 \mid_{II} + \omega_3 \mid_{I}; \tag{III.32}$$

it is precisely the distinguished set of dislocations. In case there exists a rotation between two crystallites around an axis perpendicular to the boundary surface ("twist boundary," in contrast to the "tilt boundary" that was treated above), one obtains circular screw dislocations. If the rotation axis is along, e.g., the x_1-direction (only n_1 and ω_1 non-zero) then from eq. (III.31) alone, one gets that:

$$\alpha_{22} = \alpha_{33} = \omega_1 \mid_{II} - \omega_1 \mid_{I} \tag{III.33}$$

are non-zero.

We now show that, in fact, these grain boundaries lead to no macroscopic stresses. For this, it is necessary and sufficient that the surface incompatibilities vanish. From eq. (I.87), it follows immediately that due to the constant surface density of the dislocations on the boundary surface one has $\bar{\eta} = 0$. For $\overline{\overline{\eta}}$, one has, from eq. (I.90), $\overline{\overline{\eta}} = (\bar{\boldsymbol{\alpha}} \times \boldsymbol{n})^S$. If the $\boldsymbol{i}_i$ are the basis vectors then one has for the grain boundaries of Fig. 23, with $\boldsymbol{n} = \boldsymbol{i}_1$ and $\bar{\boldsymbol{\alpha}} = \bar{\alpha}_{31} \boldsymbol{i}_3 \boldsymbol{i}_1$:

$$(\bar{\alpha}_{31} \boldsymbol{i}_3 \boldsymbol{i}_1 \times \boldsymbol{i}_1)^S = 0; \tag{III.34}$$

i.e., $\bar{\eta} = 0$, or no stresses. For the circular screw dislocations one has $\bar{\boldsymbol{\alpha}} = \alpha_{22}\, \boldsymbol{i}_2\, \boldsymbol{i}_2 + \alpha_{33}\, \boldsymbol{i}_3\, \boldsymbol{i}_3$, thus, since $\alpha_{22} = \alpha_{33} \equiv \alpha_0$:

$$[\alpha_0 (\boldsymbol{i}_2\, \boldsymbol{i}_2 + \boldsymbol{i}_3\, \boldsymbol{i}_3) \times \boldsymbol{i}_1]^S = \alpha_0 (\boldsymbol{i}_2\, \boldsymbol{i}_2 + \boldsymbol{i}_3\, \boldsymbol{i}_3)^S = 0; \tag{III.35}$$

thus, one again has $\bar{\eta} = 0$.

However, for the dislocation wall of Fig. 24, one has $\bar{\boldsymbol{\alpha}} = \bar{\alpha}_{32} \boldsymbol{i}_3 \boldsymbol{i}_2$, and one calculates from this that:

$$(\bar{\alpha}_{32} \boldsymbol{i}_3 \boldsymbol{i}_2 \times \boldsymbol{i}_1)^S = -\bar{\alpha}_{32} \boldsymbol{i}_3 \boldsymbol{i}_3; \tag{III.36}$$

i.e., $\overline{\overline{\eta}}_{33} = -\bar{\alpha}_{32}$. In this case, macroscopic elastic deformations and stresses thus appear.

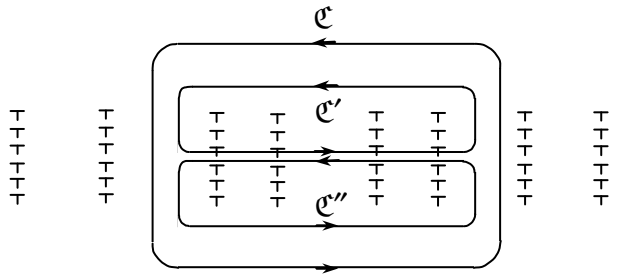

Fig. 25. Dislocation wall as a superficial incompatibility dipole.

One can easily imagine the overlap $\overline{\overline{\eta}}_{33}$ when one traces out the circuit $\oint K_{ij} dx_j$ of § 7, which indeed yields the incompatibilities. Therefore, we propose that the wall of Fig. 24 emerges from a series of grain boundaries of infinitesimal width by passing to the limit (Fig. 25). The circuit $\mathfrak{C}$ obviously gives null, but not the two circuits $\mathfrak{C}'$ and $\mathfrak{C}''$, which bring an equal and opposite rotation angle $\boldsymbol{D}$, moreover. The shows that the dislocation wall of Fig. 24 is nothing more than an overlap of surface incompatibilities.

One can flip the sign of a family of circular screw dislocations in the grain boundary and then obtain a surface incompatibility $\overline{\overline{\eta}}_{32}$ from eq. (III.35). However, an isolated family of parallel screw dislocations contributes to one half of $\overline{\overline{\boldsymbol{\eta}}}$, to the other it contributes an orientation difference. Thus all dislocation walls of constant density are accounted for.

We summarize these results for dislocation walls of constant density once more: There are essentially four different situations:

1. Edge dislocations ⊢ ⊢ ⊢ ⊢ ⊢ ⊢
grain boundary of the first type (tilt)

2. Circular screw dislocations, Burger vector and line direction parallel or anti-parallel in both cases.
grain boundary of the second type (twist)

} no long-range proper stresses

3. Edge dislocations ⊥ ⊥ ⊥ ⊥ ⊥ ⊥

4. Circular screw dislocations, Burger vector and line direction parallel in one family and anti-parallel in the other.

} source of long-range proper stresses, no orientation difference.

For non-constant dislocation densities in a wall, a surface incompatibility $\bar{\boldsymbol{\eta}}$ appears that is always linked with stresses [1].

In conclusion, we would like to treat the question that was posed in § 20 on the structure curvatures in polycrystals. One might have, perhaps, the body of Fig. 14 in a polycrystal that shall suffer the dislocation migration that brings it into the configuration of Fig. 14b when one has first cut the layers apart from each other. In reality, it remains connected and comes into the state of Fig. 1. The question is now whether the crystal in

[1] Let us mention the further work of Read and Shockley [174], as well as that of van der Merwe [100], in which the energy of a small-angle grain boundary was calculated elasticity theoretically (as a sum of proper and interaction energies of the dislocations that formed the grain boundaries) as functions of the orientation difference. For some interesting new applications of the theory of superficial configurations of dislocations, cf., Bilby [3], Bilby and Christian [6], Bullough and Bilby [14]. The last two papers mentioned include applications to the important phase migrations of martensitic type.

Furthermore, Bullough [162], with the help of the aforementioned theory, has given an explanation for the observed twinning structure in crystal lattices of diamond type.

Fig. 1 is still an ideal polycrystal or not. In the latter case, one must be able to confirm this Röntgenographically [1].

We have seen that one can regard grain boundaries as walls of dislocations. Now, this is true for not only small orientation differences, which are special only in that one can still distinguish the individual dislocations. One can meanwhile propose that, e.g., in Fig. 23, enough dislocations migrate and come to rest on the grain boundary that one ultimately obtains arbitrarily large differences in orientation. We thus employ the theorem that grain boundaries are fundamentally superficial configurations of dislocations [2]. Then the phrase "ideal polycrystal" implies a well-defined requirement for the dislocations of the body in the stress-free state. It can only be the following requirement: The Burgers circuit around an arbitrary surface element ΔF that intersects very many crystallites must give null [3]. The ideal polycrystal can be defined by this prescription perfectly. From this, it immediately follows that the body in Fig. 1 is no longer an ideal polycrystal, and this must verified, e.g., Röntgenographically [4].

For strong bending, the thickness of the dislocation wall in Fig. 1 is smaller than the mean linear dimension of a crystallite. Then, a number of such dislocation walls run through most crystallites, which leads to a mean curvature of the lattice planes. This manifests itself as an asterism in Röntgenographic absorption. Thus, it is sufficiently explained how macroscopic structure curvature can also be confirmed in polycrystals.

§ 24. The dislocation types of face-centered cubic crystals

Up to now, we have always considered primitive cubic crystals. One thus obtains a simplified picture of actual phenomena that suffices for many purposes. However, there are, in fact, no metals that crystallize into primitive cubic lattices. It is typical of metals that they aspire to fill up a large space, so it never happens that, e.g., neighboring lattice planes stand in opposition to each other, as in primitive cubic lattices, but they organize themselves into gaps, as Fig. 26 shows in the example of face-centered cubic lattice. It thus comes to pass that by far most metals crystallize into only three different lattices: the hexagonal closest packing of spheres, the face-centered cubic lattice, and the body-centered cubic lattice, of which the second occurs most frequently. In all of these crystals, one does not have such simple phenomena as in primitive cubic lattices, so we shall briefly direct our attention to at least the case of face-centered cubic lattices, since they are connected with many important problems [5].

(Fig. 26 missing)

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[1] On this, one also confers Seeger [139].

[2] In crystal physics, one generally makes no use of this theorem, since the "coarse grain boundaries" are often simpler to describe. However, many times its description as a surface distribution (pp. ?) is convenient (Bullough [162]).

[3] One proposes, perhaps hypothetically, that the volume element ΔV is an ideal crystal in an initial state, which then will be brought into the state of a polycrystal through the migration of the dislocations. The Burgers circuit may then be performed in an ideal crystal.

[4] This conclusion is also true for amorphous bodies, which will thus be likewise included in the continuum theory of dislocations.

[5] The most important cubic face-centered metals are gold, silver, copper, aluminum, brass, and certain iron alloys. However, at room temperature iron is cubic space-centered.

For the description of the crystal lattice, one introduces three Cartesian basis vectors $\boldsymbol{a}_i$, which point from, e.g., in Fig. 26, the lower left front corner atom to the lower right ($\boldsymbol{a}_1$) (upper left ($\boldsymbol{a}_2$), resp.) front corner atom, and let $\boldsymbol{a}_3 = \boldsymbol{a}_1 \times \boldsymbol{a}_2$. In order to characterize a family of parallel lattice planes, one gives, in round brackets, the components of their normal vectors, which one first rids of common factors, such that they become relatively prime whole numbers [1]. Thus, (100) are the planes $\perp \boldsymbol{a}_1$, (010) $\perp \boldsymbol{a}_2$, (001) $\perp \boldsymbol{a}_3$. These families of lattice planes are crystallographically equivalent, so if one would like to characterize them collectively the one might employ the curly brackets: {100} are all of the planes that are crystallographically equivalent to the planes (100).

In order to characterize a direction, one gives the components of one of the vectors that lie in this direction, which will likewise be made relatively prime whole numbers [1]. [100] is then the direction of $\boldsymbol{a}_1$, etc. One characterizes all directions that are crystallographically equivalent to this direction (hence, [010], [001], and further $[\bar{1}00]$, $[0\bar{1}0]$, $[00\bar{1}]$, where $\bar{1} \equiv -1$) by <100>.

The most important planes of the face-centered cubic lattice are the planes {111}, since they alone are possible as glide planes and climb planes under normal conditions. The {111}-planes are the closest-packed planes, one of which is shown in Fig. 27. A second plane can now fit into the locations *B* or *C*. A stacking sequence *ABABABAB*…(“Two-layer sequence”) yields the hexagonal closest packing of spheres, and a sequence *ABCABCABC*… (“Three-layer sequence”) yields the face-centered cubic lattice.

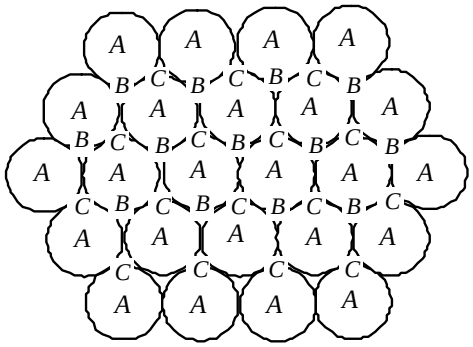

Fig. 27. Closest-packed lattice plane, from Seeger [134].

For the internal energy of a crystal, the forces between neighboring atoms are predominant. Now, if, e.g., the stacking sequence includes defects, in such way that one has, perhaps, *ABCABABCABC*…, then each atom nevertheless remains surrounded by twelve nearest-neighbors in the same way as before, while the ordering between second-to-next neighbors is no longer the same. Due to the short ranges of atomic forces, the thus-qualified increase in the internal energy is therefore relatively small, so that such “stacking faults” occur relatively often.

As we remarked in § 22, should the Burgers vector be a dislocation of the smallest possible lattice vector then in a face-centered cubic lattice it points in a <110>-direction, as one easily infers. This direction is, moreover, always the glide direction. Now, (110) is the plane perpendicular to the [110]. Meanwhile, an edge dislocation with a Burgers vector in the direction [110] is not the boundary of one inserted plane (110), but two of them, since the “thickness” of a (110)-plane is equal to one-half the atomic spacing, as follows from, e.g., Fig. 26. This is represented schematically in Fig. 28a. In practice, one now has the important process of the splitting of such a “complete” dislocation into two so-called “Shockley half-dislocations” (Fig. 28b) [2]. The notation shall thus suggest

[1] This simplified representation is valid only for cubic crystals, where covariance and contravariance must not be distinguished. For the somewhat complicated behavior in the general case, see, e.g., Jagodzinski [68].

[2] The stacking fault and the split dislocations were first described by Heidenreich and Shockley [64]. For a thorough presentation of this, cf., *inter alia*, Frank [45], Frank and Nicholas [50], Read [121], Thompson [152], Seeger [134], [136], [140].

that the Burger vector of the two half-dislocations is no longer a complete lattice vector. One writes this process of splitting as basically a "reaction equation":

$$\frac{1}{2}[110] = \frac{1}{6}[211] + \frac{1}{2}[12\bar{1}], \tag{III.37}$$

where [110]/2 stands for the Burgers vector of the complete dislocation, while the other expressions on the right in eq. (III.37) stand for the Burgers vectors of the half-dislocations; eq. (III.37) is simply an addition equation for these vectors. One easily infers that a stacking fault survives in the plane between the half-dislocations, which now raises the internal energy by the "stacking fault energy," such that an equilibrium spacing of 2η now exists for the two half-dislocations. The Burgers vector [211]/6 means that (Fig. 27), e.g., an atom at the location *B* will be displaced into the position *C* when one half-dislocation migrates in between it and the neighboring lattice plane *A*. If both half-dislocations migrate in between then it is again in the location *B*. One can then no longer distinguish whether the dislocation that has caused the relative displacement was split or not. For many purposes – above all, macroscopic problems – one can therefore ignore the splitting of dislocations.

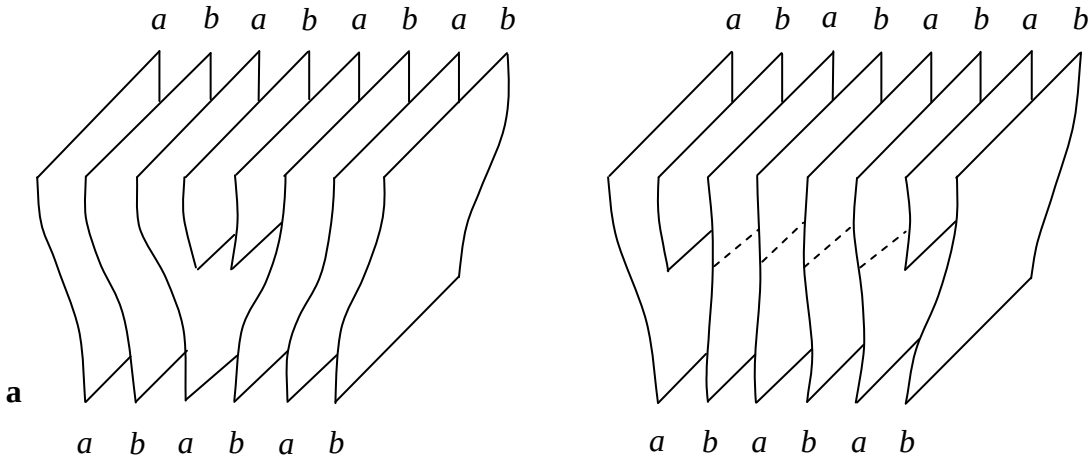

Fig. 28. Schematic representation of an edge dislocation in a face-centered cubic lattice (*a*). This splits a the formation of a stacking fault into two half-dislocations in (*b*). The notation *ababab*… means that the indicated <110>-planes represent a double-layer series. From Seeger [134].

In general, one has the theorem that each stacking fault that ends inside the interior of the crystal is bounded by an incomplete dislocation. It is not necessary that complete dislocations, and thus, half-dislocations, be straight; they can also define, e.g., closed rings in the {111}-planes, hence, run piecewise as screw dislocations, in which case, the splitting 2η is somewhat smaller at these places.

Thus, let us state without proof the following results on dislocations in face-centered cubic metals in simplified representation:

1. Complete dislocations: These run almost exclusively in the {111}-plane and therefore always split into Shockley half-dislocations there. Where they, e.g., cross over

from a {111}-plane to a neighboring plane the splitting must go back to zero; such a location is called a "jog." The complete dislocations can now glide into their stacking fault planes and climb nowhere at all. A pure screw dislocation always runs in the <110>-direction, since the line runs parallel to the Burgers vector. This direction is the line of intersection of two {111}-planes. The screw dislocation can split in both planes. Under a corresponding pressure, it can therefore cross over from one glide plane into another, so it has more possible motions than the edge dislocation, which is always bound into a glide plane.

2. Incomplete Frank dislocation [45]: This is the boundary line of an inserted (or removed) lattice plane {111}, to its Burgers vector is <111>/3. It likewise bounds a stacking fault. This dislocation is always isolated, in contrast to the Shockley half-dislocations, which normally enter in pairs. It can climb in its {111}-plane, but it has no other possibility of motion. It is therefore generally complementary to the dislocations that were mentioned in 1.

3. The combined Lomer-Cottrell dislocation [94], [24]: If two split dislocations coincide along the line of intersection of two {111}-planes then both of them can "react" on the next-lying half-dislocation, such that one obtains a combined dislocation of greater stability. One then has a stacking fault that bends around from one {111}-plane to the other. Such a Lomer-Cottrell dislocation can neither glide nor climb, and is therefore completely immobile. It represents an extremely effective obstacle to the migration of more dislocations into the glide planes in question, and therefore plays a big role in the theory of hardening [1].

The things that are possible for dislocations in face-centered cubic crystals, play a subordinate role in regard to the dislocations described in 1 to 3.

§ 25. The nonlinear treatment of the singular dislocation by Peierls

A glimpse at the dislocations in Fig. 3 and 5 shows that in the center of the dislocation the elastic deformations are certainly much too large for one to be able to calculate them with a linear theory [2]. In fact, up to the present, they have still not been calculated exactly. At the outset, one has, above all, one clue for the extension 2ζ of the dislocation center, which indeed enters definitively into the equation for the energy of the dislocation (§ 18). Peierls succeeded in calculating the extension of the dislocation center by means of an interesting combination of microscopic and macroscopic methods, at least approximately.

The basic idea of Peierls [116][3] is that the nonlinearity in the phenomena is meaningful at least in the glide plane. One thinks of the crystal as being separated into

[1] Cf., on this, the papers of Mott [104], Leibfried and Haasen [92], Cottrell and Stokes [26], Friedel [56], Seeger, Diehl, Mader, and Rebstock [143].

[2] The same is also true in the case of the splitting dislocation.

[3] Nabarro [106] especially developed the rather brief paper of Peierls. Therefore, many authors speak of the Peierls-Nabarro model.

two halves by a cut into the glide plane, which will be regarded as the elastic half-spaces ***A***: $x_2 > a/2$ and ***B***: $x_2 < -a/2$ (Fig. 29). Additionally, a nonlinear elasticity law will be employed that prevails between the two half-spaces. Thus, the special atomic arrangement will be considered in the glide plane. One obtains the simplest relationship in a primitive cubic lattice with {100}-glide planes, which was the case treated by Peierls. Thus, Leibfried and Dietze [91] have also dealt with the face-centered cubic crystal [1].

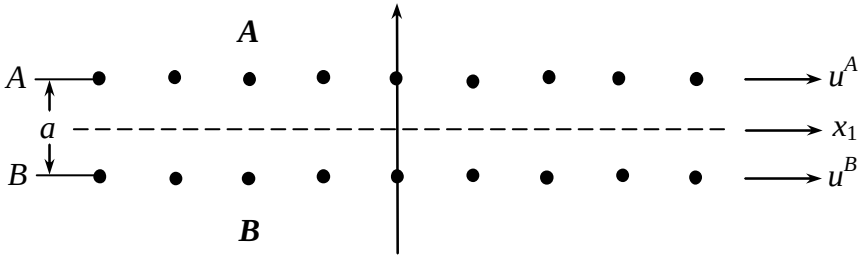

Fig. 29. The explanation for the Peierls model. The lattice planes *A* and *B* (perpendicular to the plane of the paper) bound the half-spaces ***A*** and ***B***.

The notations are explained in Fig. 29. Let us start with an ideal (primitive cubic) crystal. We denote the tangential displacement of atoms that are in opposition in the lattice planes *A* and *B* by u^A (u^B, resp.), and their relative displacement by:

$$u^{AB}(x_1) = u^A(x_1) - u^B(x_1), \tag{III.38}$$

where x_1 means the starting point. Let the crystal be infinitely extended in all directions, so $\partial/\partial x_3 = 0$. The elasticity-theoretic calculation shows, to the extent that it is correct, that $x_2 = 0$ for the edge dislocation in the glide plane, at least, in the vicinity of the dislocation center; of the stresses, only σ_{21} is non-zero (eq. (II.114)). Even at the dislocation center the other stresses may be smaller than σ_{21}. They will thus be set equal to zero in the entire glide plane. One refers to these as "Peierls assumptions," where we would like to also include the elasticity law that was employed by Peierls, and which is valid between the planes *A* and *B*. One comes to it by the following reasoning:

If one displaces the aforementioned half-space tangentially with respect to the lower one through an atomic spacing *a* then one again finds the entire crystal in equilibrium; i.e., there are no more opposing forces there. From this, it follows that the reaction to a relative displacement u^{AB} – viz., the stress σ_{21} – must be a periodic function of u^{AB} with period *a*. The simplest Ansatz is the law employed by Peierls:

$$\sigma_{21} = \frac{G}{2\pi} \sin \frac{2\pi u^{AB}(x_1)}{a}, \tag{III.39}$$

[1] Nabarro [106] has treated the interaction of two dislocations in primitive cubic crystals, and van der Merwe [100] treated planar arrangements of many dislocations. Seeger and Schöck [141], et al., could compute the energy increase of a dislocation due to the splitting into half-dislocations in Peierls. One finds a thorough summary of all of these results in Seeger [134].

where the free constant is chosen such that for small displacements one comes back to Hooke's law.

By cutting along the glide plane in the state with a dislocation, we must, when no shift of the atoms in the planes *A* and *B* is present, introduce the superficial force density σ_{21} for the lattice plane *A* and $-\sigma_{21}$ for *B*. The deformation state that belongs to these "outer surface forces" of the half-space $\boldsymbol{A}$ in the interior of the half-space is already known in elasticity theory since Boussinesq, et al. One has, for the plane *A* (cf., Leibfried and Lücke [93], eq. (12)):

$$\frac{du^A(x_1)}{dx_1} = \frac{m-1}{\pi m G}\int_{-\infty}^{\infty}\frac{\sigma_{21}(x_1')}{x_1-x_1'}dx_1', \tag{III.40}$$

and a corresponding equation with a change of sign on one side is true for *B*; i.e. [1]:

$$\frac{du^{AB}(x_1)}{dx_1} = \frac{2(m-1)}{\pi m G}\int_{-\infty}^{\infty}\frac{\sigma_{21}(x_1')}{x_1-x_1'}dx_1'. \tag{III.41}$$

On the right-hand side, one substitutes eq. (III.39):

$$\frac{du^{AB}(x_1)}{dx_1} = \frac{m-1}{\pi^2 G}\int_{-\infty}^{\infty}\frac{\sin(2\pi u^{AB/a})}{x_1-x_1'}dx_1', \tag{III.42}$$

and from the Hilbert integral theorem [2] one obtains the so-called Peierls integral equation:

$$\int_{-\infty}^{\infty}\frac{du^{AB}(x_1')/dx_1'}{x_1-x_1'}dx_1' = -\frac{m-1}{m}\sin\frac{2\pi u^{AB}(x_1)}{a}, \tag{III.43}$$

which is capable of delivering the displacements u^A and u^B in the planes *A* and *B*.

As Eshelby [37] has shown, one can derive a very similar equation:

$$\int_{-\infty}^{\infty}\frac{dw^{AB}(x_1')/dx_1'}{x_1-x_1'}dx_1' = -\sin\frac{2\pi w^{AB}(x_1)}{a} \tag{III.44}$$

for a screw dislocation in the x_3-direction with $x_2 = 0$ as the glide plane by means of a corresponding Ansatz, where w^{AB} is the relative displacement of the atoms as above, only in the x_3-direction, instead of the x_1-direction.

For Peierls, eq. (III.43) has, as one easily checks, the rigorous solution (cf., footnote 1):

---

[1] One then has $d(u^A + u^B)/dx_1 = 0$, so we set the free integration constant to zero; i.e., $u^A = -u^B$.

[2] This reads $f(\xi) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{g(x)}{\xi-x}dx$, $g(x) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(\xi)}{\xi-x}d\xi$ [128].

$$u^A = -\frac{a}{2\pi}\arctan\frac{x_1}{\zeta'}, \qquad \zeta' \equiv \frac{a}{2}\frac{m}{m-1} \approx \frac{3}{4}a, \tag{III.45}$$

which corresponds to a static edge dislocation with Burgers vector of magnitude $b = a$ in the x_1-direction (cf., *infra*). From eq. (III.39), with u^A (u^{AB}, resp.) one likewise knows σ_{21} for the atomic planes A and B; i.e., the outer surface forces of the half-space. For Leibfried and Lücke [93], the classical problem that is connected with this of finding the Airy stress function that is associated with these forces in the entire half-space has the solution:

$$\chi = -\frac{G\zeta'}{2\pi}\left(x_2 - \frac{a}{2}\right)\ln[x_1^2 + (x_3 - a/2 + \zeta')^2]. \tag{III.46}$$

Compared to the previous solution (II.113), this equation includes the additional term $-a/2$ ($-a/2 + \zeta' \approx a/4$, resp.), and thus shows that less atoms are at a distance from the dislocation center, so the Peierls solution does not differ from that of (II.113) in practical terms.

As Eshelby [37] has emphasized, and as would follow from our eq. (I.77), one can interpret:

$$\frac{du^{AB}}{dx_1} = \frac{a}{\pi}\frac{\zeta'}{x_1^2 + \zeta'^2} \tag{III.47}$$

as the superficial dislocation distribution in the glide plane, where:

$$\frac{a}{\pi}\int_{-\infty}^{\infty} dx_1 \zeta'/(x_1^2 + \zeta'^2) = a; \tag{III.48}$$

i.e., the total Burgers vector of the surface dislocations is naturally of magnitude a. The Peierls calculation thus delivers the following result: The edge dislocation in a primitive cubic crystal has a superficial extension, so one can, as before, regard it as composed of threads of strength du^{AB}. The distribution function of the threads is the bell curve (III.47) [1]; $2\zeta'$ is its half-width [2].

For Eshelby [37], the solution of eq. (III.44), which describes the static screw dislocation in a primitive cubic lattice, coincides exactly with the elasticity-theoretic solution. Obviously, in this case the Peierls method achieves less for the edge dislocation, so one does not come to a finite self-energy for the dislocation.

We now report briefly on the most important results of Leibfried and Dietze [91] for dislocations that lie in the closest-packed planes of the face-centered cubic and hexagonal crystals. Here, the simple Ansatz (III.39) is no longer applicable, so one needs an elasticity law that gives σ_{21} and σ_{23} as periodic functions of u^{AB} and w^{AB}. With it, one

[1] The fact that the extension of the dislocation turns out to be two, but not three, dimensional comes from the nature of the calculations. Whether a dislocation cannot have a three-dimensional extension in reality can still not be stated with certainty. In any case, the two-dimensional extension already leads to a finite self-energy.

[2] In the literature, ζ' mostly refers to the dislocation width.

obtains two simultaneous integral equations of Peierls type, which we shall not write down. These equations no longer have elementary solutions. For Leibfried and Dietze, one obtains simple and useful approximate solutions, in which one makes the total elastic energy per unity length:

$$T = T^A + T^B + T^{AB} \tag{III.49}$$

a minimum, where T^A and T^B relate to the two half-spaces, and T^{AB} is the mutual potential energy per unit length in the perturbed state. Leibfried and Dietze could prove that the displacements u^A and w^A that make this energy expression a minimum satisfy the Peierls integral equation, such that the stated variational method actually represents an approximation method as a consequence. Leibfried and Dietze have given the solutions for several special types of dislocations; in particular, for the important half-dislocations, as well.

We likewise give the most general (approximate) solution for the half-dislocations in the closest-packed planes [83]: Let the glide plane again be $x_2 = 0$, and the Burgers vector likewise lies in this plane (§ 24). Let β be its angle with the line direction. With the Ansatz that corresponds to (III.45):

$$u^A = -\frac{b}{2\pi}\sin\beta \arctan\frac{x_1}{\zeta'}, \qquad w^A = -\frac{b}{2\pi}\cos\beta \arctan\frac{x_1}{\zeta'}, \tag{III.50}$$

where ζ' is a free parameter, one finds the minimal energy per unit length:

$$T = \frac{Gb^2}{4\pi}\left(\frac{m}{m-1}\sin^2\beta + \cos^2\beta\right)\left(\ln\frac{R}{2\zeta'} + 1\right), \tag{III.51}$$

with:

$$\zeta' = \frac{\pi\sqrt{2}}{3\sqrt{3}}b\left(\frac{m}{m-1}\sin^2\beta + \cos^2\beta\right). \tag{III.52}$$

Thus, $2R$ is the linear dimension of the medium in the x_1-direction that goes to infinity. The formula that was given by Leibfried and Dietze for half-dislocations follows from this by specializing the angle β.

Eq. (III.51) is not suitable for the comparison with the previously-obtained eq. (II.144), since the L in the latter has no relation to the R in the former. However, we can easily compute a superficial dislocation distribution on the glide plane that corresponds to eq. (III.47). If we identify $2u^A$ and $2w^A$ with the components g_1 and g_3 of the dislocation source in eq. (I.77) then we easily obtain the dislocation density $\bar{\boldsymbol{\alpha}}$:

$$\bar{\boldsymbol{\alpha}} = -\,i_3\left(\frac{du^A}{dx_1}\boldsymbol{i}_1 + \frac{dw^A}{dx_1}\boldsymbol{i}_3\right) = \boldsymbol{i}_3\,(\sin\beta\,\boldsymbol{i}_1 + \cos\beta\,\boldsymbol{i}_3)\,\frac{b}{2\pi}\frac{\zeta'}{x_1^2 + \zeta'^2}. \tag{III.53}$$

Here, $\boldsymbol{i}_3$ gives the line direction, while the bracketed expression gives the direction of the Burgers vector of the half-dislocation. Thus, we regard the density (III.53) as an arrangement of nothing but dislocation threads with infinitesimal strengths:

$$db = \frac{b}{\pi}\frac{\zeta'}{x_1^2+\zeta'^2} = b\gamma(x_1)\,dx_1\,. \tag{III.54}$$

Thus, we obtain from eq. (II.142):

$$T = \frac{G}{4\pi}\left(\frac{m}{m-1}\sin^2\beta+\cos^2\beta\right)\frac{b^2\zeta'^2}{\pi^2}\int_{-\infty}^{\infty}\frac{dx_1}{x_1^2+\zeta'^2}\int_{-\infty}^{\infty}\frac{dx_1'}{x_1'^2+\zeta'^2}\left(\ln\frac{2L}{|x_1-x_1'|}-1\right). \tag{III.55}$$

We satisfy ourselves with an estimate of this integral, in which we now set *db* equal to:

$$db = \frac{b}{\pi\zeta'}dx_1 \quad \text{for} -\frac{\pi\zeta'}{2}\le x_1\le\frac{\pi\zeta'}{2}, \qquad 0 \text{ otherwise,} \tag{III.56}$$

in place of (III.54), which, like (III.54), satisfies the condition $\int_{-\infty}^{\infty} db = b$. (III.53) implies a constant distribution of dislocation threads in the given region. $b/\pi\zeta'$ is the height of the maximum of the bell curve $b\,\gamma_1(x_1)$. We have already calculated the integral given this way in § 18, when one replaces the ζ there with $\pi\zeta'/2$. This then gives the energy per unit length as:

$$T = \frac{Gb^2}{4\pi}\left(\frac{m}{m-1}\sin^2\beta+\cos^2\beta\right)\left(\ln\frac{2L}{\pi\zeta' e^{3/2}}-1\right). \tag{III.57}$$

This formula follows from eq. (II.145), from which one can establish that it is also valid in the case of curved dislocation lines, as long as one correctly substitutes the length ε of the cut. In § 18, we calculated the value $\varepsilon = \zeta e^{3/2}$ for ε. With $\zeta = \pi\zeta'/2$ and eq. (III.52), one then obtains:

$$\varepsilon/b = \frac{\pi^2}{\sqrt{54}e^{3/2}}\left(\frac{m}{m-1}\sin^2\beta+\cos^2\beta\right), \tag{III.58}$$

where the factor in front of the bracket is almost precisely 0.3. Therefore, we are in a position to also calculate the energy of curved half-dislocations in the closest-packed planes of the face-centered cubic and hexagonal crystals to what is certainly not a bad approximation [1].

[1] The calculations indeed include one approximation (viz., the Peierls assumptions), but on the other hand, the energy is not sensitive to changes in ε that are not too large, since it depends upon ε only logarithmically. One can still refine the calculation when one evaluates the integral (III.55) exactly, and further considers the elastic anisotropy in the half-spaces *A* and *B*, as well. Such computations were carried out by Seeger and Schöck [141] with success in the case of straight dislocations.

In these calculations, we have restricted ourselves to the Shockley half-dislocations. They play the principal role in face-centered cubic and hexagonal crystals at room temperature. In other crystals – e.g., body-centered cubic – one has other types of dislocations and must carry out a specific Peierls calculation for each type.

In summation, the Peierls method may be assessed as follows: It delivers a measure for the extension of the dislocation center by way of the dislocation width ζ, from whose magnitude it follows that the elasticity-theoretic calculation already delivers reliable results at a distance of a few atomic spacings from the dislocation line. The Peierls calculation thus gives a clue as to the cases in which one would expect an elasticity-theoretic treatment of dislocations to give sensible results, and indeed this also gives a clue as to when it is no longer applicable, due to great complications, since the treatment of the dislocation center itself (e.g., the calculation of its energy) delivers the Peierls calculation in the first approximation.

Chapter IV

Non-Riemannian geometry of dislocations [1]

Kondo, as well as Bilby, Bullough, and Smith, have independently recognized the close relationship between the geometric problems of plasticity and those of non-Euclidian geometry. One can then employ the highly-developed methods of differential geometry for the treatment of such problems; in particular, the concept of torsion. This goes back to Cartan, whose work has found a very beautiful application to real bodies. The connection between the dislocation tensor $\alpha^{\lambda\chi}$ and the torsion tensor $L^{\chi}{}_{[\mu\nu]}$ is given by equation:

$$\alpha^{\lambda\chi} = \varepsilon^{\lambda\mu\nu} L^{\chi}{}_{[\mu\nu]} \ . \tag{IV.1}$$

The difference between the theory of Kondo and that of Bilby, Bullough, and Smith is similar to the difference between our Chapters I and III: The theory of Kondo is a continuum theory, while Bilby, Bullough, and Smith develop their theory in a crystal. Further distinctions between the two theories will be discussed § 28.

§ 26. The theory of Kondo and co-workers [2]

We next recall some facts that are connected with the presentation of Kondo [74]. In a purely elastic theory one is interested in rotations on the boundary only as long as they awaken no elastic forces. The definitive quantity is then the elastic deformation. By the natural state of a volume element, one understands any stress-free state that it assumes when it experiences either external forces or a pressure from its neighboring element. Under the presence of proper stresses, the volume element can assume its natural state after being cut apart from the others, because it is linked with Euclidian space. Meanwhile, one can imagine oneself placed in a non-Euclidian space in which it can also relax without being cut apart if the pressure that was present in Euclidian space were suddenly removed. We can also define such an imaginary stress-free state in a non-Euclidian space as the natural state. One can the consider the volume elements in the natural state that were cut apart from each other as (material) Euclidian spaces that are tangent to the (material) non-Euclidian space at the point in question. Ultimately, let the final state be defined as the (Euclidian) state of the body in the condition under investigation that is marked with stresses.

In §§ 26 to 28, we distinguish between covariance and contravariance. If ds_E^2 is the square of the distance between two arbitrary points of a volume element in the final state and ds_N^2 is the square of the distance between the same points in the natural state then one has:

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[1] I am very thankful to Herren Prof. K. Kondo and Dr. B. A. Bilby for discussions on this theme. I thank Herrn Dr. J. D. Eshelby for the fact that he kindly placed the book of Kondo [74] at my disposal, which was the first time the work of Kondo was made known to me (Dec., 1956).

[2] The results reported here are all summarized in the book [74]. The greater part of them was presented by Kondo for the first time to the Second National Congress for Applied Mechanics in Japan 1952 [73].

$$ds_E^2 - ds_N^2 = (\delta_{E^{ij}} - g_{N^{ij}})\, dx^i dx^j, \qquad \text{(IV.2)}$$

where x^i are the (spatially fixed) Cartesian orthogonal coordinates of the point in the final state. $g_{N^{ij}}(x^i) = g_{N^{ji}}(x^i)$ is the metric tensor of the natural state. The metric of the final state is obviously completely determined by the quantities:

$$\varepsilon_{ij} = (\delta_{ij} - g_{ij})/2, \qquad \text{(IV.3)}$$

where the subscripts *E* and *N* have been dropped, since there is absolutely no danger of confusion. From the theory of large deformations, it is known that in the case of small compatible deformations the ε_{ij} of eq. (IV.3) assume the form:

$$\varepsilon_{ij} = \left(\frac{\partial s_j}{\partial x^i} + \frac{\partial s_i}{\partial x^j} \right) \Big/ 2; \qquad \text{(IV.4)}$$

thus ε_{ij} is identical with the deformation tensor used up to now.

The vanishing of the Riemannian curvature tensor R^i_{jkl} that is defined by the Christoffel symbol:

$$\left\{ \begin{matrix} i \\ j\,k \end{matrix} \right\} \equiv \frac{1}{2} g^{il} \left(\frac{\partial g_{kl}}{\partial x^j} + \frac{\partial g_{lj}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l} \right) \qquad \text{(IV.5)}$$

that belongs to g_{ij} is well-known to mean the same thing as the fulfillment of the compatibility conditions for the deformations. By means of:

$$R^i_{jkl} \equiv \frac{\partial}{\partial x^k} \left\{ \begin{matrix} i \\ j\,l \end{matrix} \right\} - \frac{\partial}{\partial x^l} \left\{ \begin{matrix} i \\ j\,k \end{matrix} \right\} + \left\{ \begin{matrix} i \\ m\,k \end{matrix} \right\} \left\{ \begin{matrix} m \\ j\,l \end{matrix} \right\} - \left\{ \begin{matrix} i \\ m\,l \end{matrix} \right\} \left\{ \begin{matrix} m \\ j\,k \end{matrix} \right\} = 0 \qquad \text{(IV.6)}$$

the classical theory of elasticity is then distinguished from the theory of proper stresses. This statement is also true for large deformations, while the form $\eta^{hm} = 0$ for the compatibility conditions is only true for small deformations. The totally covariant curvature tensor:

$$R_{ijkl} = g_{ih}\, R^h_{jkl} \qquad \text{(IV.7)}$$

is anti-symmetric in the first and last two pairs of indices and symmetric in the pairs *ij* and *kl*. For small deformations one has (cf., McConnell [173]):

$$R_{ijkl} = \varepsilon_{hij}\, \varepsilon_{klm}\, \eta^{hm}, \qquad \eta^{hm} = 4\, \varepsilon^{hij}\, \varepsilon^{klm} R_{ijkl}, \qquad \text{(IV.8)}$$

as one easily verifies, when one introduces the ε_{ij} of (IV.4) with the help of (IV.5) and (IV.6) into (IV.7). One then obtains:

$$\eta^{hm} = \varepsilon^{hij}\, \varepsilon^{klm}\, \frac{\partial^2 \varepsilon_{jl}}{\partial x^i\, \partial x^k}, \tag{IV.9}$$

and with (IV.4) it follows from this that $\eta^{hm} = 0$.

In a theory of elasticity that also considers proper stresses $R^i_{jkl} \neq 0$ and the description of the deformations becomes a problem in Riemannian geometry. Since the problems to be dealt with in such a theory are essentially of a geometric type, one can, with Kondo and co-workers [74], say outright: "Elasticity theory is Riemannian geometry and conversely."

Meanwhile, in the considerations up to now, we succeeded in treating the geometric questions satisfactorily – at least, for small distortions – without needing Riemannian geometry, and for that reason we would like to regard the theorem above, not as a rule for the elasticity theoretician, but as perhaps a useful clue for the especially geometrically proficient reader. Undoubtedly, the study of Riemannian and, as we will see, non-Riemannian geometry might bring more insights into the geometry of the forming of solid bodies.

Kondo now associated each (Euclidian) volume element in the natural state with its own local Cartesian coordinate system with basis vectors $\boldsymbol{e}_\lambda$ (λ = 1, 2, 3), which shall be subject to no restriction. One then has:

$$d\boldsymbol{x} = \boldsymbol{e}_\chi\; d\omega^\chi, \tag{IV.10}$$

$$d\omega^\chi = A_i^{\ \chi}\, dx^i, \qquad dx^l = A_\mu^{\ l}\, d\omega^\mu, \tag{IV.11}$$

$$\boldsymbol{e}_\lambda = A_\lambda^{\ j}\, \boldsymbol{i}_j\,, \qquad \boldsymbol{i}_j = A_j^{\ \chi}\, \boldsymbol{e}_\chi\,, \tag{IV.12}$$

$$d\boldsymbol{e}_\lambda = \boldsymbol{e}_\chi\, \Gamma^\chi_{\lambda\mu}\, d\omega^\mu, \tag{IV.13}$$

$$A_\lambda^{\ j}\, A_i^{\ \chi} = \delta_i^{\ j}, \quad A_\lambda^{\ i}\, A_j^{\ \chi} = \delta_\chi^{\ j}, \tag{IV.14}$$

where $d\boldsymbol{x} = dx^i\, \boldsymbol{i}_i$ is the difference in position between two matter points in the final state and $d\omega^\chi$ are the components of the corresponding vector (which relates to the same matter point) in the natural state. We regard $\boldsymbol{e}_\lambda$, $A_\mu^{\ l}$, $A_i^{\ \chi}$, $\Gamma^\chi_{\lambda\mu}$ as functions of x^i. For the natural system, we employ Greek indices, and for the (Cartesian) final system, Latin ones. To abbreviate, we set:

$$\partial_i \equiv \partial/\partial x^i, \qquad \partial_\nu \equiv A_\nu^{\ i}\, \partial/\partial x^i. \tag{IV.15}$$

Now, the first integrability conditions:

$$(\partial_j \partial_i - \partial_i \partial_j)\, \boldsymbol{x} = 0 \tag{IV.16}$$

must naturally be fulfilled. One has $\partial_i\, \boldsymbol{x} = \boldsymbol{e}_\chi A_i^{\ \chi}$, $\partial_j\, \boldsymbol{x} = \boldsymbol{e}_\chi\, A_i^{\ \chi}$, so:

$$\left.\begin{aligned} \partial_j \partial_i \boldsymbol{x} = A_i{}^{\lambda} \partial_j \boldsymbol{e}_\lambda + \boldsymbol{e}_\chi \partial_j A_i{}^{\chi} \quad &= \boldsymbol{e}_\chi (\Gamma^{\chi}{}_{\lambda\mu} A_i{}^{\lambda} A_j{}^{\mu} + \partial_j A_i{}^{\chi}) \\ \partial_i \partial_j \boldsymbol{x} = \quad &= \boldsymbol{e}_\chi (\Gamma^{\chi}{}_{\lambda\mu} A_j{}^{\lambda} A_i{}^{\mu} + \partial_i A_j{}^{\chi}) \end{aligned}\right\} \tag{IV.17}$$

hence:

$$\Gamma^{\chi}{}_{\lambda\mu} (A_i{}^{\lambda} A_j{}^{\mu} - A_j{}^{\lambda} A_i{}^{\mu}) + \partial_j A_i{}^{\chi} - \partial_i A_j{}^{\chi} = 0, \tag{IV.18}$$

from which, upon multiplying by $A_\nu{}^i A_\pi{}^j$ and using eq. (IV.14), it easily follows that:

$$\Gamma^{\chi}{}_{[\lambda\mu]} = \frac{1}{2} A_\lambda{}^i A_\mu{}^j (\partial_j A_i{}^{\chi} - \partial_i A_j{}^{\chi}). \tag{IV.19}$$

Square brackets will always signify that the anti-symmetric part relative to the indices in question is to be taken. After Cartan [15], one refers to the anti-symmetric part of an affine connection as the torsion. Whenever it does not vanish, one finds oneself in the realm of non-Riemannian geometry. As is well-known, $\Gamma^{\chi}{}_{[\lambda\mu]}$ is a tensor of rank three, whose Cartesian components read:

$$\Gamma^{i}{}_{[km]} = A_\chi{}^i A_k{}^{\lambda} A_m{}^{\mu} \Gamma^{\chi}{}_{[\lambda\mu]}, \tag{IV.20}$$

and with the relations (IV.14), it follows that ([74], pp. 461):

$$\Gamma^{i}{}_{[km]} = \frac{1}{2} A_\chi{}^i (\partial_m A_k{}^{\chi} - \partial_k A_m{}^{\chi}). \tag{IV.21}$$

Eq. (IV.10) and (IV.11), which define the coordinate system $\boldsymbol{e}_\lambda$, are generally Pfaffian differential forms; i.e.:

$$(\partial_i \partial_j - \partial_j \partial_i)\, \boldsymbol{e}_\lambda \neq 0. \tag{IV.22}$$

The left-hand side of this equation delivers, after simple calculations involving the $\Gamma^{\chi}{}_{\lambda\mu}$, the Riemann-Christoffel curvature tensor that relates to the natural state, which we would not like to write down explicitly, namely:

$$R^{\chi}{}_{\lambda\mu\nu} \neq 0. \tag{IV.23}$$

One can now parallel displace (relative to the connection $\Gamma^{\chi}{}_{\lambda\mu}$) a vector c^λ around an infinitesimal surface element $\Delta F^{\mu\nu} = \varepsilon^{\mu\nu\rho}\, \Delta F_\rho$ according to the rules of differential geometry [1], and thus form the integral:

$$-\oint \Gamma^{\chi}{}_{\lambda\mu} c^{\lambda} d\omega^{\mu}, \tag{IV.24}$$

and obtain (cf., Kondo) the change in c^λ in this way:

[1] See, e.g., Schouten [130].

$$\Delta c^{\lambda} = \left(\frac{1}{2} R^{\chi}{}_{\lambda[\mu\nu]} c^{\lambda} + \Gamma^{\chi}{}_{[\mu\nu]} \right) \Delta F^{\mu\nu} . \tag{IV.25}$$

Now, Kondo compared the circuit above with the Franks-Burgers circuit (§ 21). The torsion $\Gamma^{\chi}{}_{[\mu\nu]}$ gives rise to a translation:

$$\Delta b^{\chi} = \Gamma^{\chi}{}_{[\mu\nu]} \, \Delta F^{\mu\nu} = \alpha^{\rho\chi} \, \Delta F_r \, , \tag{IV.26}$$

where $\Gamma^{\chi}{}_{[\mu\nu]}$ is expressed by the associated tensor of rank two. Upon comparison with eq. (I.14), one sees that $\Gamma^{\chi}{}_{[\mu\nu]}$ is connected with our previous dislocation tensor $\boldsymbol{\alpha}$ by way of:

$$\Gamma^{\chi}{}_{[\mu\nu]} = \frac{1}{2} \, \varepsilon_{\mu\nu\rho} \, \alpha^{\rho\chi}, \qquad\qquad \alpha^{\lambda\chi} = \varepsilon^{\lambda\mu\nu} \; \Gamma^{\chi}{}_{[\mu\nu]} \, , \tag{IV.27}$$

insofar as Δb^{χ} is actually the Burgers vector. This question will be discussed in § 28.

The other contribution in eq. (IV.25) is more difficult to discuss; on this, one confers Kondo [74], pp. 466 et seq. First, let it be remarked, in addition: The tensor $R_{\iota\lambda\mu\nu} = g_{\iota\chi} \, R^{\chi}{}_{\lambda\mu\nu}$ ($g_{\iota\chi} \equiv \boldsymbol{e}_{\iota} \cdot \boldsymbol{e}_{\chi}$) associated with $R^{\chi}{}_{\lambda\mu\nu}$ is anti-symmetric only with respect to the last two indices, since $\Gamma^{\chi}{}_{\lambda\mu}$ no longer has the form $\left\{ \begin{matrix} \lambda \\ \chi \, \mu \end{matrix} \right\}$. One can, however, split off a part of $R_{\iota\lambda\mu\nu}$ that has the symmetry above, and obtain:

$$\Delta C_{\iota} = \frac{1}{2} \, R_{[\iota\lambda][\mu\nu]} \, C^{\lambda} \, \Delta F^{\mu\nu}. \tag{IV.28}$$

When $R_{[\iota\lambda][\mu\nu]}$ is substituted in eq. (IV.8), one obtains:

$$\Delta C_{\iota} = \varepsilon_{\iota\lambda\pi} \, \eta^{\pi\rho} \, C^{\lambda} \, \Delta F_{\rho} \, , \tag{IV.29}$$

or, in vector notation:

$$\Delta \boldsymbol{C} = \boldsymbol{C} \times \boldsymbol{\eta} \cdot \Delta \boldsymbol{F}. \tag{IV.30}$$

Thus, $\Delta \boldsymbol{C} \perp \boldsymbol{C}$, i.e., the vector experiences (in the case of small distortions) a pure rotation:

$$\Delta \boldsymbol{D} = - \, \boldsymbol{\eta} \cdot \Delta \boldsymbol{F}. \tag{IV.31}$$

This result shows (cf., eq. (I.64)) that, with the help of non-Riemannian geometry one not only arrives at our previous results concerning dislocations (cf., *infra*), but also the results relating to incompatibilities [1].

Besides Cartan torsion and Riemannian curvature there is yet another quantity that is important in this connection that we will now discuss. One sees the essentials most conveniently for two-dimensional matter; thus, perhaps, for curved membranes. In general, one can stretch them between two rigid planar boundaries, and they are thus

[1] The difference in sign is purely conventional.

forced into a two-dimensional Euclidian space in which they then exhibit "proper stresses." However, if the membrane forms, e.g., a thin, circular, hollow, cylinder then this is no longer true, unless one first makes a cut. For this membrane, the Riemannian curvature is zero, regardless of whether it is curved. Its extrinsic curvature in the three-dimensional Euclidian space in which it is embedded will be described by $\partial^2 \boldsymbol{X} / \partial x^i \, \partial x^j$, if, for the moment, $\boldsymbol{X}$ is the position in three-dimensional space and x^i are the coordinates in the surface.

One can now propose that our three-dimensional natural state represents a "three-dimensional membrane" in a six-dimensional Euclidian space. Let X^Λ ($A = 1 \ldots 6$) be its Cartesian coordinates. The "Euler-Schouten curvature tensor" [1] will then be defined by [130]:

$$H^\Lambda{}_{ij} = \frac{\partial^2 X^\Lambda}{\partial x^i \, \partial x^j}, \tag{IV.32}$$

where x^i once again has its previous meaning. The connection between R_{ijkl} and $H^\Lambda{}_{ij}$ is given by ([74], pp. 468):

$$R_{ijkl} = \sum_{\Lambda=1}^{6} (H^\Lambda{}_{ik} H^\Lambda{}_{jl} - H^\Lambda{}_{il} H^\Lambda{}_{jk}), \tag{IV.33}$$

in which one assumes that although when $H^\Lambda{}_{ij} = 0$, R_{ijkl} always vanishes, the converse is not true. One must also include the latter case in a complete theory.

Moreover, Kondo classified the lattice defects that appear in crystals as follows:

1. Lattice defects with incompatible metrics, which are characterized by a non-vanishing Riemannian curvature tensor in the natural state ("curvature defects").
2. "Non-Riemannian" lattice defects, which are characterized by a non-vanishing torsion tensor in the natural state ("torsion defects").
3. Lattice defects that are linked with a non-vanishing Euler-Schouten tensor.

After Kondo, it is, on the one hand, possible that of the three quantities $\Gamma^\chi{}_{[\mu\nu]}$, R_{ijkl}, and $H^\Lambda{}_{ij}$, only $\Gamma^i{}_{[jk]}$ is non-zero, but one can also have only $H^\Lambda{}_{ij}$ non-zero, and one can have $\Gamma^i{}_{[jk]} = 0$, but $R_{ijkl} \neq 0$. From the fact that the part of Δc^χ that corresponds to the curvature tensor, according to eq. (IV.25), is proportional to the vector c^λ that is being displaced, Kondo concludes that the curvature tensor also describes defect locations that extend over a large volume, while the torsion tensor is appropriate for dislocations at a smaller (more microscopic) scale. As defect locations that are described with the help of the curvature tensor, one has, above all, the build-up of dislocations in a glide plane that is provoked by foreign atoms distributed in a volume and lattice distortions through temperature fluctuations. By means of eq. (IV.33), all curvature defect locations are also described by the Euler-Schouten tensor.

If we compare these statements with our previous presentation then we see that two points still need clarification.

[1] In regard to this terminology, cf., [130], pp. 256.

a) The Riemannian curvature can also be finite in the absence of torsion – i.e., of dislocations [1]. One can show that this is not compatible with condition (IV.46) of BBS, which states that $L^{\alpha}{}_{\beta\gamma}$ has the form (IV.37). In fact, Kondo also permitted general forms for the connection. The case considered here thus goes over to the theory of BBS, as well as our theory, in which everything comes down to the dislocations. It will be a problem for future research to specify more precisely which phenomena can be attributed to such a curvature tensor with vanishing torsion.

b) If R_{ijkl} vanishes then no elastic deformation is present in the absence of external forces – at least, in the case of small distortions – so the crystal is free of proper stresses. In addition, let $\Gamma^{i}_{[jk]} = 0$. Then, from our previous presentation the only possible distortion that leaves the body compact is the plastic distortion Grad $\boldsymbol{s}^{P}$, by which dislocations form in the body, but at the end of the process they are no longer in the body. It now seems necessary to link this distortion with the tensor $H^{\Lambda}{}_{ij}$, in the case $R_{ijkl} = 0$, $\Gamma^{i}_{[jk]} = 0$, whenever (in analogy with the cylinder example above) cuts on the surface are necessary in order to bring the body from the Euler-Schouten curvature state that it is found in into the three-dimensional Euclidian state. Any cut corresponds to the migration of a dislocation through the body [2].

Certainly, the ultimate clarification of the aforementioned two points will round off the picture that we have drawn up here, such that it will be in complete harmony with our previous results. Even now, the agreement is very convincing. We further point out that eq. (IV.17) is nothing but our basic geometric equation when it is now written in the (spatially fixed) Cartesian coordinates of the final state. We will confirm this in the next paragraph.

§ 27. The theory of Bilby, Bullough, and Smith [3, 4, 5]

The states considered in the theory of Bilby and co-workers are referred to the state of the ideal crystal, the lattice vectors shall be described (§ 21) by a Cartesian coordinate system with the basis vectors $\boldsymbol{i}_{\alpha}$, and the final state was referred to by the authors as the "discolated state." In order to describe it, one chooses three independent basis vectors $\boldsymbol{e}_a(P)$ at each point that shall be the same lattice vectors everywhere [3,4]. These can be regarded as the basis vectors of the reference lattice undergoing a distortion:

$$\boldsymbol{e}_a = D_a{}^{\alpha}\,\boldsymbol{i}_{\alpha}\,, \qquad \boldsymbol{i}_{\beta} = E_{\beta}{}^{a}\,\boldsymbol{e}_a\,, \tag{IV.34}$$

where $D_a{}^{\alpha}$ is the associated tensor of the affine transformation and $E_{\beta}{}^{a}$ is the tensor reciprocal to it, so:

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[1] The following two theorems are necessary for § 27.

[2] A two-dimensional membrane can, by definition, admit only two-axis stresses. It can thus be bent into the aforementioned cylinder without stresses.

[3] For real plastic forming, the small regions that are bounded by the glide and climb planes will only be elastically formed, and neighboring atoms in them will remain neighboring during the entire process. Since these "elementary regions" are very small compared to the physical volume element (§ 20), one can regard their elastic distortions as homogeneous. One can then directly define the atomic triads under the $\boldsymbol{e}_a$ (perhaps at the center of mass of the elementary region), such that the $\boldsymbol{e}_a$ describe the lattice in the dislocated state directly.

[4] We now employ, with BBS, Greek indices for the Cartesian relative system.

$$D_a{}^{\alpha} E_{\alpha}{}^{a} = \delta_a{}^{b}, \qquad E_{\alpha}{}^{a} D_a{}^{\beta} = \delta_{\alpha}{}^{\beta}. \tag{IV.35}$$

BBS now define a new law of parallel displacement in Euclidian space, by establishing that vectors at different points shall be parallel when they have the same components in the $\boldsymbol{e}_a$-system. The actual difference C between two parallel vectors at two neighboring points P and Q will be defined as a vector attached to the point P. The authors obtain, after a simple calculation:

$$dC^{\mu} = E_{\lambda}{}^{a} \frac{\partial D_a{}^{\mu}}{\partial x^{\gamma}} C^{\lambda} dx^{\gamma}, \tag{IV.36}$$

and further consider a Euclidian space with the linear connection:

$$L^{\alpha}{}_{\beta\gamma} = - E_{\beta}{}^{\alpha} \frac{\partial D_a{}^{\mu}}{\partial x^{\gamma}} = D_a{}^{\alpha} \frac{\partial E_{\beta}{}^{a}}{\partial x^{\gamma}} \tag{IV.37}$$

(the parallel displacement law (IV.36) that belongs to it, resp.). The torsion tensor follows from:

$$L^{\alpha}{}_{[\beta\gamma]} = \frac{1}{2} D_a{}^{\alpha} \left(\frac{\partial E_{\beta}{}^{a}}{\partial x^{\gamma}} - \frac{\partial E_{\gamma}{}^{a}}{\partial x^{\beta}} \right), \tag{IV.38}$$

which corresponds to the eq. (IV.21) of Kondo. Now, one carries out the Franks-Burgers circuit. The typical element of this circuit in the dislocated crystal is [1]:

$$dx^{\alpha} \boldsymbol{i}_{\alpha} = dx^{\alpha} E_{\alpha}{}^{a} \boldsymbol{e}_a . \tag{IV.39}$$

The corresponding step in the reference lattice has, from § 21, numerically the same components in the $\boldsymbol{i}_{\alpha}$-system as the step (IV.39) in the $\boldsymbol{e}_a$-system, and can thus be written:

$$dx^{\lambda} E_{\lambda}{}^{a} \boldsymbol{i}_a \qquad \text{for} \qquad \alpha = a. \tag{IV.40}$$

From § 21, the circuit around a surface F with boundary $\mathfrak{C}$ delivers the associated Burgers vector (= dislocation flux):

$$\boldsymbol{B} \overset{n}{=} - \oint_{\mathfrak{C}} dx^{\lambda} E_{\lambda}{}^{a} \boldsymbol{i}_a \qquad \text{for} \qquad \alpha = a. \tag{IV.41}$$

Due to the sign $\overset{n}{=}$ (cf., *infra*), an application of Stokes's theorem, and going to an infinitesimal surface gives:

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[1] In the sense of footnote 2, pp ?, one may assume that one does not go from one atom to another, but from one elementary domain to another. This corresponds to the fact that in the case of macroscopic continuous dislocation distributions the dislocations are arranged between the elementary domains. For a thorough discussion of the generalization of the Burgers circuit of § 21 to continuous dislocation distributions, see the work of BBS [3], [4].

$$dB^{\alpha} \overset{n}{=} -\frac{1}{2}\left(\frac{\partial E_{\mu}{}^{a}}{\partial x^{\lambda}} - \frac{\partial E_{\lambda}{}^{a}}{\partial x^{\mu}}\right) dF^{\lambda\mu} \qquad \alpha = a. \tag{IV.42}$$

In eq. (IV.41), dx^{λ} means the difference between two points in the final state, and $E_{\lambda}{}^{a}$ shall also be taken in the final state. Henceforth, the entire right-hand side of eq. (IV.42) is referred to the final state. Therefore, eq. (IV.42) (and also the equation just before it) is no ordinary vector equation, since the $\overset{n}{=}$ says, moreover, that numerically the components on both sides of eq. (IV.42) are the same for $\alpha = a$. Due to (IV.34), the right-hand side of eq. (IV.42) is then equal to the vector in the final state, which one obtains when one constructs dB^{α} in the final state, so:

$$dL^{\alpha} = D_a{}^{\alpha}\, dB^{a} \tag{IV.43}$$

or:

$$dL^{\alpha} = \frac{1}{2} D_a{}^{\alpha}\left(\frac{\partial E_{\beta}{}^{a}}{\partial x^{\gamma}} - \frac{\partial E_{\gamma}{}^{a}}{\partial x^{\beta}}\right) dF^{\beta\gamma}. \tag{IV.44}$$

For BBS, dL^{α} is called the "local Burgers vector," while dB^{a} was referred to as the "true Burgers vector." One observes: From the standpoint of an invariant representation there is naturally only one Burgers vector; dB^{a} and dL^{α} are merely two different characterizations of these vectors. For small distortions, $D_a{}^{\alpha}$ can be replaced with $\delta_a{}^{\alpha}$ in eq. (IV.44); one no longer needs to distinguish between local and true Burgers vectors. From a comparison with eq. (I.14), one again finds the connection between the torsion tensor and the dislocation density as being given by:

$$L^{\chi}{}_{[\mu\nu]} = \frac{1}{2}\,\varepsilon_{\mu\nu\rho}\,\alpha^{\rho\chi}, \qquad \alpha^{\rho\chi} = \varepsilon^{\mu\nu\rho}\; L^{\chi}{}_{[\mu\nu]}\,. \tag{IV.45}$$

BBS speak of the local dislocation density.

The Riemann-Christoffel curvature tensor:

$$L^{\alpha}{}_{\beta\gamma\delta} = \frac{\partial L^{\alpha}{}_{\beta\delta}}{\partial x^{\gamma}} - \frac{\partial L^{\alpha}{}_{\beta\gamma}}{\partial x^{\delta}} + L^{\alpha}{}_{\lambda\gamma} L^{\lambda}{}_{\beta\delta} - L^{\alpha}{}_{\lambda\delta} L^{\lambda}{}_{\beta\gamma}, \tag{IV.46}$$

for the connection (IV.37) vanishes identically. For BBS, this is the condition for one to be able to define the local basis $\boldsymbol{e}_a$ everywhere and implies teleparallelism. This corresponds to the fact that one finds both the reference state and the final state in Euclidian space. For Kondo, by contrast, the curvature tensor (IV.23) was taken in the non-Riemannian (natural) state and thus did not vanish [1]. One obtains the same result in the BBS theory when one takes the curvature tensor relative to $\boldsymbol{e}_a \cdot \boldsymbol{e}_b$ metric (called the

[1] The two definitive curvature tensors are, from our previous standpoint, to be defined by $\Delta\beta^{G}_{\alpha\beta} \equiv L_{\alpha\beta\gamma\delta}\,\Delta F^{\gamma\delta}$, $\Delta\beta_{\alpha\beta} \equiv R_{\alpha\beta\gamma\delta}\,\Delta F^{\gamma\delta}$, where Δ has the same meaning as in eq. (IV.31).

"local metric" by BBS). This is therefore not the same curvature tensor as (IV.46). The vanishing of the curvature tensor that comes from the local metric, as well, would mean that the crystal is free of proper stresses.

It has shown that the theory we just reported on is very suitable for the investigation of the pure rotation states of Nye (§ 7), which, like the state of classical elasticity theory, is characterized by the vanishing of the incompatibilities (§ 7). We cannot reproduce in detail the calculations in question, which are very interesting as applications of non-Riemannian geometry to real bodies, but restrict ourselves to a particularly important point: the connection between the torsion tensor and the Nye curvature tensor.

The starting point is the law of parallel displacement (IV.36), which we write in form:

$$dC_a = - L_{abc}\, C^b\, dx^c. \tag{IV.47}$$

We split L_{abc} into a symmetric part $L_{(ab)c}$ and an anti-symmetric part $L_{[ab]c}$:

$$dC_a = - (L_{(ab)c} + L_{[ab]c})\, C^b\, dx^c. \tag{IV.48}$$

If one thinks of the C^b as being taken, in turn, from the lattice vectors $\boldsymbol{e}_b$ then dC_a is the change in them as they move to a neighboring point. One sees immediately that $L_{(ab)c}$ therefore refers to a pure deformation, while $L_{[ab]c}$ is a pure rotation of the lattice (one perhaps sets C^b equal to the vectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ in sequence). In the case of $L_{[ab]c}$ one then has $d\boldsymbol{C} \perp \boldsymbol{C}$ and the angle between $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ remains conserved, whereas for $L_{(ab)c}$ in general the length and angle of the $\boldsymbol{e}_a$ will be changed. If one then sets:

$$L_{[ab]c} \equiv \varepsilon_{abd}\, \chi^d{}_c\,, \qquad\qquad \chi^d{}_c = \frac{1}{2}\, \varepsilon^{abd}\, L_{abc} \tag{IV.49}$$

then the rotational part of $d\boldsymbol{C}$ is:

$$dC_a{}^{\mathrm{Rot}} = - \varepsilon_{abd}\, \chi^d{}_c\; C^b\, dx^c, \tag{IV.50}$$

or, in vector notation, $d\boldsymbol{C}^{\mathrm{Rot}} = - \boldsymbol{C} \times \boldsymbol{\chi} \cdot d\boldsymbol{x}$, from which, one obtains the rotation of the lattice under a motion from one point to another:

$$d\vec{\boldsymbol{\omega}} = \boldsymbol{\chi} \cdot d\boldsymbol{x}. \tag{IV.51}$$

By definition, $\boldsymbol{\chi}$ is then the Nye curvature tensor and (IV.49) is the connection between it and the torsion. Thus, $\boldsymbol{\chi}$ is connected with the first two indices in a similar way to the way that the dislocation tensor relates to the last two. In the case of a pure rotational state, one easily arrives at the Nye relation (I.59) by comparison with eq. (IV.45) [5] [1]. In this case, the tensors $\boldsymbol{\chi}$ and $\boldsymbol{K}$ of § 7 will also be identical [2].

BBS show further that the equations:

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[1] One thus writes eq. (IV.49) with Greek indices.

[2] For a discussion of the different curvature tensors, cf., also Eshelby [41].

$$\boldsymbol{\chi} \times \nabla = 0, \tag{IV.52}$$

which already appeared in Nye, are only valid for small rotations. These equations are indeed the conditions for the $d\vec{\boldsymbol{\omega}}$ of eq. (IV.51) to be a complete differential. Naturally, this can, however, only be true for small rotation angles.

§ 28. Discussion

In the last two paragraphs, we reported on applications of non-Riemannian geometry to continuous distributions of dislocations. The cited authors have continued to discuss in detail the connection with known problems of differential geometry – let us mention, say, Cartan's holonomy groups and structure equations and Ricci's rotation coefficients – and have thus paved the way for the non-Riemannian conception of dislocations. In the book of Kondo [74], an interesting possibility was described of treating the same problems with the help of pure Riemannian geometry in six dimensions, where one then no longer starts with the (holonomic) coordinates x^i, but must employ the anholonomic coordinates of the Riemannian space. In the opinion of this author, the method in question is particularly suitable for the treatment of plastic problems, so he has based a new theory of plasticity on it. Unfortunately, the papers of Kondo and collaborators were made know to this author just short of the completion of this report, so the aforementioned papers can be considered no further here.

In the opinion of the author, Riemannian and non-Riemannian geometry will, above all, play a role for large elastic distortions. In the first place, they are, in another connection, extraordinarily well-developed geometries that are formulated for arbitrarily large distortions from the outset. Secondly, the incompatibility tensor η_{ij} is uniquely responsible for the proper stresses only for small distortions, while the Riemannian curvature tensor is also responsible for large ones. On the other hand, one must constantly contend with tensors of rank three and four in theories that one can, at least in the case of small distortions, always replace with ones of lower rank [cf., eq. (IV.9) and (IV.27)]. As a result, the theory that was described in the first paragraph of this report in the case of small distortions seems particularly suitable when compared to one of the other theories, especially since it is developed somewhat closer to the physical processes of plastic forming.

In the introductory remarks to this chapter, we have hinted at the differing viewpoints of Kondo and BBS. Now, we shall briefly mention a second difference between the two theories that exists independently of the first one.

For a given reference state and final state of the body, the $D_a{}^{\alpha}$ and $E_{\beta}{}^{a}$ in eq. (IV.34) are uniquely determined [1], but not the $A_{\lambda}{}^{j}$ and $A_j{}^{\chi}$ in eq. (IV.12). Here, one still has a free choice of coordinate system. We would now like to first have at our disposal the freely-defined orientation of the elementary region in the natural state of Kondo, so it has the same direction everywhere. Furthermore, we imagine the final state of a virtual, ideal, point lattice, at best of the same form as the real atomic lattice in the ideal state, impressed (aufgeprägt) in such a way that, e.g., the $\boldsymbol{i}_j$ are the basis vectors of this lattice,

[1] By the convention in footnote 2, pp. ?

just as the $\boldsymbol{i}_a$ are the basis vectors of the reference lattice of BBS. If one could also establish this virtual lattice in the natural state then it would have suffered precisely the opposite (reciprocal) deformations and rotations as the actual atomic lattice under the transition from the natural state to the final state. The virtual lattice can, in the natural state (in which it is deformed), be completely described by a system of basis vectors $\boldsymbol{e}_\lambda$. Thus, if one chooses, like Kondo, the basis system $\boldsymbol{e}_\lambda$ from the present viewpoint then eq. (IV.12) takes on the same significance for the virtual lattice as eq. (IV.34) does for the real one. That is, any $A_j{}^\chi$ that take the virtual lattice from the natural state into the final state are numerically equal to the $D_a{}^\alpha$ that take the atomic lattice from the natural state or, equivalently, the reference state, to the final state. The latter is true due to the definition above of the orientation in the natural state.

From this, it follows that the components of $\Gamma^i_{[km]}$ in eq. (IV.21) and $L^\alpha{}_{[\beta\gamma]}$ in eq. (IV.38), which indeed both refer to the Cartesian coordinates of the final state, do not need to agree numerically, and it then follows that the dislocation densities $\boldsymbol{\alpha}$ computed in eq. (IV.27) and (IV.45) will generally deviate from each other. One can refer to the dislocation density calculated by Kondo as "virtual." There is no particular difficulty at all in converting the real and virtual dislocation densities into each other.

Finally, this difference in results of the two theories is a matter of convention that one should endow with no more significance than a sign convention. In the case of small distortions the difference between virtual, local, and true dislocation densities vanishes, and eqs. (IV.21) and (IV.38) then go directly into the (I.17) of the fundamental geometric equation, as we now show:

As remarked above, Kondo's $A_j{}^\chi$ and the $D_a{}^\alpha$ of BBS are numerically equal. Their meaning can be described are a distortion of the lattice from the reference state to the final state. This distortion has the form $\boldsymbol{I} + \boldsymbol{\beta}$, where $\boldsymbol{\beta}$ is identical with our previous distortion tensor ($\boldsymbol{I} \equiv$ identity tensor of rank two). For small distortions then the reciprocal distortions, which are represented by $A_\chi{}^i$ and $E_\beta{}^a$ are equal to $\boldsymbol{I} - \boldsymbol{\beta}$. If one substitutes this in eq. (IV.21) and (IV.38) and one neglects $\boldsymbol{\beta}$ compared to $\boldsymbol{I}$ in the $A_\chi{}^i$ ($D_a{}^\alpha$, resp.) that stands in front of the bracket the one obtains:

$$\Gamma^i_{[km]} = -\frac{1}{2}(\partial_m \beta_k{}^i - \partial_k \beta_m{}^i), \tag{IV.53}$$

resp.:

$$L^\alpha{}_{[\beta\gamma]} = -\frac{1}{2}\left(\frac{\partial \beta_\beta{}^\alpha}{\partial x^\gamma} - \frac{\partial \beta_\gamma{}^\alpha}{\partial x^\beta}\right). \tag{IV.54}$$

These equations are identical (the fact that Latin indices appear in one, while Greek indices appear in the other likewise arises, like the difference in sign, only from the difference between the conventions used by Kondo and BBS). Likewise, one sees upon consideration of eq. (IV.45) that eq. (IV.54) is identical with our fundamental geometrical equation (I.17) in the case of small distortions. Furthermore, one can make a proper vector equation out of eq. (IV.42) when one refers all quantities on the right-hand side to the reference state, as in § 10. One then obtains, after comparing with eq. (I.14), the fundamental geometric equation with the interpretation that is valid for large distortions. In the general case of large distortions (cf., § 10), one needs to refer the fundamental

equation to the coordinates of the points in the final state, because the equilibrium conditions themselves likewise refer to these coordinates. One then must employ the fundamental equation in the form (IV.21) ((IV.38), resp.).

Chapter V

Applications

The continuum theory of dislocations has still not been explored up to now in the context of the continuum-mechanical problems that are usually treated in classical elasticity theory, since the time since its formulation has been too short. Heretofore, the problems that were treated continuum-theoretically were of a predominately physical sort and involved, above all, isolated dislocations and atoms. An essential ambition of modern plasticity research is to understand the fundamental phenomena by starting from the microscopic picture. It has been shown that one can approximate the hardening of metals only in this way. We first bring up a few points about hardening in § 29, because we believe that the reader can be presumed to have a certain interest in such matters, in other words, to show the way in which difficult mathematical problems often come to the surface as a result of such considerations. The phenomenon of hardening is not purely mechanical, but has a complex physical nature, and here we can give, at best, an entirely brief insight into the way that such problems are treated today. For a presentation of the current state of hardening theory, refer to the new Handbuch article of *Seeger* [**135**].

In our opinion, the description of point-like lattice defects (foreign atoms, lattice vacancies, etc.) as dipoles or polarization centers takes on a fundamental meaning, so in § 31 we will treat four problems that show, in an impressive way, what sort of far-reaching problems can be coped with using the simple formulas of § 19 for such elastic singularities. It would certainly be a worthwhile problem for experimental research to measure and tabulate the dipole strength and polarizability for the largest possible number of atoms *B* embedded in a ground substance *A*, since this has long since been done for, e.g., electric and magnetic dipoles and polarizabilities.

Finally, in § 32, we shall give examples of the practical significance of the stress function tensor. We believe that the complete study of this tensor will bring to light many results of practical significance, of which we feel that the examination of the three-dimensional, and also rotationally-symmetric, boundary-value problem is especially urgent. These are only problems of an essentially mathematical nature, and § 32 may thus be regarded as an impetus to the mathematical circles, in that sense. In addition, § 32 includes the most important results on circular dislocations, which are not yet to be found in the literature.

§ 29. The hardening of face-centered, cubic metals

One of the most interesting – but, at the same time, most difficult – problems of modern solid-body physics is the hardening of metals. Fig. 30 shows the typical hardening curve for a face-centered, cubic unit crystal, as one would establish in a tensile test. One cannot derive this curve deductively from the basic equations of continuum mechanics or any laws of solid-body physics, since it largely relies upon empiricism. One makes certain assumptions about the processes that are at work inside the body during plastic forming and then examines the circumstances under which they would lead to hardening. One then performs the corresponding experiment and tests the extent to

which it is in agreement with the theoretical predictions. In that way, one has learned to distinguish the three clearly distinct hardening domains I, II, III in Fig. 30.

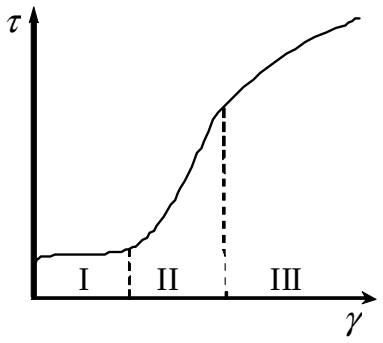

Figure 30. Typical hardening curve for a cubic face-centered metal (e.g., Cu). In the elastic domain, the curve $\tau(\gamma)$ will practically coincide with the τ-axis in our units.

In 1934, *Taylor* [**149**] first came up with the idea that proper stress fields were created under plastic deformation by the migration and compression of dislocations that tended to inhibit the further migration of the dislocations. This qualitative picture is still valid to this day.

The stress field in a crystal that originates in the applied external loads can be decomposed into its components along the glide plane and the glide directions. The shearing stress will then be greatest in one of the glide systems; we call it τ. This so-called *principal glide system* will be the first thing that we occupy ourselves with. In case the crystal is favorably oriented with respect to the strain axis, this glide system will remain up to the greatest deformation of the principal glide system, which is, for the most part, due to the plastic change of form. Fig. 31a, b show how a lengthening of the strain axis is produced by a glide in a glide system. For theoretical investigations, the external shear stress *t* will, in general, be applied against the glide in the principal glide system. This is defined to be the ratio of the (mean) relative plastic displacement of two lattice planes at a distance of *d* to *d*, which is then the plastic distortion β_{ij}^{P}, if *i* characterizes the glide plane and *j* characterizes the glide direction at the point in question on the hardening curve. Furthermore, we call this glide γ. At the same time, τ will have the meaning of a flow stress, since Fig. 30 gives the stress that one requires in order to make a crystal that has performed a forming γ flow further.

According to *Seeger* [**137**], one can split the flow stress of a pure metal into two parts by to the formula:

$$\tau = \tau_S + \tau_G ,$$

where τ_S is the part that the dislocations in the principal glide planes need in order to intersect the dislocations that traverse the other glide plane and pierce the principal glide plane (often referred to as a "forest of dislocations"). τ_G will be required to overcome the far-reaching stress field of the dislocations in the principal glide system. For many questions, τ_S will play no role in relation to τ_G, which is why we would like to restrict ourselves to the consideration of τ_G.

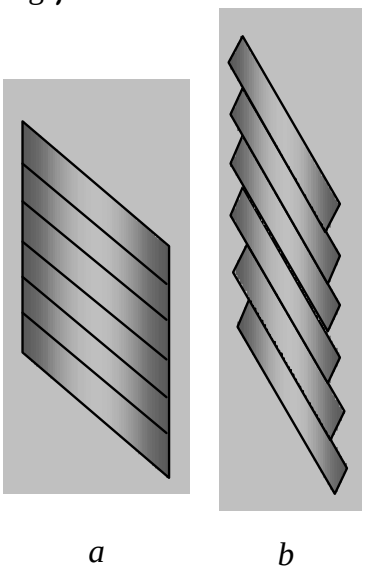

Figure 31. Model for plastic stretching of a rod. From *Schmidt-Boas* [**129**].

Frank and Read [**51**] have given a mechanism by which one can define closed rings of dislocations by means of shearing stresses that are applied to the glide

system in question. For this, one requires a not-too-short piece of a dislocation *A B* (Fig. 32a), which fixed at its ends in some way, perhaps in such a way that it defines a so-called *dislocation node* there with other dislocations (this is not indicated in the figure), which is often immobile then. The dislocation then bulges under a suitable shearing stress (Fig. 32b). From eq. (II.148), the force on a dislocation in a stress field will always be perpendicular to the dislocation line, so it will successively bulge into the forms that are indicated by Fig. c, d. The curve segments at *C* have the same Burgers vector, but opposite line directions, so the dislocations will have opposite signs, so, from § 18, they will draw together until they might annihilate, such that ultimately a new ring (*e*) is formed, and the original line *A B* remains. The process can then start all over again. This formation of a dislocation ring is completely analogous to the creation of soap bubbles, in which the line stress takes on the role of the outer surface stress for the soap bubble. The required starting lines *A B* are also present in an undeformed crystal in sufficient number, since a "network" of dislocations is already constructed in the crystal by its growth ([1]). For many metals, the number of dislocations that pierce 1 cm^2 is on the order of 10^7, and under deformation, it will increase by several more orders of magnitude.

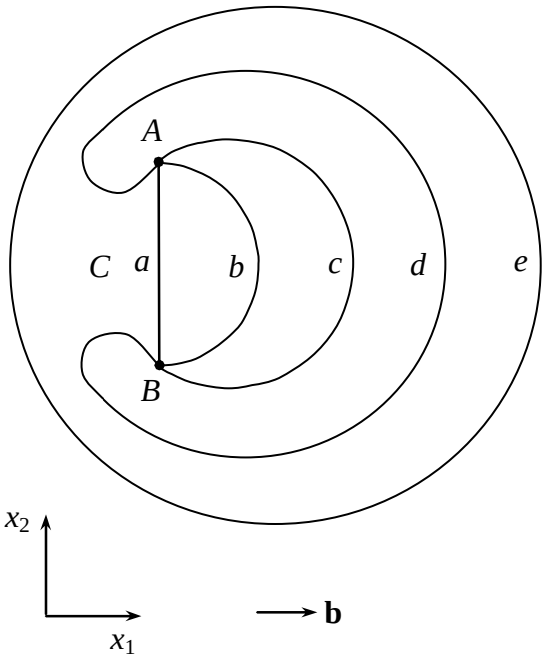

Fig. 32. Generating a dislocation ring with the help of the Frank-Read mechanism

In the hardening domain I – which is usually referred to as the "easy-glide domain" – the so-called hardening coefficient $d\tau / d\gamma$ is relative small, so the dislocations can form and wander without much obstruction. From the length of the electron-microscopically visible glide lines on the outer surface of the crystal (which was polarized before the deformation), one can conclude (*Mader* [**96**]) that the dislocations define paths here that are comparable to the cross-sectional dimensions of the crystal (mm).

([1]) Dislocations play a principal role in the theory of crystal growth. One might confer *Frank* [**49**], *Verma* [**154**], *Dekeyser* and *Amerlinckx* [**32**].

The essentially stronger hardening coefficient in the hardening domain II will be attributed to the appearance of *Lomer-Cottrell* dislocations (§ 24) that shrink the glide path of the dislocations noticeably. One of the dislocations that emanate from a *Frank-Read* source might possibly coalesce with a dislocation that moves in a second glide system, such that a *Lomer-Cottrell* dislocation reaction will take place. None of the other dislocations of the aforementioned source will get past this obstruction now, but they will pile up against it and will perhaps define a dislocation wall of the kind in Fig. 24 (but with variable distances between the dislocations). *Seeger*, *Diehl*, *Mader*, and *Rebstock* [**143**] have investigated the processes in hardening domain II theoretically and experimentally by starting with this picture, and could explain the linear rise in the hardening curve in domain II, at least quantitatively.

The hardening coefficient is even smaller in domain III of the hardening curve. This is currently explained by saying that now the possibility emerges of going around the obstruction by the increased external shearing stress of the piled-up dislocations. In order for that to happen, in any case, the breakdown of the dislocation lines (§ 24) in the glide plane must decrease over a distance of several atomic distances. This cannot come about by means of the stresses that originate at the location of the dislocations alone, since that would require an increase in the free energy on the order of eV at the location in question (cf., footnote 1 on pp. 7). The distance between the semi-dislocations will then be reduced somewhat by the resulting external and internal stresses at the location of the dislocation, while the rest of the energy that one needs in order to make the breakdown decrease completely (viz., the so-called *activation energy* Q) must be produced by temperature fluctuations. Obviously, Q will depend upon the stress, and it is only when that is sufficiently large (so Q will be sufficiently small) that the activation energy will actually result from temperature fluctuations.

For that reason, step dislocations will indeed still not have more possible glides than before, since, from § 24, only *one* glide plane orientation will exist for them. By contrast, screw dislocations that, from § 24, extend in the <110> directions, which always define the line of intersection of two {111}-planes, can, after the splitting dies out, split in the other {111}-plane, which will be the so-called “transverse glide plane” for it. Fig. 33 illustrates this process. The newly-obtained degree of motion for the screw dislocations will lead to the observed reduction of the hardening rise.

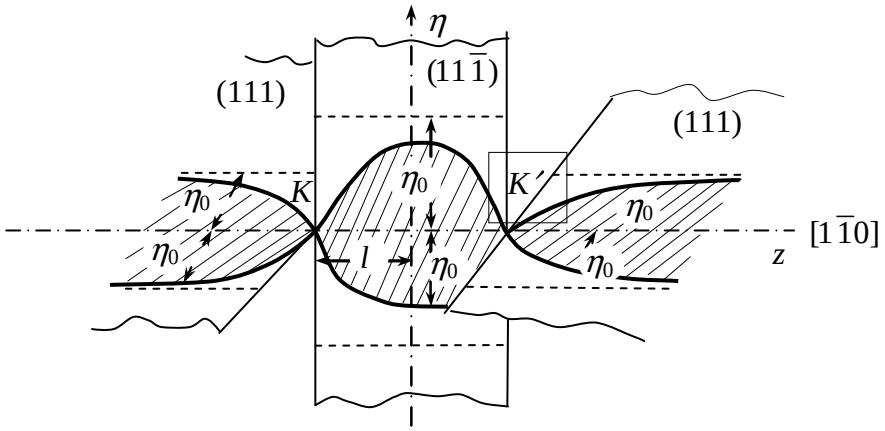

Figure 33. Transverse guiding in a screw dislocation. Hatched region: stacked defect, $z \equiv x_3$. From *Seeger* [**134**].

Seeger, *Diehl*, *Mader*, and *Rebstock* [**143**] have verified the validity of this picture conclusively by observing the outer surfaces of polarized crystal with an electron microscope. However, the complete rounding-off of the picture of hardening in domain III belongs to the treatment of a new problem statement that still cannot presently be resolved satisfactorily, due to great mathematical difficulties ([1]).

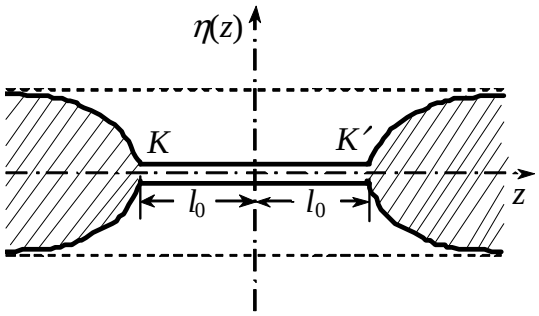

Figure 34. Calculating the activation energy for the transverse guiding in a screw dislocation. The distance between the dashed lines is $2\eta_0$. From *Seeger* [**134**].

For the time being, we shall base our discussion upon that of *Schoeck* and *Seeger* [**133**] or *Seeger* [**134**] (pp. 610, *et seq.*), resp. If the length $2l$ over which the splitting dies off is very large compared to the width $2\eta_0$ (Fig. 34) of the splitting then one will find the dislocation to be almost in a state of unstable equilibrium, since the probability of it splitting in the principal glide planes is not essentially larger than the probability of it splitting in the transverse glide planes. The shorter the segment $2l$ is, the more preferable the principal glide plane will generally be. However, if a shearing stress τ_Q exists in the transverse glide plane then it will be preferred, since the dislocation will then flatten and glide in it, so the shearing stress τ_Q can thus perform work ([2]). From this, one can conclude that any such shearing stress will be associated with an equilibrium length $2l_0$, for which, the splitting in the principal and transverse glide planes will be equally probable. At that length, the splitting of the concentrated dislocations in the principal glide system must, at the least, diminish when the external pressure is produced by a jump in the transverse glide system. The activation energy Q of this process will be the energy that the temperature fluctuations must give rise to in order to exhibit two parallel semi-dislocations at a distance of 2η in the configuration of Fig. 34.

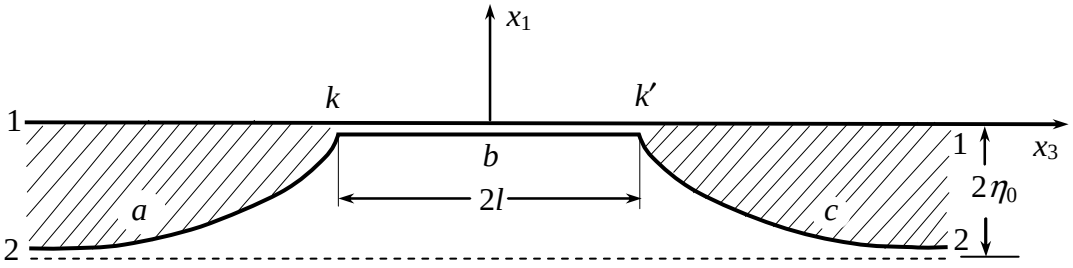

Figure 35. A somewhat simpler model for the calculation of the activation energy for transverse guiding.

One can deduce a value for the so-called *specific stacked defect energy* from experiments, which is the energy that is required for the formation of a stacked defect that extends completely through the crystal, measured per cm^2 of stacked defect surface. The

([1]) This problem is currently being worked on.

([2]) One observes: The resultant force on the concentrated dislocations is approximately zero in the principal glide plane, since otherwise it would, in fact, glide.

activation energy Q then takes on the following parts: The (positive) energy E_{12} that one needs in order to overcome the mutual repulsion of the two semi-dislocations under bending together and the (positive) energies E_{11} and E_{22} that one needs in order to bring the semi-dislocations alone into the position of Fig. 34 (in which they will be longer than before). Finally, the (negative) stacked defect energy E_{St} that one gets due to the fact that the stacked defect surface will be reduced.

The exact calculation of the activation energy raise great difficulties, since the form of the curve for the dislocations is not known at all, but will first be ascertained by the calculus of variations. For this, only direct methods will be at issue in practice. Since the asymmetry of the arrangement in Fig. 34 is not established physically, we would like to depict the foregoing in the somewhat simpler arrangement of Fig. 35. We take the free parameters to be, e.g., the transverse glide length $2l$, the splitting width 2η, and the rise in the curve at the nodes K and K'. The calculation of the part E_{St} is trivial. The part E_{12}, which previously raised great difficulties, will follow very simply from formula (II.136). The main work provokes the energy E_{22} . The line 2 consists of three pieces a, b, c. We write $E_{22} = E_{aa} + E_{bb} + E_{ab} + E_{ac} + E_{bc}$. Thus, E_{bb} is very easy to obtain from eq. (II.144), and likewise, E_{aa} and A_{ac} are relatively easy, since one of the two line integrals in eq. (II.128) proves to be elementary in that case. Since $E_{aa} = E_{bb}$, the main problem is then the calculation of E_{aa} and E_{ac} . Meanwhile, the branches a and c are relatively far apart, such that E_{ac} certainly makes a small contribution to E_{22} that is not entirely precise. One can then deal with this part with a simple approximate calculation. Thus, what essentially remains will be the energy E_{aa} that one needs in order to bring the branch a into the precise form that is given in Fig. 35. We will treat the problem of the self-energy of curve dislocations in the next paragraph and will see that one can manage quite well today with not-too-complicated curve forms.

One is then currently in a position to treat successfully the activation energy problems of the aforementioned kind that show up very often in solid-body physics with the methods of the continuum theory and to confirm or contradict by comparison with experiment the conclusions about elementary processes in solid bodies that were deduced in other ways.

In aluminum, in contrast to copper, the stacked defect energy is relatively large, so the splitting, and simultaneously the activation energy, will be small, such that one should expect that here hardening domain III will begin with essentially smaller stresses than it does for copper. This is confirmed quite well by experiment. It is very satisfying that one can presently understand the differences in hardening behavior of cubic face-centered metals almost quantitatively that has been quite puzzling for some years. The hardening of cubic space-centered metals is understood much less well.

§ 30. An approximation method for the calculation of the self-energy of singular dislocations

The self-energy of bent dislocations is important for many problems of solid-body physics. Compared to the older methods, in which one must evaluate, at the very least, a line integral of a function that was given as a surface integral in order to obtain the self-

energy of the dislocation, the reduction of the problem to the double line integral (II.145) represents a great advance. Still, it may be evaluated exactly only in the simplest cases. From *Kröner* [**83**], one reaches one's destination in the complicated cases with the help of the convergent approximation method that we will now describe.

The starting point is eq. (II.145), in which the cutoff length ε will have to be assumed as given. For the semi-dislocations that appeared in the last paragraph, one takes ε as in eq. (III.58). The integral in eq. (II.145) will have either the form:

$$\int_L \int_{L'_\varepsilon} \frac{dL' dL}{|\boldsymbol{x} - \boldsymbol{x}'|} \tag{V.1}$$

or the forms that one obtains when one adds either the expression $(x_1 - x'_1)(x_3 - x'_3) / (\boldsymbol{x} - \boldsymbol{x}')^2$ or the expression $(x_3 - x'_3)^2 / (\boldsymbol{x} - \boldsymbol{x}')^2$ to the integrand of (V.1) as a factor. The calculation of the latter integral is not essentially different from that of eq. (V.1), so we would like to restrict ourselves here to the treatment of (V.1). One will now have:

$$d\boldsymbol{L} = dx_1\, \boldsymbol{i}_1 + dx_3\, \boldsymbol{i}_3\,, \qquad d\boldsymbol{L}' = dx'_3\, \boldsymbol{i}_1 + dx'_1\, \boldsymbol{i}_3\,, \tag{V.2}$$

such that one must compute integrals of the form:

$$\iint \frac{dx_i dx'_j}{|\boldsymbol{x} - \boldsymbol{x}'|}, \qquad i, j = 1, 3. \tag{V.3}$$

Thus, in the expression:

$$|\boldsymbol{x} - \boldsymbol{x}'| = \sqrt{(x_1 - x'_1)^2 + (x_3 - x'_3)^2}\,, \tag{V.4}$$

one must always replace one of the two quantities x_1, x_3 (x'_1, x'_3, resp.) with the other one according to the curve equation $x_1 = x_1(x_3)$. If the dislocation is piecewise straight then the evaluation of the integral is elementary. For curves of degree two, one obtains elliptic integrals; in other cases, one can evaluate the integral numerically. In the case of § 29, where the curve equation includes free parameters, such a process would be much too involved. The following method then leads to our destination in many cases: We set:

$$|\boldsymbol{x} - \boldsymbol{x}'| = \left|x_3 - x'_3\right| \sqrt{1 + s}\,, \qquad s \equiv \left(\frac{x_1 - x'_1}{x_3 - x'_3} \right)^2 \tag{V.5}$$

and develop the root $w(s)$ in a Legendre polynomial in S in the domain $0 \le s \le S$. Since $w(s)$ is a parabola, one can compute with good convergence, as long as one takes S to be not much greater than, say, 3. Due to the simple form of $w(s)$, the estimation of the error will involve no great effort. If the dislocation moves, e.g., in such a way that its tangent vector never defines an angle $< 30^\circ$ with the $\pm x_1$ direction then obviously $0 \le s \le 3$, and the development of $w(s)$ will converge quite well along the entire dislocation. In the

event that, by contrast, the dislocation includes approximately vertical places, along with the approximately horizontal ones, one will employ:

$$| \boldsymbol{x} - \boldsymbol{x}' | = \left| x_1 - x_1' \right| \sqrt{1+1/s}\,, \qquad s \equiv \left(\frac{x_1 - x_1'}{x_3 - x_3'} \right)^2 , \tag{V.6}$$

in addition to (V.5) and represent $w(1/s)$ as a *Legendre* polynomial in $1/s$. However, it would be good to investigate whether one can, in the relevant case, avoid the complication that is linked with it, namely, that one adds higher terms in the development of the development of $w(s)$, by which, the domain $0 \leq s \leq S$ can be enlarged, in such a way that one might perhaps no longer need $w(1/s)$. The foregoing is recommended, e.g., for the calculation of the energy E_{aa} .

The scope of the calculations will be, for the most part, determined by the form of the curve for the dislocation. If $x_1(x_3)$ is a polynomial then that will make the integrations elementary. The same thing will be true for, e.g., the hyperbolas $x_1 \pm a = c \,/\, (x_3 \pm b)$, and indeed this statement will also be true for finite segments of hyperbolas. One can obviously describe the line segments *a*, *c* in Fig. 35 with such a hyperbola and calculate the energy in an elementary way, moreover. The scope of this calculation is tolerable.

Naturally, one can also apply the same processes to the calculation of interaction energies of two bent dislocations using eq. (II.128). Thus, one can then also calculate the activation energy that belongs to the arrangement in Fig. 34.

The τ_3 part of the flow stress will be required for the reciprocal intersecting of two dislocations. According to *Heidenreich* and *Schockley* [**64**], the splitting must die off around the point of intersection of both dislocations. One will then obtain a so-called *constriction* (*Stroh* [**148**]), which will follow from Figs. 34 and 35 when one sets $l = 0$ in them. *Schöck* [**132**] and *Schöck* and *Seeger* [**133**] calculated the activation energy for the intersection of dislocations in some cases that were in satisfactory agreement with the experimentally-measured activation energies (cf., also *Seeger* [**136**]).

§ 31. Foreign atoms as elastic dipoles and polarization centers

One speaks of “foreign” atoms when one finds isolated atoms of a second sort *B* in a crystal that consists of atoms of one sort *A*. A foreign atom can either replace the regular lattice site of an atom *A* (viz., a substitution) or it can occupy a so-called “intermediate” lattice site. This will happen especially when the atom *B* is very small in comparison to the atoms *A*. Very small sets of such foreign atoms can already influence the so-called “structural sensitivity” properties of matter very strongly (*Smekal* [**145**]). It is known that there are strong variations of the properties that iron exhibits when carbon (*C*) has been dissolved in it in a dilute concentration. Here, the *C* atoms will sit on intermediate lattice sites. From *Cottrell* [**23**] and *Cottrell* and *Bilby* [**25**], the elastic interaction of the *C* with the dislocations in iron is responsible for, e.g., the known “yield-point effect” of steel.

Fig. 36a shows how such a *C* atom is installed in the cubic space-centered lattice of iron. In order for it to have sufficient room, it must push the neighboring atoms apart. One will obviously obtain the same state of distortion of the lattice in the immediate

vicinity when one applies any force of magnitude P to the location of the C atom that pushes the one atom up and the other one down. If a is the distance between atoms in the normal state then one will have a force dipole here that has one non-zero component $P_{22} = a\,P$.

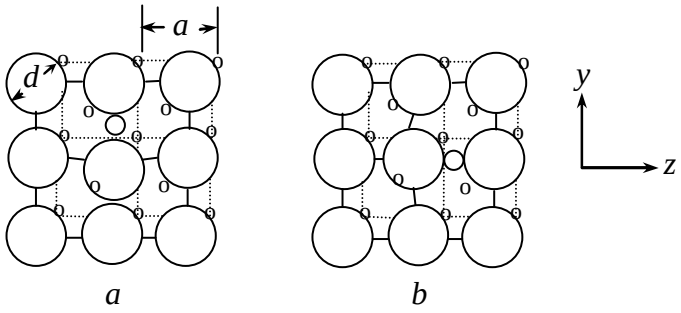

Fig. 36a) Intermediate lattice sites in cubic space-centered crystal as a model for carbon atoms in an iron lattice. For clarity, only the required atoms are indicated in their entire magnitude. For $d = a\sqrt{3/2}$, each atom will contact its eight nearest neighbors.

b) The same thing, after switching the places of the intermediate lattice atoms. (Read x_2 for y and x_1 for z.)

The picture that was just described is a bit too simplistic. For a more precise discussion, one must examine which of these atoms hold together; that will be determined by the special type of electron distribution that exists. One can show that with the inclusion of the C atom, not only will the force ratios between two neighboring atoms change, but other neighboring atoms will be drawn into the sphere of influence. One must then reckon that collectively the C atom does not act as just an isolated dipole P_{22}, but that other dipole components, as well as components of higher poles will also play a role. However, experience has shown that the direct interaction between atoms already dies off so quickly at a scale of one atomic distance that one can indeed describe the C in the situation of Fig. 36a by its dipole components P_{22}, as well as P_{11} and P_{33}, to good approximation.

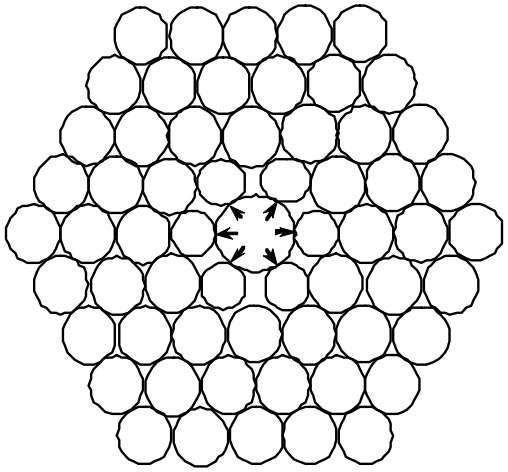

Fig. 37. Substitution of a foreign atom in closest-packing plane. The atomic rows are bent slightly.

An example of the substitution of a foreign atom is shown in Fig. 37. Here, one can think of the foreign atom as being replaced by a number of force dipoles that are rotated through 60^o from each other. One can show that the displacement field that is provoked

by such a dipole arrangement will be that of a dilatation center P_{ii} ([1]). In fact, in a continuum a dilatation center will correspond to small constrained sphere.

For many problems, it is important to know the energy with which a foreign atom is bound to a dislocation. In general, a dislocation will exert a force on a foreign atom by means of its elastic deformation field according to eq. (II.156). If one moves it from a site with zero deformation to a site with a deformation ε_{ij} in the neighborhood of the dislocation then one can arrive at an energy in the given case that would follow from eq. (II.165). The wandering of such an atom will first become possible with the aid of temperature fluctuations. By imposing a shearing stress (at least, at room temperature), a dislocation can now glide with an essentially larger velocity than that of the foreign atom, which the dislocation naturally strives to maintain, There then exists a tendency for the applied shearing stress to separate the dislocation and the foreign atom. However, extra energy will be required for that, whose magnitude will be equal to the "binding energy" of the foreign atom to the dislocation. According to *Cottrell*, in the normal state, any dislocation will be surrounded by an entire "cloud" of *C* atoms, such that collectively a seemingly large extra energy will be necessary in order to rip the dislocation loose from the cloud and help it gain its actual mobility. This leads to the well-known yield-point effect ([2]).

We would now like to examine the interaction of the *C* atom with a screw dislocation by appealing to the method of representation in *Cochardt, Schöck,* and *Wiedersich* [**16**] ([3]).

In iron, screw dislocations move in the <111> direction. We next restrict ourselves to calculating the interaction of a force dipole P_{11} with a screw dislocation in the [111] direction. The starting point is the two equations:

$$K_k = P_{ij}\, V_k\, \varepsilon_{ij}, \tag{V.7}$$

$$U = -\, P_{ij}\, \varepsilon_{ij} \tag{V.8}$$

of § 19. In polar coordinates ρ, φ, z, with [111] for the z-direction and $\varphi = 0$ in the $[\bar{2}11]$-direction, the deformation field that follows from the stress field (II.118) will have only the components:

$$\varepsilon_{z\varphi} = \varepsilon_{\varphi z} = \frac{b}{4\pi\rho}. \tag{V.9}$$

([1]) This statement is true in the case of a cubic face-centered lattice, but now, however, for the hexagonal lattice, with no further assumptions, since in the latter the elastic compliance that relates to the isolated dipoles will depend upon their direction, so one will need differing dipole strengths in order to displace opposing atoms from each other by the same amount.

([2]) According to the most recent argument of *Seeger* [**135**], this basis must be modified.

([3]) *Cottrell* and *Bilby* have described the *C* atom as essentially a center of dilatation P_{ii}, and thus obtained no interaction with a screw dislocation, since ε_{ii} will vanish for a screw dislocation from eq. (II.118). As these authors themselves remarked, and as was emphasized by *Crussard* [**27**] and *Nabarro* [**107**] especially, this will follow from a consideration of the tetragonality of the distortions as a result of the interactions of a *C* atom with screw dislocations. This was first investigated quantitatively by *Cochardt, Schöck,* and *Wiedersich* [**16**]. They employed no force dipole explicitly, but their method is very similar to our own.

From eq. (V.8), a force will be exerted on a dipole in the field of such a screw dislocation only when it likewise has a $z\varphi$ (φz, resp.) component. As a tensor, the force dipole can now be transformed into the ρ, φ, z system by the usual rules, and one will get:

$$P_{z\varphi} = P_{\varphi z} = (\boldsymbol{i}_\varphi \cdot \boldsymbol{i}_1)\,(\boldsymbol{i}_z \cdot \boldsymbol{i}_1)\,P_{11}\,. \tag{V.10}$$

If we set $\boldsymbol{i}_\varphi = \boldsymbol{i}_{\varphi_0}\cos\varphi + \boldsymbol{i}_{\varphi_{90}}\sin\varphi$ then $\boldsymbol{i}_{\varphi_0}$ will be the unit vector that is perpendicular to the vectors $\boldsymbol{i}_z = \left[\sqrt{\tfrac{1}{3}}, \sqrt{\tfrac{1}{3}}, \sqrt{\tfrac{1}{3}}\right]$ and $\boldsymbol{i}_{\varphi_{90}} = \left[\sqrt{\tfrac{2}{3}}, -\sqrt{\tfrac{1}{6}}, -\sqrt{\tfrac{1}{6}}\right]$ ([1]). $\boldsymbol{i}_{\varphi_0} = \boldsymbol{i}_z \times \boldsymbol{i}_{\varphi_{90}}$ will have no component in the x_1-direction, as one easily confirms, such that one will ultimately obtain:

$$P_{z\varphi} = P_{\varphi z} = \frac{\sqrt{2}}{3} P_{11} \sin\varphi \tag{V.11}$$

for eq. (V.10). Thus, from eq. (V.7), since $\varepsilon_{\varphi z}$ depends upon only ρ, the dipole will experience a force:

$$K_\rho = \frac{-b}{3\sqrt{2\pi}} \frac{\sin\varphi}{\rho^2}\, P_{11}\,. \tag{V.12}$$

This will be a force of tension in the domain $0 < \varphi < 180^\circ$, and a compression otherwise. If one repeats the same argument for P_{22} and P_{33} then one will obtain formulas that follow from eq. (V.12) when one replaces φ with $\varphi + 180^\circ$ ($\varphi + 240^\circ$, resp.) in it. This symmetry will be a natural consequence of the fact that the [111] direction defines the same angles with the x_i-axes.

One obtains the energy of the dipole P_{11} in the elastic field of the dislocation:

$$U = -\frac{b\,P_{11}}{3\sqrt{2\pi}} \frac{\sin\varphi}{\rho} \tag{V.13}$$

from eq. (V.8), along with (V. 9) and (V.11).

The application of these formulas to the case of the *C* atom is hindered by the fact that the dipole strength is not known for the *C* atom. At present, one can still supply it only experimentally, which means that one needs a theory of how to couple the dipole strengths with the experimentally-measured values, which it admits. There is still no process that is applicable in every case, as of yet.

Eshelby [**39**] has given the following method for measuring the dipole strengths of dilatation centers: One dissolves a number of atoms of type *B* in a pure metal *A* (e.g., Al in Cu) and measures the changes in the lattice constants that then appear. They will depend upon the concentration of the atom *B* in *A* that is of issue and the strengths of the dilatation centers. *Eshelby* has given the necessary formulas for the determination of P_{ii} from the changes in the lattice constants.

([1]) One sees immediately that $\boldsymbol{i}_z$ and $\boldsymbol{i}_{\varphi_{90}}$ are unit vectors in the [111] and $[2\bar{1}\bar{1}]$ directions, resp.

We would now like to say that this method is capable of being extended to the case of arbitrary force dipoles. If one distributes a number of force dipoles statistically in a previously homogeneous pure body *A* then that will change its external form and size, in general. If one chooses certain physical volume elements that contain very many such dipoles (but are, on the other hand, very small compared to the external dimensions of the body) then one can say that every volume element will have been imprinted with a mean (macroscopic) deformation ε_{ij}^{Q} (§ 6) as a result of the dipole distribution. If we assume that the concentration is constant as a function of the volume element (which one can generally achieve experimentally) then the connectivity of the body will not be disturbed by these imprinted deformations – i.e., no (macroscopic) elastic deformations will be necessary in order to keep the body compact. Therefore, the imprinted deformation will already be the macroscopically-observed total deformation $\boldsymbol{\varepsilon}^{G}$. The macroscopic stresses will vanish.

Now, it is clear that one cannot distinguish macroscopically whether one has, say, 10^6 dipoles of strength A_{ij} or 10^7 dipoles of strength A_{ij} / 10 in a volume element; in other words, one can think of the *N* dipoles of strength P_{ij} as being replaced with a constant dipole density p_{ij} that is determined from the condition:

$$\iiint_V p_{ij}\, dV = p_{ij}\, V = N\, P_{ij}\,. \tag{V.14}$$

Rieder [**123**] called such a dipole density, when taken to be negative, an "extra stress," and the term "imprinted stress" was also meaningful. It will be connected with the imprinted deformation only by way of the equation (cf., [**123**], [**122**]):

$$p_{ij} = c_{ijkl}\, \varepsilon_{kl}^{Q}\,, \tag{V.15}$$

as one easily sees ([1]). Since p_{ij} is known, one will likewise have the total deformation $\varepsilon_{ij}^{G} = \varepsilon_{ij}^{Q}$ of the body. Conversely, one will immediately obtain the dipole strength:

$$P_{ij} = (V / N)\, c_{ijkl}\, \varepsilon_{kl}^{G} \tag{V.16}$$

for a known concentration N / V and total deformation ε_{ij}^{G}.

Due to the assumption of *Hooke's law* (V.15), the method will be true only for small deformations. One can conveniently measure them as changes in the lattice constants ([2]). Once more, in general, one does not employ entirely simple arguments in order to convince oneself that the changes in the lattice constants that one establishes Röntgenographically will actually be directly equal to the macroscopic deformation.

([1]) One imagines that outer surface forces that initially produce no deformation are combined with the dipole density at the same time. One can then excise the volume element and measure the forces $p_{ij}\, dF_i$ that one must add if no distortions are to be present. The deformation that is associated with the relaxation of tension will naturally be coupled to these stresses by *Hooke's law*. One likewise sees that one must work with small p_{ij} – i.e., small concentrations – since otherwise eq. (V.15) would no longer be true.

([2]) Since one must generally melt the probe in order to introduce the dipole, one can compare their external dimensions with and without dipoles only very poorly.

Naturally, one will very well have microscopic stresses and fields of elastic deformations that change signs over distances whose order of magnitude is the mean distance between dipoles. The investigations of *Miller* and *Russel* [**101**], *Huang* [**66**], *Teltow* [**150**], and *Eshelby* [**39**] regarding this subject seem to have answered this question in the positive ([1]).

In cubic, space-centered iron-carbon the *C* atoms are distributed statistically over the three possible positions – we would like to call them, *mutatis mutandis*, the 1, 2, and 3 positions. One can establish only one uniform dilatation of the lattice and obtain only one intersection statement about the P_{ij} in this case. By contrast, the *C* atoms in martensite are arranged tetragonally (e.g., all of them in the 1-position), so one would expect a strong dipole P_{11} and two weak dipoles P_{22} and P_{33} .

Kurdjumov and *Kaminski* [**85**] obtained an increase in the lattice constant *c* in the 1 direction from 2.86 to 2.96 Å for 1 Gew.-% of *C* in Fe (corresponding to $V / N = 2.58 \cdot 10^{-22}$ cm^3), while, at the same time, the ratio c / a rose from 1.0 to 1.04. (a = lattice constant in the 2 or 3 direction, resp.) This means that there are rotations $\varepsilon_{11}^G = 0.035$, $\varepsilon_{22}^G = \varepsilon_{33}^G = -0.0048$. With $c_{1111} = 2.37 \cdot 10^{12}$ dyne/cm^2, $c_{1122} = 1.41 \cdot 10^{12}$ dyne/cm^2 ([2]), it will follow easily that ([3]):

$$P_{11} = 11.2 \text{ eV}, \qquad P_{22} = P_{33} = 4.6 \text{ eV}. \tag{V.17}$$

One will then obtain the energy of the total dipole that represents the *C* atom at a distance of *b* from the screw displacement when $\varphi = 90^o$ (where $| U |$ is maximal) as 0.5 eV ([4]) from eq. (V.13) and the corresponding formulas for P_{22} and P_{33} .

We shall briefly give an application to the important *Snoek* effect [**146**] ([5]). If one imposes a stress σ_{11} in the crystal of Fig. 36a such that it is extended in the x_1-direction then the *C* atoms would like to cross over to the 1-positions (Fig. 36b) since they would have more room there. The total deformation will now be composed of an elastic part $\varepsilon_{ij} = s_{ijkl}\, \sigma_{kl}$ and an additional quasi-plastic deformation $\varepsilon_{ij}^Q = s_{ijkl}\, p_{kl}$ [eq. (V.15)], where p_{kl} is the change in the dipole density that appears as a result of the transition of *n* dipoles from 2 and 3 position into the 1 position. The total deformation will then be:

$$\varepsilon_{ij}^G = s_{ijkl}\, (\sigma_{kl} + p_{kl}). \tag{V.18}$$

([1]) Cf., the discussion of *Eshelby* [**39**].

([2]) One now has $c_{1111} = c_{2222} = c_{3333} \equiv c_{11}$, $c_{1122} = c_{1133} = c_{2211} = (\text{etc.}) \equiv c_{11}$, c_{11} and c_{12} are both taken from *Zener* [**158**], pp. 17.

([3]) 1 eV = $1.60 \cdot 10^{-12}$ erg.

([4]) The discrepancy with the value of 0.75 that was obtained by *Cochardt* and collaborators then originates in the fact that these authors employed the isotropic *E* modulus. This is a practically meaningful and instructive example of how large the difference can be when one does not consider the elastic anisotropy of the crystal.

([5]) The *Snoek* effect is much used for the determination of the smallest *C* content in Fe [**75**]. *Zener* [**158**] has treated this effect theoretically in a very satisfying way, so we shall use his presentation, in part. What is new and different here is the fact that we will be attributing the relaxation of the elastic coefficients s_{ijkl} to the force dipole that the *C* atom represents.

In this, p_{kl} shall now be expressed in terms of known quantities. Let P_{ij}^1, P_{ij}^2, P_{ij}^3 be the dipoles in the three positions. Due to the fact that P_{ij}^2 and P_{ij}^3 have the same value, it will then follow from eq. (V.14) that:

$$p_{kl} = \frac{n}{V}(P_{kl}^1 - P_{kl}^2)\,. \tag{V.19}$$

According to *Zener* [**158**], one will obtain n in this very simply from *Boltzmann statistics:*

$$n = -\frac{2}{9}N\frac{U_1 - U_2}{kT}, \qquad \text{for} \qquad \frac{|U_1 - U_2|}{kT} \ll 1, \tag{V.20}$$

where U_1 is the elastic energy of the dipole in the 1 position, and analogously for U_2. It follows from eq. (V.8) that:

$$U_1 - U_2 = -\,(P_{ij}^1 - P_{ij}^2)\,\varepsilon_{ij}\,, \tag{V.21}$$

from which, one will get:

$$p_{kl} = \frac{2}{9}\frac{N}{VkT}(P_{kl}^1 - P_{kl}^2)(P_{ij}^1 - P_{ij}^2)\,\varepsilon_{ij}\,. \tag{V.22}$$

We write $\varepsilon_{ij} = s_{ijmn}\,\sigma_{mn}$ in this, and then substitute p_{kl} in eq. (V.18). That will then give:

$$\varepsilon_{ij}^G = (s_{ijkl} + \Delta s_{ijkl})\,\sigma_{kl}\,, \tag{V.23}$$

with

$$\Delta s_{ijkl} = \frac{2}{9}\frac{N}{VkT}Q_{ij}Q_{kl}\,, \qquad Q_{ij} \equiv s_{ijmn}\,(P_{mn}^1 - P_{mn}^2)\,. \tag{V.24}$$

One calls $s_{ijkl} + \Delta s_{ijkl}$ the *relaxed* elastic coefficients, and one measures them statically as the ratio of the total deformation to the applied stress. By contrast, one can measure the *unrelaxed* elastic coefficients s_{ijkl} in oscillating rods for which the period of oscillation is short enough that the rearrangement of the dipoles (viz., the *relaxation*), which always takes a finite length of time, cannot take place. *Zener*, whose numerical results are essentially the same as ours, has further described the way in which Δs_{ijkl} is a measure of the magnitude of the observed damping. Confer *Zener* [**158**] for this and the comparison with experimental results, which proves to be quite satisfactory.

In conclusion, we shall briefly go into polarizability (§ 19). We assume that the probe contain no dipoles, but only centers that can be polarized. An important example of this is the lattice vacancies in many face-centered, cubic crystals ([1]). One can refer to such a material as *dia-elastic*, in analogy with the behavior in electrodynamics, and by contrast we would like to call a body with dipoles *para-elastic*. The para-elastic bodies always have a certain dia-elasticity.

([1]) The fact that the dipole strength of these lattice vacancies is approximately zero was recently calculated by *Seeger* and *Bross* [**142**] using electron theory.

If one imposes, e.g., a homogeneous stress σ_{ij} on a dia-elastic rod then P_{ij}^{ind} dipoles with a density of p_{ij}^{ind} will be induced in it. One will then get:

$$\sigma_{ij} + p_{ij}^{\text{ind}} = c'_{ijkl}\, \varepsilon_{kl}\,, \tag{V.25}$$

in place of the usual *Hooke* law:

$$\sigma_{ij} = c_{ijkl}\, \varepsilon_{kl} \tag{V.26}$$

that one would have in the absence of dia-elasticity. Combining the two equations will yield:

$$p_{ij}^{\text{ind}} = (c'_{ijkl} - c_{ijkl})\, \varepsilon_{kl}\,, \tag{V.27}$$

and

$$c_{ijkl} = c'_{ijkl} - c_{ijkl} \tag{V.28}$$

will be the *elastic susceptibility* of the probe. Since $p_{ij}^{\text{ind}} = P_{ij}^{\text{ind}}\, N / V$, the elastic polarizability of eq. (II.169) will be given by:

$$R_{ijkl} = r_{ijkl}\, V / N\,. \tag{V.29}$$

c'_{ijkl} and c_{ijkl} can be measured quite well in many cases as elastic moduli by a probe with and without polarization centers, such that polarizability of a defect site is often obtained relatively simply ([1]). The interaction of the dia-elastic lattice defect sites is determined essentially by the polarizability. This is a short-range intersection (the force is proportional to the – 6 power of distance), since the force between dipoles goes with the – 4 power.

§ 32. Applications of the stress function tensor χ' to rotationally-symmetric and three-dimensional problems

Let ρ, φ, z be cylindrical coordinates, and let $\boldsymbol{i}_\rho$, $\boldsymbol{i}_\varphi$, $\boldsymbol{i}_z$ be the associated basis vectors (with magnitude 1). The components of the stress function tensor χ' might not depend upon φ. One can then write the auxiliary conditions $\nabla_i \chi'_{ij} = 0$ of § 12, as one easily verifies:

$$\frac{\partial(\rho\, \chi'_{\rho\rho})}{\partial \rho} - \chi'_{\varphi\varphi} + \rho \frac{\partial\, \chi'_{\rho z}}{\partial z} = 0, \tag{V.30}$$

([1]) The effective moduli c'_{ijkl} of a probe with polarization centers appear to be reduced or increased in comparison with the moduli of the probe without centers according to the sign of $p_{ij}^{\text{ind}} / \varepsilon_{kl}$. According to *Zener* [**159**], all proper stress sources produce yet another effect that always diminishes the moduli. It goes back to an increase in the entropy of oscillation of the body with increasing elastic deformation, which is – in contrast to our effect – essentially temperature-dependent. Thus, it can be isolated experimentally.

$$\frac{1}{\rho^2}\frac{\partial(\rho^2\,\chi'_{\rho\varphi})}{\partial\rho}+\frac{\partial\,\chi'_{z\varphi}}{\partial z}=0, \tag{V.30′}$$

$$\frac{1}{\rho}\frac{\partial(\rho\,\chi'_{\rho z})}{\partial\rho}+\frac{\partial\,\chi'_{zz}}{\partial z}=0. \tag{V.30″}$$

If they are fulfilled then the stresses will follow most simply from eq. (II.23):

$$\boldsymbol{\sigma}=2G\left[\Delta\boldsymbol{\chi}'+\frac{m}{m-1}(\nabla\nabla-\Delta I)\chi'_I\right],$$

with

$$(\sigma_{\rho\rho}+\sigma_{\varphi\varphi})/2G=\Delta\chi'_+-\frac{m}{m-1}\left(\Delta+\frac{\partial^2}{\partial z^2}\right)\chi'_I,$$

$$(\sigma_{\rho\rho}-\sigma_{\varphi\varphi})/2G=\left(\Delta+\frac{4}{\rho^2}\right)\chi'_-+\frac{m}{m-1}\left(\frac{\partial^2}{\partial\rho^2}-\frac{1}{\rho}\frac{\partial}{\partial\rho}\right)\chi'_I,$$

$$\sigma_{zz}/2G=\Delta\chi'_{zz}-\frac{m}{m-1}\left(\Delta-\frac{\partial^2}{\partial z^2}\right)\chi'_I,$$

$$\sigma_{\rho z}/2G=\left(\Delta-\frac{1}{\rho^2}\right)\chi'_{\rho z}+\frac{m}{m-1}\frac{\partial^2}{\partial\rho\,\partial z}\chi'_I,$$

$$\sigma_{\varphi z}/2G=\left(\Delta-\frac{1}{\rho^2}\right)\chi'_{\varphi z},\qquad \sigma_{\rho\varphi}/2G=\left(\Delta-\frac{1}{\rho^2}\right)\chi'_{\rho\varphi}.$$

Now, the Cartesian components of χ'_{ij} satisfy eq. (II.20):

$$\Delta\Delta\chi'_{ij}=\eta_{ij}. \tag{V.31}$$

It will follow from this directly that:

$$\Delta\Delta\chi'_+=\eta_+,\quad \Delta\Delta\chi'_{zz}=\eta_{zz}. \tag{V.32}$$

One derives the remaining equations somewhat more tediously:

$$\left(\Delta-\frac{1}{\rho^2}\right)\left(\Delta-\frac{1}{\rho^2}\right)\chi'_-=\eta_-,\qquad \left(\Delta-\frac{1}{\rho^2}\right)\left(\Delta-\frac{1}{\rho^2}\right)\chi'_{\rho\varphi}=\eta_{\rho\varphi}, \tag{V.32′}$$

$$\left(\Delta-\frac{1}{\rho^2}\right)\left(\Delta-\frac{1}{\rho^2}\right)\chi'_{\rho z}=\eta_{\rho z},\qquad \left(\Delta-\frac{1}{\rho^2}\right)\left(\Delta-\frac{1}{\rho^2}\right)\chi'_{\varphi z}=\eta_{\rho\varphi}. \tag{V.32″}$$

Since $\nabla_i\ \eta_{ij} = 0$, η_{ij} will naturally be subject to the restrictions that correspond to (V.30). One sees that initially the components of $\boldsymbol{\chi}'$ are not coupled to each other by the differential equations, although $(\chi'_{\rho\rho}, \chi'_{\varphi\varphi}, \chi'_{\rho z})$, $(\chi'_{\rho\varphi}, \chi'_{\varphi z})$, and $(\chi'_{zz}, \chi'_{\rho z})$ will be linked with each other by the conditions (V.30). One can often say from the outset that $\sigma_{\varphi z} = \sigma_{\varphi\rho} = 0$, so one can ignore $\chi'_{\varphi z}$ and $\chi'_{\varphi\rho}$. Now, in the event that one further has $\eta_{\varphi z} = \eta_{zz} = 0$, one will require only $\chi'_{\rho\rho}$ and $\chi'_{\varphi\varphi}$ in order to arrive at the particular integral. Even though we can still find no rigorous proof, it is very likely that all stress states in which Div $\boldsymbol{\sigma}$, $\boldsymbol{\eta}_i$, $\sigma_{\varphi z}$, $\sigma_{\varphi\rho}$ vanish can be expressed in terms of $\chi'_{\rho\rho}$ and $\chi'_{\varphi\varphi}$ alone. One will then be dealing with the stress states for which one will generally use *Love's* displacement function in order to calculate them. We would like to consider the case of $\chi'_{\varphi z} = \chi'_{\varphi\rho} = \chi'_{\rho z} = \chi'_{zz} = 0$ by itself later on.

Remarkably, from eq. (V.30), $\chi'_{\varphi\varphi}$ can now be expressed in terms of $\chi'_{\rho\rho}$. One can also write the conditions (V.30) in terms of $\chi'_{\rho\rho}$ and χ'_{+} (χ'_{-}, resp.), and correspondingly for $\boldsymbol{\eta}$:

$$\chi'_{+} = \frac{1}{\rho}\frac{\partial(\rho^2\chi'_{\rho\rho})}{\partial\rho}, \qquad \chi'_{-} = -\rho\frac{\partial\chi'_{\rho\rho}}{\partial\rho}, \qquad \eta_{+} = \frac{1}{\rho}\frac{\partial(\rho^2\eta_{\rho\rho})}{\partial\rho}, \qquad \eta_{-} = \rho\frac{\partial\eta_{\rho\rho}}{\partial\rho}. \quad \text{(V.33)}$$

Moreover, the stresses can be written:

$$\left.\begin{aligned}
(\sigma_{\rho\rho} + \sigma_{\varphi\varphi})/2G &= \Delta\chi'_{+} - \frac{m}{m-1}\left(\Delta + \frac{\partial^2}{\partial z^2}\right)\chi'_{+},\\
(\sigma_{\rho\rho} - \sigma_{\varphi\varphi})/2G &= \left(\Delta - \frac{4}{\rho^2}\right)\chi'_{-} + \frac{m}{m-1}\left(\frac{\partial^2}{\partial\rho^2} - \frac{1}{\rho}\frac{\partial}{\partial\rho}\right)\chi'_{+},\\
\chi'_{-} &= 2\chi'_{\rho\rho} - \chi'_{+},\\
\sigma_{zz}/2G &= -\frac{m}{m-1}\left(\Delta - \frac{\partial^2}{\partial z^2}\right)\chi'_{+},\\
\sigma_{\rho z}/2G &= \frac{m}{m-1}\frac{\partial^2}{\partial z\,\partial\rho}\chi'_{+}.
\end{aligned}\right\} \quad \text{(V.34)}$$

By adding the eqs. (V.32) and (V.32′), which refer to η_{+} and η_{-}, one will obtain with (V.33)

$$\Delta'\,\Delta'\,\chi_{\rho\rho} = \eta_{\rho\rho}, \qquad \Delta' \equiv \Delta + \frac{2}{\rho}\frac{\partial}{\partial\rho}. \quad \text{(V.35)}$$

Now, is $\chi'_{+} = 0$ then it will follow from eq. (V.33) that $\rho^2\chi'_{\rho\rho}$ can only have the form $f(z)$. If one has $\boldsymbol{\eta} = 0$ then $f(z)$ can only be a polynomial of degree three [since otherwise eq. (V.35) would not be fulfilled]. The stresses that belong to $f(z)$ will follow easily from eq.

(V.34) as $\sigma_{\rho\rho} = -\sigma_{\varphi\varphi} = (c_0 + c_1 z) / \rho^2$, while all other components will vanish. Insofar as this relatively trivial stress state is not present in the body in question ([1]) (which will be true for, e.g., convex bodies that are subject to outer surface forces), the function χ'_+ (and therefore also χ'_-) will have the same value as $\chi'_{\rho\rho}$. One can then calculate – say, χ'_+ – and then calculate $\chi'_{\rho\rho}$ from that with the help of the formula:

$$\chi'_{\rho\rho} = \frac{1}{\rho^2} \int \rho \chi'_+ \, d\rho , \tag{V.36}$$

which follows from eq. (V.33), and with that the stresses will be obtained from eq. (V.34) by differentiating it twice. Insofar as one calculates $\chi'_{\rho\rho}$ from the outset, one needs to differentiate it three times in order to get the stresses. We shall not actually write down eq. (V.34) in terms of $\chi'_{\rho\rho}$.

Presently, the still-unsolved problem is that of how one can express the boundary conditions advantageously in terms of $\chi'_{\rho\rho}$ or χ'_+. Any bi-harmonic function $\chi'_{\rho\rho}$ or χ'_+ will yield a possible stress state, such that the boundary-value problem will be doubly harmonic. One must conclude from this that the totality of stress states that are encompassed by *Love* functions (with Div $\boldsymbol{\sigma} = 0$) are also actually encompassed by $\chi'_{\rho\rho}$ (χ'_+, resp.). Since one gets somewhat closer to the stresses with the latter function, the solution of the boundary-value problem by these functions will have some practical significance.

These considerations show the great variety of possibilities that the stress function tensor implies, such one can adapt a given problem to a great extent. This is especially true for the use of components for the stress function tensor that belong to curvilinear coordinates. Here, we have chosen the auxiliary condition $\nabla_i \chi'_{ij} = 0$ from the outset, but there are still numerous possibilities, about which, almost nothing is known at present.

We would now like to give an application of the above to circular dislocations. One of will lie in the plane $z = 0$ with its center at the origin and a radius of R. The stress function field of this dislocation will be given essentially by eq. (II.107) as the integral:

$$\oint x \, dL'_i , \tag{V.37}$$

which one can easily show will have the form:

$$\boldsymbol{F}(\rho, z) \, \boldsymbol{i}_\varphi . \tag{V.38}$$

From *Franz* and *Kröner* [**53**], one will then have:

([1]) In the other case, one can probably also remove it from the total state somehow. One will find most of the results in this § for the case of $\eta = 0$ in *Marguerre* [**98**].

$$\boldsymbol{F} = \frac{s^3}{3k}[2k'\boldsymbol{K} - (2 - k^2)\,\boldsymbol{E}]. \tag{V.39}$$

In this, $\boldsymbol{K} \equiv \boldsymbol{K}(k)$ and $\boldsymbol{E} \equiv \boldsymbol{E}(k)$ mean the complete elliptic integrals of genus 1 and 2, respectively. Moreover, one has:

$$k \equiv \frac{4\rho R}{s^2}, \qquad s^2 \equiv z^2 + (R + \rho)^2, \qquad k'^2 \equiv 1 - k^2. \tag{V.40}$$

As one easily verifies, the stress functions of eq. (II.107):

$$\chi'_{\rho\rho} = \frac{b}{4\pi}\frac{\boldsymbol{F}}{\rho}, \qquad \chi'_{\varphi\varphi} = \frac{b}{4\pi}\frac{\partial \boldsymbol{F}}{\partial \rho} \tag{V.41}$$

will follow from $\boldsymbol{F}$, while the remaining components of χ' will vanish. This equation is true for the case in which the *Burgers* vector (magnitude b) of the dislocation points in the z-direction, and it only is in that case that the problem will be rotationally-symmetric.

The author has treated an arrangement of parallel, equidistant, circular dislocations by starting from eq. (V.41) [**79**]. This arrangement is very similar to the well-known current coil in electrical engineering. If one takes the coil to be very long in comparison with the radius R then one can employ the same approximation that one uses for the current coil, so one will obtain:

$$\sigma_{\rho\rho} = \sigma_{\varphi\varphi} = \frac{-G}{m-1}\,\nu\, b, \qquad \sigma_{zz} = \frac{-2Gm}{m-1}\,\nu\, b \tag{V.42}$$

inside of the coil and:

$$\sigma_{\rho\rho} = -\,\sigma_{\varphi\varphi} = \frac{-G}{m-1}\,\nu\, b\,\frac{R^2}{\rho^2} \tag{V.42′}$$

outside of it (ν = number of windings per unit length). All of the remaining components will vanish in this approximation. The external state is precisely the state that was written above with $\chi'_+ = 0$. The energy per unit of coil volume will then be:

$$e = \frac{mG}{m-1}\,\nu^2\, b^2. \tag{V.43}$$

There are no major difficulties associated with treating this problem exactly with the use of elliptic integrals ([1]).

Calculations of this kind are significant for some problems in the physics of metals. One sees the essence of them in the following problem: In the cold hardening of the very

([1]) One can attribute the stresses in the known problem of the stretching of a hollow cylinder onto a solid cylinder with a somewhat larger radius to an arrangement of dislocations in the boundary surface. However, from eq. (I.77) these dislocations will move in the z-direction and have their *Burgers* vector in the φ-direction. Our problem will then correspond to the welding together of two cylinders, as above, but where the inner one is elastically-extended in the longitudinal direction compared to the external one.

important aluminum-copper alloy (Duraluminum), there are, *inter alia*, the two following states: One has superficial accumulations of Cu atoms in the {100} planes of the Al lattice, and in fact, possible monoatomic layers of copper. These are distributed statistically in one state (which is then called the *Guinier-Preston zone* I), and in the other one they are arranged into a number of complexes that are parallel to them (viz., *Guinier-Preston zones* II) ([1]).

One can describe the process of Cu enrichment by saying that a partial net plane of atoms is taken out of the Al lattice and replaced with Cu atoms. The Cu layer will now be "thinner," so, in effect, a layer is missing whose density is equal to the difference between the net plane separation in Al (d^{Al}) and Cu (d^{Cu}). Meanwhile, the connection will be once more represented by the atomic forces of cohesion, so one will obtain elastic reactions, and the Cu layer will act like a dislocation line with a *Burgers* vector of magnitude $b = d^{Al} - d^{Cu}$ in the z-direction, if that direction is perpendicular to the layer. One now treats the "quasi-dislocation" (they are not complete step dislocations, since the net plane does not, in fact, belong to them) as being approximately circular. Experimentally, one will observe that (at least, in certain temperature regimes) they do, in fact, arrange themselves into a "coil." Since the *Burgers* vectors of the individual dislocations run parallel to each other, one should expect that the opposite situation will be more likely, since dislocations of the same type with parallel *Burgers* vectors will repel each other, according to § 18.

One can understand the converse effect, according to *Franz* and *Kröner* [**53**], by saying that perhaps one of the intermediate Cu layers of the complex will be bounded by a complete dislocation with an opposite *Burgers* vector; i.e., this layer will not advance like an Al net plane from the outside. Since the *Burgers* vector of this dislocation will be essentially larger than that of the quasi-dislocation, it can attract a large number of quasi-dislocations, until the sum of all *Burgers* vectors of the complex will be zero. Thus, the far-reaching, energy-consuming, stress field will be formed. As a probable arrangement of the layers in the *Guinier-Preston* zones II, one has today: Any fourth net plane is a copper-rich one ([2]). This will yield six as the number of layers in a complex [**53**], so the vertical arrangement of a zone is equal to 21 net planes, corresponding to perhaps over 40Å, which is satisfactory agreement with the experiments that can, at present, be interpreted only with coarse precision.

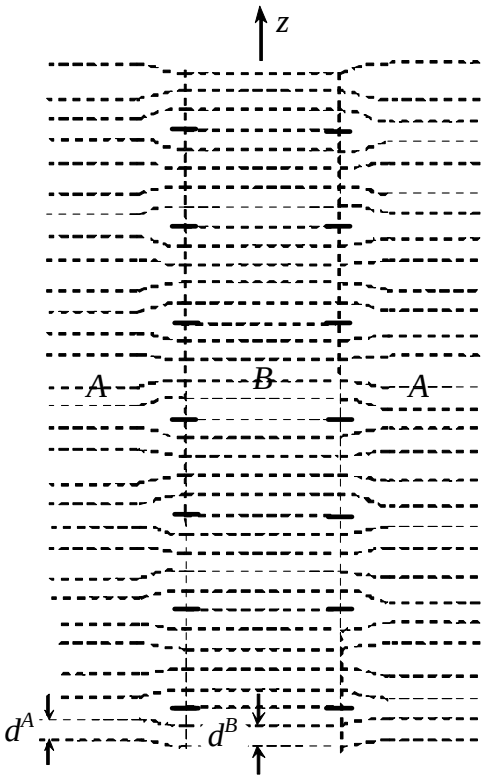

Fig. 38. Phase boundaries as superficial arrangements of dislocations. Dashed lines: net planes. (Longitudinal section)

([1]) Cf., e.g., *Gerold* [**59**] or *Hardy* and *Heal* [**63**].

([2]) If only every second atom in a layer is a Cu atom then that would roughly double the vertical extension. Experimentally, this has still not been decided to date.

One can also treat the phase boundaries between two pure metals *A* and *B* in an analogous way. For the sake of simplicity, we assume that the lattice constants of *A* and *B* differ only in the *z*-direction and define the phase *B* inside of *A* to be a circular cylinder with its axis in the *z*-direction. If the aforementioned lattice constant d^B of *B* is smaller that that of *A* then from time to time a net plane must belong to *B*, since in the other case, one would obtain the larger energy (V.43). One can again regard each net plane of *B* as a quasi-dislocation with a *Burgers* vector with magnitude $d^A - d^B$. If, e.g., $d^A - d^B = d^B / 5$ then a complete dislocation must always appear after five net planes. Thus, one can describe this phase boundary by the arrangement of dislocations and quasi-dislocations that is indicated in Fig. 38. Whereas the dislocations and quasi-dislocations are strong sources of proper stresses (§ 23) in their own right, both of them will act together essentially as a surface distribution of "dislocation dipoles" (or "incompatibility quadrupoles," § 23), so their elastic effects, and therefore their elastic energy, will be small. If one would like to calculate them, then one would have to solve a boundary-value problem that relates to the boundary surface and then consider the different elastic constants inside and outside.

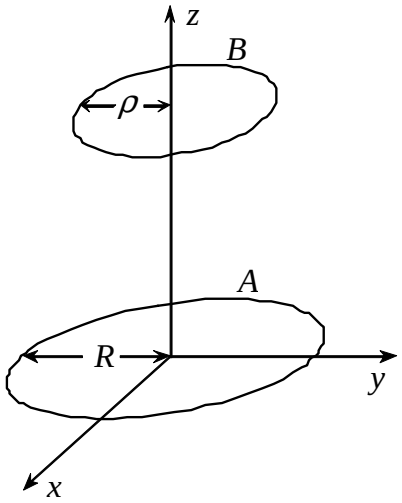

Fig. 39. The calculation of the interaction energy of circular dislocations.

In order to calculate the energy gain during the transition from the *Guinier-Preston* zones I to the zone II, naturally, formula (V.43) will no longer suffice. To that end, *Kröner* and *Franz* [**53**] have calculated the interaction energy of two coaxial circular dislocations with *Burgers* vectors that are perpendicular to the dislocation planes exactly with the use of elliptic integrals using formula (II.128). *Pfleiderer* [**117**] has treated the intersection of circular dislocations somewhat more generally, and likewise started with eq. (II.128). We summarize these results, which are presently surprisingly simple, as the following:

Let:

$$\boldsymbol{F}' \equiv -\left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}-\frac{1}{\rho^2}\right)\boldsymbol{F}$$

$$= -\frac{k^2}{2\rho\sqrt{\rho k}}\left\{2(\rho^2+R^2)\boldsymbol{K}-\left[(\rho+R)^2+\frac{(R-\rho)^2}{k'^2}\right]\boldsymbol{E}\right\}, \qquad \text{(V.44)}$$

$$\boldsymbol{F}'' \equiv \frac{\partial^2\boldsymbol{F}}{\partial z^2} = -2\sqrt{\frac{R}{\rho}}\left\{\left[\frac{2}{k}\left(1+\frac{z^2}{s^2}\right)-k\right]\boldsymbol{K}-\left[\frac{2}{k}\left(1+\frac{z^2}{s^2}\right)+\frac{k\,z^2}{k'^2 s^2}\right]\boldsymbol{E}\right\}. \qquad \text{(V.44}'\text{)}$$

Thus, from *Pfleiderer*, one will have for two dislocations *A* and *B* in the arrangement of Fig. 39:

$$E^{AB} = H_{11} + H_{22} + H_{33}\,, \qquad \text{(V.45)}$$

where:

$$H_{11} = \frac{G}{8} b_1^A b_1^B \rho \left(-\frac{3m-1}{m-1} \boldsymbol{F}'' + \boldsymbol{F}' \right), \qquad H_{33} = \frac{-m}{2(m-1)} G b_3^A b_3^B \rho \boldsymbol{F}'. \quad \text{(V.46)}$$

H_{22} follows from H_{11} by the replacement of $b_1^A b_1^B$ with $b_2^A b_2^B$. One should now understand ρ to mean the radius of the dislocation *B*. b_i^A (b_i^B, resp.) are the Cartesian components of the *Burgers* vectors of the lines *A* and *B*. The formulas simplify essentially when both rings lie in the plane *z* (= x_3) = 0, or when both of them have the same radius. One will then have:

$$\left.\begin{aligned}
H_{11}\big|_{\rho=R} &= \frac{G b_1^A b_1^B R}{2(m-1)k} \left\{ [(1-4m)k^2 + 2(3m-1)] \boldsymbol{K} - [-mk^2 + 2(3m-1)] \boldsymbol{E} \right\}, \\
H_{33}\big|_{\rho=R} &= \frac{mG}{m-1} b_1^A b_1^B R\, k\, [\boldsymbol{K} - \boldsymbol{E}], \\
H_{11}\big|_{z=0} &= \frac{2m-1}{2(m-1)} G b_1^A b_1^B (R+\rho) \left[\left(1 - \frac{k^2}{2} \right) \boldsymbol{K} - \boldsymbol{E} \right], \\
H_{33}\big|_{z=0} &= \frac{m}{m-1} G b_1^A b_1^B (R+\rho) \left[\left(1 - \frac{k^2}{2} \right) \boldsymbol{K} - \boldsymbol{E} \right].
\end{aligned}\right\} \quad \text{(V.47)}$$

It follows from eq. (V.45, 46) that dislocations whose *Burgers* vectors are perpendicular to each other do not affect each other. We already obtained this result in § 18 for straight dislocations. For the aforementioned problem, the energy of two dislocations with $\rho = R$ and a *Burgers* vector (magnitude *b*) in the *z*-direction will be important. One refers then to:

$$\frac{m}{m-1} G\, b^2\, R\, k\, (\boldsymbol{K} - \boldsymbol{E}). \quad \text{(V.48)}$$

For other purposes, the case in which both dislocations have their *Burgers* vectors in the x_1-direction is especially interesting and lie in the plane $z = 0$ (this is then the glide plane, so the relevant formulas can serve for the calculation of the energy of concentrated dislocations). One will then have the likewise-very-simple expression:

$$\frac{2m-1}{2(m-1)} G b^2 (R+\rho) \left[\left(1 - \frac{k^2}{2} \right) \boldsymbol{K} - \boldsymbol{E} \right]. \quad \text{(V.49)}$$

If one makes the distances between the dislocations very small here – perhaps, equal to twice the length ε that was cut away in formula (II.145) – then one will also obtain the self-energy of the dislocation in the approximation that was described in § 18 and § 25. This self-energy was first calculated by *Nabaro* [**110**] for the dislocation that was implied in (V.49) in a different way [starting with eq. (II.122)]. One will have $k \approx 1$ in that case

(when $\varepsilon \ll R$), so one will get the approximation formulas $\boldsymbol{E} \approx 1$, $\boldsymbol{K} \approx \ln(4/k')$ [**69**], and will thus obtain:

$$\frac{2m-1}{2(m-1)} G b^2 R \left(\ln \frac{4R}{\varepsilon} - 2 \right) \tag{V.50}$$

for the self-energy of this dislocation, in agreement with *Nabarro*.

All of these energy formulas were calculated upon the foundation of an infinitely-extended medium. Such a dislocation loop will act like a force dipole at great distances; i.e., the far-reaching displacement field of the dislocation will go like $1 / r^2$ ($r \equiv$ distance from the origin), so the stress field will go like $1 / r^3$, and the energy density $\varepsilon_{ij}\, \sigma_{ij} / 2$ will go like $1 / r^6$. The component of the energy in the infinite medium that is localized outside of a sphere of radius r_0 will then go like $1/r_0^3$. Thus, when the dimensions of a body are sufficiently large compared to R, which will be true in almost all applications, one will not need to observe its outer surface – i.e., the formulas above will also give the energy for the finite medium with practically the same approximation.

The stress function tensor was only indirectly involved in these energy calculations [indeed the basic formula (II.128) was derived with its help]. In conclusion, we shall give the formulas for the dislocation with a *Burgers* vector in the circle plane (i.e., the glide plane) that are derived from the stress function field $\boldsymbol{\chi}'$ (*Keller*, [**70**]). Let $(x, y, z) \equiv (x_1, x_2, x_3)$. One will then have [cf., (V.40)]:

$$\left.\begin{aligned}
\sigma_{11} &= \alpha \frac{m}{m-1} (3B - 2C + Dx^2) xz, \\
\sigma_{22} &= \alpha \left[2C + \frac{m}{m-1} (B - 2C + Dy^2) \right] xz, \\
\sigma_{33} &= \alpha \frac{m}{m-1} (C + Fz^2) xz, \\
\sigma_{12} &= \alpha \left[-C + \frac{m}{m-1} (B + Dy^2) \right] yz, \\
\sigma_{23} &= \alpha \left[-B + \frac{m}{m-1} (B + Ez^2) \right] xy, \\
\sigma_{31} &= \alpha \left\{ A + By^2 + \frac{m}{m-1} [A + Cz^2 + (B + Ez^2) x^2] \right\},
\end{aligned}\right\} \tag{V.51}$$

$$\left.\begin{aligned} A &\equiv \frac{1}{\rho s k^2}[2\boldsymbol{E} - (2-k^2)\boldsymbol{K}), & B &\equiv -\frac{1}{2\rho^2}(a^2 C + 3A), \\ C &\equiv \frac{1}{\rho s^6 k\, k'^2}\boldsymbol{F}, & D &\equiv -\frac{1}{\rho^2}(4B + C + Fz^2), \\ E &\equiv -\frac{1}{2\rho^2}(a^2 F + 5C), & F &\equiv -\frac{1}{\rho s^5 k'^4}(\rho s k'^2 A - 2k^2 \boldsymbol{E}), \\ \alpha &\equiv -RbG/\pi, & a^2 &\equiv z^2 + R^2 - \rho^2. \end{aligned}\right\} \qquad \text{(V.51′)}$$

One observes: The stresses are not rotationally-symmetric, so the problem that is solved here with the aid of *Keller's* stress function tensor will be truly three-dimensional.

All that remains of these stresses at the origin will be:

$$\sigma_{31} = \frac{Gb}{4}\frac{2m-1}{m-1}\frac{1}{R}. \qquad \text{(V.52)}$$

APPENDIX

The decomposition of a tensor field of rank 2

We apply the vector symbolism of Gibbs [1] piecewise, so we mostly calculate in the traditional index notation, including the Einstein summation convention [2]. I. e.:

$$\left.\begin{array}{lll} \mathbf{ab} & \text{or } a_i b_j & \text{is the dyadic} \\ \mathbf{a}\cdot\mathbf{b} & \text{or } a_i b_i & \text{is the scalar} \\ \mathbf{a}\times\mathbf{b} & \text{or } \varepsilon_{ijk} a_j b_k & \text{is the vectorial} \end{array}\right\} \text{ product of two vectors } \mathbf{a} \text{ and } \mathbf{b}.$$

Therefore, one has $\varepsilon_{123} = \varepsilon_{231} = 1$, $\varepsilon_{132} = \varepsilon_{321} = \varepsilon_{213} = -1$, while all of the remaining components of the totally anti-symmetric ε-tensor vanish. The following formulas will be used frequently ([34], Bd. I, pp. 74):

$$\varepsilon_{ijk}\,\varepsilon^{lmn} = \begin{vmatrix} \delta_i^l & \delta_i^m & \delta_i^n \\ \delta_j^l & \delta_j^m & \delta_j^n \\ \delta_k^l & \delta_k^m & \delta_k^n \end{vmatrix}. \tag{A.1}$$

From this, it follows for $n = k$ that:

$$\varepsilon_{ijk}\,\varepsilon^{lmk} = \delta_i^l \delta_j^m - \delta_i^m \delta_j^l, \tag{A.2}$$

and when one also has $m = j$:

$$\varepsilon_{ijk}\,\varepsilon^{lmk} = 2\delta_i^l. \tag{A.3}$$

Let [3]:

$$\left.\begin{array}{lll} \text{Grad}\,\mathbf{a} & \equiv \nabla\mathbf{a} & \equiv (\nabla_i a_j) \\ \text{Div}\,\boldsymbol{\tau} & \equiv \nabla\cdot\boldsymbol{\tau} & \equiv (\nabla_i \tau_{ij}) \\ \text{Rot}\,\boldsymbol{\tau} & \equiv \nabla\times\boldsymbol{\tau} & \equiv (\varepsilon_{ijk}\nabla_j \tau_{kl}). \end{array}\right\} \tag{A.4}$$

(One reads "gradient of **a**," "divergence of $\boldsymbol{\tau}$," and "rotation of $\boldsymbol{\tau}$.")

One can uniquely decompose any tensor field $\boldsymbol{\tau}$ that vanishes at infinity in an infinitely-extended medium according to the formula:

$$\boldsymbol{\tau} = \nabla\mathbf{a} + \nabla\times\boldsymbol{\alpha}, \tag{A.5}$$

where $\boldsymbol{\alpha} \equiv (\alpha_{ij})$. Likewise, for an arbitrary tensor $\boldsymbol{\alpha}$, one has the unique decomposition:

$$\boldsymbol{\alpha} = \mathbf{b}\nabla + \boldsymbol{\beta}\times\nabla, \tag{A.6}$$

---

1 This was briefly advocated by the International Union for Pure and Applied Physics [67].

2 One finds this notation publicized in the books of Duschek and Hochrainer [34], especially.

3 We write the first symbols large in order to imply that we are concerned with tensor fields.

with $\boldsymbol{\beta} \equiv (\beta_{ij})$. When this is substituted into eq. (A.5), one obtains, with $\nabla \times \mathbf{b} \equiv \mathbf{c}$ (i.e., div $\mathbf{c} = 0$):

$$\boldsymbol{\tau} = \nabla \mathbf{a} + \mathbf{c}\nabla + \nabla \times \boldsymbol{\beta} \times \nabla, \tag{A.7}$$

In this, one has:

$$(\nabla \times \boldsymbol{\beta} \times \nabla)_{il} \equiv \varepsilon_{ijk}\, \varepsilon_{lmn}\, \nabla_j\, \nabla_n\, \beta_{km}\,. \tag{A.8}$$

Symbolically, we also then write [1]:

$$(\mathrm{Ink}\, \boldsymbol{\beta})_{il} \tag{A.9}$$

(read "the incompatibility of $\boldsymbol{\beta}$"). In the event that $\boldsymbol{\beta}$ is symmetric, one can switch i and l in (A.8), such that:

$$\nabla \times \boldsymbol{\beta}^S \times \nabla = (\nabla \times \boldsymbol{\beta} \times \nabla)^S, \tag{A.10}$$

where S means that the symmetric part is to be taken. If $\boldsymbol{b}$ is anti-symmetric then one can switch i and l in (A.8) with a change of sign; i.e.:

$$\nabla \times \boldsymbol{\beta}^A \times \nabla = (\nabla \times \boldsymbol{\beta} \times \nabla)^A, \tag{A.11}$$

where A means "the anti-symmetric part of."

The symmetric part of eq. (A.7) then reads, when we write $\mathbf{a} + \mathbf{c} \equiv \mathbf{g}$, $\mathbf{a} - \mathbf{c} \equiv \mathbf{h}$:

$$\boldsymbol{\tau}^S = \frac{1}{2}(\nabla \mathbf{g} + \mathbf{g}\nabla) + \nabla \times \boldsymbol{\beta}^S \times \nabla, \tag{A.12}$$

and the anti-symmetric part reads:

$$\boldsymbol{\tau}^A = \frac{1}{2}(\nabla \mathbf{h} - \mathbf{h}\nabla) + \nabla \times \boldsymbol{\beta}^A \times \nabla. \tag{A.13}$$

We also write eq. (A.12) symbolically as [2]:

$$\boldsymbol{\tau}^S = \mathrm{Def}\, \mathbf{g} + \mathrm{Ink}\, \boldsymbol{\beta}^S. \tag{A.14}$$

Since (A.5) and (A.6) were unique decompositions, (A.14) is a unique decomposition of a symmetric tensor field. One easily verifies the identity relations:

$$\left.\begin{aligned} \mathrm{Ink\ Def} &\equiv 0 \\ \mathrm{Def\ Ink} &\equiv 0. \end{aligned}\right\} \tag{A.15}$$

---

[1] The notation shall thus recall the fact that Ink $\boldsymbol{\varepsilon} = 0$ are the St. Venant compatibility conditions. They are fulfilled when the "incompatibility of $\boldsymbol{\varepsilon}$" vanishes.

[2] For Def, one reads "deformation of." The notation shall then recall that $\boldsymbol{\varepsilon} = \mathrm{Def}\, \boldsymbol{s}$ is the connection between the deformation $\boldsymbol{\varepsilon}$ and the displacement field $\boldsymbol{s}$ ([52], Bd. I, pp. 97).

This says that a tensor $\boldsymbol{\tau}^S$ that is subject to the restriction Div $\boldsymbol{\tau}^S = 0$ is an *incompatibility* tensor, while it is a *deformator* (and thus derived from a vector field) when one has Ink $\boldsymbol{\tau}^S = 0$. The meaning of the operations Ink and Def for elasticity theory lies in the fact that the state of an elastic body that is stressed only at the boundary is completely determined by the equations:

$$\text{Ink } \boldsymbol{\varepsilon} = 0, \qquad \text{Div } \boldsymbol{\sigma} = 0, \tag{A.16}$$

in which the Hooke law and the equation for the elastic energy density are assumed.

In eq. (A.13), one can replace $\boldsymbol{\beta}^A$ (like any anti-symmetric tensor [34]) by an equivalent vector according to the formula:

$$\beta_{ij}^A = \varepsilon_{ijk}\beta_k^A, \quad \beta_k^A = \frac{1}{2}\varepsilon_{ijk}\beta_{ij}^A. \tag{A.17}$$

Thus, it follows from this by a routine calculation that:

$$\tau_{ij}^A = \varepsilon_{ijk}(\varepsilon_{klm}\nabla_l h_m + \nabla_k \lambda), \qquad \lambda \equiv -\nabla_l \nabla_i \beta_i^A, \tag{A.18}$$

or, corresponding to eq. (A.17):

$$\tau_k^A = \varepsilon_{klm}\nabla_l h_m + \nabla_k \lambda \equiv (\text{rot } \mathbf{h} + \text{grad } \lambda)_k. \tag{A.19}$$

I.e., the decomposition of the anti-symmetric tensor field corresponds to the well-known decomposition of the associated vector field into a source field and a vortex field.

In eq. (A.5), one can add a gradient tensor to $\boldsymbol{\alpha}$ without changing $\boldsymbol{\tau}$ [1]. Correspondingly, one can add a deformator to $\boldsymbol{\beta}^S$ in eq. (A.14) without changing $\boldsymbol{\tau}^S$. As a result, one must impose certain auxiliary conditions on $\boldsymbol{\alpha}$ in eq. (A.5) ($\boldsymbol{\beta}^S$ in eq. (A.14), resp.). E. g., Div $\boldsymbol{\alpha} \equiv 0$ (Div $\boldsymbol{\beta}^S = 0$, resp.) is always a "supplementary" condition; i.e., one can represent any $\boldsymbol{\tau}$ ($\boldsymbol{\tau}^S$, resp.) by eq. (A.5) ((A.14), resp.) when $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}^S$ are subjected to the stated restrictions [77]. If $\boldsymbol{\gamma}$ and $\mathbf{q}$ are the incompatibilities (sources, resp.) of $\boldsymbol{\tau}^S$ then in the case Div $\boldsymbol{\beta}^S = 0$ one obtains from eq. (A.14), as one easily checks:

$$\text{Ink } \boldsymbol{\tau}^S = \text{Ink Ink } \boldsymbol{\beta}^S = \Delta\Delta\boldsymbol{\beta}^S = \boldsymbol{\gamma}. \tag{A.20}$$

From this, the associated $\boldsymbol{\beta}^S$ follows uniquely from:

$$\boldsymbol{\beta}^S = -\frac{1}{8\pi}\iiint_\infty \boldsymbol{\gamma}(\mathbf{x}')\,|\,\mathbf{x}-\mathbf{x}'\,|\,dV', \tag{A.21}$$

up to an uninteresting function that depends upon $\mathbf{x}$ linearly [2].

On the other side, it follows from eq. (A.14):

---

[1] Naturally, one has the identities Rot Grad ≡ 0, Div Rot ≡ 0.

[2] We recall: $\boldsymbol{\tau}^S$ shall vanish at infinity. One easily verifies that (A.21) fulfills the auxiliary condition.

$$\text{Div } \boldsymbol{\tau}^S = (\Delta \mathbf{g} + \nabla\nabla \cdot \mathbf{g})/2 = \mathbf{q}. \tag{A.22}$$

By repeated application of the divergence, one obtains:

$$\Delta \text{ div } \mathbf{g} = \text{div } \mathbf{q}, \tag{A.23}$$

from which div **q** follows, up to a constant. Thus, one can easily obtain **g** from eq. (A.22), up to an uninteresting constant. It is thus shown how the decomposition (A.14) is actually to be performed in an infinite space.

Addendum

We present two more theorems about a medium that contains only proper stresses.

1. One has, for arbitrary elastic homogeneity and anisotropy:

$$\iiint_V \sigma_{ij} dV = 0,$$

which is integrated over the entire volume (that is in the proper stress state).

2. The total change in volume of the medium for a nonlinear law of elasticity is:

$$\Delta V = t_{ijkl} \iiint_V \sigma_{ij}\sigma_{kl} dV + \text{terms of higher order}$$

with the material constants:

$$t_{ijkl} \equiv \frac{1}{2} \frac{\partial^2 \Theta}{\partial \sigma_{ij} \partial \sigma_{kl}}\bigg|_{\sigma=0}.$$

Theorem 1 follows from equilibrium considerations [179]. Theorem 2 follows from Theorem 1 when one develops the differential change of volume Θ in powers of σ_{ij}. Theorem 2 would change somewhat in comparison with the particular form that were found and tested by the current experiments of Zener [178] for elastic isotropy, extended by Seeger [176] to cubic crystal symmetry, and applied to dislocations. The tensor t_{ijkl} has the same symmetry and number of components as the elasticity tensor c_{ijkl} for the medium in question.

The "volume theorem" of Colonetti that was stated in § 1 follows immediately from Theorem 1 by applying Hooke's law.

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