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The fundamental connection between dislocation density and stress functions

By

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Abstract: It will be shown that asymmetric distortion tensors ε (but only symmetric stress tensors σ) must be employed for the complete description of the state of a body that is endowed with dislocations that are distributed in the form of a tensor density α in the given case. The antisymmetric part of e describes lattice rotations (curvatures, resp.), as one might verify Röntgenographically for, e.g., plastically-formed media. For the elastic-theoretic treatment of this, the ST. VENANT compatibility conditions must be replaced with a new equation of very general validity [eq. (11)]. The stress function tensor φ that is defined by the equations σ = Rot φ, Div φ = 0 can be interpreted as a (tensorial) potential for dislocations. φ is especially suited to the calculation of the stress that are coupled to a dislocation distribution α. A summary of the most important concepts and discussion will be given in § 6.

§ 1. Introduction. Previous results.

In present paper, stress states will be considered that come about without the influence of forces; in other words, ones for which one has Div σ = 0 when $\sigma(\mathfrak{r})$ is the stress tensor. The topic in the theory of elasticity that is of issue will generally be referred to as the "theory of internal stresses" or also the "theory of proper stress" ([1]). Various influences come under consideration as the origins of such stresses – e.g., magnetostriction, defect arrangements in lattice structures of crystals, as might perhaps be produced by plastic deformation, temperature fluctuations, *et al.* The elastic effect of all these influence can be described by the so-called "incompatibility tensor" (η), which can also be regarded as the origin of the internal stresses, in its own right. The determination of the internal stresses decomposes into two parts: First, one must ascertain the incompatibility tensor η as a function of the position $\mathfrak{r}$ from the physical givens. This is a problem of a predominantly physical nature. Once it is solved, one will then come to the mathematical part, namely, the determination of the internal stresses from the now-given incompatibility tensor. This problem can always be resolved with the help of the spatial stress functions that were introduced by the author ([1]).

([1]) For recent papers on this subject, cf., e.g., J. D. ESHELBY, Phil. Trans. Roy. Lond., Ser. A **244** (1951), 87 and E. KRÖNER, Z. angew. Phys. (1955), 249.

The following considerations are, above all, concerned with the physical questions in the theory of internal stresses. ESHELBY ([1]) has treated the most general case of internal stresses (spatially-distributed incompatibilities) and confirmed the physical reality of the incompatibilities. NABARRO ([2]) recognized that such an incompatibility distribution is equivalent to a continuous distribution of dislocations. The precise connection, however, remains open. The author ([1]) showed that every discrete dislocation line is associated with a well-defined incompatibility. The relevant formula reads ([*]):

$$\boldsymbol{\eta} = \tfrac{1}{2}(\mathfrak{t}\ \mathfrak{b} \times \nabla - \nabla \times \mathfrak{b}\ \mathfrak{t}), \tag{1}$$

where $\mathfrak{t}$ is the unit tangent vector to the line in question and $\mathfrak{b}$ is the associated BURGERS vector. $\mathfrak{t}$ will be differentiated ([**]).

One can thank NYE ([3]) for some further results. He considered families of discrete dislocation lines in crystals that were laid so close to each other that it became reasonable to take the mean over a large number of them. NYE's results – to the extent that they are of interest to us here – read: Any such distribution of dislocation lines can be described by an asymmetric position-dependent tensor (α) of rank 2. A well-defined average curvature of the lattice of the crystal in question is linked with every distribution of dislocations. It will not change when one increases the density of the dislocations (say, n_i per cm^2 and BURGERS vector $\mathfrak{b}_i$) and simultaneously decreases the BURGERS vector $\mathfrak{b}_i$ in such a way that the product $n_i\ \mathfrak{b}_i$ remains constant. When one lets the $\mathfrak{b}_i$ go to 0 in this way, the elastic energy will vanish for entirely specialized dislocation distributions, while the lattice curvature will remain the same. The ambition to define such arrangements of dislocations led, e.g., to the known phenomena of polygonization and the formation of fine-grain boundaries. NYE could give the connection between the dislocation tensor α and the lattice curvature tensor $\kappa(\mathfrak{r})$ for such minimal-energy arrangements of dislocations. It read ([***]):

$$\kappa = \alpha - \tfrac{1}{2}\,\alpha_I\, I. \tag{2}$$

In this, α_I is the scalar (tensor trace) of α, and I is the unit tensor of rank 2. The curvature tensor κ will be defined by the equation $d\boldsymbol{\Phi} = \kappa \cdot d\mathfrak{r}$, where $\boldsymbol{\Phi}$ is the axial vector that described the rotation of the individual volume elements. The definition:

$$\kappa \equiv \boldsymbol{\Phi}\ \nabla \tag{3}$$

([2]) F. R. N. NABARRO, Adv. Phys. **1** (1952), 269.

([*]) The notations of M. LAGALLY (*Vorlesungen über Vektorrechnung*, Leipzig, Akademische Verlagsgesellschaft, 1928) will be employed for the requisite tensor analysis.

([**]) Cf., the Appendix for this.

([3]) J. F. NYE, Acta Met. **1** (1953), 153.

([***]) More precisely, NYE employed a tensor that was the transpose of α, although it meant the same thing physically.

is equivalent to it. NYE's results lie outside of the previously-developed theories of internal stresses (*). In a sufficiently general theory, however, they should be obtained in precisely the same way as the determination of the proper stresses. Such a theory must produce the connection between the dislocation density α and curvature κ for arbitrarily general dislocation distributions, and thus not just for ones of minimal elastic energy, and at the same time produce the associated stresses. A theory that does that much will be given in what follows.

§ 2. The basic equations.

For the presentation of the theory, we let ourselves be guided by the author's (**) rigorously-exhibited analogy between the theory of internal stresses and the theory of the magnetic fields of stationary currents. If one first considers a linear current in the latter theory then one can characterize it by the length L along which it flows and its associated current strength i. Upon going to continuously-distributed currents, one proposes that the number of linear currents must continually increase while the current strength simultaneously decreases in such a way that the total current will remain finite everywhere. One arrives at the concept of the current density $\mathfrak{j}$ in that way, which is defined, perhaps, by one of the two equations:

$$\mathfrak{j} = di \,/\, d\mathfrak{f}, \qquad i = \iint_F d\mathfrak{f} \cdot \mathfrak{j},$$

where $d\mathfrak{f}$ means any vectorial surface element. The right-hand equation gives the total current flux i that goes through an arbitrary surface F. Now, it is known that the current that goes through all possible surfaces F will be the same as long as these surfaces all have the same boundary R. For that reason, the surface integral must be converted into a line integral. For this, it is necessary and sufficient that $\mathfrak{j}$ be a rotor, so div $\mathfrak{j} = 0$.

In a completely corresponding way, we go from an individual dislocation line L with a BURGERS vector $\mathfrak{b}$ to the dislocation density α, which we define through one of the equations:

$$\alpha = d\mathfrak{b} \,/\, d\mathfrak{f}, \qquad \mathfrak{b} = \iint_F d\mathfrak{f} \cdot \alpha \,. \tag{4}$$

The dislocation density $\alpha(\mathfrak{r})$ will be described by an asymmetric tensor field of rank 2. The right-hand equation in (4) is closely related to the vector $\mathfrak{b}$ that is called the "BURGERS vector" in the case of discrete dislocation lines, and is given the name of *dislocation flux*. Should this flux have be the same for all surfaces F that have the same boundary, then the surface integral in (4) could be converted into a line integral. However, α must then be a rotor tensor – i.e., it must have the form Rot $\beta \equiv \nabla \times \beta$. It follows from this that Div $\alpha \equiv \nabla \cdot \alpha = 0$. This equation also appeared in NYE.

(*) Cf., the cited papers of ESHELBY and KRÖNER.
(**) See footnote 1, pp. 1.

We are now close to replacing the second basic equation of magnetic field theory – viz., rot $\mathfrak{H} = \mathfrak{j}$ – with the second basic equation of the theory of internal stresses:

$$\text{Rot}\,\varepsilon = \alpha. \tag{5}$$

[The first basic equation is known to be div $\mathfrak{B} = 0$, (Div $\sigma = 0$, resp.)] Now, from a decomposition formula that was recently treated quite rigorously by the author ([4]), one can decompose the symmetric tensor ε into a deformation tensor $\varepsilon_1 = \frac{1}{2}(\nabla\mathfrak{s} + \mathfrak{s}\nabla)$ and an incompatibility tensor $\varepsilon_2 = \nabla \times \iota \times \nabla$. $\mathfrak{s}(\mathfrak{r})$ is therefore the displacement vector field in the special case of $\iota = 0$. Whereas ε_2 will be zero wherever dislocations are present, Rot ε_1 will also be non-vanishing outside of the dislocation lines. Eq. (5) cannot be correct then, due to the physical meaning of α.

As our next possibility, we offer the equation:

$$\text{Rot}\,\varepsilon_2 = \alpha. \tag{6}$$

This is also in complete analogy with the theory of magnetic fields. Namely, if we decompose $\mathfrak{H}$ into $\mathfrak{H}_1 = \text{grad}\,V$ and $\mathfrak{H}_2 = \text{rot}\,\mathfrak{A}$ then rot $\mathfrak{H} = \mathfrak{j}$, since rot grad $\equiv 0$ is nothing but rot $\mathfrak{H} = \mathfrak{j}$. It will remains for us to establish whether eq. (6) fulfills all requirements. In order to do that, we now ask what one must substitute for the tensor α of an isolated dislocation line (L, $\mathfrak{b}$). As we will soon prove, the answer reads:

$$\alpha = \mathfrak{t}\,\mathfrak{b}, \tag{7}$$

if $\mathfrak{t}$ is the unit tangent vector that belongs to L; the multiplication is intended to be dyadic. Naturally, α will be non-zero only along L here. With (4), one gets:

$$\mathfrak{b} = \iint_F d\mathfrak{f} \cdot \mathfrak{t}\mathfrak{b} = \left(\iint_F d\mathfrak{f} \cdot \mathfrak{t}\right)\mathfrak{b}, \tag{8}$$

in which one can place the constant vector $\mathfrak{b}$ outside of the integral. One must set the remaining integral equal to 1, as one infers most simply from the formulas that are analogous to (8):

$$i = \iint_F d\mathfrak{f} \cdot \mathfrak{j} = \iint_F d\mathfrak{f} \cdot i\mathfrak{t} = \left(\iint_F d\mathfrak{f} \cdot \mathfrak{t}\right) i\,.$$

Since one also has Div($\mathfrak{t}\mathfrak{b}$) = (div $\mathfrak{t}$) $\mathfrak{b}$ = 0, due to the constancy of $\mathfrak{b}$, (7) will obviously give the correct expression.

For the tensor (7), one generally has:

([4]) E. KRÖNER: Z. Physik **139** (1954), 175.

$$\text{Div}\ \tilde{\alpha} = \mathfrak{b} \cdot \nabla\, \mathfrak{t} \neq 0$$

and

$$\alpha_I = \mathfrak{t} \cdot \mathfrak{b} \neq 0$$

now, with $\tilde{\alpha}_{ij} \equiv \alpha_{ji}$. Both statements contradict eq. (6). Since Div $\varepsilon_2 \equiv 0$, this equation next demands that one must also have Div $\tilde{\alpha} = 0$. One further demands that $\alpha_I = 0$, as one will see when one calculates (Rot $\varepsilon)_I$. One will get:

$$(\text{Rot}\ \boldsymbol{\varepsilon})_I = \frac{\partial}{\partial x}(\varepsilon_{yz} - \varepsilon_{zy}) + \frac{\partial}{\partial y}(\varepsilon_{zx} - \varepsilon_{xz}) + \frac{\partial}{\partial z}(\varepsilon_{xy} - \varepsilon_{yx})\,, \qquad (9)$$

and it will follow that (Rot $\varepsilon_2)_I \equiv 0$, since ε_2 is symmetric.

The stated contradiction can be resolved only when one allows an asymmetric distortion tensor ε. The following paragraph shall be directed towards the physical meaning of taking such a step.

§ 3. The asymmetric state of distortion

VOIGT ([5]) has already given a thorough analysis of the asymmetric state of distortion, so we can summarize it briefly here. With VOIGT, we assume that we have an elastic body that is composed of nothing but small elementary masses. Now, a distortion shall mean that these elementary masses are not only displaced with respect to each other, but are also rotated with respect to each other. We shall initially leave open the question of whether this rotation does or does not produce a stress.

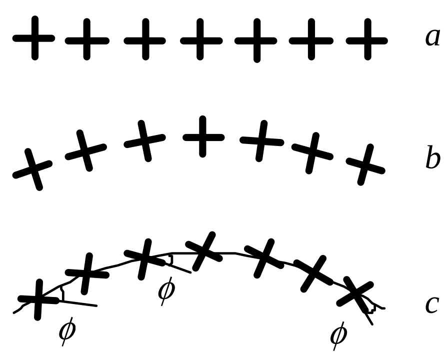

Figure 1a-c. The elementary masses are indicated by crosses, which we imagine are rigidly coupled with the masses.

We consider a sequence of elementary masses that was originally straight (Fig. 1a). Under a displacement of the particles, they will go to a curved line (Fig. 1b). In addition, the particles shall then all be rotated through an angle of $\phi(\mathfrak{r})$ (Fig. 1c). Now, according to VOIGT, the symmetric part of this total distortion $\boldsymbol{\varepsilon}$ will describe the distortion 1b precisely, while the anti-symmetric part will describe the additional distortion – i.e., the rotation in 1c. One observes that the "symmetric" distortion 1b is already linked with a rotation of the elementary masses. This is known from the ordinary theory of elasticity (rot $\mathfrak{s}$). By contrast, what is new and different is the "incompatible" rotation ϕ. The picture that is shown here will also be true for three-dimensional distortions with a corresponding conversion.

([5]) W. VOIGT: *Lehrbuch der Kristallphysik*, pp. pp. 596, *et seq.* Berlin: J. B. Teubner, 1910.

The assertion that we shall now make is that such "incompatible" rotations will exist everywhere there are dislocations. We consider, e.g., a grain boundary according to the well-known model of BURGERS and BRAGG ([6]). Such a grain boundary consists of a sequence of dislocations that act in such a way that two grains will be rotated with respect to each other. Indeed, the places at which elementary masses are rotated with respect to each other will be precisely the ones at which the dislocations are present. The incompatible rotations will be non-zero at these places, and only at them. Fig. 2a shows a crystal in a polygonized state. Here, we have several grain boundaries of the kind that was just described. The number of boundaries has doubled in Fig. 2b, so the strengths of the dislocation lines have been halved, and thus the rotations of the neighboring masses will also be only half as large. If we now think of the grain boundaries are becoming gradually denser then we will gradually come to a continuous distribution of dislocations and a continuous rotation of the elementary masses with respect to each other. Macroscopically, this will already appear to us to be a continuous curvature of the crystal lattice for a not-completely-continuous distribution of dislocations.

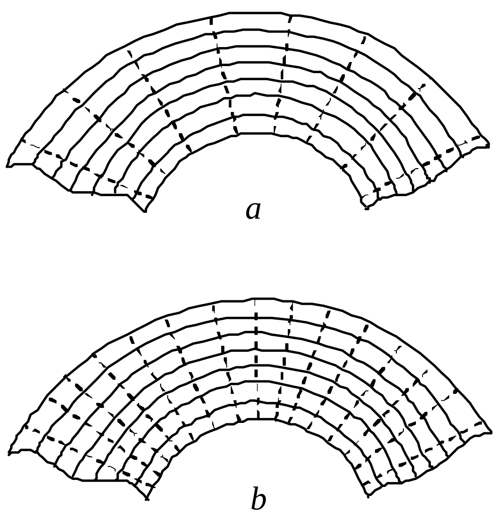

Figure 2a and b. A bent crystal after polygonization. Dashed lines are dislocation walls, solid lines are traces of net planes. The mean curvature of the net planes is the same in both cases.

We have thus arrived at the new and interesting result: The anti-symmetric part of the distortion tensor describes precisely the lattice rotations (lattice curvature, resp.) that are observed on the basis of the presence of dislocations – e.g., on grain boundaries and after polygonization.

The question of whether the lattice rotations produces stresses can also be answered. We must first exclude asymmetric stress tensors, since they contradict the laws of equilibrium in the theory of elasticity. Indeed, asymmetric stress tensors only come under consideration when a distribution of rotational moments acts upon the body *externally*, which is excluded here. The energy density shall also be equal to $\sigma \cdot\cdot \varepsilon/2$ here. Now, twice the scalar product of an anti-symmetric tensor with a symmetric one will always be zero. Therefore, the anti-symmetric part of the distortion – which we would like to call ε_a – will contribute nothing to the energy; i.e., it can also produce no stress.

Now, in order for the stated lattice rotations to actually take place, real dislocations must be present. A body will admit the existence of real dislocations when it responds to an external pressure – at least, partially – with rotations that produce no stresses, and for that reason are also coupled with no elastic energy.

([6]) J. M. BURGERS: Proc. Kon. Ned. Akad. Wetensch. **42** (1939), 293. – W. L. BRAGG: Proc. Phys. Soc. Lond. **54** (1940).

Like any anti-symmetric tensor, ε_a is also equivalent to an axial vector, which we will call $\boldsymbol{\varepsilon}_a$; it is the vector that describes the direction and magnitude of the rotation. The vector $\boldsymbol{\varepsilon}_a$ will then obviously be identical to the NYE rotation vector $\boldsymbol{\phi}$ in eq. (3). We will then have to set:

$$\kappa \equiv \boldsymbol{\varepsilon}_a \nabla \tag{10}$$

for the lattice curvature tensor κ.

§ 4. Incorporation of the previous results.

Now that we have invoked asymmetric distortion tensors, which are obviously physically meaningful, we are finally in a position to formulate the desired basic equations in a manner that is free of contradictions and in complete analogy to the theory of magnetic fields. One will have (*)(**):

$$\mathrm{Rot}\,(\varepsilon_2 + \varepsilon_a) = \alpha. \tag{11}$$

In this paragraph, it will be shown, among other things, that the basic equation (11) is in harmony with NYE's results and with the theory of the calculation of internal stresses that is constructed with the use of the incompatibility tensor. If one introduces $\varepsilon_a \equiv I \times \boldsymbol{\varepsilon}_a$ into eq. (11) then that will yield:

$$\mathrm{Rot}\,\varepsilon_2 + \boldsymbol{\varepsilon}_a\nabla - (\mathrm{div}\,\boldsymbol{\varepsilon}_a)\,I = \alpha, \tag{12}$$

from a known decomposition formula, from which, it will, in turn, follow that $\alpha_I = -2$ div $\boldsymbol{\varepsilon}_a$. If one then eliminates div $\boldsymbol{\varepsilon}_a$ from (12) and considers (10) then one will get:

$$\mathrm{Rot}\,\varepsilon_2 + \kappa = \alpha - \tfrac{1}{2}\,\alpha_I\,I. \tag{13}$$

This is NYE's eq. (2), up to the first term. Now, since NYE expressly referred to distributions of dislocations that had minimal energy – which are obviously ones for which Rot ε_2 will vanish in the mean – (13) will not contradict (2), if one naturally assumes that there is actually such a distribution of dislocations, to begin with. However, the last question is answered in the affirmative, since α alone is subject to the restriction that div $\alpha = 0$.

(*) The conclusion of eq. (11) is also compulsory in the absence of any consideration of analogues when one demands linearity – i.e., when one restricts oneself to small deformations and assumes that the dislocation density is obtained from any sort of distortion quantities by a first-order differentiation. Since rot $\mathfrak{s}$ will drop out of eq. (11) for the same reason as ε_1, a linear combination of ε_2 and ε_a will remain in the bracket in (11). The fact that it will be precisely $\varepsilon_2 + \varepsilon_a$ will follow from the agreement with previously-proved eqs. (2) and (15). $\varepsilon_2 + \varepsilon_a$ will contain six independent components, as it must, and thus the same number as α. One concludes the fact that the stated assumption is fulfilled from the fact that the distortion field of an isolated straight dislocation line will vanish at infinity like $1 / \rho$ (ρ = distance from the dislocation line).

(**) **Remark by the editor:** B. A. BILBY has just given a formulation that is apparently equivalent to eq. (11). (Rep. Conf. Def. in Cryst. Solids, Bristol, 1955, pp. 124).

NYE's dislocation state can then be defined by the equation:

$$\alpha_{\text{NYE}} = \text{Rot}\ \varepsilon_a\ ,$$

in which Rot ε_a is to be considered as a spatial mean for discontinuous distributions of dislocations. Eq. (2) will yield the associated average curvature.

The stresses for an isolated dislocation line can be ascertained with the help of the incompatibility tensor (1). If we introduce $\mathfrak{t}\,\mathfrak{b} = \alpha$ into (1) then we will get an expression that is also true for continuous distributions of dislocations, as would emerge from the following:

$$\eta = \tfrac{1}{2}(\alpha\times\nabla - \nabla\times\tilde{\alpha}) = -SY(\text{Rot}\,\tilde{\alpha}) \tag{14}$$

with $SY \equiv$ "symmetric part of." One next convinces oneself by substitution in (14) that the Rot ε_a part of α contributes nothing to η. Therefore, it will produce no stresses, as was required in § 3. We thus understand why ε_a did not enter into the previous theory of internal stresses. We also recognize that HOOKE's law sufficed in the older form, since only a coupling of the stress tensor with the symmetric part of the distortion tensor is necessary.

Thus, only the Rot ε_a part of α will contribute to the incompatibility, so we are in complete agreement with the theory up to now. Since the incompatibility (Ink) of an anti-symmetric tensor does not vanish, but yields an anti-symmetric tensor, from the earlier theory (*), one must set:

$$\text{Ink}\ \varepsilon_2 \equiv \nabla\times\varepsilon_2\times\nabla = \eta. \tag{15}$$

This equation is identical with eq. (14) when one substitutes α in it from eq. (11). Eq. (15) was discussed thoroughly by the author. Its integration by the author was facilitated especially by the use of the tensor of stress functions χ ([4]), which is introduced by means of the equations $\sigma = \text{Ink}\ \chi$, $(m + 2)\ \text{Div}\ \chi = \text{grad}\ \chi_I$.

§ 5. The integration of the basic equation (11).

This raises the question of whether one cannot employ eq. (11) directly for the calculation of the internal stresses, instead of starting with (15). This is, in fact, often possible.

In the theory of the magnetic fields of stationary currents, the magnetic energy of an infinitely-extended medium is given by $\frac{1}{2}\iiint \mathfrak{j}\cdot\mathfrak{A}\,d\tau$, where $\mathfrak{A}(\mathfrak{r})$ is the vector potential whose introduction will make the basic equation div $\mathfrak{B} = 0$ be satisfied identically.

If one analogously introduces the asymmetric stress function tensor $\varphi(\mathfrak{r})$ with the help of $\sigma = \text{Rot}\ \varphi$, which will then fulfill the basic equation Div $\sigma = 0$, then one will get the elastic energy:

(*) See footnote 1, pp. 1.

$$E = \tfrac{1}{2}\iiint \tilde{\alpha}\cdot\cdot\varphi\, d\tau = \tfrac{1}{2}\iiint \alpha\cdot\cdot\tilde{\varphi}\, d\tau \tag{16}$$

in an infinitely-extended medium. The proof of this follows from the known energy formula $\tfrac{1}{2}\iiint \varepsilon\cdot\cdot\sigma\, d\tau$, in which one introduces Rot φ and – after partial integration – α according to eq. (11). This is possible, since, as one can show, the ε_1 part of this integral contributes nothing.

The tensor φ is closely connected with the symmetric stress function tensor χ that was defined at the end of the last paragraph. One has $\varphi = \chi\times\nabla$, from which, the equations $\varphi_I \equiv 0$ [cf., (9)] and Div $\tilde{\varphi} \equiv 0$ will follow immediately. The auxiliary condition Div $\varphi = 0$ will follow further from the second defining equation for χ.

We conclude from the simple form of the energy equation (16) that φ is especially suited to the calculation of internal stresses for a given dislocation density. In order to formulate the differential equation for φ, we define the tensor $\alpha_2 \equiv$ Rot ε_2 . We will then have:

$$\Delta(\tilde{\varphi} + m\varphi) = -\,2G\,(m + 1)\,\alpha_2$$

in an infinite medium, or:

$$\Delta\varphi = -\frac{2G}{m-1}(m\alpha_2 - \tilde{\alpha}_2)\,. \tag{17}$$

which is equivalent to that. One can convert these equations into the known differential equations (4) for the stress functions χ with the help of eq. (14) and $\varphi = \chi\times\nabla$ and then deduce the proof of eq. (17) backwards from that. The fact that it is, in fact, α_2 that enters into it, and not α, is based in just the fact that only the α_2 part of α will contribute to the stresses.

The determination of α_2 from α is, from (13), usually not possible without an integration that the curvature tensor κ has to supply. We deduce the equation to be integrated from (12) and (13):

$$\Delta\boldsymbol{\varepsilon}_a = \text{Div}\ \tilde{\alpha} - \tfrac{1}{2}\text{grad}\ \alpha_I\,.$$

This integration is inapplicable when the dislocations are distributed in such a way that they collectively create no curvature ($\alpha_2 = \alpha$), which is obviously possible. If one is dealing with a finite body then there will be a boundary-value problem to solve, in addition to the summation problem.

Finally, the possibility of employing φ for the determination of stresses that are produced by boundary forces in a medium is worthy of attention. The equation $\Delta\Delta\varphi = 0$ will be valid for that case.

We shall go into the behavior of bodies of finite extent more thoroughly in another place.

§ 6. Further problem statements.

For many purposes, the theory of internal stresses that is constructed from the incompatibility tensor will suffice in the context that the author (*) has recently described. For example, one can calculate the temperature stresses for a given temperature distribution, the magnetostrictive stresses for a given magnetization (**), and much more. By contrast, that theory does not suffice for most of the problems that are related to plastic deformation. In order to have an example in mind, we imagine any crystalline material as being plastically deformed from a normal state by a certain amount. It will then be found in a completely well-defined state; this state cannot be described completely by the incompatibility tensor. (This says nothing about, e.g., the curvature of the lattice.) However, the statements that the incompatibility tensor does make will also be less intuitive. For that reason, the possibility of replacing that tensor with the very intuitive tensor of dislocation density α is a significant advance. Beyond that, α will describe the state of the medium to a large extent, at least to the extend that one can average the properties of the dislocations that are represented by α over microscopically larger domains. Naturally, one is dealing with mean stresses in the calculation of stresses for an averaged, but in reality discontinuous, distribution of dislocations.

With the knowledge of the tensor α, the formalism of the last paragraph can be brought into play, and the state of the body in isolation can be established. The most difficult, and generally still unsolved, problem is, however, the determination of α from the influences that act upon the body.

The problem that was broached here, namely, that of determining the state of a body as a result of plastic deformation, is the basic problem of macroscopic plasticity theory. The dislocation density tensor did not enter into this theory, up to now. We believe that the introduction of the tensors α and φ into this theory can bring essentially progress. This says, among other things, that the dislocation density is *the natural* quantity for a process in which dislocations play the main role. One must further note that the stress functions also keep their meaning in a plastically-formed medium, since Div $\sigma = 0$, while a displacement field can no longer be defined there. However, the most important thing seems to be that the introduction of dislocations into the macroscopic theory of plasticity is probably the best bridge to the results of the atomistic theory of plasticity.

We have refrained from a rigorous analysis of the dislocation tensor α and the curvature tensor κ. Otherwise, the dislocation density has been a concept in the (atomistic) theory of plasticity for some time now, so we can refer to the pertinent papers.

§ 7. Summary of the most important concepts and equations.

Internal stresses – even ones that are produced by temperature fluctuations or magnetization – can also be interpreted as a consequence of a (possibly continuous) tensorial distribution α of dislocations. One thinks of a continuous distribution of dislocations as arising from, perhaps, an ever denser family of discrete dislocations. One

(*) See footnote 1, pp. 1.
(**) From unpublished research of G. RIEDER, Stuttgart.

then employs the fact that the dislocation flux $\iint_F d\mathfrak{f} \cdot \alpha$ through an arbitrary surface F is equal to the total BURGERS vector $\mathfrak{b}$ of all of the dislocation lines that are encircled by the boundary line of F.

One will require an asymmetric distortion tensor ε for the complete description of the state of distortion of a medium that is endowed with dislocations. The anti-symmetric part ε_a of e describes the "incompatible" rotation of the volume element of the body, which are rotations by which these elements will be rotated with respect to each other. One will always have to deal with such rotations whenever actual dislocations appear. One might exhibit such volume elements directly by two grains of a polycrystal that are separated by a grain boundary. These grains are rotated with respect to each other and, according to BURGERS and BRAGG, the rotation will be accomplished by dislocations that define the ends of the latter grain boundaries.

The elasticity-theoretic part of the total problem consists of calculating the stresses and distortions for a given dislocation density α. In order to do this, ordinary elasticity theory must be extended. This will happen when one replaces the ST. VENANT compatibility conditions, which are valid only in special cases, with a new fundamental equation of broader validity. However, the second basic equation will still be assumed to be true, which is the equilibrium condition in the form Div $\sigma = 0$. (Let the volume forces be excluded.) The new basic equation reads:

$$\text{Rot}\,(\varepsilon_2 + \varepsilon_a) = \alpha, \tag{11}$$

where $\varepsilon_2 + \varepsilon_a$ is *the* part of the distortion that is coupled locally to the dislocations, and thus vanishes everywhere that there are no dislocations running through. ε_2 describes the incompatible distortions (excluding the rotations) of the body in question. The integration of the basic equation can be achieved with the help of the symmetric stress function tensor χ that was previously given by the author. However, it is often better to replace it with a new stress function tensor φ that must fulfill the equations:

$$\sigma = \text{Rot}\,\varphi, \qquad \text{Div}\,\varphi = 0,$$

which insure the equilibrium conditions. φ, like α, is asymmetric, and it will have the remarkable property that when it is multiplied by α, that will yield an elastic energy. One can then regard φ as a potential for the dislocations. In addition, φ will have the advantage that it already gives the stresses by a single differentiation. In an infinite medium, the tensor φ will satisfy the differential equation ($\tilde{\alpha}_{ij} \equiv \alpha_{ji}$):

$$\Delta\varphi = -\frac{2G}{m-1}(m\alpha_2 - \tilde{\alpha}_2) \tag{17}$$

with $\alpha_2 \equiv \text{Rot}\,\varepsilon_2$.

In order to employ this equation, it will be necessary to first calculate α_2 in terms of α. In many cases, one must then calculate the rotations that are described by ε_a . If $\boldsymbol{\varepsilon}_a$ is the axial vector that is equivalent to $\boldsymbol{\varepsilon}_a$ – viz., the vector that described the direction and magnitude of the rotations – then the differential equation that must be solved will read:

$$\Delta\boldsymbol{\varepsilon}_a = \mathrm{Div}\,\tilde{\alpha} - \tfrac{1}{2}\,\mathrm{grad}\,\alpha_I\,.$$

However, α_2 is also given at the same time as $\boldsymbol{\varepsilon}_a$ [eq. (11)]. Nothing else stands in the way of solving the problem that was just posed with the help of eq. (17). One calculates φ, then σ, and then $\varepsilon - \varepsilon_a$, with the help of HOOKE's law. The stress state, like the distortion state will then be determined completely.

Appendix

Eq. (1) was originally introduced by the author is a somewhat different way. The values (1) of η shall be substituted in the particular solution:

$$\chi'(\mathfrak{r}) = -\frac{1}{8\pi}\oint \eta(\mathfrak{r}')\,|\,\mathfrak{r} - \mathfrak{r}'\,|\,dl'$$

of the elastic differential equations that are valid in an infinite medium, and the differentiations are performed on $|\,\mathfrak{r} - \mathfrak{r}'\,|$, but not on $\mathfrak{t}$; χ' stands for $\left(\chi - \frac{1}{m+2}\chi_I I\right)\Big/2G$ in this. If one now integrates over the infinite medium, instead of along the line (which amounts to the same thing, since η is non-zero only along the line), then one will get:

$$\chi'(\mathfrak{r}) = -\frac{1}{4\pi}\iiint (\mathfrak{t}\,\mathfrak{b}\times\nabla - \nabla\times\mathfrak{b}\,\mathfrak{t})\,|\,\mathfrak{r} - \mathfrak{r}'\,|\,d\tau'\,. \tag{18}$$

The difference between this integral and one that is described in the same form – except that now one differentiates $\mathfrak{t}$, instead of $|\,\mathfrak{r} - \mathfrak{r}'\,|$ – amounts to precisely an outer surface integral (GAUSS's law!), and it will vanish when the dislocations lie at finite points. A comparison with eq. (14), which is correct in any case, will show the physical reality of the differentiation of $\mathfrak{t}$ in (1). However, this will change nothing about the fact that for actual stress calculations, it is preferable to differentiate $|\,\mathfrak{r} - \mathfrak{r}'\,|$ in (18).

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