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# The stress functions of three-dimensional, isotropic, elasticity theory

By

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The isotropic integration problem of elastostatics will be treated with the help of the tensor of stress functions. It will be proved that it is biharmonic when no volume forces or incompatibilities are present. In particular, the stress functions of MAXWELL and MORERA prove to be harmonic. It will be shown that the MAXWELL stress functions, and thus the total stress state of an elastic body, can be described by three harmonic stress functions, with which – in contrast to the PAPCOVITCH-NEUBER functions – it is not the compatibility conditions, but the equilibrium conditions that are fulfilled identically. The stated stress functions also seem suitable for the practical treatment of elastic problems.

## *Preface*

The notation of M. LAGALLY [1] was chosen for the necessary tensor calculations. In particular:

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|---|---|
| $\mathfrak{i}, \mathfrak{j}, \mathfrak{k}$ | are Cartesian unit vectors, |
| $\mathfrak{a} \cdot \mathfrak{b}$ | is the scalar product, |
| $\mathfrak{a} \times \mathfrak{b}$ | is the vectorial product, and |
| $\mathfrak{a}\ \mathfrak{b}$ | is the dyadic product |

of two vectors $\mathfrak{a}$ and $\mathfrak{b}$. Corresponding statements are true for the multiplication of tensors, as well as tensors and vectors. The first scalar of a tensor will always be denoted by the index *I*.

## *§ 1. Introduction. Distortion functions and stress functions.*

One can formally decompose any symmetric tensor field $\tau(\mathfrak{r})$ of rank 2 into two special symmetric tensor fields according to the formula [2]:

[1] LAGALLY, M. *Vorlesungen über Vectorrechnung.* Leipzig: Akademische Verlagsgesellschaft 1928.

[2] A proof of the decomposition formula will be given in an appendix. Whether it was already expressed in this form is not known to me. In any case, consequences of (1) have already been used before; cf., the cited papers, footnote 2, pp. 179, footnote 3, pp. 179, footnote, pp. 186.

$$\tau = \tfrac{1}{2}(\nabla\mathfrak{a} + \mathfrak{a}\nabla) + \nabla\times\varphi\times\nabla. \tag{1}$$

In this, $\mathfrak{a}(\mathfrak{r})$ is a vector field that is determined by $\tau$, $\varphi(\mathfrak{r})$ a symmetric tensor field of rank 2 of that nature, as well. The tensor $\tau_1 \equiv \tfrac{1}{2}(\nabla\mathfrak{a} + \mathfrak{a}\nabla)$ has the same character as the deformation tensor $\varepsilon$ of the elasticity theory, since $\varepsilon$ is derived from the displacement vector field in the same way that $\tau_1$ is derived $\mathfrak{a}$. Therefore, tensors of this type were previously referred to as *deformators*, and in order to describe the connection between $\tau_1$ and $\mathfrak{a}$ the operation Def (read: "deformation of") is defined with the help of [1]:

$$\text{Def}\,\mathfrak{a} \equiv \tfrac{1}{2}(\nabla\mathfrak{a} + \mathfrak{a}\nabla).$$

Tensors of the form $\tau_2 \equiv \nabla\times\varphi\times\nabla$ are also not new in elasticity theory. It is known that the compatibility equations of St. VENANT can be written in the form $\nabla\times\varepsilon\times\nabla = 0$ (cf., M. LAGALLY, *loc. cit.*). This equation may also be translated into words: The incompatibility of $\varepsilon$ is null. Tensors of the form $\nabla\times\varphi\times\nabla$ will thus be referred to as *incompatibility tensors*. For the treatment of such tensors, the operation Ink (read: "the incompatibility of") will now be introduced by way of [1]:

$$\text{Ink}\,\varphi \equiv \nabla\times\varphi\times\nabla.$$

This formula plays a role in the mathematical treatment of symmetric tensor fields that is similar to that of the known decomposition formula, by which one decomposes a vector field into a gradient and a rotor field, does in the treatment of vector fields. Due to this, the identity relations:

$$\text{Ink Def} \equiv 0, \quad \text{Div Ink} \equiv 0 \tag{2}$$

are easy to verify. In them, Div is the operation "divergence [1]" as it is applied to a tensor. Due to (2), $\varphi$ is determined in (1) only up to a deformator. For that reason, one can prescribe certain auxiliary conditions on the components of $\varphi$ that will play an important role in what follows.

One recognizes the importance of the decomposition (1) for elasticity theory immediately when one writes down the elastic differential equation for the distortions $\varepsilon$ (the stresses $\sigma$, resp.). In the simplest case, one has the compatibility equations:

$$\text{Ink}\,\varepsilon = 0, \tag{3}$$

and the equilibrium conditions:

$$\text{Div}\,\sigma = 0, \tag{4}$$

[1] One finds the operators Def and Div explained in FRANK-v.MISES, among others. The operator Ink was introduced by H. SCHAEFER, *loc. cit.*, under the name of "symmetric rotation." I cannot agree with this terminology, since the physical meaning of Ink $\varphi$ is not that of a rotation of $\varphi$. One obtains such a thing only by the application of a differential operator of first order. A component representation of the incompatibility tensors in included in § 8.

in which, from HOOKE's law, one has:

$$\sigma = 2G\left(\varepsilon + \frac{\varepsilon_I I}{m-2}\right), \qquad \varepsilon = \frac{1}{2G}\left(\sigma - \frac{\sigma_I I}{m+1}\right) \tag{5}$$

($m$ = transverse contraction number, $G$ = shear modulus).

From eq. (3) to (5), it follows – perhaps along the lines of the BELTRAMI equations [cf. (9)] – that $\sigma_I$, $\varepsilon_I$ are harmonic, so the components of $\sigma$ and $\varepsilon$ are biharmonic. One does not use this knowledge very much in the solution of boundary-value problems, since the total of the biharmonic stress (distortion, resp.) tensor would be greater than the tensors that are allowed by (3) to (5). Thus, all known integration theorems begin with either eq. (3) or eq. (4) being satisfied identically. New functions will always be introduced for that purpose, which will be referred to as *distortion functions* when (3) is satisfied identically with them, as compared to *stress functions* when (4) is fulfilled identically. From (2), the Ansatz for the distortion functions $V_i$ has the form:

$$\varepsilon = \mathrm{Def}[f(V_i)],$$

and with stress functions $S_i$ :

$$\sigma = \mathrm{Ink}[f(S_i)],$$

where $f$ can be any function, or also a differential operator. The distortion functions obey certain differential equations, whose fulfillment is equivalent to that of the equilibrium conditions. Corresponding statements are true for the differential equations of the stress functions, whose fulfillment guarantees that of the compatibility conditions.

From the definitions of distortion (stress, resp.) functions, the former can also describe states of elastic bodies that are subject to volume forces, but not, by contrast, strain states that are provoked by incompatibilities. The converse is true for strain functions. For that reason, we further remark: In a simply-connected body that is stressed with proper stresses, it is fundamental that the compatibility equations are not fulfilled. One can scarcely do without the stress functions for the treatment of such proper stresses (cf., also § 6).

The most important distortion functions are the components of the displacement vector $\mathfrak{s}(\mathfrak{r}) = (u, v, w)$, with the help of which, from (2), eq. (3) will be satisfied identically with the Ansatz $\varepsilon = \mathrm{Def}\ \mathfrak{s}$. Moreover, the harmonic and biharmonic displacement functions [1], which will be partially defined in the following paragraph, belong to this class of distortion functions.

The optimal formulation of the integration problem of elastostatics with the help of distortion functions has already been found. In summary, one may say: For two-dimensional isotropic problems (so it is also true for axial symmetry) one always arrives at the representation of the stress state that obeys (3) to (5) with the help of a biharmonic function (MAGUERRE, LOVE) [1], or also with two harmonic functions (specialized

[1] BIEZENO-GRAMMEL, *Technische Dynamik*, Berlin, Springer 1939. The authors use the collective term "displacement functions" for functions, from which one obtains the components of displacements by differentiation.

PAPKOVITCH-NEUBER functions) [1]. The three-dimensional problem may be treated completely with the aid of a triharmonic function [2] or also with three harmonic functions (BURGATTI , PAPKOVITCH-NEUBER ) [3]. The latter representation has proved to be especially advantageous due to its symmetry. The three distortion functions are the components of a harmonic vector $\mathfrak{b}(\mathfrak{r})$ and will be introduced by way of:

$$\left.\begin{aligned} u = \mathfrak{b}_x + \partial F/\partial x,\ v = \mathfrak{b}_y + \partial F/\partial y,\ w = \mathfrak{b}_z + \partial F/\partial z \\ F = -\frac{m}{m-1}(x\mathfrak{b}_x + y\mathfrak{b}_y + z\mathfrak{b}_z), \end{aligned}\right\} \tag{6}$$

such that the distortion state will be described by:

$$\varepsilon = \operatorname{Def}\left(\mathfrak{b} - \frac{m/4}{m-1}\operatorname{grad}(\mathfrak{r}\cdot\mathfrak{b})\right). \tag{6′}$$

For $\sigma$, one then calculates with (5):

$$\frac{1}{G}\sigma = \nabla\mathfrak{b} + \mathfrak{b}\nabla + \frac{1}{m-1}\nabla\cdot\mathfrak{b}I - \frac{m/2}{m-1}\nabla\nabla(\mathfrak{r}\cdot\mathfrak{b}) - \frac{m/2}{(m-1)(m-2)}\mathfrak{r}\cdot\Delta\mathfrak{b}I\,. \tag{6″}$$

From now on, it will be assumed that we have arrived at similar simple results with the help of stress functions. This assumption has been confirmed for a long time for the two-dimensional problem (AIRY functions). By contrast, the formulation of the spatial integration problem of elastostatics with the aid of stress functions was achieved so simply. SLOBODJANSKI [4] and BLOCH [5], achieved the presentation of a biharmonic tensor of stress functions whose utilization is thus contradicted by the fact that the stress tensor for them is the result of four differentiations. The Ansätze of MAXWELL, MORERA [6], and BLOCH [4] lead to non-obvious simultaneous differential equations of fourth order for the stress functions whose integration seems hopeless [7]. All of these Ansätze began from (2) with the demand that (4) is satisfied identically by:

$$\sigma = \operatorname{Ink}\chi. \tag{7}$$

[1] PAPKOVITCH, P. F., C. R. Acad. Sci. Paris **195**, 513, 754, 836 (1932). – NEUBER, H., Z. angew. Math. u. Mech. **14**, 203 (1934). – *Kerbspannungslehre*, Berlin, Springer, 1937.

[2] GEBBIA, M, Ann. di Math. (III) **10**, 157 (1904).

[3] Cf., e.g., MALKIN, I., Z. angew. Math. u. Mech. **10**, 182 (1930).

[4] SLOBODJANSKI, M. G., "Stress functions for the spatial problem of elasticity theory." Utsch. spisski Mosk. (Russ.) **24**, 184 (1938).

[5] BLOCH, W. I, "Die Spannungsfunktionen in der Elastizitätstheorie," Prikl. Mat. i Mech. (Russ.) **14**, 415 (1950). Here, further groups of three stress functions will be given that come about by specializing $\chi$ in (9), but seem to offer no advantage over the Ansätze of MAXWELL and MORERA.

[6] MORERA, G., Accad. Lincei Rend. Roma, Ser. V **1** (1892). The relations between the MAXWELL and MORERA stress functions will be mentioned in this volume of MORERA and BELTRAMI. Cf., also FINZI, B., Accad. Lincei, Rend. Roma, Ser. VI **19**, 578 (1934).

[7] One finds these differential equations, in part, in LOVE, A. E. H, *A Treatise on the Mathematical Theory of Elasticity*, pp. 136, Cambridge 1952.

As will be shown in the sequel, for a certain choice of the aforementioned auxiliary conditions for the tensor, its stress functions $\chi(\mathfrak{r})$ are biharmonic. The reduction from six to three biharmonic stress functions follows as the next step. They may be completely exhibited by three harmonic stress functions.

### *§ 2. The differential equations of the stress functions $\chi$ (compatibility equations)* [1].

The set of all stress states that are described by the differential equations:

$$\text{Ink}\ \varepsilon = 0, \qquad \text{Div}\ \sigma = 0 \tag{8}$$

will now be considered. As is known, the equation Ink $\varepsilon = 0$, when expressed in terms of the stresses, reads, with Div $\sigma = 0$ (Beltrami):

$$\Delta\sigma + \frac{m}{m+1}\nabla\nabla\sigma_I = 0. \tag{9}$$

The equation Div $\sigma = 0$ will now be satisfied identically by means of the Ansatz:

$$\sigma = \text{Ink}\ \chi = \Delta\chi - \nabla\nabla\cdot\chi + \chi\cdot\nabla\nabla + \nabla\cdot\chi\cdot\nabla I + \nabla\nabla\cdot\chi_I - \Delta\chi_I\, I. \tag{10}$$

From this, one easily calculates:

$$\sigma_I = \nabla\cdot\chi\cdot\nabla - \Delta\chi_I .$$

Thus, from (9), one has:

$$\Delta\Delta\chi - \Delta(\nabla\nabla\cdot\chi + \chi\cdot\nabla\nabla) + \frac{m}{m+1}\nabla\nabla\nabla\cdot\chi\cdot\nabla + \frac{m}{m+1}\nabla\nabla\Delta\chi_I = 0. \tag{11}$$

The necessary and sufficient conditions for $\chi$ to be biharmonic thus read:

$$-\Delta(\nabla\nabla\cdot\chi + \chi\cdot\nabla\nabla) + \frac{m}{m+1}\nabla\nabla\nabla\cdot\chi\cdot\nabla + \frac{m}{m+1}\nabla\nabla\Delta\chi_I = 0. \tag{12}$$

One now asks whether these auxiliary conditions for the components of $\chi$ are superfluous – i.e., is it possible that the totality of stress states $\sigma$ can be obtained from stress functions $\chi$ for which the auxiliary equations (12) are not necessary? For the response to this question, one considers perhaps the stronger auxiliary conditions:

$$(m + 2)\ \nabla\cdot\chi - \Delta\chi_I = 0, \tag{13}$$

whose fulfillment is likewise guaranteed by (12), as one easily verifies. As one now shows, however, (13), and therefore also (12), become superfluous auxiliary conditions.

[1] In this §, all of the NABLA operators to the right of $\chi$ will also act by differentiation.

Let an arbitrary given stress state $\sigma$ be described by $\sigma$ = Ink $\chi'$. Then, from (2), Ink($\chi'$ + Def $\mathfrak{a}$) yields the same stress tensor. Now, one can always determine $\mathfrak{a}$ such that (13) is fulfilled for:

$$\chi = \chi' + \text{Def } \mathfrak{a}. \tag{14}$$

Namely, one obtains equations for $\mathfrak{a}$:

$$\frac{m+2}{2}\Delta\mathfrak{a} + \frac{m}{2}\nabla\nabla\cdot\mathfrak{a} = -(m+2)\nabla\cdot\chi' + \nabla\chi'_I \equiv f(\mathfrak{r}) \tag{15}$$

from (13) and (14) that have the same type as the inhomogeneous differential equations for the displacements. The right-hand side here is a known function of position, so a particular integral always exists that therefore delivers the desired $\mathfrak{a}$. Thus, the extraneous nature of the conditions (13) is proved, and thus, from (12), also not only for the stress states according to (8), but due to the arbitrariness of $f$ in (15), indeed for the general stress states that were considered in § 6.

The result of the computations up to now is then: With the totality of tensors $\chi$ that fulfill the auxiliary conditions (12), one can describe any stress state that obeys (8) by means of (10). All of these tensors are biharmonic.

## *§ 3. Reduction from six to three biharmonic functions.*

Obviously, all biharmonic tensors $\chi$ that differ only by a biharmonic deformator describe the same stress state; i.e., they all fulfill (12). The multiplicity of these tensors is further restricted by additional conditions. It will now be shown that a condition, among others, that is in addition to (12) can be posed by the requirements:

$$\chi_{xx} = \chi_{yy} = \chi_{zz} = 0. \tag{16}$$

Once again, let $\sigma$ be given by Ink $\chi'$, so $\Delta\Delta\chi' = 0$. One then also has $\sigma$ = Ink($\chi'$ + Def $\mathfrak{a}$), and the question is then whether one can determine $\mathfrak{a}$ such that:

$$\frac{\partial a_x}{\partial x} = -\chi'_{xx}, \qquad \frac{\partial a_y}{\partial y} = -\chi'_{yy}, \qquad \frac{\partial a_z}{\partial z} = -\chi'_{zz}, \tag{17}$$

when $\mathfrak{a} = (a_x, a_y, a_z)$, in which Def $\mathfrak{a}$ shall likewise be biharmonic. In fact, (16) would then be fulfilled for $\chi = \chi'$ + Def $\mathfrak{a}$ and $\chi$ would be biharmonic. From (17), one obtains:

$$a_x = -\int \chi'_{xx}\, dx + A(y, z), \tag{18}$$

and correspondingly $a_y$, $a_z$ . $A(y, z)$ is arbitrary to begin with. Furthermore, it follows from (17) and (18), since $\chi_{xx}$ , $\chi_{yy}$, $\chi_{zz}$ are biharmonic, that:

$$\Delta\Delta\int \chi'_{xx} dx = a(y, z), \tag{19}$$

where $a(y, z)$ is a given function. If one applies $\Delta\Delta$ to (18) then this yields, with (19):

$$\Delta\Delta a_x = -a(y, z) + \Delta\Delta A(y, z),$$

where one can always determine $A$ such that $a_x$ will be biharmonic. Corresponding equations are true for $a_y$ , $a_z$ .

However, that means: For a given biharmonic $\chi'$ one can always determine a biharmonic vector $\mathfrak{a}$ such that (16) is true for $\chi = \chi' + \text{Def}\ \mathfrak{a}$, where $\Delta\Delta\chi = 0$. In other words: One may impose auxiliary conditions on the components of the MORERA stress function tensor:

$$\begin{pmatrix} 0 & \chi_{xy} & \chi_{zx} \\ \chi_{xy} & 0 & \chi_{yz} \\ \chi_{zx} & \chi_{yz} & 0 \end{pmatrix}$$

that make $\chi_{xy}$ , $\chi_{yz}$, $\chi_{zx}$ biharmonic.

The proof is analogous for the MAXWELL tensor:

$$\begin{pmatrix} \chi_{xx} & 0 & 0 \\ 0 & \chi_{yy} & 0 \\ 0 & 0 & \chi_{zz} \end{pmatrix}.$$

The result that $\chi_{xx}$ , $\chi_{yy}$, $\chi_{zz}$ are biharmonic then follows directly. It is known that the MAXWELL Ansatz:

$$\sigma_{xx} = -\frac{\partial^2\chi_{yy}}{\partial z^2} - \frac{\partial^2\chi_{zz}}{\partial y^2}, \qquad \sigma_{yy} = -\frac{\partial^2\chi_{zz}}{\partial x^2} - \frac{\partial^2\chi_{xx}}{\partial z^2}, \qquad \sigma_{zz} = -\frac{\partial^2\chi_{xx}}{\partial y^2} - \frac{\partial^2\chi_{yy}}{\partial x^2}, \tag{20}$$

$$\sigma_{xy} = \frac{\partial^2\chi_{zz}}{\partial x\,\partial y}, \qquad \sigma_{yz} = \frac{\partial^2\chi_{xx}}{\partial y\,\partial z}, \qquad \sigma_{zx} = \frac{\partial^2\chi_{yy}}{\partial z\,\partial x}. \tag{21}$$

(21) easily yields that the MAXWELL Ansatz for:

$$\left.\begin{array}{c} \psi_{xx} = \chi_{xx} + A(x, y) + B(z, x), \quad \psi_{yy} = \chi_{yy} + C(y, z) + D(x, y), \\ \psi_{zz} = \chi_{zz} + E(z, x) + F(y, z), \end{array}\right\} \tag{22}$$

leads to the same stresses when one subjects the otherwise arbitrary functions $A(x, y)$, etc., from (20), to the conditions:

$$\frac{\partial^2 C}{\partial z^2}+\frac{\partial^2 F}{\partial y^2}=0,\qquad \frac{\partial^2 E}{\partial x^2}+\frac{\partial^2 B}{\partial z^2}=0,\qquad \frac{\partial^2 A}{\partial y^2}+\frac{\partial^2 D}{\partial x^2}=0. \tag{23}$$

Since the components of $\sigma$ are biharmonic, one next has, due to (21) [1]:

$$\left.\begin{aligned}\Delta\Delta\chi_{xx}=a(x,y)+b(z,x),\quad \Delta\Delta\chi_{yy}=c(y,z)+d(x,y),\\ \Delta\Delta\chi_{zz}=e(z,x)+f(y,z)\end{aligned}\right\} \tag{24}$$

as the differential equations for the stress functions $\chi_{xx}$, $\chi_{yy}$, $\chi_{zz}$ . From (20), it easily follows that the otherwise unknown functions $a(x, y)$, etc., must satisfy the conditions:

$$\frac{\partial^2 c}{\partial z^2}+\frac{\partial^2 f}{\partial y^2}=0,\qquad \frac{\partial^2 e}{\partial x^2}+\frac{\partial^2 b}{\partial z^2}=0,\qquad \frac{\partial^2 a}{\partial y^2}+\frac{\partial^2 d}{\partial x^2}=0. \tag{25}$$

Substituting (22) into (24) yields:

$$\begin{aligned}\Delta\Delta\psi_{xx}&=\Delta\Delta A(x,y)+\Delta\Delta B(z,x)+a(x,y)+b(z,y),\\ \Delta\Delta\psi_{yy}&=\Delta\Delta C(y,z)+\Delta\Delta D(x,y)+c(y,z)+d(x,y),\\ \Delta\Delta\psi_{zz}&=\Delta\Delta E(z,x)+\Delta\Delta F(y,z)+e(z,x)+f(y,z).\end{aligned}$$

Due to the identical forms of the conditions (23) and (25), one always determines $A$, $B$ in such a way that $\psi_{xx}$, $\psi_{yy}$, $\psi_{zz}$ become biharmonic. Since the MAXWELL Ansatz yields the same stresses for $\psi$ and $\chi$, one can also express the results of the calculation as follows:

All stress states that obey (8) may be described with the help of three biharmonic stress functions of MAXWELLian type by means of (20) and (21).

## *§ 4. Reduction of the three biharmonic to three harmonic stress functions.*

The starting point for the further simplifications is defined by the biharmonic MAXWELL stress functions. One can repeat the last computation for the MAXWELL functions once again, when one employs eq. (9), instead of the equation $\Delta\Delta\sigma = 0$. In precisely the same way above, one obtains:

$$\Delta\chi_{xx}+\frac{m}{m+1}\sigma_I=0,\quad \Delta\chi_{yy}+\frac{m}{m+1}\sigma_I=0,\quad \Delta\chi_{zz}+\frac{m}{m+1}\sigma_I=0, \tag{26}$$

$$\sigma_I=-\Delta\chi_I+\frac{\partial^2\chi_{xx}}{\partial x^2}+\frac{\partial^2\chi_{yy}}{\partial y^2}+\frac{\partial^2\chi_{z}}{\partial z^2}\ . \tag{27}$$

[1] One thinks of $\Delta\Delta$ as being applied to (20) and (21).

If one assumes, for the moment, that $\sigma_I$ is known then one obtains the general representation of the stress functions $\chi$:

$$\chi_{xx} = \psi_{xx} + \text{particular integral of (26)},$$
$$\chi_{yy} = \psi_{yy} + \text{particular integral of (26)},$$
$$\chi_{zz} = \psi_{zz} + \text{particular integral of (26)},$$

where the stress functions $\psi_{xx}$, $\psi_{yy}$, $\psi_{zz}$ are harmonic. Since $\chi_{xx}$, $\chi_{yy}$, $\chi_{zz}$ are, on the other hand, biharmonic, there is a closely-associated Ansatz of the form $x\varphi_{xx} + y\varphi_{yy} + z\varphi_{zz}$, in which the functions $\varphi$ are harmonic. From this, one easily finds that (26) and (27) can be satisfied by means of:

$$\left.\begin{aligned} \chi_{xx} = \psi_{xx} + H,\ \chi_{yy} = \psi_{yy} + H,\ \chi_{zz} = \psi_{zz} + H, \\ H = \frac{m/2}{m-1}\left( x\frac{\partial \psi_{xx}}{\partial x} + y\frac{\partial \psi_{yy}}{\partial y} + z\frac{\partial \psi_{zz}}{\partial z}\right). \end{aligned}\right\} \tag{28}$$

With (28), the biharmonic MAXWELLian stress functions lead back to three harmonic stress functions. Collectively, the stress state of an elastic body may then be described by:

$$\sigma = \operatorname{Ink}\left( \psi + \frac{m/2}{m-1}\mathfrak{r}\nabla\cdots\psi I \right), \tag{29}$$

in which $\psi$ is the harmonic tensor:

$$\psi = \begin{pmatrix} \psi_{xx} & 0 & 0 \\ 0 & \psi_{yy} & 0 \\ 0 & 0 & \psi_{zz} \end{pmatrix}.$$

With that, we have found a representation of the stress state by means of the stress functions that approaches the PAKOVITCH-NEUBER representation (6′) of the distortion state by means of distortion functions in simplicity, as the comparison of (28) with (6) shows.

By substituting (29) into (6), one shows that eq. (8) are also fulfilled by an *arbitrary* biharmonic tensor $\psi$ by means of the Ansatz (29). With the help of (10), when developed, (29) reads, when one then sets $\psi_I = 0$:

$$\sigma = -\nabla\nabla\cdot\ \psi - y\cdot\nabla\nabla - \frac{1}{m-1}\nabla\cdot(\nabla\cdot\psi)I + \frac{m/2}{m-1}\nabla\nabla\mathfrak{r}\cdot(\nabla\cdot\psi) + \Delta\psi, \tag{30}$$

equations that certainly satisfy the equilibrium conditions identically. If one now replaces $\Delta\psi$ in (30) with $\dfrac{m/2}{(m-1)(m-2)}\mathfrak{r}\cdot\Delta\nabla\cdot\psi I$ then one obtains an expression for $\sigma$ that indeed no longer fulfills the equilibrium conditions identically, although they do satisfy the compatibility conditions. Since one sets $\Delta\psi = 0$ in the absence of volume forces and incompatibilities, both Ansätze come from the same one, in practice. In this special case, these special distortion and stress functions then become equal. One then remarks that $\psi$ only enters into the second Ansatz in the form $\nabla \cdot \psi$, which can be set equal to a vector $-G\mathfrak{b}$. One thus obtains precisely the PAPKOVITCH-NEUBER Ansatz (6″) with the distortion functions. An interpretation of these functions as stress functions is obviously not possible [1].

## *§ 5. The boundary conditions.*

From the considerations up to now, one might say: If one has found any biharmonic stress functions $\chi$ by which the boundary conditions are fulfilled then one obtains the correct stress tensor from these stress functions. One does need to observe the auxiliary conditions (12) in all of the calculations.

In the majority of cases, one will make a development in harmonic functions for the stress functions, or also an integral theorem of the FOURIER type. In these cases, what remains is to calculate $\psi$ or $\partial\psi / \partial n$ on the boundary, perhaps from the prescribed values of the boundary forces $\mathfrak{A}(\mathfrak{r})$. It suffices to calculate with the derivatives of $\chi$ ($\psi$, resp.) that are given on the boundary using the boundary conditions:

$$\mathfrak{n} \cdot \mathrm{Ink}\, \chi = \mathfrak{A}.$$

For the AIRY stress function $F$, the determination of the boundary value of $F$ and $\partial F / \partial n$ is achieved by undetermined integrations around the boundary line. It is very apparent that correspondingly the determination of the boundary values of $\chi$, $\partial\chi / \partial n$, $\psi$, $\partial\psi / \partial n$ are achieved by means of undetermined integrations over the boundary surface. For boundary surfaces that are pieces of Cartesian coordinate surfaces, one finds the integrations in question for the MAXWELL functions almost immediately. These suggestions might suffice, for the moment.

## *§ 6. The stress functions in the presence of volume forces or incompatibilities*

As is known, the determination of the stresses that are provoked by simultaneous volume and boundary forces may always be divided into a summation problem and a boundary problem; the latter can be resolved as above. The question is then whether the summation problem, namely, to determine the particular integrals that belong to the volume forces, may be resolved with the help of stress functions. In principle, that is indeed possible in the cases in which the volume forces have a potential, so the method seems very artificial in comparison to its usefulness, so it therefore acquires no significance, whatsoever. In the absence of general volume forces, the stress functions may no longer be defined, at all.

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[1] By a completely difficult argument, H. SCHEFER, *loc. cit.*, came to the PAPKOVITCH-NEUBER functions along a detour from the stress function tensor.

There are no such complications for the appearance of incompatibilities. Here, one again satisfies (4) by means of the Ansatz (7), by which one enters into the extended compatibility equations Ink $\varepsilon = \eta$. The physical meaning of the incompatibility tensor $\eta(\mathfrak{r})$, like that of the stress function tensor, has still not been clarified sufficiently. An uncomplicated calculation yields, when $\chi$ is subject to the conditions (12):

$$\Delta\Delta\chi = 2G\left(\eta + \frac{1}{m-1}\eta_I I\right). \tag{34}$$

A reduction to three stress functions is no longer possible now. The corresponding reduction in § 3 is, in fact, based upon the fact that $\chi'$ was biharmonic from the outset, which is no longer the case now. The stresses (viz., proper stresses) that are attributed to $\eta$ give the particular integral:

$$\chi(\mathfrak{r}) = \frac{G}{4\pi}\iiint |\,\mathfrak{r} - \mathfrak{r}'\,|\left[\eta(\mathfrak{r}') + \frac{1}{m-1}\eta_I(\mathfrak{r}')I\right] d\tau', \tag{32}$$

in which the integral is taken over the volume of the elastic medium. The solution of the summation problem is thus found. The boundary-value problem can perhaps be solved as in § 4. If one adds the MAXWELL stress functions thus obtained to the particular integrals (32) then one obtains the resulting stress function tensor, and thus, the stress state.

One can show that the elastic energy that is associated with proper stresses in an infinitely extended medium $M$ is given by [1]:

$$\tfrac{1}{2}\iiint_M \chi \cdots \eta \, d\tau, \tag{33}$$

in which the connection between $\chi$ and $\eta$ is mediated by (32) [2]. (33) once more underscores the close coupling of stress functions and incompatibilities. This suggests the conjecture that precisely the stress functions would enable one to treat the problem of proper stresses effectively, as was already suggested in § 1.

## *§ 7. Generalization to anisotropic media*

In the calculation at the end of § 4, the assumption of biharmonic stress components could be replaced by a law of the form:

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[1] The starting point for the derivation of this formula is the energy expression $\frac{1}{2}\iiint \sigma \cdots \varepsilon \, d\tau$, in which $\sigma$ = Ink $\chi$, and after two partial integrations, one introduces Ink $\varepsilon = \eta$.

[2] For the close connection between these quantities, cf., also SCHAEFER, H., Z. angew. Math. u. Mech. **33**, 356 (1953). There, the problem of stress functions was treated by analogy with potential theory, and in particular, he sought to find the covariant form of the integration circumstances that would be appropriate to the auxiliary conditions. The possibility of introducing biharmonic stress functions was not known, there.

$$f\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)\sigma = 0, \tag{34}$$

where $f$ shall be subject to only the restriction that the differentiations of $f$ with respect to $\partial/\partial x$, $\partial/\partial y$, $\partial/\partial z$ are commutable. This then yields, as above, that the stress function tensor [1] $\chi$ satisfies the equations:

$$f\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)\chi = 0. \tag{35}$$

Now, for any crystal, one has eq. (34), where $f$ is a scalar, linear, homogeneous of order six, differential operator, and is characteristic of any crystal system [2]. Eqs. (35) can perhaps serve as the foundations for the generalization of the connection (32) between $\chi$ and $\eta$ to the case of anisotropic media.

## § 8. *The incompatibility tensor in component representation.*

In conclusion, let the components of the incompatibility tensor:

$$\sigma = \operatorname{Ink} \chi$$

be given in Cartesian and curvilinear orthogonal coordinates. They are:

$$\sigma_{xx} = -\frac{\partial^2 \chi_{yy}}{\partial z^2} - \frac{\partial^2 \chi_{zz}}{\partial y^2} + 2\frac{\partial^2 \chi_{yz}}{\partial y\,\partial z}$$

$$\sigma_{yy} = -\frac{\partial^2 \chi_{zz}}{\partial x^2} - \frac{\partial^2 \chi_{xx}}{\partial z^2} + 2\frac{\partial^2 \chi_{zx}}{\partial z\,\partial x}$$

$$\sigma_{zz} = -\frac{\partial^2 \chi_{xx}}{\partial y^2} - \frac{\partial^2 \chi_{y}}{\partial x^2} + 2\frac{\partial^2 \chi_{xy}}{\partial x\,\partial y}$$

$$\sigma_{yz} = -\frac{\partial}{\partial x}\left(-\frac{\partial \chi_{yz}}{\partial x} + \frac{\partial \chi_{zx}}{\partial y} + 2\frac{\partial \chi_{xy}}{\partial z}\right) + \frac{\partial^2 \chi_{xx}}{\partial y\,\partial z},$$

$$\sigma_{yz} = -\frac{\partial}{\partial y}\left(-\frac{\partial \chi_{zx}}{\partial x} + \frac{\partial \chi_{xy}}{\partial y} + 2\frac{\partial \chi_{yz}}{\partial z}\right) + \frac{\partial^2 \chi_{yy}}{\partial z\,\partial x}$$

$$\sigma_{xy} = -\frac{\partial}{\partial z}\left(-\frac{\partial \chi_{xy}}{\partial z} + \frac{\partial \chi_{yz}}{\partial x} + 2\frac{\partial \chi_{zx}}{\partial y}\right) + \frac{\partial^2 \chi_{zz}}{\partial x\,\partial y}.$$

---

[1] First the MAXWELLian one, but then, as before, also the general tensor $\chi$. The auxiliary conditions that enter in place of (12) are still not known.

[2] For the operator $f$, cf., e.g., KRÖNER, E., Z. Physik **136**, 402 (1953).

Fur curvilinear orthogonal coordinates, one has [1]:

$$\sigma^{ij} = -\,\varepsilon^{ilm}\,\varepsilon^{jrs}\{\chi_{r,ms} - \left(\Gamma^{p}_{rm}\right)_{,s}\chi_{lp} - \Gamma^{p}_{rm}\chi_{lp,s} - \Gamma^{t}_{sl}\chi_{tr,m} + \Gamma^{t}_{sl}\Gamma^{p}_{mt}\chi_{pr} + \Gamma^{t}_{sl}\Gamma^{p}_{rm}\chi_{tp} - \Gamma^{t}_{sm}\chi_{lr,t}$$
$$+\ \Gamma^{t}_{sm}\Gamma^{p}_{tl}\chi_{pr} + \Gamma^{t}_{sm}\Gamma^{p}_{rt}\chi_{lp}\}\,,$$

where the quantities $\Gamma^{\alpha}_{\beta\gamma}$ are the CHRISTOFFEL three-index symbols and $\varepsilon^{\alpha\beta\gamma}$ are the components of the anti-symmetric LEVI-CIVITA tensor. Differentiations are suggested in the usual way by a comma.

## Appendix

*Proof of the decomposition formula* (1)

For this, one starts perhaps with the known representation of a symmetric tensor field $\tau$ by three vector fields $\mathfrak{t}_\nu$:

$$\tau = \mathfrak{t}_1\,\mathfrak{i} + \mathfrak{t}_2\,\mathfrak{j} + \mathfrak{t}_3\mathfrak{k},$$

where $\mathfrak{t}_1$ is the vector with the Cartesian components $\tau_{xx}$, $\tau_{yx}$, $\tau_{zx}$, and correspondingly for $\mathfrak{t}_2$, $\mathfrak{t}_3$. If, for example, $\tau$ is the stress tensor then the $\mathfrak{t}_\nu$ are the well-known stress vectors. One now decomposes the vector fields $\mathfrak{t}_\nu$ into their gradients and rotor fields:

$$\mathfrak{t}_1 = \nabla a_1 + \nabla\times\mathfrak{b}_1, \qquad \mathfrak{t}_2 = \nabla a_2 + \nabla\times\mathfrak{b}_2, \qquad \mathfrak{t}_3 = \nabla a_3 + \nabla\times\mathfrak{b}_3\,.$$

Thus, the vectors $\mathfrak{b}$ are defined only up to a gradient. One then obtains for $\tau$:

$$\tau = \nabla(a_1\mathfrak{i} + a_2\mathfrak{j} + a_3\mathfrak{k}) + \nabla\times(a_1\mathfrak{i} + a_2\mathfrak{j} + a_3\mathfrak{k}),$$

and when one sets the bracket expression equal to a vector $\mathfrak{a}$ (an asymmetric tensor $\beta$ of rank 2, resp.), one has:

$$\tau = \nabla\mathfrak{a} + \nabla\times\beta. \tag{a}$$

The right-hand side of this equation represents a symmetric tensor. Here, one must be able to split $\nabla\times\beta$ into a part $\mathfrak{a}\nabla$ that extends $\nabla\mathfrak{a}$ to a symmetric tensor. In general, $\nabla\mathfrak{a}$ cannot be symmetric, since one can prescribe no conditions on the components of $\mathfrak{a}$ without the reducing the generality of the representation (a). What then remains is:

$$\tau = \nabla\mathfrak{a} + \mathfrak{a}\nabla + \nabla\times\gamma. \tag{b}$$

---

[1] This formula is completely due to M. BRDIČKA. It represents the essential result of his paper "The compatibility equations and stress functions in tensor form" [Czech. J. physics (Russ.) **3**, 36 (1953)].

where $\nabla\times\gamma$ is a symmetric tensor. By the demand of the symmetry of $\nabla\times\gamma$, as one confirms by checking, one must prescribe that $\gamma$ must satisfy $(\gamma - \gamma_I\, I)\cdot\nabla = 0$, which implies that $\gamma - \gamma_I\, I = \delta\times\nabla$. One then has, $-2\gamma_I = (\delta\times\nabla)_I$, $\gamma = \delta\times\nabla - \frac{1}{2}(\delta\times\nabla)_I\, I$, and:

$$\nabla\times\gamma = \nabla\times\delta\times\nabla - \tfrac{1}{2}\nabla\times I\,(\delta\times\nabla)_I\,.$$

If one decomposes $\delta$ into a symmetric part $\delta^s$ and an anti-symmetric part $\delta^a$ then one obtains:

$$\nabla\times\gamma = \nabla\times\delta^s\times\nabla, \tag{c}$$

since $(\delta\times\nabla)_I \equiv 0$, and the two parts of $\delta^a$ drop out, as one easily shows with the help of the formulas $\delta^a = \boldsymbol{\delta}^a \times I$, $(\delta^a\times\nabla)_I = -2\,\nabla\cdot\boldsymbol{\delta}^a$, $\boldsymbol{\delta}^a\times I\times\nabla = \nabla\boldsymbol{\delta}^a - \nabla\cdot\boldsymbol{\delta}^a\, I$. In this, $\boldsymbol{\delta}^a$ is the vector with the components $-\delta^a_{yz}$, $-\delta^a_{zx}$, $-\delta^a_{xy}$. Eq. (1) is proved with (b) and (c).

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