"Zur vierdimensionalen Formulierung der undulatorischen Mechanik," Ann. Phys. (Leipzig) (4) **81** (1926), 632-636.

On the four-dimensional formulation of wave mechanics

By **J. Kudar**

Translated by D. H. Delphenich

In his first communication ([1]) on wave mechanics, **Schrödinger** introduced the substitution:

(1) $$W = \frac{h}{2\pi} \log \psi$$

into **Hamilton**'s partial differential equation:

$$H\left(q_i, \frac{\partial W}{\partial q_i}\right) - E = 0$$

and derived the wave equation from the variational principle:

$$\delta \iiint \left[H\left(q_i, \frac{h}{2\pi}\frac{1}{\psi}\frac{\partial \psi}{\partial q_i}\right) - E \right] \psi \, dx \, dy \, dz = 0 ,$$

In the second communication ([2]), he dropped the relation (1) and used a different one that was based upon an adaptation of **de Broglie**'s ideas about wave mechanics. The starting point was the relation:

$$(\text{grad } W)^2 = 2\,(E - V) ,$$

which represents **Hamilton**'s partial differential equation in classical mechanics (E = energy, V = static potential). Based upon the construction of the surfaces W = const., that gave the phase velocity:

$$u = \frac{E}{\sqrt{2(E-V)}} .$$

In the associated wave equation:

([1]) **E. Schrödinger**, Ann. Phys. (Leipzig) **79** (1926), pp. 361.
([2]) **E. Schrödinger**, Ann. Phys. (Leipzig) **79** (1926), pp. 489.

$$\Delta\psi - \frac{1}{u^2}\frac{\partial^2\psi}{\partial t^2} = 0 ,$$

one substitutes:

$$\psi = e^{2\pi i E t/h}\cdot\psi_1,$$

in which ψ_1 is independent of time. It will then follow that:

$$\Delta\psi_1 + \frac{8\pi^2}{h^2}(E - V)\psi_1 = 0 .$$

As **Schrödinger** also emphasized, it does not seem that one can find an analogue of that process in relativistic mechanics in the presence of the four-potential.

V. Fock (1) had also exhibited the wave equation for three-dimensional mechanics. He substituted the Ansatz:

$$\frac{\partial W}{\partial t} = - E , \quad \frac{\partial W}{\partial q_i} = - E\frac{\dfrac{\partial\psi}{\partial q_i}}{\dfrac{\partial\psi}{\partial t}} ,$$

in

$$H\left(q_i, \frac{\partial W}{\partial q_i}\right) + \frac{\partial W}{\partial t} = 0$$

and applied the variational principle:

$$\delta\int\left[H\left(q_i, -E\frac{\dfrac{\partial\psi}{\partial q_i}}{\dfrac{\partial\psi}{\partial t}}\right) - E\right]\left(\frac{\partial W}{\partial t}\right)^2 d\Omega = 0 ,$$

in which $d\,\Omega = dt\; dq_1\; dq_2\; dq_3$.

In what follows, we will derive the wave equation for relativistic mechanics in the general case where a four-potential exists by adopting the viewpoint of **Schrödinger**'s first method.

The metric (in rectangular spatial coordinates) is:

(2) $$ds^2 = dx_0^2 - (dx_1^2 + dx_2^2 + dx_3^2) ,$$

in which $x_0 = c\ t$. The equations of motion of the electron can be derived from the Lagrange function (2):

(1) **V. Fock**, Zeit. Phys. **38** (1926), pp. 242.

(2) e = charge of the electron, m – rest mass. The Ansatz:

$$L = \tfrac{1}{2} m c^2 \left[\left(\frac{dx_0}{ds} \right)^2 - \sum_{i=1}^{3} \left(\frac{dx_i}{ds} \right)^2 \right] + e \sum_{i=1}^{3} \varphi_k \frac{dx_k}{ds} ,$$

in which φ_i means the components of the four-potential. If we start with the Hamiltonian function:

(3)
$$H = \tfrac{1}{2} m c^2 \left[\left(\frac{dx_0}{ds} \right)^2 - \sum_{i=1}^{3} \left(\frac{dx_i}{ds} \right)^2 \right]$$

and introduce the impulse coordinates:

(4)
$$\frac{\partial L}{\partial \frac{dx_0}{ds}} = mc^2 \frac{dx_0}{ds} + e \varphi_0 = \frac{\partial W}{\partial x_0} ,$$

(5)
$$\frac{\partial L}{\partial \frac{dx_i}{ds}} = - mc^2 \frac{dx_i}{ds} + e \varphi_i = \frac{\partial W}{\partial x_i} \qquad (i = 1, 2, 3),$$

then **Hamilton**'s partial differential equation will become:

(6)
$$\frac{1}{2mc^2} \left[\left(\frac{\partial W}{\partial x_0} - e \varphi_0 \right)^2 - \sum_{i=1}^{3} \left(\frac{\partial W}{\partial x_i} - e \varphi_i \right)^2 \right] + \frac{\partial W}{\partial s} = 0 .$$

Since the proper time s does not occur explicitly, one has $\frac{\partial W}{\partial s}$ = const., and indeed from (2) and (3):

$$L = mc^2 \sqrt{ \left(\frac{dx_0}{ds} \right)^2 - \left[\left(\frac{dx_1}{ds} \right)^2 + \left(\frac{dx_2}{ds} \right)^2 + \left(\frac{dx_3}{ds} \right)^2 \right] } + e \sum_{i=0}^{3} \varphi_i \frac{dx_i}{ds}$$

is not applicable, since the associated Hamiltonian function:

$$H = \sum_{i=0}^{3} \frac{\partial L}{\partial \frac{dx_i}{ds}} \frac{dx_i}{ds} - L$$

vanishes identically. After eliminating proper time, one will have:

$$L = - mc^2 \sqrt{ 1 - \frac{1}{c^2} \left[\left(\frac{dx_1}{ds} \right)^2 + \left(\frac{dx_2}{ds} \right)^2 + \left(\frac{dx_3}{ds} \right)^2 \right] } + e \left(\varphi_0 + \frac{1}{e} \sum_{i=0}^{3} \varphi_i \frac{dx_i}{ds} \right) .$$

Cf. my note: Phys. Zeit. **26** (1925), pp. 207.

(7) $$\frac{\partial W}{\partial s} = -\tfrac{1}{2} m c^2 .$$

When the four-potential does not include the cosmic time x_0, there will exist an energy integral:

(8) $$\frac{\partial W}{\partial x_0} = \text{const.} = E .$$

We will consider this case later. For the time being, the law of energy will not be assumed.

In order to arrive at the wave equation, we set:

(9) $$W = \frac{eh}{2\pi i} \log \psi \qquad (i = \sqrt{-1}).$$

It follows (7) that:

(10) $$\frac{eh}{2\pi i} \frac{1}{\psi} \frac{\partial \psi}{\partial s} = -\tfrac{1}{2} m c^2 ,$$

so

(11) $$\psi = e^{-2\pi i \varepsilon s/ch} \cdot \psi_0 (x_0, x_1, x_2, x_3) ,$$

in which ψ_0 is independent of s, and $\varepsilon = \tfrac{1}{2} m c^2$. From (9) and (10), we get:

(12) $$\frac{\partial W}{\partial x_i} = -\varepsilon \frac{\dfrac{\partial \psi}{\partial q_i}}{\dfrac{\partial \psi}{\partial t}} \qquad (i = 0, 1, 2, 3).$$

After substituting (7), (12) in (6), we will get:

(13) $$\left(\varepsilon \frac{\partial \psi}{\partial x_0} + e\varphi_0 \frac{\partial \psi}{\partial s} \right)^2 - \sum_{i=1}^{3} \left(\varepsilon \frac{\partial \psi}{\partial x_i} + e\varphi_i \frac{\partial \psi}{\partial s} \right)^2 - 4\varepsilon^2 \left(\frac{\partial \psi}{\partial s} \right)^2 = 0 .$$

Corresponding to **Schrödinger**'s line of reasoning ([1]), instead of that differential equation, we shall consider the variational principle:

(14) $$\delta \int Q \, d\omega = 0 ,$$

in which $d\omega = dx_0\, dx_1\, dx_2\, dx_3\, ds$, and Q denotes the quadratic form on the left-hand side of (13).

After an easy calculation (by partial integration), we will get the following wave equation from (14):

([1]) **E. Schrödinger**, *loc. cit.*, I. Mitteilung.

(15) $$\varepsilon^2 \,\mathrm{Div}\,\mathrm{grad}\,\psi + \varepsilon\, e\,\mathrm{Div}\,\Phi \cdot \frac{\partial \psi}{\partial s} + 2\varepsilon\, e \left(\Phi \,\mathrm{Grad}\, \frac{\partial \psi}{\partial s} \right) + (e^2\,\Phi^2 - 4\varepsilon^2)\frac{\partial^2 \psi}{\partial s^2} = 0 \,,$$

in which:

$$\mathrm{Div}\ \mathrm{grad}\ \psi = \frac{\partial^2 \psi}{\partial x_0^2} - \sum_{i=1}^{3} \frac{\partial^2 \psi}{\partial x_i^2} \,,$$

$$\mathrm{Div}\ \Phi = \frac{\partial \varphi_0}{\partial x_0} - \sum_{i=1}^{3} \frac{\partial \varphi_i}{\partial x_i} \,,$$

$$\left(\Phi \ \mathrm{Grad}\, \frac{\partial \psi}{\partial s} \right) = \varphi_0 \frac{\partial^2 \psi}{\partial s\, \partial x_0} - \sum_{i=1}^{3} \varphi_i \frac{\partial^2 \psi}{\partial s\, \partial x_i} \,,$$

$$\Phi^2 = \varphi_0^2 - \sum_{i=1}^{3} \varphi_i^2 \,,$$

$$\varepsilon = \tfrac{1}{2} m c^2 \,.$$

We now consider the special case in which $\varphi_1 = \varphi_2 = \varphi_3 = 0$, and $e\ \varphi_0 = V\ (x_1,\, x_2,\, x_3)$ is independent of x_0. If we set:

$$\psi = e^{-2\pi i \varepsilon s / ch}\, e^{2\pi i E x_0 / ch} \cdot \bar{\psi}(x_1, x_2, x_3)$$

then (8) will also be fulfilled formally when we recall (9). The wave equation (15) will reduce to:

(16) $$\Delta \bar{\psi} + \frac{4\pi^2}{c^2\, h^2} [(E - V)^2 - m^2 c^4] \bar{\psi} = 0 \,.$$

We can also arrive at this latter equation by the second **Schrödinger** method ([1]). Namely, **Hamilton**'s partial differential equation is:

$$(\mathrm{grad}\ W)^2 = (E - V)^2 - m^2 c^4 \,,$$

in this case, where grad W is taken to be three-dimensional. Here, E is normalized such that the rest energy of the electron will amount to mc^2, according to (8), (4). If we put $mc^2 + E$ in place of E and set $V = -\, e^2 / r$ then (16) will go to equation (27) in **Fock** ([2]). It should be remarked that the phase velocity that enters into (16) agrees with the **de Broglie** Ansatz ([3]) for the index of refraction.

Budapest, August 1926.

(Received on 30 August 1926)

([1]) **E. Schrödinger**, II. Mitteilung. – **Oskar Klein** [Zeit. Phys. **37** (1926), pp. 895] has derived our equation (16) in the context of the five-dimensional theory of relativity.

([2]) **V. Fock**, *loc. cit.*, pp. 247.

([3]) **L. de Broglie**, Journal de Physique, January 1926.