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On the fourth-degree surfaces with sixteen singular points

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Hr. Kummer read about the fourth-degree surfaces with sixteen singular points.

The Fresnel wave surface is a fourth-degree surface that has 16 singular points, four of which are real and lie in a principal plane, eight of which are imaginary and lie in the other two principal planes, and the remaining four lie in a plane at infinity. One then sees from this that fourth-degree surfaces with 16 singular points can actually exist. However, fourth-degree surfaces cannot have more than 16 singular points. The reciprocal polar surfaces of the general fourth-degree surface would then be of degree 36, although that degree would be reduced by two units by each singular point, so for more than 16 singular points the degree of the reciprocal polar would drop to two, or even less, which is impossible.

In order to study the properties of fourth-degree surfaces with 16 singular points, we will consider their enveloping cones. For each fourth-degree surface, that cone will be of degree 12, but when its vertex falls upon a singular point of the fourth-degree surface, it will be of degree only six. If the fourth-degree surface has 16 singular points, at one of which, the vertex of the enveloping cone lies, then the 15 straight lines that go from it to the remaining 15 singular points must be double edges of the enveloping cone. However, an irreducible cone of degree four can have no more than ten double edges, so it must decompose into cones of lower degree, and in order for these cones of lower degree to collectively have 15 edges, they must necessarily consist of only six planes that go through the same point, and in fact, the 15 lines of intersection of each two of them will serve for that purpose. The six planes that comprise the enveloping cone whose vertex lies at one of the 16 singular points must, as enveloping planes, contact the fourth-degree surface in curves that are necessarily conic sections, so six singular tangential planes that contact the surface in conic sections must go through each of the 16 singular points of the surface. Moreover, since the 15 lines of intersection of the six singular tangential planes that go through one and the same singular point go through the remaining 15 singular points, it will then follow that six singular points lie in each singular tangential plane, and this will imply that 16 singular tangential planes must be present, in all. Therefore:

Every fourth-degree surface with 16 singular points has, at the same time, 16 singular tangential planes, and these points and planes lie in such a way that each of the 16 planes will contain 6 of the points, and 6 of the planes will go through each of the 16 points.

The six singular points that lie in a plane always have the special position that a conic section can be laid through them. They will then necessarily belong to the points that the singular tangential plane has in common with the surface, and thus, to the points of the contact conic section. Likewise, the six singular tangential planes that go through a singular point will have the property that they are tangential planes to a certain second-degree cone, and indeed to the second-degree cone that osculates the fourth-degree surface at the singular point. As singular tangential planes to the surface that contacts that cone in curves that go through the singular point, these planes must also be tangential planes to the cone that osculates the surface at that point.

One recognizes that this relationship between the 16 points and the 16 planes is obviously a reciprocal polar relationship that has its basis in the fact that the reciprocal polar surface to a fourth-degree surface with 16 singular points is itself a fourth-degree surface with 16 singular points, and that every singular point of the reciprocal polar surface will be assigned to a singular tangential plane, and conversely.

In order to define the general equation of all fourth-degree surfaces with 16 singular points, one selects four of the 16 singular tangential planes such that the four vertices of the tetrahedron that they define are likewise four of the sixteen singular points, which is a condition that can always be satisfied in several ways. If one denotes the equations of the four singular tangential planes that are chosen in that way by:

$$p = 0, \quad q = 0, \quad r = 0, \quad s = 0,$$

where p, q, r, s are entire, linear functions of the three coordinates then one can regard the desired general equation of this kind of fourth-degree surface as one that is homogeneous in the four variables p, q, r, s. With the assumption that the four vertices of the tetrahedron that is defined by the planes $p = 0$, $q = 0$, $r = 0$, $s = 0$ should be singular points of the surface, this homogeneous function must now have degree four, and it when it is set to zero, it will represent the equation of the desired surface, when one also sets its first four derivatives with respect to $p, q, r,$ and s equal to zero, as long as three of those variables are set to zero. It follows from this that no terms can enter into the equation of the surface that contain the fourth powers of the variables and that no terms can enter into it that contains the cube of one of these variables. Namely, if the equation contained – e.g., a term Ap^4 – then it would not be fulfilled if, at the same time, one were to set $p = 0$, $q = 0$, $s = 0$, while if it contained a term $Bp^3 q$ then the first derivative with respect to q would not be equal to zero if, at the same time, one were to set $p = 0$, $q = 0$, $s = 0$. The equation of the surface would then be of degree only two in each of the four variables p, q, r, s. Moreover, since the aforementioned four base planes should be singular tangential planes of the fourth-degree surface, and each singular tangential plane should cut out overlapping conic sections, it will follow that for $p = 0$ the equation of the surface must become a complete square of a homogeneous function of degree two of the remaining three variables, and the same thing would also be true for $q = 0$, $r = 0$, and $s = 0$. If one now chooses the most general form for a homogeneous equation in the four variables p, q, r, s that is only of degree two in each of the variables when they are taken separately, and determines the undetermined coefficients of this form according to the given conditions then one will obtain the following, most-general, form for the equation:

$$a^2 q^2 r^2 + b^2 r^2 p^2 + c^2 p^2 q^2 + d^2 p^2 s^2 + e^2 q^2 s^2 + f^2 r^2 s^2 + 2bcp^2qr$$
$$+ 2\varepsilon acpq^2r + 2\varepsilon abpqr^2 + 2\varepsilon' cdp^2qs + 2\varepsilon'\varepsilon'' cepq^2s + 2\varepsilon'' depqs^2$$
$$+ 2bdp^2rs + 2bfpr^2s + 2dfprs^2 + 2aeq^2rs + 2afqr^2s + 2efqrs^2 - 4gpqrs = 0,$$

where a, b, c, d, e, f, and g are arbitrary constants, while ε, ε', ε'' are just three units that can each take on the two values ± 1. This general form contains all surfaces of degree four that have four singular tangential planes and four singular points that lie at the vertices of the tetrahedron thus-defined, which includes the surfaces with sixteen singular points. In order to single out the latter, it is that remarkable that a further specialization by condition equations in the constants is not required, since it is, moreover, entirely sufficient to make a correct choice of the three undetermined units ε, ε', ε'', namely, the one for which all three of them are equal to – 1; any other choice of these units would yield only surfaces with less than 16 singular points.

The most general equation of all fourth-degree surfaces with 16 singular is then:

$$\begin{aligned} &a^2 q^2 r^2 + b^2 r^2 p^2 + c^2 p^2 q^2 + d^2 p^2 s^2 + e^2 q^2 s^2 + f^2 r^2 s^2 \\ &+ 2bcp^2qr - 2acpq^2r - 2abpqr^2 - 2cdp^2qs + 2cepq^2s - 2depqs^2 \\ &+ 2bdp^2rs + 2bfpr^2s + 2dfprs^2 + 2aeq^2rs + 2afqr^2s + 2efqrs^2 - 4gpqrs = 0. \end{aligned} \tag{1}$$

Of the seven constants a¸ b, c, d, e, f, g, one can omit four of them in such a way that one couples them with the four remaining linear functions p, q, r, s, so only four of them will remain as essential constants. Since 15 constants are included in p, q, r, s, in addition, it will then follow that the general equation of the surface with 16 singular points will contain 18 constants.

The general equation (1) can be easily put into the following form:

$$\phi^2 = 4pq\psi, \tag{2}$$

where

$$\phi = aqr + brp + cpq + dps + eqs + frs,$$
$$\psi = abr^2 + des^2 + acqr + cdps + g'rs,$$
$$g' = g + \tfrac{1}{2}(ad + bc + cf),$$

and in the same way, there are five more completely corresponding forms in which the second part of the equation has the products pr, qr, qs, rs, in place of pq.

If one adds the quantity $4kpqf + 4k^2p^2q^2$ to both sides of equations (2) then one will get:

$$(\phi + 2kpq)^2 = 4pq\,(\psi + k\phi + k^2pq).$$

If one now determines the undetermined constant k such that the second-degree equation:

$$\psi + k\phi + k^2pq = 0$$

represents a conic surface then one will get a sixth-degree equation for k that is only the complete square of the following third-degree equation:

(3) $$cfk^3 + \left(g - \frac{ad}{2} - \frac{be}{2} - \frac{3cf}{2}\right)k^2 + \left(g - \frac{3ad}{2} + \frac{be}{2} + \frac{cf}{2}\right)k - ad = 0,$$

but if one gives k one of the three values that satisfy this cubic equation then:

$$\psi + k\phi + k^2 pq = 0$$

will not merely be the equation of a second-degree conic surface but, in fact, this expression will decompose into two linear factors, which shall be denoted by p' and q'. The general equation of the fourth-degree surface with 16 singular points will then also assume the following form:

(4) $$[aqr + brp + c\,(1 + 2k)\,pq + dps + eqs + frs]^2 - 4k\,(k + 1)\,pqp'q' = 0,$$

where:

(5) $$p' = cq + \frac{br}{k+1} + \frac{ds}{k}, \qquad q' = cp + \frac{ar}{k} + \frac{es}{k+1}.$$

This equation can be put into corresponding forms in five other different ways, in which, when one sets:

(6) $$r' = fs + \frac{bp}{k+1} - \frac{aq}{k}, \qquad s' = fr - \frac{dp}{k} + \frac{eq}{k+1},$$

in addition, the products of four factors $rsr's'$, $qsq's'$, $prp'r'$, $psp's'$, $qrq'r'$ will enter in place of the product of four linear factors $pqp'q'$.

It is also especially remarkable that the irrational form of this equation for the surface:

(7) $$\sqrt{kpp'} + \sqrt{(k+1)qq'} + \sqrt{-rr'} = 0,$$

becomes entirely identical to (4) when makes it rational by giving p', q', r' the expressions that are given by (5) and (6). The four equations:

$$p' = 0,\ q' = 0,\ r' = 0,\ s' = 0$$

represent twelve of the 16 singular tangential planes of the surface, namely, when one gives the k that it contains its three values that satisfy the cubic equation (3); the remaining four singular tangential planes are:

$$p = 0, \quad q = 0, \quad r = 0, \quad s = 0.$$

In these equations, which do not contain the constant g directly, one can regard that constant as being completely replaced with the new k, such that a, b, c, d, e, f, k represent the seven constants of the surface that are not included in p, q, r, s. The other two roots of the cubic equation, which will be denoted by k_1 and k_2, will then be determined as the roots of the following quadratic equation:

$$cfk_1^2 + \left(\frac{ad}{k} - \frac{be}{k+1} + cf\right)k_1 + \frac{ad}{k} = 0,$$

or – what amounts to the same thing – by the two equations:

$$(8) \qquad k_1\, k_2 = \frac{ad}{cfk}, \qquad (k_1 + 1)(k_2 + 1) = \frac{be}{cf(k+1)}.$$

One can also represent equation (7) in the following form, in which the traces of their special way of being generated in (1), which still carries them in its coefficients, have been removed completely:

$$(9) \qquad \sqrt{p(\beta q + \gamma r + \delta s)} + \sqrt{q(\alpha' p + \gamma' r + \delta' s)} + \sqrt{r(\alpha'' p + \beta'' q + \delta'' s)} = 0,$$

along with the two condition equations:

$$\alpha'\gamma + \alpha''\beta - \beta\gamma = 0,$$
$$\alpha''\gamma' + \beta''\gamma - \alpha''\beta'' = 0.$$

Without these two condition equations in the coefficients, equation (9) would be equivalent to:

$$\sqrt{pp'} + \sqrt{qq'} + \sqrt{rr'} = 0,$$

where p, q, r, p', q', r' are six completely arbitrary linear functions of the three coordinates, and would give a surface that had only 14 singular points.

Finally, another form conversion might be mentioned here that one can perform on the equation of these surfaces. If one chooses the four singular tangential planes:

$$p = 0, \quad q = 0, \quad p' = 0, \quad q' = 0$$

that are included in the form (4) to be the fundamental planes, so p, q, p', q' will be the four homogeneous coordinates, and one accordingly denotes the last two by r and s, then one will get the following form for the equation:

(10) $$\phi^2 = 16\, Kpqrs,$$
where
$$\phi = p^2 + q^2 + r^2 + s^2 + 2a\,(qr + ps) + 2b\,(rp + qs) + 2c\,(pq + rs)$$
$$K = a^2 + b^2 + c^2 - 2abc - 1,$$

in which the seven constants *a*, *b*, *c*, *d*, *e*, *f*, *k* in that form are restricted to the correct number of three constants *a*, *b*¸ *c*. If one chooses the linear expressions *p*, *q*, *r*, *s* in this form to be real and the three constants *a*, *b*, *c* to likewise be real and assumes that all three of them are greater than one (except for the sign) then one will get only surfaces in which the sixteen singular points are all real, and likewise the sixteen singular tangential planes, along with their sixteen contact conic sections, will also be real.

In order to give the positions of these 16 points, 16 planes, and 16 conic sections the clearest possible picture, I have represented them in custom-made wire models. In these models, the four fundamental planes *p*, *q*, *r*, *s* are chosen in such a way that they define the four faces of a regular tetrahedron, and in order to make the regularity of the models complete, the three constants *a*, *b*, *c* are chosen to all be equal to each other – namely, to 2. The four contact conic sections that belong to the fundamental planes are circles that lie in one and the same plane, while the remaining 12 contact conic sections that belong to singular tangential planes are hyperbolas. Of the 16 singular tangential planes, 12 of them lie at the endpoints of the six edges of the regular tetrahedron, which have been lengthened by equal line segments, and also the spherical surface that contains the four contact circles, and the four remaining singular points lies on the four altitudes of the tetrahedron that have be erected over its vertices. Six of the contact conic sections go through each of the 16 singular points, so six of the wires that represent them. The surface itself consists of 12 separate parts that are connected to each other only by means of the 16 singular points. Four of these parts, which are completely separated from each other, contact four other parts at three of these points, so the basis for each of them will closely approach one of the four faces of the tetrahedron, and as the distance from that basis increases, they will become ever thicker and each of them will extend to infinity. Four other parts of the surface are finite, and their forms come close to three-sided pyramids, each of which is linked with three of the previously-described parts by a singular point, and with one of the remaining four sub-faces at a singular point, in addition. Each of the latter appears spherical, *in toto*, but it is linked with the remaining parts at only a single point, and extends from that point to infinity. Each of the four contact circles will be divided into six parts by the six singular points that they contain, three of which will lie on three of the originally-described sub-surfaces without lying on their boundaries, but the remaining three will lie on three finite parts that have the form of pyramids. Of the 12 contact hyperbolas, one branch will contain four singular points, while the other one will contain two of them, and each of the four branches that contain singular points will lie on one of the originally-described four sub-surfaces, on two of the other kind, and on two of the third kind, with which, it goes to infinity. Each of the two branches of the hyperbola that contain singular points, however, will lie on one part of the second kind, and will then go to infinity on both sides that lie on two parts of the first kind.

The fourth-degree surfaces with 16 singular points, despite containing only 18 essential constants – and thus, 16 fewer than the most general fourth-degree surface – still occupy a level of generality such that all plane curves of degree four can be found in them, or conversely:

One can lay a fourth-degree surface with 16 singular points through any given plane curve of degree four.

In order to prove this, I shall take the given plane curve of degree four to have the form:

$$(11)\qquad \sqrt{p_0(Ap_0+A_1q_0+A_2r_0)}+\sqrt{q_0(Bp_0+B_1q_0+B_2r_0)}+\sqrt{r_0(Cp_0+C_1q_0+C_2r_0)} = 0,$$

where p_0, q_0, r_0 are arbitrary linear functions of the two coordinates x and y. *Hesse* showed that the most general fourth-degree curve can be represented in this form in his treatise on the double tangents to the curves of order four in Crelle's Journal, v. 49, where he developed the rational form that is equivalent to this on page 301 and presented it in equation (47). I shall further take the equation of the fourth-degree surface with 16 singular points:

$$p = p_0 + az,\qquad q = q_0 + bz,\qquad r = r_0 + cz,\qquad s = p_0 + mq_0 + nr_0 + dz$$

to have the form that was given by (9). The intersection of that surface with the plane $z = 0$ will then be identical with the given curve when the following eleven equations are true:

$$(12)\qquad \begin{array}{lll} A = \delta, & B = \delta' + \alpha', & C = \delta'' + \alpha'', \\ A_1 = m\delta + \beta, & B_1 = m\delta', & C_1 = m\delta'' + \beta'', \\ A_2 = n\delta + \gamma, & B_2 = n\delta', & C_2 = n\delta'', \end{array}$$

$$\alpha'\gamma + \alpha''\beta - \beta\gamma = 0,\quad \alpha''\gamma' + \beta''\gamma - \alpha''\beta'' = 0,$$

from which the eleven quantities $\beta, \gamma, \delta, \alpha', \gamma', \delta', \alpha'', \beta'', \delta''$, m, and n will be determined, while the four coefficients a, b, c, d will remain completely arbitrary. In order to carry out the solution of these equations in the simplest possible way, I will introduce an auxiliary quantity u, which I set equal to:

$$(13)\qquad u = -\frac{\gamma}{\alpha''},$$

so the last two of the eleven equation in (12) will then give:

$$\frac{u+1}{u} = \frac{\alpha'}{\beta},\quad u + 1 = \frac{\gamma'}{\beta''},$$

and when one substitutes the values of $\beta, \gamma, \alpha', \gamma', \alpha'', \beta''$ that arise immediately from first nine equations into this, one will obtain the three equations:

$$(14)\qquad \begin{aligned} &An^2 - (A_2 + Cu)\, n + C_2\, u = 0, \\ &A\,(u + 1)\, m^2 - (A_2\,(u + 1) - Bu)\, m - B_1\, u = 0, \\ &C_2\,(u + 1)\, m^2 - (C_1\,(u + 1) - B_2)\, mn - B_1\, n^2 = 0. \end{aligned}$$

If one eliminates the two quantities m and n from them then one will get a cubic equation for the determination of u that can be represented in the form of a determinant thus:

$$(15)\qquad \begin{vmatrix} 2uA & uB-(u+1)A & A_2+uC \\ uB-(u+1)A & -2(u+1)B_1 & B_2-(u+1)C_1 \\ A_2+uC & B_2-(u+1)C_1 & 2C_2 \end{vmatrix} = 0.$$

Two values of n belong to each of the three values of u that this equation gives, and furthermore, one value of m, and likewise one value of $\beta, \gamma, \delta, \alpha', \gamma', \delta', \alpha'', \beta'', \delta''$, belongs to each of these six values of n, but the constants a, b, c, d remain completely arbitrary. The theorem that was stated above can thus be made more precise in the following way:

One can lay six different four-fold infinite families of fourth-degree surfaces with 16 singular points through every given fourth-degree curve.

If a fourth-order surface with 16 double points goes through a fourth-degree curve in such a way that the curve lies in the surface then each singular tangential plane of the surface will cut a double tangent to the curve from the plane of the curve. The sixteen singular tangential planes of the surface will then yield sixteen of the 28 double tangents to the curve. If one lays any other one of the aforementioned six surfaces through that curve then the 16 singular tangential planes to that surface will likewise cut 16 double tangents from the plane of the curve that are partially the same and partially different. If one lays all six surfaces through the fourth-degree curve then all 28 double tangents to the curve will be cut from the 96 singular tangential plane to those surfaces, and in fact – as a closer examination that I would not like to go into here has shown me – six double tangents will be cut out six times, six double tangents, two times, and 16 double tangents, three times.

A very remarkable, seemingly deep, property of fourth-degree surfaces with 16 singular points is connected with this, which will be explained as follows: It is known that 12 straight lines go through each point of space that doubly contact a general fourth-degree surface, so if one now considers all of the straight lines that contact the surface twice then they will define a ray system that has the surface for its focal surface, and which I will refer to as a *ray system of order twelve*, since twelve rays of the system will go through each point of space. If the fourth-degree surface were cut by an arbitrary plane then the curves that would be thus cut out would have 28 double tangents that would collectively constitute all of the rays of the system that lie in that plane, and for that reason, I call it a *ray system of class 28*. Now, if the fourth-degree focal surface of the ray system one has 16 singular points (and consequently, 16 tangential planes, as well) then the class of that ray system will be reduced by 16 units when one removes all rays that lie in the 16 singular tangential planes and fill it up completely, since each straight line that is drawn arbitrarily in a singular tangential plane is a doubly-contacting line of the surface. The ray system will then become one of order twelve and class twelve. If one now considers twelve rays that lie in an arbitrary plane $t = 0$ that are twelve of the double tangents that are cut out of the focal surface by that plane then one can construct them in such a way that one lays through the fourth-degree plane curve, in addition to the one surface with 16 double points (which already goes through it), also the five other ones that are associated with it, whose singular tangential planes cut these rays out of the plane $t = 0$. Among these five other surfaces, one of them can be expressed *rationally* in terms of the coefficients of the given surface and the coefficients of the intersecting plane. Namely, a root u of the third-degree equation upon which the existence of the six surfaces depends is given at the same time as one is given the surface, and indeed, it can be expressed rationally in terms of the coefficients of the surface and the plane. Now, if one chooses a second surface to be the one that belongs to the same root of the cubic equation and the other value of the quadratic equation – which will be rational, in any case, here, since the first value is rational – then as the complete execution of all calculations will show, the 16 singular tangential planes to that second surface will cut out eight of the twelve rays from the plane. It will further show that these eight rays – just like the singular tangential planes that they cut out pair-wise (i.e., any two are given by a single equation) – will be expressed rationally in terms of the coefficients of the given surface and the intersecting plane. It follows from this that the ray system of class twelve decomposes into four special ray systems of class two and one of class four that are individually representable by equations. If one considers the polar ray system to this then one will immediately recognize that the four ray systems of class two must also be ones of order two, and that the ray system of class four must likewise be one of order four. One then has the following theorem:

The complete ray system of order 12 and class 28 that has a general fourth-degree surface for its focal surface will consist of:

1. *Sixteen ray systems, each of which consists of only all straight lines that lie in a plane,*
2. *Four ray systems of order two and class two,*
3. *A ray system of order four and class four,*

when that fourth-degree focal surface has 16 singular points.

I was first made aware of the importance of these fourth-degree surfaces by this special property of fourth-degree surfaces with 16 singular points, which I discovered in my investigations into algebraic rays systems (which I will publish later) and proved in an entirely different way. One can also consider this property from a completely different standpoint – e.g., when one lets the arbitrary plane in which lie the 12 rays of the four systems of class two and the system of class four be a tangential plane of the focal surface, so every two of these 12 rays will coalesce into one. The same thing will be true for the six tangents that contact the surface at one and the same point when each of them contacts it at yet another point. Thus:

For fourth-degree surfaces with 16 singular points, the sixth-degree equation by which one determines the six tangents to the general fourth-degree surface that have a common contact point and contact the surface another time, in addition, decomposes into four factors of degree one and one factor of degree two that can be expressed rationally in terms of the coordinates of the common contact point.

If one calls the two contact points of one and the same double tangent to the surface *associated points* of it then it will further follow that:

On any fourth-degree surface with 16 singular points, one can, in four different ways, associate each point of the surface with another one in such a way that the coefficients of the associated points can be expressed rationally in terms of the given ones, and likewise, the coordinates of the given points can be expressed rationally in terms of the associated ones.

These theorems become simpler when applied to the Fresnel wave surface, since for them, and all surfaces that are collinear to them, the ray system of order four and class four will further decompose into two ray systems of order two and class two.

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