“,” Monats. d. König. Preuss. Akad. Wiss. Berlin (1860), 469-474; *Collected Papers*, v. 2., ed. A. Weil, Springer, Berlin, 1975.

Three models of general, infinitely-thin, rectilinear ray sheaves that are composed of filaments

By E. E. Kummer

Translated by D. H. Delphenich

Hr. Kummer presented three models of general, infinitely-thin, rectilinear ray sheaves that are composed of filaments and coupled that to the following publication:

As I proved in a paper that appeared in volume 57 of *Borchardt's* mathematical Journal and was presented to this class on 17 October of the previous year, the general, infinitely-thin sheaves of rays are bounded by rectilinear surfaces whose generating lines will always go through two straight lines that are perpendicular to the axis of the ray sheaf, and will simultaneously go through an infinitely-small, closed curve that surrounds the axis. In the present models, this small, closed curve will be chosen to be a circle whose plane is perpendicular to the axis and whose center lies on the axis. The bounding surface of the ray sheaf will thus be a rectilinear surface of degree four whose sections that are perpendicular to the axis will always be ellipses, two of which will degenerate into straight lines in the first and second models of ray sheaves that will be represented. The two straight guiding lines that are perpendicular to the axis, and which correspond to the two rectilinear sections of the ray sheaf, and along with them, the two planes that are laid through the axis and either of the straight guiding lines – which I call the *focal planes* of the ray sheaf – define a right angle in the first model and an acute angle in the second one, but in the third one they will be imaginary and define an imaginary angle, but in such a way that the ray sheaf and its bounding surface remain real. The three kinds of ray sheaf that are represented by these models and their bounding surfaces – namely, the conical and cylindrical surfaces, as I proved in the cited paper – are the only ones that are mathematically possible. Since then, I have also examined the question of whether, and under what circumstances, these sheaves can and must actually occur in Nature as optical ray sheaves, and I have found a very general and simple theorem in regard to that, which gives the complete answer to the question, and indeed, not only for the simply-refracting media whose wave surface is a sphere, the uniaxial crystals whose wave surface is the sphere and an ellipsoid of rotation, and the biaxial crystals, which belong to Fresnel wave surfaces, but in fact for all possible transparent media or crystals that might belong to any other wave surface of light. The theorem is the following one:

Theorem:

Any infinitely-thin, optical ray sheaf in the interior of a homogeneous medium has the property that its two focal planes cut two curves that intersect in conjugate directions out of the wave surface of light that belongs to this medium, and the midpoints of these *curves will be chosen to lie on the axis of the ray sheaf. Moreover, every ray sheaf that has that property will actually be optically-representable.*

Among the conjugate direction on the wave surface, one will find the directions of two conjugate diameters of the infinitely-small *Dupin* conic section that belongs to the point of the wave surface in question – namely, the *indicatrix* – which is a conic section that is either an ellipse or a hyperbola according to whether the surface is convex-convex or convex-concave at that location, respectively.

For each well-defined direction in the crystal, and for each point of intersection of that direction with the radius vector of the wave surface that is parallel to it, one can choose the position of the one focal plane arbitrarily, and the position of the other focal plane will then be determined completely, from the theorem that was given. There will always be a certain position of the first focal plane for which the second focal plane defines a right angle with it, such that the ray sheaf of the first kind – whose focal planes are perpendicular to each other – will exist in any crystal for any arbitrary direction of its axis, but only for a certain position of itsfocal planes, in general.

Now, first of all, when the wave surface is convex-convex at the point at which radius vector meets it – so the indicatrix will be an ellipse – and one rotates the first focal plane, starting with the position in which it is perpendicular to the second one, around the radius vector as axis then the angle between the two focal planes will become smaller and it will attain a well-defined minimum for which the two focal planes lie in such a way that the angle between the mutually-perpendicular focal planes will be bisected. If one lets α denote the angle through which the first focal plane is rotated from given, initial position and one denotes the angle between the two associated focal planes by γ then the smallest value of γ will occur when $\gamma = 2\alpha$.

Secondly, if the wave surface is convex-concave at the endpoint of the radius vector in question – so the indicatrix will be a hyperbola – and one rotates the first focal plane from the position in which the second one is perpendicular to it then the angle γ between the two focal planes will become smaller and attain the value zero at a well-defined position, and if one goes to that position then the angle γ will again increase to 90° and then decrease to zero a second time. The two positions of the focal planes for which $\gamma = 0$ correspond to the directions of the infinitely-large radius of curvature of the wave surface, or – what amounts to the same thing – the directions of the asymptotes of the hyperbolic indicatrix. Since the hyperbolas possess imaginary, conjugate diameters, in addition to their real, conjugate diameters, it will then follow that for the directions in which the radius vector enters a convex-concave part of the wave surface, the infinitely-thin ray sheaf of the third kind – which has imaginary focal planes – will also actually come about.

If the transparent medium is a simply-refracting one, so its wave surface is the outer surface of a ball, then all indicatrices will be circles, and it will follow that all conjugate directions will only be mutually-perpendicular, and that since the radius vectors are everywhere perpendicular to the wave surface here, it will then follow that focal planes of the ray sheaves will also be everywhere mutually-perpendicular. In a simply-refracting medium, no other optical ray sheaves can exist besides ones of the first kind whose focal planes are mutually-perpendicular.

If the transparent medium is an optically-uniaxial crystal whose irregular rays have an ellipsoid of rotation for their wave surface then the indicatrices will only be ellipses. The direction in which the first focal plane must lie in order for the second one to be perpendicular to it here is always the one in which the optical axis lies. If the semi-axis of rotation of the wave ellipsoid is equal to c, the semi-axis of that ellipsoid that is perpendicular to the latter axis is a, and ω is the angle that the axis of the ray sheaf makes with the optical axis, moreover, and:

$$\rho = \frac{ac}{\sqrt{a^2 \cos^2 \omega + c^2 \sin^2 \omega}}$$

is the radius vector that corresponds to that direction then the smallest angle γ between the two focal planes of the ray sheaf that lies in that direction will be given by the formula:

$$\tan \frac{\gamma}{2} = \frac{c}{\rho} \qquad \text{or} \qquad \tan \frac{\gamma}{2} = \frac{\rho}{c},$$

according to whether $c < a$ or $c > a$ – i.e., according to whether the uniaxial crystal is a negative or positive one, respectively. For $\omega = 90^\circ$ – i.e., for the position that is perpendicular to the optical axis – one will obtain the ray sheaf with the smallest angle between the focal planes that can even exist in such a crystal, namely:

$$\tan \frac{\gamma}{2} = \frac{c}{a} \qquad \text{or} \qquad \tan \frac{\gamma}{2} = \frac{a}{c},$$

according to whether $c < a$ or $c > a$.

For Iceland spar, for which:

$$\frac{1}{a} = 1.483, \qquad \frac{1}{c} = 1.654,$$

one will then get:

$$\gamma = 83^\circ\, 45'\, 50''.$$

Inside Iceland spar, one will then find no other ray sheaves, as such, for which the angle of the two focal planes lies between 90° and $83^\circ\, 45'\, 50''$. For the ray sheaves that are perpendicular to two parallel, natural surfaces of the rhombohedral Iceland spar and define an angle of $44^\circ\, 36'\, 30''$ with the optical axis, one will find that the smallest angle between the focal planes is $\gamma = 87^\circ\, 5'$.

In the optically-biaxial crystals, which belong to the Fresnel wave surface, one finds not only the ray sheaves of the first and second kind (and indeed for all angles between the two focal planes from a right angle to zero), but also the ray sheaves of the third kind, which have imaginary focal planes. In fact, the Fresnel wave surface has four locations on its outer shell at which it is convex-concave, which are locations that will be bounded by the known four circles at which the contact of the singular tangential planes with the surface takes place. The ray sheaves of the third kind and the ones for which the angle between the focal planes drops to zero exist only in the directions in the crystal whose corresponding radius vectors meet the wave surface inside these circles; for each of the direction that are included between these limits, there is a certain minimum of the angle γ between the two focal planes that will become ever larger as the radius vector once more grows further away from the aforementioned four circles. The value of the angle γ, as a function of the direction of the axis of the ray sheaf and the direction of the first focal plane, as well as the value of the minimum of γ for every given direction of the axis of the ray sheaf, can be give without any difficulty, although the expressions will become complicated, so I would not like to go into them here.

If one lets a ray sheaf of the first, second, or third kind that exists inside of a crystal leave that crystal into a simply-refracting medium – e.g., into air – then it will always be converted into a ray sheaf of the first kind with mutually-perpendicular focal planes, and for that reason, one can, conversely, optically generate any ray sheaf that is possible in a crystal in such a way that one lets a suitable ray sheaf of the first kind fall upon the crystal.

One can generate a ray sheaf of the first kind with arbitrarily-given distances between the two mutually-perpendicular rectilinear sections most simply by a convex, spherical lens, through which one lets the light that emanates from a point go, and in addition, the light must go through a narrow opening in order for the ray sheaf to be sufficiently thin. If one directs the lens in such a way that its axis lies in the direction of the incident light itself then one will get only the conical ray sheaf, for which the two rectilinear sections coalesce into a single point – viz., the *focal point*; however, if one rotates the lens in such a way that its axis defines an acute angle with the direction of the incident light then the two rectilinear sections will diverge from each other, and their separation will grow larger as that angle grows smaller; likewise, the two rectilinear sections will also increase in length proportionately. A piece of white paper that is held perpendicular to the axis of the ray sheaf at various distances will make the various sections visible, among which, the two that are rectilinear and mutually-perpendicular will also emerge quite clearly.