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The screwing motion, the null system, and the linear complex

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The present exposition can be recommended especially for the purpose of lecturing at technical institutes. I have published its foundations in a treatise on the variation of the positions of rigid systems in volume 6 of *Schlömilch's* Zeitschrift für Mathematik und Physik.

1. A screwing motion is given by its axis C, a translation $d\sigma$ in the direction of C, and a rotation $d\tau$ around C. Let the perpendicular $ac = x$ be dropped from an arbitrary point a in space to C. a will describe a helix in a certain sense whose tangent A at a will define an angle φ with C such that:

$$x = \frac{d\sigma}{d\tau} \cdot \tan \varphi .$$

The *non-zero constant* $d\sigma / d\tau = k$ is called the *parameter of the motion*. A is perpendicular to ca, and is called the *translation ray of the point a*.

If one considers the translation rays of all points of ca then one will see that they define a ruled family of equilateral paraboloids T_0 that has C, ac for its vertex lines and k for its distribution parameter. T_0 is the *translation surface of the lines ac.*

If a, with a translation ray A, were subjected to a finite screwing motion then the displaced ray A would remain the translation ray for the displaced point a.

We now draw a plane E through a that is perpendicular to A and call it the *null plane of a*, which we give the notation of $\mathfrak{e}$. In this way, any point $\mathfrak{e}$ of space will be assigned a plane E that goes through it that contains the perpendicular $\mathfrak{e}c$ from $\mathfrak{e}$ to C that is possible. Conversely, an arbitrary plane E that cuts C – say, at c – will be the null plane of any of its points $\mathfrak{e}$. There is a line ca in E that goes through c and is perpendicular to C. From what was just said, $\mathfrak{e}$ must be on it. If one imagines the translation surface T_0 that belongs to ca then one and only one of its lines that are perpendicular to ca will also be perpendicular to E, and it will then meet E at the point $\mathfrak{e}$, for which E will be the null plane, and $\mathfrak{e}$ will be its *null point*. If E and $\mathfrak{e}$ are at the basis of the screwing motion then the displaced point $\mathfrak{e}$ will always remain the null point of the displaced plane E. *The null system consists of the points of space and their null planes.*

The null system is obviously well-defined when the screw axis C is known, along with the translation of any point. Now, if only the translation ray of a were given then the helix that a describes would indeed be known, but the sense in which would be traversed by a would still be double-valued. Which of them one would like to choose is irrelevant for the determination of the null system.

2. Conjugate lines G, $\mathfrak{G}$ in the null system. The planes that are perpendicular to C will be cut by C at its null points. We choose any of them to be the horizontal projection plane and pose the problem: *Find the null points $\mathfrak{e}$ for all planes E that go through a line G that is skew to C!* Let $e_0\,c_0$ be the shortest transversal of G, C and let the horizontal plane be drawn through it, while the vertical plane is perpendicular to $e_0\,c_0$. Let G' be the horizontal projection of G, and let G'' be the vertical one. E_0 will be the plane that has $e_0\,c_0$ for its horizontal trace and is inclined with respect to the horizontal plane by the angle w_0. Its null point $\mathfrak{e}_0$ must lie on $e_0\,c_0$, and can be chosen arbitrarily on it, so the null system will, in turn, be determined in such a way that the translation of $\mathfrak{e}_0$ is perpendicular to E_0. E is drawn through G, its trace is E', and w is the angle that it defines with the horizontal plane. Its null point $\mathfrak{e}$ is found on a line that cuts C perpendicularly, and whose horizontal projection will then go through e_0 and must be parallel to E'. The point e where this line meets G immediately provides its vertical projection $e''\,c''$. One must now deal with the position of the horizontal projection $\mathfrak{e}'$ of the null point $\mathfrak{e}$ in question. One now shows that $\mathfrak{e}$ is on the line $\mathfrak{G}'$ that is drawn through $\mathfrak{e}_0$ parallel to G'. If $\mathfrak{x}$ denotes the unknown length, and c, $\mathfrak{e}'$, $\mathfrak{x}$ are the known c_0 e_0 then one must have:

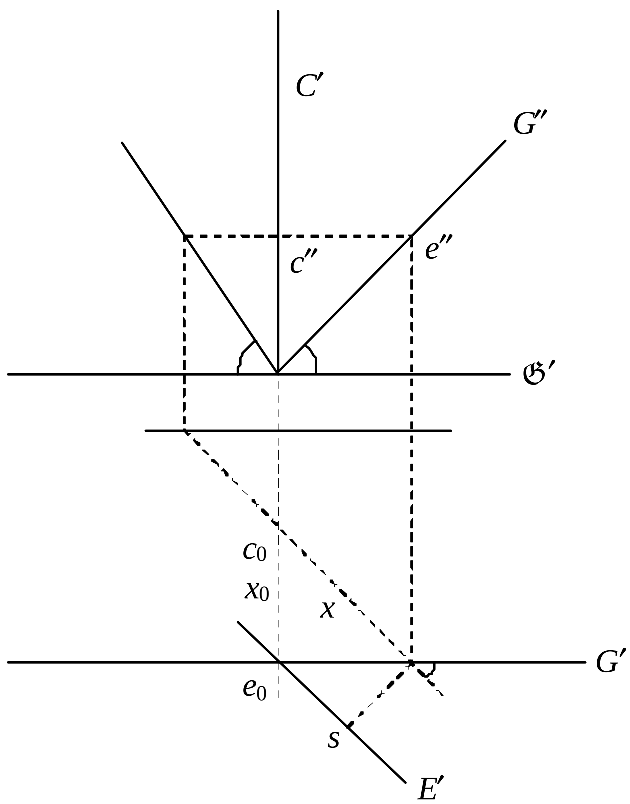

Figure 1.

$$\frac{\mathfrak{x}}{\mathfrak{x}_0} = \frac{\tan w}{\tan w_0}.$$

Obviously, if $\mathfrak{e}'$ is thought of as being on G' then $c_0\,\mathfrak{e}'$ will fulfill this condition, as one will recognize when one ascertains w by means of the perpendicular $e\,s$ that is dropped from e to E'.

The fact that the locus of $\mathfrak{e}$ is now a line $\mathfrak{G}$ will easily come to light when one observes that $e\ c$ describes an equilateral paraboloid when e is assumed to vary on G: This paraboloid has a line $\mathfrak{G}$ that belongs to the same family as C, G in common with the plane that is erected vertically to $\mathfrak{G}'$, and that will be the desired locus. If one calls the angle of inclination of $\mathfrak{G}$ over the horizontal plane $\mathfrak{w}_0$ then one will see that:

$$\frac{\tan \mathfrak{w}_0}{\tan w_0} = \frac{c''e''}{c''\mathfrak{e}''} = \frac{x_0}{\mathfrak{x}_0}.$$

It follows from this that the plane $\mathfrak{E}_0$ that that goes through $\mathfrak{G}$ and $c_0\ e_0$ has its null point at e_0 . If one then bases the argument that was just carried out upon the $\mathfrak{G}$ that was just found, instead of G then that must imply that G is the locus of the null points e of the planes $\mathfrak{E}$ that possibly go through $\mathfrak{G}$, which is where we started. G, $\mathfrak{G}$ are called *conjugate lines because any axis is a pencil of planes that have their null points in the others.*

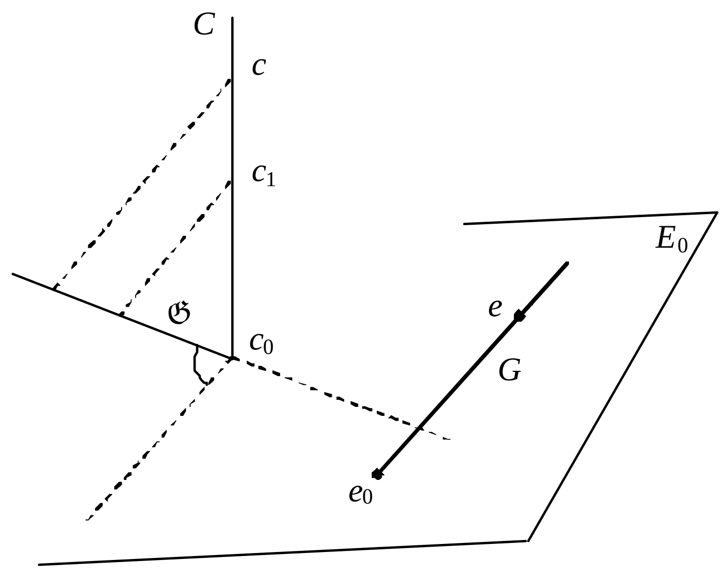

Figure 2.

Here, however, one must especially take note when $\angle\ w_0 = 0$, because the paraboloid that facilitates the proof would no longer exist then. As a horizontal plane, E_0 would also have its null point at c_0 now, such that the null system would first be determined when one assigned a null point $\mathfrak{e}_1$ to a second plane E_1 that goes through G.

If E_1 meets the axis C at c_1 and if $c_1\ \mathfrak{e}_1$ is parallel to G then the null point of E_1 must lie anywhere on the line $c_1\ \mathfrak{e}_1$, so it will also be arbitrary as long as the null system is not already assumed to be given. If E denotes a third plane through G, $\mathfrak{e}$, its null point, and w its inclination angle above the horizontal then one will have:

$$\frac{\tan w}{\tan w_1} = \frac{c\,\mathfrak{e}}{c_1\,\mathfrak{e}_1}.$$

If will then follow that $\frac{c\,\mathfrak{e}}{c_1\,\mathfrak{e}_1} = \frac{c_0\,c}{c_0\,c_1}$; i.e., the locus of $\mathfrak{e}$ is a line $\mathfrak{G}$ that goes through c_0 and is perpendicular to $c_0\,e_0$. One will have $\tan\mathfrak{w}_0 = \frac{c_0\,c}{c\,\mathfrak{e}}$ for them, and since $\tan w = \frac{c_0\,c}{e_0\,c_0}$, one will have $\frac{\tan\mathfrak{w}}{\tan w} = \frac{c_0\,e_1}{c\,\mathfrak{e}}$.

One sees from this that e_0 is the null point of the plane $\mathfrak{E}_0$ that contains $\mathfrak{G}$, c_0, e_0. Finally, if $\mathfrak{E}$ is drawn through $\mathfrak{G}$ arbitrarily, and $\mathfrak{w}$ is its angle with the horizontal plane then one will have $\frac{\tan\mathfrak{w}}{\tan\mathfrak{w}_0} = \frac{c_0\,e}{c_0\,e_0}$, as above, if e means the point of intersection of G and $\mathfrak{G}$. However, e_0 is the null point of $\mathfrak{E}_0$, so e is that of $\mathfrak{E}$.

3. The foregoing discussion immediately yields these consequences:

a) Two conjugate G, $\mathfrak{G}$ are skew.

b) If the angle between G, C is not $\pi / 2$ then the same thing will be true for $\angle\,\mathfrak{G}, C$, and G, $\mathfrak{G}$, C will determine an equilateral paraboloid that will have C for its vertex line and $e_0\,\mathfrak{e}_0$ for the shortest transversal of G, $\mathfrak{G}$.

c) By contrast, if $\angle\,G, C = \pi / 2$ then $\mathfrak{G}$, C will intersect, and conversely, if G meets the axis C then one must have $\angle\,\mathfrak{G}, C = \pi / 2$.

d) If $\angle\,G, \mathfrak{G} = \pi / 2$ then G, $\mathfrak{G}$ must be the translation rays of the points e_0, $\mathfrak{e}_0$, since the null plane of e_0 that goes through $\mathfrak{G}$ will be perpendicular to G. Conversely, if G is the translation ray of e_0 then $\mathfrak{G}$ must be rectangular to G, since $\mathfrak{G}$ will lie in the null plane of e_0. *The translation rays will then be paired as rectangular conjugates lines.*

e) If G is parallel to C then the translation rays will be parallel to the points of G, so they will also be their null planes; i.e., $\mathfrak{G}$ will be at infinity. This also sheds light upon the fact that the null points to parallel planes are found on lines that are parallel to C. Thus, if two points have parallel translations then their connecting lines C must be parallel.

f) Let E be an arbitrary plane that intersects C, such that its null point $\mathfrak{e}$ lies at a finite point. Any line G that is thought of as being in E, but not going through $\mathfrak{e}$ will have a conjugate G that goes through $\mathfrak{e}$, but does not lie in E. *It will then follow that the null planes of all points of E will go through* $\mathfrak{e}$, *and all possible planes that go through* $\mathfrak{e}$ *will have their null points in E.*

Now, when G is drawn through $\mathfrak{e}$, the null planes of its points must all contain G, or else G would have to coincide with its conjugates. The pencil of rays ($\mathfrak{e}$) that is present in E consists of such self-conjugate lines – viz., the so-called *complex rays* – and any point in space will be the center of such a pencil of rays.

g) If E is parallel to C then one will draw a G that is rectangular to C in E and consider its conjugate $\mathfrak{G}$. Any point e on G will have a null plane $\mathfrak{E}$ that goes through $\mathfrak{G}$. If one then draws a parallel to C through e that lies in E then any point of it must possess a null plane that is parallel to $\mathfrak{G}$. Thus, an arbitrary line of the plane E will have a conjugate that is parallel to $\mathfrak{G}$, which will then also run parallel to E. More briefly:

The null point of E is to be thought of as being at infinity in the direction that goes through $\mathfrak{G}$*, and the complex rays that are found in E will all have that direction.*

4. The translation surface for the lines G in space.

a) Let G be parallel to C. The parallel translations of the points of G will trace out a plane.

b) Let G be the translation ray of the point e_0. Its conjugate $\mathfrak{G}$ will, in turn, likewise be the translation ray for the point $\mathfrak{e}_0$, and the plane $\mathfrak{e}_0$ G will be the translation surface of G. Since the translation of a point e that is chosen on G will be perpendicular to the plane that joins $\mathfrak{G}$ with e, *the translation rays that are present in the plane* $\mathfrak{e}_0 G$ *will envelop a parabola* whose focal point will be $\mathfrak{e}_0$. Should any translation whatsoever be in a plane E whose null point is $\mathfrak{e}$ – say, the point e – then the null plane of e must obviously include the translation ray of $\mathfrak{e}$.

c) Let G be either parallel to C, a translation, or a complex ray, and let $\mathfrak{G}$ be its conjugate.

The angle G, $\mathfrak{G}$ is either 0 or $\pi / 2$. From a known theorem, since they will be perpendicular to the planes $\mathfrak{G}e$, the translation rays of the points e that are found on G will be *the lines of a common (not equilateral) paraboloid* whose vertex lines are G, and the translation ray will be its axis C at the next-lying point e_0 . For that reason, these two lines cannot define a right angle, since otherwise G, $\mathfrak{G}$ would have to be parallel.

d) Let G be a complex ray. Its translation surface is always an *equilateral* paraboloid. If one then drops the perpendicular $e\ c$ from any point e to the C then $G\ c$ will be the null plane of e. However, the line $e\ c$ will describe a paraboloid as e varies that will contact the plane $G\ c$ at e. If one constructs the normal paraboloid that belongs to G for it then one will obtain the translation surface in question.

Thus, the complex ray can be defined as a line that occurs as the translation surface of an equilateral paraboloid or also as one from which a point has a translation that is perpendicular to it; *any other point on it must then behave similarly* (*).

It is important for the kinetic theory of curvature to stress the difference that exists between a screwing motion and a rotation: For the former, a complex ray will always describe a *skew* element. For the latter, by contrast, it will describe a *planar* one, and the

(*) The property of the change in position of a rigid line that is emphasized here can make the connection between the lines of curvature of parallel surfaces immediately recognizable.

point of intersection of infinitely-close complex rays will yield a point of the rotational axis.

5. The quadratic complex of translation rays.

Let A_1 be the translation ray of the point a_1. What is the locus of the possible translations A through a_1, and where does the point a to which they belong lie? *Chasles* first answered this question, and thus took a step that led to the tetrahedral complex by a closely-related generalization. One will arrive at the ray A in a very simple way when one looks for it in the planes E that are drawn through A_1. One has a certain parabola in E to which A_1 is tangent, and which admits a tangent that goes through a_1; it will then be A. One now exhibits the shortest transversal ac of A, C, which meets the conjugate to A at right angles, and must therefore be cut by the translations of all points of the line A, and thus those of A_1, as well. ac is therefore easy to find: It lies in the plane E, and is perpendicular to C. Once one has drawn ac, one drops a perpendicular to it from a_1 whose foot will be a. Now, in order to get the locus of A, one needs only to take a parallel C_1 to C and the plane-pair C_1A, A_1A : Since ca, as well as C, is rectangular to A, and cuts A_1, moreover, the two aforementioned planes will define a right angle, and their line of intersection A will generate a second-degree cone with edges A_1, C_1.

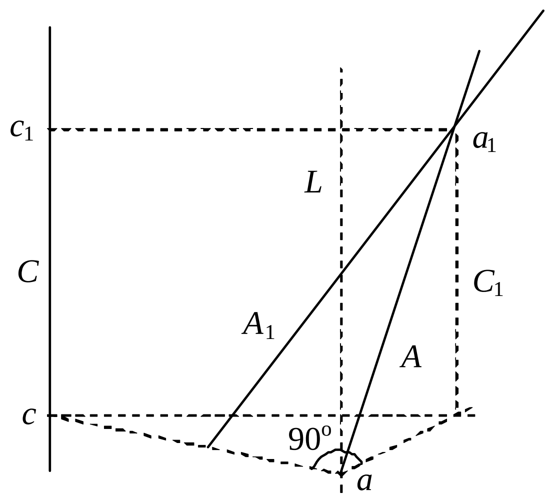

Figure 3.

Finally, if one thinks of the line $\mathfrak{C} \parallel C$ as going through a then the planes $C\mathfrak{C}$, $C_1\mathfrak{C}$ will intersect at right angles along it, and it will follow that $\mathfrak{C}$ lives on a cylinder of rotation with the edges C, C_1. The cylinder and the cone will intersect each other in a space curve of order three – viz., the locus of a. One also immediately confirms that this space curve is found on the translation paraboloid that belongs to the complex ray $a_1 c_1$.

6. The fundamental ways of determining the null system (or linear complex).

The first manner of determination that we shall use us the one for which the axis C is given, along with a point a and its translation ray A.

Second. Two conjugate lines G, $\mathfrak{G}$ are given, and the translation ray A of any point a_1 in their shortest transversal $e_0\,\mathfrak{e}_0$. In fact the translation rays of the three points e_0, $\mathfrak{e}_0$, a_1 will thus be known. They will determine an equilateral paraboloid whose one vertex line lies along e_0, $\mathfrak{e}_0$, while the second one must, in turn, be the axis C. Conversely, if this is assumed then the complex will be established by a_1, A_1 alone.

Third. Two conjugates G, $\mathfrak{G}$, and the translation ray A of any point a in space can be taken to be the determining data, assuming that A is perpendicular to the transversal t that is possible from a to G, $\mathfrak{G}$.

The fact that the extra condition must be fulfilled will follow from the fact that the transversal $\mathfrak{G}$ that we spoke of will meet the plane Ga at the null point, and will thus be a complex ray. Moreover, the assumption of a, A comes down to the same thing as when one assigns an arbitrary plane to a null point that is found anywhere – here, at a – on the transversal t to G, $\mathfrak{G}$ that is contained in E. If E then cuts the shortest transversal of G, $\mathfrak{G}$ at a_1 then a_1 will be the null point of the plane E_1 that joins that shortest transversal with a. For that reason, if one determines (from 2) the complex in such a way that the translation of a_1 is perpendicular to E_1 then E will include the null point a, since a will appear as the point of intersection of two complex rays a_1a, t that are found in E.

Fourth. One is given G, $\mathfrak{G}$ as conjugates and a complex ray l that does not cut G, $\mathfrak{G}$: An arbitrary plane E through l will contain a transversal t over G, $\mathfrak{G}$. The point of intersection tl would be the null point of E in the possible complex, and (from 3), one can assume this.

Fifth. Three skew complex rays a, b, c are given, along with a plane E and its null point e, if E does not contact the hyperboloid abc. There is a transversal to a, b that lies in E, which will be called G, and a second $\mathfrak{G}$ will contain the point $\mathfrak{e}$. If one observes that any complex ray that meets an arbitrary line must also meet its conjugate then one will see that G, $\mathfrak{G}$ will be conjugate in the possible complex; from 4, it will be determined by the ray c. Since any transversal of G, $\mathfrak{G}$ is now a complex ray, the entire ruled family abc will belong to the complex rays, while the guiding family will consist of pair-wise conjugate lines.

We further infer that if four skew a, b, c, d admit two and only two transversals then they will be conjugate in all complexes that contain those four as rays. Any fifth ray $\mathfrak{e}$ that does not cut G, $\mathfrak{G}$ will suffice to determine that complex.

Now, if five skew lines a, b, c, d, e are present, and no line exists that cuts four of them then there will also be, in general, a single complex that has the five of them as rays: Let the hyperboloids abc, cde be denoted by H_1, H_2, respectively. A plane that goes through c will contain a transversal G to a, b, c and a transversal Γ to c, d, e. G, Γ intersect at a point 1 that lies in H_1, as well as H_2 . A line δ of the family cde will go through 1. Either δ will also belong to the family abc or it will not. In the first case, one can take G to be conjugate to any other transversal of $abc\delta$, and d to be determined by a complex that obviously also contains e as a ray, since it will contain three lines $c\delta d$ of the family cde. In the second case, δ will cut the hyperboloid H_1 at perhaps the point 2, in addition to 1. Let the transversal that goes through 2 over abc be $\mathfrak{G}$. The ray δ must be present in the complex that corresponds to our requirement, so G, $\mathfrak{G}$ must be conjugate in it. If one assumes this then it will be determined in such a way that it will possess the ray d, as well. However, both families *abc*, *cde* will, in turn, be present in it.

It is not by any means trivial to also draw one's attention to the fact that δ contacts the surface H_1 at the point 1. We will be led to consider the complex ray from a new viewpoint by this in itself.

7. The complex ray as the union of two infinitely-close (i.e., neighboring) conjugates.

Let G be an arbitrary line and let $\mathfrak{G}$ be its conjugate, where the shortest transversal of the two has a finite length. If a, b, c, d then mean any four complex rays that are skew to 2 and meet G, $\mathfrak{G}$, but do not lie hyperboloidally then the hyperboloid *abc* will be cut by d on G and $\mathfrak{G}$. By contrast, if G is a complex ray then d must obviously be tangent to that hyperboloid at the point of intersection d, G – i.e., $\mathfrak{G}$ will now be the line of the hyperboloid that is skew to G and lies infinitely close to it. One must infer from this what one means by the phrase "infinitely-neighboring skew lines G, $\mathfrak{G}$": If one thinks of any hyperboloid through G then the line on it that belongs to the same family as G will be its neighbor. Any point e of G is the center of a pencil of rays in whose plane $\mathfrak{G}$ is thought to be; that plane is, in fact, the tangential plane to H at e. Thus, if a hyperboloid osculates H along G then it will likewise contain the neighboring $\mathfrak{G}$. Since it suffices for this that the hyperboloid possess the same tangential planes at three different points of G, it will easily follow that ∞^3 hyperboloids through G and a well-defined neighbor $\mathfrak{G}$ are possible. If 1, 2, 3 are three points in space then they will determine three planes $G1$, $G2$, $G3$ that contact H at e_1, e_2, e_3 on G. The lines $e_1 1$, $e_2 2$, $e_3 2$ will then determine a new hyperboloid H_1 that will, as a result of the definition, contain the same neighboring line to G to as H does.

Now, the question arises of whether the determination of a complex by two conjugates and a plane with its null point will still be valid when the conjugates G, $\mathfrak{G}$ are neighboring.

We assume that G is a complex ray and the infinitely-close conjugate $\mathfrak{G}$ is known; in other words, one is given the null points of three planes that go through G or also three complex rays a, b, c that go through different points on G. From what was just said, the null points of all points of G will be the associated tangential planes of the hyperboloid *abc*.

Now, if the arbitrary plane E cuts G at the point e, and $\mathfrak{G}$ is the null plane of e then the null point of E must obviously lie on the line of intersection $E\mathfrak{E}$. If it were chosen arbitrarily here and joined with the point Ga through the line G_1 then the line of intersection $\mathfrak{G}_1$ of the planes E, Ga would have to belong to G_1 as its conjugate. If one assumes this and assigns the plane Gb to the null point Gb then a complex will be determined (from 3) that is the only one that will satisfy the requirements that were posed. It is clear that one can also choose a complex ray that is skew to G in order to establish it.

A complex is thus given by four skew rays a, b, c, d, and a fifth one G that is cut by three of them.

If G encounters only two of them a, b then the complex will likewise be determined. In order to see this, one takes the hyperboloid Gcd, so the plane Ga will have the line G_1 in common with it, and its conjugate that goes through the point Ga must be a line $\mathfrak{G}_1$ that is different from G. If this were assumed then one could make the plane Gb correspond to the point of intersection G, b as a null point, with which, the complex in question would then be found.

Finally, if a alone were cut by G then, from 5, the complex would be established by the rays b, c, d, and the plane $G\,a$ with its null point Ga. The case that was pointed out in 6, in which the auxiliary line δ is tangent to the hyperboloid H_1 at the point 1, will also find its resolution by our process.

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